Probab. Theory Relat. Fields 103, 381–407 (1995)

Probability Theory and Related Fields

Time dependent critical fluctuations of a one dimensional local mean field model

J. Fritz[1], **B. Rüdiger**[2]

[1] Department of Probability and Statistics, Eötvös Lóránd University of Sciences, H-1088 Budapest, Múzeum Krt. 6–8, Hungary
(e-mail: jofri@ludens.elte.hu)
[2] Dipartimento di Matematica, Universitá di Roma II. Tor Vergata, Via della Ricerca Scientifica, I-00133 Roma, Italy
(e-mail: rudiger@mat.utovrm.it)

Received: 2 December 1994 / Revised form: 20 April 1995

Summary. One-dimensional stochastic Ising systems with a local mean field interaction (Kac potential) are investigated. It is shown that near the critical temperature of the equilibrium (Gibbs) distribution the time dependent process admits a scaling limit given by a nonlinear stochastic PDE. The initial conditions of this approximation theorem are then verified for equilibrium states when the temperature goes to its critical value in a suitable way. Earlier results of Bertini-Presutti-Rüdiger-Saada are improved, the proof is based on an energy inequality obtained by coupling the Glauber dynamics to its voter type, linear approximation.

Mathematics Subject Classification (1991): 60K35, 82A05, 82B40

1 Introduction

The main purpose of this paper is to investigate the scaling limit of a stochastic Ising ferromagnet with a Kac type, local mean field interaction. This is an extremely simplified model of the following physical phenomenon, see [HH] for a more authentic explanation. In a vicinity of the critical temperature the equilibrium fluctuations blow up, while the time dependent fluctuations slow down and they are described by a nonlinear stochastic PDE, a Ginzburg-Landau equation. From a mathematical point of view we study a family of Markov processes in an infinite product space $(\Sigma, \mathscr{B}_\Sigma)$, where $\Sigma := \{-1, +1\}^{\mathbb{Z}}$ is the set of spin configurations $\sigma = \sigma(k)$ on the one-dimensional integer lattice $\mathbb{Z}$, and $\mathscr{B}_\Sigma$ denotes the associated Borel field. The so called local mean field, $h_\gamma = h_\gamma(k,\sigma)$ is defined for $k \in \mathbb{Z}$, $\sigma \in \Sigma$ and $\gamma > 0$ as

$$h_\gamma(k,\sigma) := \sum_{j\neq k} J_\gamma(k-j)\,\sigma(j)\,, \tag{1.1}$$

where J_γ is a symmetric probability distribution on $\mathbb{Z}$ such that $J_\gamma(k) = 0$ if $k = 0$ or $\gamma|k| > 1$. Let $*$ denote the convolution in the space $\ell(\mathbb{Z})$ of real functions $\varphi : \mathbb{Z} \mapsto \mathbb{R}$, then $h_\gamma = J_\gamma * \sigma$. Technical conditions expressing that J_γ is an asymptotically uniform distribution are to be listed in the next section, typical examples of J_γ are such that $J_\gamma(k) = z_\gamma J(\gamma k)$, where J is a nonnegative symmetric function on the interval $[-1,1]$, and $z_\gamma \approx \gamma$ is the normalization.

For $\gamma, \beta > 0$ let $\lambda_\gamma^\beta \sim e^{-\beta H_\gamma}$ denote the Gibbs state at inverse temperature $\beta > 0$ with energy H_γ,

$$H_\gamma(\sigma) := -\frac{1}{2}\sum_{k\in\mathbb{Z}} \sigma(k) h_\gamma(k,\sigma)\,. \tag{1.2}$$

In one space dimension λ_γ^β is a unique Borel probability on Σ specified by the DLR equations

$$\lambda_\gamma^\beta[\sigma(k)|\sigma(j) : j \neq k] = \tanh(\beta h_\gamma(k,\sigma)) \quad \text{for all } k \in \mathbb{Z}\,, \tag{1.3}$$

where $\lambda[\cdot|\cdot]$ denotes the conditional expectation. It is well known that this model exhibits a phase transition at $\beta = 1$ as $\gamma \to 0$, see [KUH],[LP],[COP]. In fact, λ_γ^β converges weakly in Σ to the uniform Bernoulli measure as $\gamma \to 0$ if $0 < \beta \leq 1$, while its limit distribution is a superposition of two product measures of mean spin $\pm\kappa_\beta$ if $\beta > 1$, where $\kappa_\beta = \tanh \beta\kappa_\beta$. A much richer picture is seen from a perspective of continuum limit when the spacing of the lattice goes to zero and we are interested in the limit distribution of the properly scaled local mean field process. This means that we have a limiting process $\xi = \xi(x)$ with continuous parameter $x \in \mathbb{R}$ such that $h_\gamma(k,\sigma) \approx \delta_\gamma \xi(\varepsilon_\gamma k)$ in a weak sense as $\gamma, \delta_\gamma, \varepsilon_\gamma \to 0$ in a proper way. At this level there is a transition from white noise ($\beta \ll 1$) to a non-Gaussian limit process ($\beta \approx 1$), the so called $P(\phi^4)_1$ Euclidean field, see [Sim]. At an intermediate scale the Ornstein-Uhlenbeck process appears as the continuum limit of the local mean field. As it was pointed out by [BPRS], this critical behavior is reflected also in a time dependent situation.

A stochastic spin flip dynamics with jump rates $c = c(k,\sigma)$ is generated by an operator L acting on local functions $f : \Sigma \mapsto \mathbb{R}$ as

$$(Lf)(\sigma) \equiv Lf(\sigma) := \sum_{k\in\mathbb{Z}} c(k,\sigma)\big(f(\sigma^k) - f(\sigma)\big)\,, \tag{1.4}$$

where $\sigma^k \in \Sigma$ is defined by $\sigma^k(k) = -\sigma(k)$ and $\sigma^k(j) = \sigma(j)$ if $j \neq k$. An associated quadratic (Dirichlet) form $D = D_L$ characterizes the intensity of the noise, it is defined for local functions f, g by $D_L(f,g) := L(fg) - fLg - gLf$, thus from (1.4)

$$D_L(f,g) = \sum_{k\in\mathbb{Z}} c(k,\sigma)\big(f(\sigma^k) - f(\sigma)\big)\big(g(\sigma^k) - g(\sigma)\big)\,. \tag{1.5}$$

Notice that $Lh_\gamma = -2J_\gamma * (\sigma c)$ and $D_L(f_k, f_j) = 0$ if $f_k(\sigma) \equiv \sigma(k)$ and $j \neq k$, while $D_L(f_k, f_k) = 4c(k,\sigma)$. The evolved configuration at time $t \geq 0$ will be denoted by $\sigma_t = (\sigma_t(k))_{k\in\mathbb{Z}}$. A familiar version of the Glauber dynamics associated with H_γ at inverse temperature $\beta > 0$ is specified by the jump rates c_γ,

$$c_\gamma(k,\sigma) := \frac{1}{2}\big(1 - \sigma(k)\tanh(\beta h_\gamma(k,\sigma))\big)\,, \tag{1.6}$$

the generator of this process will be denoted by L_γ. Let us remark that λ_γ^β is a stationary and reversible measure of the spin flip evolution defined by L_γ, see [Lig].

Motivated by (1.3) and (1.6) we claim that the local mean field h_γ plays a crucial role here, first of all we evaluate $L_\gamma h_\gamma$. Since $\tanh x \approx x - x^3/3 + 2x^5/15 - \cdots$ near zero, and h_γ is supposed to vanish in view of the law of large numbers as $\gamma \to 0$, expanding $L_\gamma h_\gamma$ up to its first nonlinear term we get

$$L_\gamma h_\gamma \approx \Delta_\gamma h_\gamma + (\beta - 1)J_\gamma * h_\gamma - \frac{\beta^3}{3} J_\gamma * h_\gamma^3 + \cdots\,,$$

where $\Delta_\gamma \varphi := J_\gamma * \varphi - \varphi$ for $\varphi \in \ell(\mathbb{Z})$. Observe now that the asymptotic behavior of this expression changes in a very radical way at $\beta = 1$ because there the leading term $(\beta - 1)J_\gamma * h_\gamma$ diminishes, thus the diffusive effect of Δ_γ can not be neglected any more. Let $\varepsilon > 0$ be the macroscopic size of the lattice and interpret a smooth function $\varphi : \mathbb{R} \mapsto \mathbb{R}$ as a sequence $\varphi_\varepsilon \in \ell(\mathbb{Z})$, $\varphi_\varepsilon(k) := \varphi(\varepsilon k)$, then

$$\begin{aligned}\Delta_\gamma \varphi_\varepsilon(k) &= \frac{\varepsilon^2 \varphi''(\varepsilon k)}{\gamma^2} \sum_{j\in\mathbb{Z}} J_\gamma(k-j)(\gamma k - \gamma j)^2 + o(\varepsilon^2/\gamma^2) \\ &= a\frac{\varepsilon^2\varphi''(\varepsilon k)}{\gamma^2} + o(\varepsilon^2/\gamma^2)\end{aligned}$$

if $\varepsilon = o(\gamma)$ as $\gamma \to 0$, where $a > 0$ is the asymptotic variance of J_γ, see condition (2.1) below. This means that Δ_γ turns out to be a lattice approximation of mesh ε/γ to the differential operator $a\partial_x^2$. Suppose that there is a macroscopic field $\xi = \xi(x,\tau)$, $x \in \mathbb{R}$, $\tau \geq 0$ such that $h_\gamma(k,\sigma_t) \approx \delta_\gamma \xi(\varepsilon_\gamma k, \alpha_\gamma t)$ in a weak sense as $\gamma, \alpha_\gamma, \delta_\gamma, \varepsilon_\gamma \to +0$. Then the rate of change of $\mathbf{E}h_\gamma$ and the orders of the linear terms $\Delta_\gamma h_\gamma$ and $(1-\beta)J_\gamma * h_\gamma$ of $L_\gamma h_\gamma$ are the same if $\alpha_\gamma = \varepsilon_\gamma^2/\gamma^2$ and $1 - \beta = \rho\alpha_\gamma$, where ρ is a constant and $\mathbf{E}$ denotes the expectation. The space-time intensity of the noise remains positive and finite if $\alpha_\gamma\delta_\gamma^2 = \varepsilon_\gamma$, see (1.5) and the calculation of the quadratic variation of the martingale part of the evolution in Section 3. Therefore the contribution of the nonlinear term $J_\gamma * h_\gamma^3$ diminishes if $\delta_\gamma^2 = o(\alpha_\gamma)$, and a nontrivial scaling limit is obtained if $\alpha_\gamma = \delta_\gamma^2$, whence

$$\delta_\gamma = \gamma^{1/3}\,, \quad \alpha_\gamma = \gamma^{2/3}\,, \quad \beta = \beta_\gamma = 1 - \rho\gamma^{2/3}\,, \quad \varepsilon_\gamma = \gamma^{4/3}\,. \tag{1.7}$$

Define now the scaled field $\xi = \xi(x,\tau)$ for $x \in \mathbb{R}$ and $\tau \geq 0$ by

$$\xi_\gamma(x,\tau) := \delta_\gamma^{-1} h_\gamma(k, \sigma_{\tau/\alpha_\gamma}) \quad \text{if} \quad k\varepsilon_\gamma - \varepsilon_\gamma/2 \le x < k\varepsilon_\gamma + \varepsilon_\gamma/2 . \tag{1.8}$$

In view of the multiscale analysis above, we expect that ξ_γ converges in a weak sense to a macroscopic field $\xi = \xi(x,\tau)$ specified as a weak solution to a stochastic partial differential equation of type

$$d\xi(x,\tau) = \left(a\partial_x^2\xi(x,\tau) - \rho\xi(x,\tau) - b\xi^3(x,\tau)\right) d\tau + \sqrt{2}\, dw(x,\tau), \tag{1.9}$$

where $b = 0$ if $\delta_\gamma^2 = o(\alpha_\gamma)$, while $b = 1/3$ if a scaling (1.7) is adopted. Therefore if $|1-\beta|^{-1} = o(\gamma^{-2/3})$ and the linear terms and the intensity of the noise are of the same order, i.e. $\alpha_\gamma = \rho|1-\beta|$, $\varepsilon_\gamma = \gamma\alpha_\gamma^{1/2}$ and $\delta_\gamma = \gamma^{1/2}\alpha_\gamma^{-1/4}$, then we see Gaussian fluctuations at a time scale α_γ as $b = 0$ in (1.9). However, non-Gaussian fluctuations take place if $1 - \beta = O(\gamma^{2/3})$ and the scaling parameters are given by (1.7); they are governed by (1.9) with $b = 1/3$. In this paper we focus on the second, more interesting case. In view of the physical interpretation, it is reasonable to distribute the initial configuration of the microscopic process by a Gibbs state λ_γ^β such that $\beta = 1 - \rho\gamma^{2/3}$.

In our basic reference [BPRS] it is shown that if $\beta = 1$ and the initial distribution of σ is the uniform Bernoulli measure then the scaling rule (1.7) results in a macroscopic equation (1.9) with $\rho = 0$, $b = 1/3$ and $\xi(x,0) \equiv 0$. Since the proof is based on Girsanov's formula, this nice result is restricted to finite volumes. In fact, it is assumed that the system is periodic with period $[2\gamma^{-4/3}]$, where $[u]$ denotes the integer part of $u \in \mathbb{R}$. Using a method of parabolic energy inequalities we can handle the large scale description (1.7)-(1.9) of model (1.6) with $1 - \beta = O(\gamma^{2/3})$ in an infinite volume, and on the initial distribution we only need a moment condition of type $\mathbf{E}h_\gamma^4(k,\sigma_0) = O(\delta_\gamma^4)$. We also show that the Gibbs states λ_γ^β satisfy this moment condition, thus (1.9) really describes the time dependent fluctuations of the Kac-Glauber model (1.6) near its critical point. Since (1.9) has a unique stationary and reversible state, a $P(\phi^4)_1$ field, the stationary distribution of ξ_γ should converge to his object. From a technical point of view, our basic result is an approximation theorem for (1.9), see Theorem 2.10 below, and Theorem 2.12 of [FR] for a more general formulation. Even the study of the equilibrium states λ_γ^β is based on dynamical properties of the system.

2 Conditions and main results

First we list the conditions we need on J_γ and on the initial distribution. Throughout this paper we assume that J_γ satisfies

$$\lim_{\gamma\to 0} \frac{\gamma^2}{2} \sum_{k\in\mathbb{Z}} J_\gamma(k)k^2 = a , \tag{2.1}$$

$$\limsup_{\gamma\to 0} \frac{1}{\gamma} \sum_{k\in\mathbb{Z}} J_\gamma^2(k) < +\infty , \tag{2.2}$$

$$1 - \hat{J}_\gamma(\omega) \geq \frac{a'}{\gamma^2} \min\{\omega^2, \gamma^2 \sqrt{\alpha_\gamma}\} \tag{2.3}$$

for all $\omega \in [-\pi, +\pi]$, where $0 < a' < a$ are given constants, $\alpha_\gamma > 0$ is the macroscopic unit of microscopic time, and

$$\hat{J}_\gamma(\omega) := \sum_{k \in \mathbb{Z}} e^{\imath \omega k} J_\gamma(k)$$

denotes the Fourier transform of J_γ, i.e. $\imath := \sqrt{-1}$. Obviously, $\hat{J}_\gamma$ is a 2π-periodic real function such that $\hat{J}_\gamma(0) = 1$ and $|\hat{J}_\gamma(\omega)| < 1$ otherwise; (2.1) and (2.2) can also be formulated in terms of $\hat{J}_\gamma$. The above conditions can easily be verified if $J_\gamma(k) \approx \gamma J(\gamma k)$, cf. [BPRS]. Here and in what follows α_γ, δ_γ, ε_γ, denote the scaling parameters, i.e. $h_\gamma(k, \sigma_t) \equiv \delta_\gamma \xi_\gamma(\varepsilon_\gamma k, \alpha_\gamma t)$ and $\beta = \beta_\gamma = 1 - \rho\alpha_\gamma$ with some fix $\rho \in \mathbb{R}$.

The initial configuration of the microscopic evolution is distributed by a family of probability measures $\{\mu_\gamma : \gamma > 0\}$ on $(\Sigma, \mathscr{B}_\Sigma)$ satisfying the following moment condition

$$\limsup_{\gamma \to 0} \sum_{k \in \mathbb{Z}} \exp(-q\varepsilon_\gamma |k|) \int h_\gamma^4(k, \sigma)\, \mu_\gamma(d\sigma) < +\infty \tag{2.4}$$

for all $q > 0$. Since $\varepsilon_\gamma = \delta_\gamma^4$, cf. (1.7), (2.4) is actually a statement on the scaled magnetization $\xi_\gamma = \delta_\gamma^{-1} h_\gamma$. The initial value of the macroscopic equation (1.9) is a locally integrable function $\xi_0 : \mathbb{R} \mapsto \mathbb{R}$ such that

$$\lim_{\gamma \to 0} \frac{\varepsilon_\gamma}{\delta_\gamma} \sum_{k \in \mathbb{Z}} \varphi(\varepsilon_\gamma k) h_\gamma(k, \sigma) = \int_{-\infty}^{\infty} \varphi(x) \xi_0(x)\, dx \tag{2.5}$$

in probability with respect to μ_γ for all test functions φ from the real Schwartz space $\mathscr{D}(\mathbb{R})$ of infinitely differentiable functions with compact supports.

Solutions to (1.9) are to be interpreted in a very weak sense. Let $\mathbb{R}_+ := [0, +\infty)$ and $\mathbb{R}_+^2 := \mathbb{R} \times \mathbb{R}_+$, the second coordinate of $(x, \tau) \in \mathbb{R}_+^2$ is interpreted as time, the first one is the space variable. The spaces of integrable or locally integrable functions on $\mathbb{X} = \mathbb{R}, \mathbb{R}_+^2, \mathbb{R}^2$ will be denoted by $\mathbb{L}^p(\mathbb{X})$ or by $\mathbb{L}^p_{\text{loc}}(\mathbb{X})$, respectively; they are considered as subsets of the corresponding Schwartz space of distributions, $\mathscr{D}'(\mathbb{X})$. For example, $\mathscr{D}(\mathbb{R}^2)$ denotes the space of infinitely differentiable and compactly supported $\varphi : \mathbb{R}^2 \mapsto \mathbb{R}$, its dual space is $\mathscr{D}'(\mathbb{R}^2)$. If $\xi : \mathbb{R}_+^2 \mapsto \mathbb{R}$ is jointly measurable and $\xi(\cdot, \tau) \in \mathbb{L}^1_{\text{loc}}(\mathbb{R})$ for all $\tau \geq 0$ then

$$X(\varphi, \tau) := \int_{-\infty}^{\infty} \varphi(x) \xi(x, \tau)\, dx \qquad \text{for } \varphi \in \mathscr{D}(\mathbb{R}) \tag{2.6}$$

defines a trajectory in $\mathscr{D}'(\mathbb{R})$. The Skorohod space of right continuous functionals $X : \mathbb{R}_+ \mapsto \mathscr{D}'(\mathbb{R})$ having left limits at every $\tau \in \mathbb{R}_+$ will be denoted by $\mathbf{D}[\mathbb{R}_+, \mathscr{D}'(\mathbb{R})]$, see [HS] or [EK], and $X(\varphi, \tau)$ is the value of $X \in \mathbf{D}[\mathbb{R}_+, \mathscr{D}'(\mathbb{R})]$ at $\varphi \in \mathscr{D}(\mathbb{R})$ and $\tau \geq 0$. The scaled magnetization ξ_γ shall be interpreted as a random element of $\mathbf{D}[\mathbb{R}_+, \mathscr{D}'(\mathbb{R})]$, cf. (2.6),(2.9), and we are going to describe its limit distribution on this space as $\gamma \to 0$.

Suppose now that we are given a white noise on $\mathbb{R}_+^2$, that is a Borel probability $\mathscr{Q}$ on $\mathscr{D}'(\mathbb{R}^2)$ such that the substitution functional $W(\phi, w) \equiv w(\phi)$ is a Gaussian random variable of mean zero and variance

$$\int W^2(\phi, w)\,\mathscr{Q}(dw) = \int_0^\infty d\tau \int_{-\infty}^\infty \phi^2(x,\tau)\,dx$$

for all $\phi \in \mathscr{D}(\mathbb{R}^2)$. Intuitively, the random functional W can be considered as a stochastic integral: $W(\phi, w) = \int_0^\infty \int_{-\infty}^\infty \phi(x,\tau)\,w(dx, d\tau)$.

2.7 Definition. *Consider the following formal SPDE:*

$$d\xi(x,\tau) = \big(a\partial_x^2 \xi - \xi B(\xi)\big)\,d\tau + \sqrt{A}\,dw(x,\tau), \qquad \text{(i)}$$

where $B : \mathbb{R} \mapsto \mathbb{R}$ is a continuously differentiable function, $a, A > 0$ are given constants, and w is the white noise, i.e. $w \in \mathscr{D}'(\mathbb{R}^2)$ is distributed by $\mathscr{Q}$ as specified above. A jointly measurable map $\xi = \xi(x,\tau,w)$ of $\mathbb{R}_+^2 \times \mathscr{D}'(\mathbb{R}^2)$ into $\mathbb{R}$ is called a weak solution to (i) with initial value $\xi_0 \in \mathbb{L}_{loc}^1(\mathbb{R})$ if both ξ and $\xi B(\xi)$ belong to $\mathbb{L}_{loc}^1(\mathbb{R}_+^2)$ for $\mathscr{Q}$-a.e. w, and

$$\int_0^\infty d\tau \int_{-\infty}^\infty \xi(x,\tau,w)\big(\partial_\tau + a\partial_x^2 - B(\xi(x,\tau,w))\big)\phi(x,\tau)\,dx + \int_{-\infty}^\infty \xi_0(x)\phi(x,0)\,dx + \sqrt{A}\,W(\phi,w) = 0 \qquad \text{(ii)}$$

$\mathscr{Q}$-a.s. simultaneously for all $\phi \in \mathscr{D}(\mathbb{R}^2)$.

Since the diffusion coefficient A does not depend on ξ, we need not refer to an underlying filtration. In fact, we have a very good uniqueness result for weak solutions allowing us to identify the solution we obtain as the weak limit of the scaled field ξ_γ, see Section 6.

2.8 Theorem. *Suppose that we have two constants $p \geq 0$ and $d > 0$ such that*

$$-d \leq C(x,y) := \frac{xB(x) - yB(y)}{x - y} \leq d\,(1 + |x| + |y|)^p \qquad \text{(i)}$$

for all $x, y \in \mathbb{R}$. Then (2.7) has at most one weak solution $\xi = \xi(x,\tau,w)$ satisfying

$$\int_0^T d\tau \int_{-\infty}^\infty e^{-q|x|}|\xi(x,\tau,w)|^{p+1}\,dx < +\infty \quad w\text{-a.s} \qquad \text{(ii)}$$

for all $T, q > 0$.

To prove existence of solutions we have to assume that the drift has the right sign. The condition $d + dB(x) \geq |x|^p$ is certainly sufficient, but we need not go into details of this issue because the existence of weak solutions to equation (1.9) is implied also by our approximation procedure as follows. Consider the scaled field $\xi_\gamma = \xi_\gamma(x,\tau)$ defined by (1.8) for $x \in \mathbb{R}$, $\tau \geq 0$, $\gamma > 0$ and set

$$X_\gamma(\varphi,\tau) := \int_{-\infty}^{\infty} \varphi(x)\,\xi_\gamma(x,\tau)\,dx \quad \text{for } \varphi \in \mathscr{D}(\mathbb{R}). \tag{2.9}$$

We interpret X_γ as a random element of $\mathbf{D}[\mathbb{R}_+,\mathscr{D}'(\mathbb{R})]$, its law will be denoted by $\mathscr{P}_\gamma$. By means of the a priori bounds of Sections 4 and 5 we show that the family $\{\mathscr{P}_\gamma : 0 < \gamma \leq 1\}$ is tight on $\mathbf{D}[\mathbb{R}_+,\mathscr{D}'(\mathbb{R})]$, which is a crucial step towards proving

2.10 Theorem. *Suppose (1.7) and (2.1)-(2.5), then $\mathscr{P}_\gamma$ converges weakly to a probability measure $\mathscr{P}$ on $\mathbf{D}[\mathbb{R}_+,\mathscr{D}'(\mathbb{R})]$ such that $\mathscr{P}$ is the distribution of a jointly measurable process $\xi = \xi(x,\tau,w)$ satisfying (1.9) with initial value ξ_0, i.e. $B(x) = \rho + x^2/3$ and $A = 2$ in (2.7). Moreover*

$$\int_0^T d\tau \int_{-\infty}^{\infty} e^{-q|x|}\mathbf{E}\xi^4(x,\tau)\,dx < +\infty$$

for all $T, q > 0$, thus ξ is uniquely specified in view of Theorem 2.8.

Let us remark that the initial value of the limiting process need not be a deterministic function. Without any change of the argument we can replace (2.5) by an assumption that P_γ, the distribution of $\xi_\gamma(\cdot,0)$, converges in $\mathscr{D}'(\mathbb{R})$ to a limit P as $\gamma \to 0$. Of course, (2.4) implies that the macroscopic initial distribution P will be concentrated on measurable functions.

Suppose first that the initial distribution μ_γ is chosen as the homogeneous Bernoulli measure of mean zero, then (2.4) is obviously satisfied and we have (2.5) with $\xi_0(x) = 0$ for all $x \in \mathbb{R}$; this is the problem discussed by [BPRS] with periodic boundary conditions. Consider now the equilibrium dynamics, that is $\mu_\gamma = \lambda_\gamma^\beta$ with $\beta = 1 - \rho\alpha_\gamma$ and $\rho \in \mathbb{R}$. The distribution of ξ_γ in this stationary and reversible regime will be denoted as $\mathscr{P}_\gamma^\beta$; its marginal at any fixed time is P_γ^β, that is the distribution of $\xi_\gamma(\cdot,\tau)$ for each $\tau \geq 0$. We shall see that the moment condition (2.4) holds true also in this case, but the initial value of the limiting process shall not be a deterministic function any more. The limit distribution of P_γ^β can be specified as follows. Let $Q_{r,a}^{x,y}$ denote the distribution of the Wiener bridge of variance $2a$ on the interval $[-r,r]$ with boundary conditions $\chi(-r) = x$, $\chi(r) = y$, Q is a Borel probability on the space $\mathscr{C}(\mathbb{R})$ of continuous functions $\chi : \mathbb{R} \mapsto \mathbb{R}$.

2.11 Definition. *A Borel probability P on $\mathscr{C}(\mathbb{R})$ is called a $P(\phi^4)_1$ field with parameters $a > 0$, $b \geq 0$ and $\rho \in \mathbb{R}$ if its two-sided conditional distributions satisfy the DLR equation*

$$\frac{P[d\chi|\mathscr{A}_r^c]}{Q_{r,a}^{x,y}(d\chi)} = \frac{\exp\left(-\mathscr{H}_r^{b,\rho}(\chi)\right)}{Z_r(x,y,a,b,\rho)} \quad P\text{-a.s.}$$

for all $r > 0$ and $x,y \in \mathbb{R}$ whenever $\chi(-r) = x$, $\chi(r) = y$, where Z_r is the normalization,

$$\mathscr{H}_r^{b,\rho}(\chi) := \frac{1}{4}\int_{-r}^{r}\left(2\rho\chi^2(u) + b\chi^4(u)\right)du\,,$$

and $\mathscr{A}_r^c$ *denotes the* σ*-field of* $\mathscr{C}(\mathbb{R})$ *generated by the projections* $\{\chi(v) : |v| \geq r\}$.

If $b = 0$ and $\rho > 0$ then (2.11) defines a familiar Ornstein-Uhlenbeck process. For $b > 0$ and $\rho \in \mathbb{R}$ the existence of a $P(\phi^4)_1$ field is proven by [Sim]; in fact it is specified as a diffusion process with a unique stationary distribution. Let P^* denote the law of this diffusion process; since P^* is ergodic, it is easy to verify that there is no other $P(\phi^4)_1$ field. [Iwa2] identifies the stationary and reversible measures of (1.9) as $P(\phi^4)_1$ Gibbs states, see also [Fun2] for a more general treatment of such fields including their characterization as reversible measures of Ginzburg-Landau equations. In our case $b = 1/3$ and $a > 0$ is given by (2.1). The equilibrium dynamics is characterized by

2.12 Theorem. *Suppose (1.7),(2.1),(2.2),(2.3), and let* $\mu_\gamma = \lambda_\gamma^\beta$ *with* $\beta = 1 - \rho\alpha_\gamma$, *then* $\mathscr{P}_\gamma^\beta \Longrightarrow \mathscr{P}^*$ *in* $\mathbf{D}[\mathbb{R}_+, \mathscr{D}'(\mathbb{R})]$ *as* $\gamma \to 0$.

If $\rho > 0$ then a straightforward coupling trick (energy inequality, see Section 3) shows that P^* is the unique stationary state of (1.9), but a more sophisticated entropy argument is needed to prove the same statement when $\rho \leq 0$. Here we claim only that every reversible measure is a Gibbs state.

3 The macroscopic equation and its derivation

In this section we prove Theorem 2.8 and outline the derivation of equation (1.9).

Proof of Theorem 2.8. We use some standard methods of parabolic equations, see e.g. [Frie]. Suppose that ξ and $\bar{\xi}$ are weak solutions with a common initial value, then $\zeta := \xi - \bar{\xi}$ satisfies

$$\int_0^\infty d\tau \int_{-\infty}^{\infty} \zeta(x,\tau,w)\left(\partial_\tau + a\partial_x^2 - C(\xi,\bar{\xi})\right)\phi(x,\tau)\,dx = 0 \tag{3.1}$$

w-a.s. for all $\phi \in \mathscr{D}(\mathbb{R}^2)$, and (3.1) extends to a much wider class of test functions by continuity.

Let $0 \leq \psi \in \mathscr{D}(\mathbb{R}^2)$ vanishing outside of the strip $\mathbb{R}\times[0,T]$, fix $w \in \mathscr{D}'(\mathbb{R}^2)$ such that condition (ii) holds true for both solutions, and consider the classical solution ϕ_Γ to

$$\partial_\tau\phi(x,\tau) + a\partial_x^2\phi(x,\tau) - \Gamma(x,\tau)\phi(x,\tau) = \psi(x,\tau) \quad \text{for } \tau \in [0,T] \tag{3.2}$$

specified by $\phi_\Gamma(x,T) = 0$ for each $x \in \mathbb{R}$, where $\Gamma : \mathbb{R}_+^2 \mapsto \mathbb{R}$ is uniformly continuous and bounded together with its first and second partial derivatives. We shall see that ϕ_Γ vanishes exponentially fast at infinity, thus (3.1) turns into

$$\int_0^T d\tau \int_{-\infty}^{\infty} \zeta(x,\tau,w)\psi(x,\tau)\,dx \tag{3.3}$$
$$= \int_0^T d\tau \int_{-\infty}^{\infty} \zeta(x,\tau,w)\big(C(\xi,\tilde{\xi}) - \Gamma(x,\tau)\big)\phi_\Gamma(x,\tau)\,dx\,.$$

It is certainly clear that we want to approximate C by Γ to conclude that the left hand side vanishes for each ψ, thus $\Gamma \geq -d$ may be assumed. In view of the maximum principle, (or by the Feynman-Kac formula, see [Sim]), $0 \leq \phi_\Gamma \leq \phi_{-d}$, where ϕ_{-d} denotes the solution corresponding to $\Gamma \equiv -d$. Since

$$\phi_{-d}(x,\tau) = e^{d\tau} \int_\tau^T d\vartheta\, e^{-d\vartheta} \int_{-\infty}^{\infty} \Pi_{\vartheta-\tau}(x-y)\psi(y,\vartheta)\,dy \tag{3.4}$$

for $\tau \leq T$, where Π is a heat kernel,

$$\Pi_\tau(x) := \frac{1}{\sqrt{4\pi a\tau}} \exp\Big(\frac{-x^2}{4a\tau}\Big) \quad \text{if } \tau > 0\,,$$

we have an explicit bound for ϕ_Γ allowing us to derive (3.3) and conclude that the left hand side of (3.3) is zero by sending Γ to C. Since ψ is an arbitrary element of a rich class of functions, this completes the proof. □

The study of SPDE (2.7) can be reduced by a coupling trick to its particular case of $B(x) \equiv \tilde{\rho} > 0$ and $A = 2$, this idea will then be used to prove Theorem 2.10. Observe that this Gaussian solution $\tilde{\xi}$ with initial value ξ_0 can be represented as

$$\tilde{\xi}(x,\tau,w) := e^{-\tilde{\rho}\tau} \int_{-\infty}^{\infty} \Pi_\tau(x-y)\xi_0(y)\,dy \tag{3.5}$$
$$+ \sqrt{2}e^{-\tilde{\rho}\tau} \int_0^T e^{\tilde{\rho}\vartheta} \int_{-\infty}^{\infty} \Pi_{\tau-\vartheta}(x-y)\,w(dy,d\vartheta)\,,$$

where w is the white noise and Π is the heat kernel (3.4). Since $\Pi \in \mathbb{L}^2(\mathbb{R} \times [0,T])$ for all $T > 0$, the stochastic integral on the right hand side is a well defined extension of the random functional $W(\phi,w)$, and $\tilde{\xi}$ is a jointly continuous function of $\tau > 0$ and $x \in \mathbb{R}$, see e.g. [Iwa1], [Wal] or [Fun1]. It is plain that $\tilde{\xi}$ is the unique weak solution to (2.7) with $B \equiv \tilde{\rho}$ and initial value ξ_0, cf. Theorem 2.8. It will be useful to interpret $\tilde{\xi}$ as a random element of $\mathbf{D}[\mathbb{R}_+, \mathscr{D}'(\mathbb{R})]$ by introducing $\tilde{X}(\varphi,\tau) := (\varphi, \tilde{\xi}(\cdot,\tau,w))$ for $\varphi \in \mathscr{D}(\mathbb{R})$, where $(\varphi,\xi) := \int \varphi\xi\,dx$ denotes the usual scalar product in $\mathbb{L}^2(\mathbb{R})$, cf. (2.9); $\tilde{X}(\cdot,\tau)$ is actually a continuous trajectory in $\mathscr{D}'(\mathbb{R})$. Let $\{\mathscr{F}_\tau : \tau > 0\}$ denote the natural filtration in $\mathbf{D}[\mathbb{R}_+, \mathscr{D}'(\mathbb{R})]$, that is $\mathscr{F}_\tau$ is the σ-field generated by the projections $\{X(\varphi,\vartheta) : \vartheta \leq \tau,\ \varphi \in \mathscr{D}(\mathbb{R})\}$, then $\tilde{X}$ is the only process such that

$$\tilde{W}(\varphi,\tau) := \tilde{X}(\varphi,\tau) - \int_0^\tau \tilde{X}(a\varphi'' - \tilde{\rho}\varphi, \vartheta)\,d\vartheta$$
$$\text{and} \quad \tilde{W}^2(\varphi,\tau) - 2(\varphi,\varphi)\tau \tag{3.6}$$

are $\mathscr{F}_T$-martingales for each $\varphi \in \mathscr{D}(\mathbb{R})$, see [HS1] or [Iwa1]. This characterization will be used when we prove the weak convergence of the linearized version (voter model) of the Kac-Glauber evolution to the generalized Ornstein-Uhlenbeck process $\tilde{\xi}$ given by (3.5). The distribution of $\tilde{X}$ will be denoted by $\tilde{\mathscr{P}}$.

Let us summarize now some basic information on the microscopic systems. An auxiliary process, the voter model is specified by the jump rates $\tilde{c}_\gamma$,

$$\tilde{c}_\gamma(k,\sigma) := \frac{1}{2}\big(1 - \tilde{\beta}\sigma(k)h_\gamma(k,\sigma)\big), \tag{3.7}$$

where $0 < \tilde{\beta} = 1 - \tilde{\rho}\alpha_\gamma \leq 1$, i.e. $\tilde{\rho} \geq 0$ and $\tilde{\rho}\alpha_\gamma \leq 1$; the corresponding generator will be denoted as $\tilde{L}_\gamma$. Related questions are discussed by [MT]. Since

$$\tilde{L}_\gamma h_\gamma(k,\sigma) = \Delta_\gamma h_\gamma(k,\sigma) - (1-\tilde{\beta})J_\gamma * h_\gamma(k,\sigma), \tag{3.8}$$

the voter process can really be interpreted as a lattice approximation of the Ornstein-Uhlenbeck process. In case of the original model we write $\beta = 1 - \rho\alpha_\gamma$ with $\rho \in \mathbb{R}$, thus

$$\begin{aligned} L_\gamma h_\gamma(k,\sigma) = \Delta_\gamma h_\gamma(k,\sigma) - (1-\beta)J_\gamma * h_\gamma(k,\sigma) \\ - J_\gamma * \Omega(\beta h_\gamma(k,\sigma)), \end{aligned} \tag{3.9}$$

where $\Omega(u) := u - \tanh u$ is the nonlinear term of the drift. Observe that $L_\gamma h_\gamma = \tilde{L}_\gamma h_\gamma - J_\gamma * \Omega$. A common form of the stochastic evolution equations reads as

$$h_\gamma(k,\sigma_t) = h_\gamma(k,\sigma_0) + \int_0^t L_\gamma h_\gamma(k,\sigma_s)\,ds + m_\gamma(k,t), \tag{3.10}$$

where $m_\gamma(k,t)$ is a martingale for each $k \in \mathbb{Z}$. If necessary then objects related to the voter process shall be distinguished by a mark "tilde". For instance, in (3.10) we ought to write $\tilde{L}, \tilde{\sigma}, \tilde{m}$ instead of L, σ, m in case of the voter model. The mean intensity of the cross variation of the martingales $m_\gamma(k,t)$ and $m_\gamma(j,t)$ is $A_\gamma(k,j,\sigma) := D_{L_\gamma}(h_\gamma(k,\cdot), h_\gamma(j,\cdot))$; from (1.5)

$$A_\gamma(k,j,\sigma) = 4\sum_{i\in\mathbb{Z}} c_\gamma(i,\sigma)J_\gamma(i-k)J_\gamma(i-j) \tag{3.11}$$

and the same expression holds true for the voter model, too. Notice that $A_\gamma(k,j,\sigma)$ vanishes if $|k-j| > 2/\gamma$, and $A_\gamma(k,j,\sigma) \approx 2\gamma$ otherwise, provided that h_γ is really a vanishing quantity. The scaling rule $\alpha_\gamma\delta_\gamma^2 = \varepsilon_\gamma$ is motivated by this observation. More precisely, in the next two sections we shall show that $\mathbf{E}h_\gamma^2 = O(\delta_\gamma^2)$ in both cases, therefore the intensity of the noise at site $k \in \mathbb{Z}$ is just $\sum_{j\in\mathbb{Z}} \mathbf{E}A_\gamma(k,j,\sigma) \approx 2$, which is magnified by a factor $\alpha_\gamma^{-1}\delta_\gamma^{-2}$ in the macroscopic picture. Since ε_γ is the macroscopic width of one site, a nontrivial scaling requires $\varepsilon_\gamma^{-1} = \alpha'\alpha_\gamma^{-1}\delta_\gamma^{-2}$, and if $\alpha' = 1$ then a constant $A = 2$ is obtained as the intensity of the macroscopic noise, see the calculations below.

Let $\tilde{\xi}_\gamma = \tilde{\xi}_\gamma(x, \tau)$ denote the scaled local mean field of the voter process, see (1.8), and define also $\tilde{X}_\gamma = \tilde{X}_\gamma(\varphi, \tau)$, cf. (2.9). Of course, $\tilde{X}_\gamma$ is a function of the evolved configuration $\tilde{\sigma}_t \in \Sigma$ at $t = \tau/\alpha_\gamma$, thus its rate of change is governed by the scaled generator $\tilde{\mathscr{L}}_\gamma := \alpha_\gamma^{-1}\tilde{L}_\gamma$. Since $1 - \tilde{\beta} = \tilde{\rho}\alpha_\gamma$ by assumption, from (3.8)

$$\begin{aligned}\tilde{\mathscr{L}}_\gamma \tilde{X}_\gamma(\varphi, \tau) &= \tilde{X}_\gamma\big(\Delta_\gamma \varphi - \tilde{\rho}\mathscr{J}_\gamma * \varphi, \tau\big) \\ &= \int_{-\infty}^{\infty} \tilde{\xi}_\gamma(x, \tau)\big(\Delta_\gamma\varphi(x) - \tilde{\rho}\mathscr{J}_\gamma * \varphi(x)\big)\, dx\,,\end{aligned} \tag{3.12}$$

where $f * g$ denotes the convolution also in $\mathbb{L}^1_{\text{loc}}(\mathbb{R})$, Δ_γ is the macroscopic version of Δ_γ, that is

$$\Delta_\gamma \varphi := \frac{\mathscr{J}_\gamma * \varphi(x) - \varphi(x)}{\alpha_\gamma} \quad \text{for } \varphi \in \mathbb{L}^1_{\text{loc}}(\mathbb{R})\,, \tag{3.13}$$

where $\mathscr{J}_\gamma \in \mathbb{L}^2(\mathbb{R})$ is defined by $\mathscr{J}_\gamma(x) := J_\gamma([2^{-1} + x\varepsilon_\gamma^{-1}])$.

Notice that if $\varphi \in \mathscr{D}(\mathbb{R})$ then $\mathscr{J}_\gamma * \varphi \to \varphi$ and $\Delta\varphi \to a\partial_x^2\varphi$ uniformly in $x \in \mathbb{R}$ as $\gamma \to 0$. In view of (3.11) the quadratic form associated with (3.12) reads as

$$\begin{aligned}\tilde{\mathscr{D}}_\gamma(\varphi, \tau) &:= \tilde{\mathscr{L}}_\gamma \tilde{X}_\gamma^2(\varphi, \tau) - 2\tilde{X}_\gamma(\varphi, \tau)\tilde{\mathscr{L}}_\gamma \tilde{X}_\gamma(\varphi, \tau) \\ &= 2\int_{-\infty}^{\infty} \big(\mathscr{J}_\gamma * \varphi(x)\big)^2\big(1 - \beta\delta_\gamma \tilde{\mathrm{s}}_\gamma(x, \tau)\tilde{\xi}_\gamma(x, \tau)\big)\, dx\,,\end{aligned} \tag{3.14}$$

where $\mathrm{s}_\gamma(\cdot, \tau) := \sigma_t(k)$ for $\tau = t\alpha_\gamma$ and $k\varepsilon_\gamma - 2^{-1}\varepsilon_\gamma \le x < k\varepsilon_\gamma + 2^{-1}\varepsilon_\gamma$.

The a priori bounds of the next section imply that the right hand side of (3.14) converges in mean square to $2(\varphi, \varphi)$. Let $\tilde{\mathscr{P}}_\gamma$ denote the distribution of $\tilde{\xi}_\gamma$, by means of the martingale approach we prove in Section 6

3.15 Proposition. *Let $\tilde{\beta} = 1 - \tilde{\rho}\alpha_\gamma$ with $\tilde{\rho} \ge 0$ and suppose (1.7) and (2.1)-(2.5), then $\tilde{\mathscr{P}}_\gamma$ converges in $\mathbf{D}[\mathbb{R}_+, \mathscr{D}'(\mathbb{R})]$ to $\tilde{\mathscr{P}}$ as $\gamma \to 0$.*

The martingale approach can also be applied to prove Theorem 2.10, but we prefer a more direct way. Let $\eta = \xi - \tilde{\xi}$, where ξ and $\tilde{\xi}$ denote the solutions to problem (2.7) with a general B, and with $B(x) \equiv \tilde{\rho} > 0$, respectively. If $\xi(\cdot, 0) = \tilde{\xi}(\cdot, 0)$ then (ii) of (2.7) turns into

$$\begin{aligned}&\int_0^\infty d\tau \int_{-\infty}^{\infty} \eta(x, \tau)(\partial_\tau + a\partial_x^2)\phi(x, \tau)\, dx \\ &\quad = \int_0^\infty d\tau \int_{-\infty}^{\infty} (\xi B(\xi) - \tilde{\rho}\tilde{\xi})\phi(x, \tau)\, dx\end{aligned} \tag{3.16}$$

for $\phi \in \mathscr{D}'(\mathbb{R})$. Since the existence of the Gaussian solution $\tilde{\xi} = \tilde{\xi}(x, \tau, w)$ is known, we can consider (3.16) as a deterministic equation for ξ. In view of Theorem 2.8, under a mild regularity condition, there is at most one solution of this problem. This solution can be found by means of a standard compactness argument based on a parabolic energy inequality, see the end of this section for explanation. The possibility of such a simplified treatment is due to the existence of an effective coupling of ξ and $\tilde{\xi}$ obtained by identifying the realizations of their white noise fields. Now we suggest that an analogous, although a bit more sophisticated coupling argument is available also at the microscopic level.

A coupled process $(\sigma_t, \tilde{\sigma}_t)$ can be defined as follows, we assume that $\sigma_0(k) = \tilde{\sigma}_0(k)$ for each $k \in \mathbb{Z}$. If $\sigma_t(k) \neq \tilde{\sigma}_t(k)$ at $t \geq 0$ then they flip independently with rates $c_\gamma(k, \sigma_t)$ and $\tilde{c}_\gamma(k, \tilde{\sigma}_t)$, respectively, while identical spins change simultaneously at the largest possible rate, see Chapter III of [Lig]. Let G_γ denote the generator of the coupled process, it is acting on local functions $f : \Sigma \times \Sigma \mapsto \mathbb{R}$ as

$$G_\gamma f(\sigma, \tilde{\sigma}) = \sum_{k\in\mathbb{Z}} c_\gamma(k,\sigma)\big(f(\sigma^k,\tilde{\sigma}) - f(\sigma,\tilde{\sigma})\big) + \sum_{k\in\mathbb{Z}} \tilde{c}_\gamma(k,\tilde{\sigma})\big(f(\sigma,\tilde{\sigma}^k) - f(\sigma,\tilde{\sigma})\big)$$
$$+ \sum_{k:\sigma(k)=\tilde{\sigma}(k)} \min\{c_\gamma(k,\sigma), \tilde{c}_\gamma(k,\tilde{\sigma})\} T_k f(\sigma,\tilde{\sigma}), \tag{3.17}$$

where

$$T_k f := f(\sigma^k, \tilde{\sigma}^k) - f(\sigma^k, \tilde{\sigma}) - f(\sigma, \tilde{\sigma}^k) + f(\sigma, \tilde{\sigma}).$$

Of course, if $\beta < 1$ then it is reasonable to set $\tilde{\beta} = \beta$, while $|1 - \beta| \leq 1 - \tilde{\beta}$ otherwise.

Observe that $T_k(\sigma(j) - \tilde{\sigma}(j)) = 0$ even if $j = k$, and

$$G_\gamma\big(\sigma(j) - \tilde{\sigma}(j)\big)^2 + \big(c_\gamma(j,\sigma) + \tilde{c}_\gamma(j,\tilde{\sigma})\big)\big(\sigma(j) - \tilde{\sigma}(j)\big)^2$$
$$= |c_\gamma(j,\sigma) - \tilde{c}_\gamma(j,\tilde{\sigma})|\big(\sigma(j) + \tilde{\sigma}(j)\big)^2. \tag{3.18}$$

To get further information on the effectivity of the coupling, let us consider also the Dirichlet form (1.5) for G_γ, and denote $Q_\gamma(k, j, \sigma, \tilde{\sigma})$ its value if $f = \sigma(k) - \tilde{\sigma}(k)$ while $g = \sigma(j) - \tilde{\sigma}(j)$. From (3.17) and (1.5) we see that $Q_\gamma(k, j, \sigma, \tilde{\sigma}) = 0$ if $j \neq k$, while

$$Q_\gamma(k,k,\sigma,\tilde{\sigma}) = 4c_\gamma(k,\sigma) + 4\tilde{c}_\gamma(k,\tilde{\sigma}) - 2\min\{c_\gamma(k,\sigma), \tilde{c}_\gamma(k,\tilde{\sigma})\}\big(\sigma(k) + \tilde{\sigma}(k)\big)^2$$
$$\leq -G_\gamma\big(\sigma(k) - \tilde{\sigma}(k)\big)^2 + 2|c_\gamma(k,\sigma) - \tilde{c}_\gamma(k,\tilde{\sigma})|\big(\sigma(k) + \tilde{\sigma}(k)\big)^2$$
$$\leq -G_\gamma\big(\sigma(k) - \tilde{\sigma}(k)\big)^2 + 4\beta|h_\gamma(k,\sigma)| + 4\tilde{\beta}|h_\gamma(k,\tilde{\sigma})| \tag{3.19}$$

in view of (3.18) and $|\tanh x| \leq |x|$.

Consider now the coupled process at the macroscopic level, let $\eta_\gamma := \xi_\gamma - \tilde{\xi}_\gamma$ and $\mathscr{G}_\gamma := \alpha_\gamma^{-1} G_\gamma$. Since ξ and $\tilde{\xi}$ coincide at $\tau = 0$ by assumption,

$$\eta_\gamma(x,\tau) = \int_0^\tau \mathscr{G}_\gamma \eta(x,\vartheta)\,d\vartheta + M_\gamma(x,\tau) \tag{3.20}$$

for each $x \in \mathbb{R}$, where M_γ is a family of martingales, and

$$\mathscr{G}_\gamma \eta_\gamma = \Delta_\gamma \eta_\gamma - \rho \mathscr{J}_\gamma * \xi_\gamma + \tilde{\rho} \mathscr{J}_\gamma * \tilde{\xi}_\gamma - \gamma^{-1} \mathscr{J}_\gamma * \Omega(\beta\delta_\gamma\xi_\gamma),$$

thus multiplying by $\partial_\tau \phi$ and integrating by parts we get

$$\int_0^\infty d\tau \int_{-\infty}^\infty \eta_\gamma(x,\tau)(\partial_\tau + \Delta_\gamma)\phi(x,\tau)\,dx + \int_{-\infty}^\infty dx \int_0^\infty M_\gamma(x,d\tau)\,\phi(x,\tau)$$
$$= \int_0^\infty d\tau \int_{-\infty}^\infty \left(\gamma^{-1}\Omega(\beta\delta_\gamma\xi_\gamma(x,\tau)) + \rho\xi_\gamma(x,\tau) - \tilde{\rho}\tilde{\xi}_\gamma(x,\tau)\right)\mathscr{J}_\gamma * \phi(x,\tau)\,dx \tag{3.21}$$

simultaneously for each $\phi \in \mathscr{D}(\mathbb{R}^2)$ with probability one.

We are going to compare (3.16) and (3.21). By means of (3.19) in Section 6 we show that the mean square of the martingale term on the left hand side of (3.21) vanishes as $\gamma \to 0$, thus the equivalence of the left hand sides will be more or less obvious. The evaluation of the right hand side of (3.21) needs a bit more work because we have to determine the limit of a nonlinear function of a weakly convergent sequence. While the tightness of the joint distribution of X_γ and $\tilde{X}_\gamma$ follows immediately by moment estimates, at this step we also need a property of spatial smoothness of ξ_γ. The necessary a priori bounds will be proven in Sections 4 and 5.

To expose the derivation of the a priori bounds at an intuitive level, let us now return to the macroscopic process and assume for convenience that the system is periodic with period one, $B(x) = \rho + x^2/3$ and $|\rho| < \tilde{\rho}$. From (3.16) with $\phi(x,\vartheta) \approx 2\eta(x,\vartheta)$ for $0 \le \vartheta \le \tau$ and $\phi = 0$ otherwise, by a standard approximation procedure, or directly from (1.9), we get an energy inequality,

$$\int_0^1 \eta^2(x,\tau)\,dx + 2a\int_0^\tau d\vartheta \int_0^1 (\partial_x\eta)^2\,dx + \frac{1}{4}\int_0^\tau d\vartheta \int_0^1 \xi^4(x,\vartheta)\,dx$$
$$\le \frac{1}{4}\int_0^\tau d\vartheta \int_0^1 \xi^4(x,\vartheta)\,dx + \frac{2}{3}\int_0^\tau d\vartheta \int_0^1 (\xi - \tilde{\xi})(3\tilde{\rho}\tilde{\xi} - 3\rho\xi - \xi^3)\,dx \tag{3.22}$$
$$\le \frac{1}{4}\int_0^\tau d\vartheta \int_0^1 (2\tilde{\xi}^4 + 16\tilde{\rho}\xi^2 - \xi^4)\,dx \le \frac{1}{2}\int_0^\tau d\vartheta \int_0^1 (\tilde{\xi}^4 + 32\tilde{\rho}^2)\,dx$$

where $2uv \le u^2 + v^2$ and $4uv^3 \le u^4 + 3v^4$ were used. Since the Gaussian process $\tilde{\xi}$ is well controlled, we have got a bound for the space-time integrals of ξ^4 and $(\partial_x\eta)^2$. In Section 5 we materialize this argument at the microscopic level; the second estimate concerns the space-time integral of $-\eta_\gamma\Delta_\gamma\eta_\gamma$ implying the desired spatial smoothness of η_γ, thus also that of ξ_γ. In this way we complete the proof of Theorem 2.10 in Section 6.

The first step of the derivation of Theorem 2.12 from Theorem 2.10 is the verification of the moment condition (2.4). We start the microscopic process with the uniform Bernoulli distribution and consider the time averaged distribution $\bar{\mu}^t_\gamma$ as $t \to +\infty$ while γ and β are fixed,

$$\bar{\mu}^t_\gamma := \frac{1}{t}\int_0^t \mu^s_\gamma\,ds\,, \tag{3.23}$$

where μ_γ^s denotes the distribution of the evolved configuration σ_s. Since $\bar{\mu}_\gamma^t \Longrightarrow \lambda_\gamma^\beta$ in Σ as $t \to +\infty$, it is sufficient to show that $\bar{\mu}_\gamma^t$ satisfies the desired moment condition, and the bound does not depend on t and γ.

The identification of the continuum limit of the equilibrium states, that is the proof of $P_\gamma^\beta \Longrightarrow P^*$ needs some more work. Since λ_γ^β is a reversible measure of the Kac-Glauber evolution, we have

$$\begin{aligned}&\int f(\sigma)L_\gamma g(\sigma)\,\lambda_\gamma^\beta(d\sigma)\\&\quad= -\frac{1}{2}\sum_{k\in\mathbb{Z}}\int c_\gamma(k,\sigma)\big(f(\sigma^k)-f(\sigma)\big)\big(g(\sigma^k)-g(\sigma)\big)\,\lambda_\gamma^\beta(d\sigma)\end{aligned}$$

for local functions f and g. Let $g(\sigma) = \Sigma_{j\in\mathbb{Z}}\psi(j)h_\gamma(j,\sigma)$, then we get

$$\begin{aligned}&\sum_{k\in\mathbb{Z}}\int f(\sigma)h_\gamma(k,\sigma)\big(\Delta_\gamma\psi(k) - B_\beta(h_\gamma(k,\sigma))J_\gamma * \psi(k)\big)\,\lambda_\gamma^\beta(d\sigma)\\&\quad=\sum_{k\in\mathbb{Z}}\int c_\gamma(k,\sigma)\sigma(k)\big(f(\sigma^k)-f(\sigma)\big)J_\gamma * \psi(k)\,\lambda_\gamma^\beta(d\sigma),\end{aligned} \tag{3.24}$$

where $B_\beta(u) := 1 - u^{-1}\tanh\beta u$. This is an integration by parts formula (KMS condition) characterizing the Gibbs states λ_γ^β. We are going to evaluate the continuum limit of (3.24).

Define now the scaled equilibrium magnetization $\chi_\gamma = \chi_\gamma(x,\sigma)$ as $\chi_\gamma(x) := \delta_\gamma^{-1}h_\gamma(k,\sigma)$ if $[x\varepsilon_\gamma^{-1}+2^{-1}] = k$, its distribution P_γ^β is considered as a Borel probability on $\mathbb{L}^2_{\mathrm{loc}}(\mathbb{R})$. Let $f(\sigma) = F(\chi_\gamma)$, where $F(\chi) := \Phi((\varphi_1,\chi),(\varphi_2,\chi),\ldots,(\varphi_n,\chi))$ for $\chi \in \mathbb{L}^2_{\mathrm{loc}}(\mathbb{R})$, $\Phi \in \mathscr{D}(\mathbb{R}^n)$ and $\varphi_i \in \mathscr{D}(\mathbb{R})$, $i = 1,2,\ldots,n$. The functional derivative of F can be written as

$$\nabla F(\chi,\psi) := \sum_{i=1}^n \Phi_i(\chi)(\varphi_i,\psi),$$

thus a formal continuum limit procedure transforms (3.24) into

$$\begin{aligned}&\int F(\chi)\nabla\mathscr{H}(\chi,\psi)P(d\chi) = \int \nabla F(\chi,\phi)P(d\chi), \quad \text{where}\\&\nabla\mathscr{H}(\chi,\psi) := -\int_{-\infty}^{\infty}\chi(x)\big(a\partial_x^2 - \rho - 3^{-1}\chi^2(x)\big)\psi(x)\,dx.\end{aligned} \tag{3.25}$$

By means of the a priori bounds we show that every limit point of the family P_γ^β satisfies (3.25), thus the argument is completed by noting that this KMS condition implies the DLR equation (2.11), see [Iwa2].

4 A priori bounds for the voter model

In this section we derive some basic probability estimates for the voter process defined by the jump rates $\tilde{c}_\gamma$ with $0 < \tilde{\beta} \le 1$, see (3.7). As in the previous section, the mark 'tilde' refers to this model. The usual scalar product in $\ell^2(\mathbb{Z})$ will be denoted as $\langle \cdot , \cdot \rangle$, and $\|\varphi\| := \langle \varphi, \varphi \rangle^{1/2}$, $\|\varphi\|_1 := \langle 1, |\varphi| \rangle$ are the norms of $\ell^2(\mathbb{Z})$ and $\ell^1(\mathbb{Z})$, respectively. From (3.10)

$$h_\gamma(k, \tilde{\sigma}_t) = h_\gamma(k, \tilde{\sigma}_0) + \int_0^t \big(\Delta_\gamma h_\gamma(k, \tilde{\sigma}_s) + (\tilde{\beta} - 1) J_\gamma * h_\gamma(k, \tilde{\sigma}_s)\big) ds + \tilde{m}_\gamma(k, t) \,, \tag{4.1}$$

whence by an elementary Fourier calculus we obtain

$$\langle \psi, h_\gamma(\cdot, \tilde{\sigma}_t) \rangle = \langle \psi, p_{\gamma,t} * h_\gamma(\cdot, \tilde{\sigma}_0) \rangle + \tilde{R}_{\gamma,t}(\psi) \tag{4.2}$$

whenever $\psi \in \ell^1(\mathbb{Z})$, where

$$\tilde{R}_{\gamma,t}(\psi) := \sum_{k \in \mathbb{Z}} \int_0^t \psi * p_{\gamma,t-s}(k) \, \tilde{m}_\gamma(k, ds)$$

and

$$p_{\gamma,t} = p_{\gamma,t}(k) := \frac{1}{2\pi} \int_{-\pi}^{\pi} \exp\big(\imath \omega k - t + \tilde{\beta} t \hat{J}_\gamma(\omega)\big) \, d\omega \,.$$

Notice that if $\tilde{\beta} = 1$ then $p_{\gamma,t}$ is just the transition probability of a random walk on $\mathbb{Z}$ with continuous time and jump rates J_γ.

First we investigate the martingale term $\tilde{R}$ on the right hand side of (4.2), we prove

4.3 Lemma. *Suppose (2.2),(2.3) and $\alpha_\gamma \le 1$, then we have a universal constant $\tilde{K}_0$ such that*

$$\mathbf{E} \tilde{R}^4_{\gamma,t}(\psi) \le \tilde{K}_0 \|\psi\|_1^4 \gamma^2 \min\{t + \alpha_\gamma^{-1}, (1 - \tilde{\beta})^{-1}\} \,.$$

Proof. Let $\tilde{\mathscr{F}}_t$ denote the natural filtration of the process and introduce

$$\tilde{R}_{\gamma,t}(\psi, u, v) := \sum_{k \in \mathbb{Z}} \int_u^v \psi * p_{\gamma,t-s}(k) \, \tilde{m}_\gamma(k, ds)$$

for $0 \le u < v \le t$. Since the jump rates are less than one, from (3.11) by the Plancherel theorem it follows that

$$
\begin{aligned}
&\mathbf{E}[\big(\tilde{R}^2_{\gamma,t}(\psi,u,v)|\tilde{\mathscr{F}}_u]\\
&=\sum_{k\in\mathbb{Z}}\sum_{j\in\mathbb{Z}}\int_u^v \big(\psi * p_{\gamma,t-s}(k)\big)\big(\psi * p_{\gamma,t-s}(j)\big)\mathbf{E}[\tilde{A}_\gamma(k,j,\sigma_s)|\tilde{\mathscr{F}}_u]\,ds\\
&\le 4\sum_{k\in\mathbb{Z}}\sum_{j\in\mathbb{Z}}\sum_{i\in\mathbb{Z}}\int_u^v \big(\psi * p_{\gamma,t-s}(k)\big)\big(\psi * p_{\gamma,t-s}(j)\big)J_\gamma(k-i)J_\gamma(j-i)\,ds\\
&=\frac{2}{\pi}\int_{-\pi}^{\pi} d\omega\int_u^v |\hat{\psi}(\omega)\hat{J}_\gamma(\omega)|^2\exp\big((2t-2s)(\tilde{\beta}\hat{J}_\gamma(\omega)-1)\big)\,ds\\
&\le \frac{\|\psi\|_1^2}{\pi}\int_{-\pi}^{\pi}\hat{J}^2_\gamma(\omega)\frac{\exp\big((2v-2t)(1-\tilde{\beta}\hat{J}_\gamma)\big)-\exp\big((2u-2t)(1-\tilde{\beta}\hat{J}_\gamma)\big)}{1-\tilde{\beta}\hat{J}_\gamma(\omega)}\,d\omega\,,
\end{aligned}
\tag{4.4}
$$

consequently

$$
\limsup_{v\to u+0}\frac{1}{v-u}\mathbf{E}[\tilde{R}^2_{\gamma,t}(\psi,u,v)|\tilde{\mathscr{F}}_u]\le\|\psi\|_1^2\,F_{\gamma,t}(u) \tag{4.5}
$$

almost surely, where

$$
F_{\gamma,t}(u):=\frac{2}{\pi}\int_{-\pi}^{\pi}\hat{J}^2_\gamma(\omega)\exp\big((2u-2t)(1-\tilde{\beta}\hat{J}_\gamma)\big)\,d\omega\,;
$$

moreover

$$
\mathbf{E}\tilde{R}^2_{\gamma,t}(\psi,0,s)\le\|\psi\|_1^2\int_0^s F_{\gamma,t}(u)\,du\,. \tag{4.6}
$$

From (2.3) with $z=2t-2s\ge 0$

$$
\begin{aligned}
\exp\big(-z(1-\tilde{\beta}\hat{J}_\gamma(\omega))\big)&\le e^{-z(1-\tilde{\beta})}\exp(-a'\tilde{\beta}z\omega^2\gamma^{-2})\\
&\quad+e^{-z(1-\tilde{\beta})}\exp(-a'\tilde{\beta}z\alpha_\gamma^{1/2})\,,
\end{aligned}
$$

whence by a direct calculation using (2.2) we get

$$
F_{\gamma,t}(s)\le K_1\gamma\big(\gamma(t-s)^{-1/2}+\exp(2a'(s-t)\alpha_\gamma^{-1/2})\big)\,e^{(2t-2s)(\tilde{\beta}-1)}\,, \tag{4.7}
$$

consequently as $\alpha_\gamma\le 1$ by assumption,

$$
\mathbf{E}\tilde{R}^2_{\gamma,t}(\psi,0,s)\le K_2\|\psi\|_1^2\,\gamma\,\min\{(s+\alpha_\gamma^{-1})^{1/2},\,(1-\tilde{\beta})^{-1/2}\}\,, \tag{4.8}
$$

where K_1 and K_2 are universal constants.

To estimate the fourth power of $\tilde{R}$ let $R_n:=\tilde{R}_{\gamma,t}(\psi,0,nt/m)$ for $0\le n\le m\in\mathbb{N}$ and observe that

$$
\begin{aligned}
R^4_{n+1}-R^4_n-4(R_{n+1}-R_n)R^3_n&\le 6(R_{n+1}-R_n)^2(R^2_{n+1}+R^2_n)\\
&=12(R_{n+1}-R_n)^2R^2_n+12(R_{n+1}-R_n)^3R_n+6(R_{n+1}-R_n)^4\\
&\le 18(R_{n+1}-R_n)^2R^2_n+12(R_{n+1}-R_n)^4\,.
\end{aligned}
\tag{4.9}
$$

We have to sum for $n\le m$ and take the expectation of both sides of (4.9) to obtain a bound for $\mathbf{E}\tilde{R}^4$. Then $(R_{n+1}-R_n)R^3_n$ vanishes in view of the martingale property of $\tilde{R}$, and the expectation of the contribution of the first term on the right hand side can be estimated by means of (4.5)-(4.8). On the other hand, the flipping times of $\sigma(i)$ can be selected from the set of points of independent Poisson process, N_i of unit intensity, thus

$$|R_{n+1} - R_n| \leq \frac{K_3(t,\gamma)}{m} + \sum_{i\in\mathbb{Z}} \int_{tn/m}^{tn/m+t/m} A_i(s) N_i(ds),$$

where K_3 does not depend on m and

$$A_i(s) := 2 \sum_{k\in\mathbb{Z}} \sum_{j\in\mathbb{Z}} |\psi(k)| p_{\gamma,t-s}(k-j) J_\gamma(j-i).$$

Therefore by a direct calculation

$$\sum_{n=0}^{m} \mathbf{E}|R_{n+1} - R_n|^4 \leq \frac{K_4(t,\gamma)}{m} + \sum_{i\in\mathbb{Z}} \int_0^t A_i^4(s)\, ds$$
$$\leq \frac{K_4(t,\gamma)}{m} + \gamma K_5 \|\psi\|_1^4 \int_0^t F_{\gamma,t}(s)\, ds \tag{4.10}$$

with some universal constant K_5. Indeed, as $A_i \leq 2\|\psi\|_1 \max J_\gamma$ and $\max J_\gamma = O(\gamma^{1/2})$ in view of (2.2), by the Plancherel theorem

$$\sum_{i\in\mathbb{Z}} A_i^4(s) \leq \gamma K_5 \|\psi\|_1^2 \sum_{i\in\mathbb{Z}} A_i^2(s) \leq \gamma K_5 \|\psi\|_1^4 F_{\gamma,t}(s),$$

which completes the proof by sending $m \to +\infty$ via (4.7)-(4.8). □

A local bound for the deterministic part $p_{\gamma,t} * h_{\gamma,0}$ of the right hand side of (4.2) can be derived by means of a clever weight function θ_r, see e.g. [Frit]. Let $\theta_r(x) := \theta(rx)$ for $x \in \mathbb{R}$ and $r > 0$, where $\theta : \mathbb{R} \mapsto (0,1]$ is an infinitely differentiable and exponentially decreasing symmetric function such that $\theta(x) = 1$ if $|x| \leq 1$ and $\theta(x+y) \leq (1+|y|)\,\theta(x)$ whenever $|y| \leq 1$. For example, we can choose $-\log\theta$ as a symmetric and convex function such that $-\log\theta(x) = 0$ if $|x| \leq 1$ and $-\log\theta(x) = |x|\log 2$ if $|x| \geq 2$. Notice that θ_r is a smooth version of the indicator function of the interval $[-1/r, 1/r]$ thus $r = O(\varepsilon_\gamma)$ when we are looking for a macroscopic bound. In the calculations below $\theta_r = \theta_r(k)$, $k \in \mathbb{Z}$ is considered as an element of $\ell^2(\mathbb{Z})$, i.e., $\langle\theta_r, \varphi\rangle \equiv \Sigma_{k\in\mathbb{Z}}\theta_r(k)\varphi(k)$ whenever $\varphi \in \ell^2(\mathbb{Z})$, and so on.

4.11 Proposition. *Conditions (2.2) and (2.3) imply the existence of a constant $\tilde{K}$ such that for all $t > 0$, and $r > \gamma > 0$ we have*

$$\mathbf{E}\langle\theta_r, \tilde{h}_\gamma^4(\cdot,\sigma_t)\rangle \leq 8\exp(r^2 t/\gamma^2)\mathbf{E}\langle\theta_r, \tilde{h}_\gamma^4(\cdot,\sigma_0)\rangle$$
$$+ \frac{\tilde{K}\gamma^2}{r} \min\{t + \alpha_\gamma^{-1}, (1-\tilde{\beta})^{-1}\}.$$

Proof. Let $g \in \ell(\mathbb{Z})$ be nonnegative and observe that $g_t := p_{\gamma,t} * g$ satisfies $\partial_t g_t = \Delta_\gamma g_t + (\tilde{\beta} - 1) J_\gamma * g_t$, therefore as $\tilde{\beta} \leq 1$ and

$$\theta_r(x+y)+\theta_r(x-y)-2\theta_r(x)\le y^2r^2\theta_r(x) \quad \text{if } r|y|\le 1$$

we have

$$\begin{aligned}\partial_t\langle\theta_r,g_t\rangle &\le \sum_{\ell>0}J_\gamma(\ell)\sum_{k\in\mathbb{Z}}g_t(k)\big(\theta_r(k+\ell)+\theta_r(k-\ell)-2\theta_r(k)\big)\\ &\le r^2\gamma^{-2}\langle\theta_r,g_t\rangle\,,\end{aligned}$$

which implies by the Gronwall inequality that

$$\langle\theta_r,p_{\gamma,t}*g\rangle\le\langle\theta_r,g\rangle\exp\big(\frac{tr^2}{\gamma^2}\big)\,. \tag{4.12}$$

Choosing $g=h_\gamma^4(\cdot,\sigma_0)$ we obtain a bound also for $(p_{\gamma,t}*g)^4$ by convexity, which completes the proof by Lemma 4.3. Indeed, set $\psi=\psi_k$ such that $\psi_k(k)=1$ and $\psi_k(j)=0$ if $j\neq k$. Since $\tilde{R}_{\gamma,t}$ does not depend on $k\in\mathbb{Z}$ and $\|\theta_r\|_1=O(r^{-1})$, we really have the desired bound. □

Proposition 4.11 yields an effective bound for the scaled field $\tilde{\xi}_\gamma$ because $O(\gamma^2t)=O(\delta_\gamma^4)$ if $t=\tau\alpha_\gamma^{-1}=\tau\gamma^{-2/3}$ and $r=q\varepsilon_\gamma=q\gamma^{4/3}$ with $\tau,q>0$.

The asymptotic evaluation of the right hand side of (3.21) is not immediate because of the presence of a nonlinear term, Ω. Since we have tightness of the scaled process only in $\mathbf{D}[\mathbb{R}_+,\mathscr{D}'(\mathbb{R})]$, we have to mollify it by a convolution kernel Ψ_ℓ. More exactly, we replace ξ_γ by $\Psi_\ell*\eta_\gamma+\Psi_\ell*\tilde{\xi}_\gamma$, where $\eta_\gamma:=\xi_\gamma-\tilde{\xi}_\gamma$, $\tilde{\xi}_\gamma$ denotes the scaled voter process, $\Psi\in\mathscr{D}(\mathbb{R})$ is a symmetric probability density, $\Psi_\ell(x):=\ell\Psi(\ell x)$ for $\ell>0$, and $*$ denotes the convolution operator also in $\mathbb{L}^1_{\mathrm{loc}}$. At this step the following property of spatial smoothness of the voter process will be needed.

4.13 Proposition. *Suppose (1.7), (2.1)-(2.4) and* $\tilde{\beta}=1-\tilde{\rho}\alpha_\gamma$ *with* $\tilde{\rho}\ge 0$, *then for all* $q,\tau>0$

$$\lim_{\ell\to\infty}\limsup_{\gamma\to 0}\int_{-\infty}^{\infty}\theta_q(x)\mathbf{E}\big(\tilde{\xi}_\gamma(\tau,x)-\Psi_\ell*\tilde{\xi}_\gamma(\tau,x)\big)^2\,dx=0\,.$$

Proof. Our starting point is again the identity (4.2), first we handle its deterministic part $p_{\gamma,t}*h_{\gamma,0}$. We split $\tilde{\xi}_\gamma(\cdot,0)=\xi_\gamma(\cdot,0)$ as $\tilde{\xi}_\gamma=\theta_{r,\gamma}\tilde{\xi}_\gamma+(1-\theta_{r,\gamma})\tilde{\xi}_\gamma$, where $\tilde{\xi}_\gamma$ and $\theta_{r,\gamma}$ are step functions of step size ε_γ; $\theta_{r,\gamma}(x):=\theta_r([x\varepsilon_\gamma^{-1}+2^{-1}])$. In view of (4.12) and (2.4) the contribution of the second term vanishes uniformly in γ and ℓ as $r\to 0$ while $q>0$ is fix. To treat the first term let $\zeta_{\gamma,r}(x,\tau):=\Pi_{\gamma,\tau}*(\theta_{r,\gamma}\tilde{\xi}_\gamma(x,0))$, where $\Pi_{\gamma,\tau}(x):=\varepsilon_\gamma^{-1}p_{\gamma,t}([x\varepsilon_\gamma^{-1}+2^{-1}])$ at $\tau=t\alpha_\gamma$, and introduce $\Psi_{\gamma,\ell}\in\ell^2(\mathbb{Z})$ by

$$\Psi_{\gamma,\ell}(k):=\ell\int_{\varepsilon_\gamma k-\varepsilon_\gamma/2}^{\varepsilon_\gamma k+\varepsilon_\gamma/2}\Psi(\ell x)\,dx\,.$$

Rewriting the integral of a step function as a sum, we get

$$\begin{aligned}&\int_{-\infty}^{\infty} \theta_q(x)(\zeta_{\gamma,r}(x,\tau) - \Psi_\ell * \zeta_{\gamma,r}(x,\tau))^2\,dx\\ &\le \int_{-\infty}^{\infty} (\zeta_{\gamma,r}(x,\tau) - \Psi_\ell * \zeta_{\gamma,r}(x,\tau))^2\,dx\\ &\le \frac{\varepsilon_\gamma}{2\pi}\int_{-\pi}^{\pi} \big(1-\hat{\Psi}_{\gamma,\ell}(\omega)\big)^2 \exp\big(-2\tau\alpha_\gamma^{-1}(1-\hat{J}_\gamma(\omega))\big)|\hat{g}_{\gamma,r}(\omega)|^2\,d\omega\,,\end{aligned} \tag{4.14}$$

where $g_{\gamma,r}\in\ell^2(\mathbb{Z})$ as $g_{\gamma,r}(k):=\delta_\gamma^{-1}\theta_r(\varepsilon_\gamma k)h_\gamma(k,\sigma_0)$. It is easy to verify that

$$0\le 1-\hat{\Psi}_{\gamma,\ell}(\omega)\le \min\{2,\frac{a''\omega^2}{\ell^2\varepsilon_\gamma^2}\}\le \frac{a''+2\sqrt{\alpha_\gamma}\ell^2}{a'\ell^2}\frac{1-\hat{J}_\gamma(\omega)}{\alpha_\gamma} \tag{4.15}$$

with some constant a'' depending only on Ψ. Since (2.4) implies

$$\limsup_{\gamma\to 0}\frac{\varepsilon_\gamma}{2\pi}\int_{-\pi}^{\pi}|\hat{g}_{\gamma,r}(\omega)|^2\,d\omega<+\infty \tag{4.16}$$

for each $q,r>0$, in view of (4.15) the left hand side of (4.14) vanishes as $\gamma\to 0$ and then $\ell\to+\infty$.

To estimate the random term on the right hand side of (4.2) define $\psi=\psi_{\gamma,\ell}\in\ell^2(\mathbb{Z})$ by $\psi_{\gamma,\ell}(k):=\Psi_{\gamma,\ell}(k)$ if $k\ne 0$, while $\psi_{\gamma,\ell}(0):=\Psi_{\gamma,\ell}(0)-1$, where $\Psi_{\gamma,\ell}$ is the same as above. In view of the scaling rule (1.7), we have to show that

$$\lim_{\ell\to+\infty}\limsup_{\gamma\to 0}\delta_\gamma^{-2}\mathbf{E}\tilde{R}^2_{\gamma,\tau/\alpha_\gamma}(\psi_{\gamma,\ell})=0\,.$$

Since $\delta_\gamma^2=\alpha_\gamma$ and $|\hat{J}_\gamma|\le 1$, from (4.4)

$$\begin{aligned}&\delta_\gamma^{-2}\mathbf{E}\tilde{R}^2_{\gamma,\tau/\alpha_\gamma}(\psi_{\gamma,\ell})\\ &\quad\le\frac{1}{\pi}\int_{-\pi}^{\pi}(1-\hat{\Psi}_{\gamma,\ell}(\omega))^2\frac{1-\exp\big(-2\tau\alpha_\gamma^{-1}(1-\hat{J}_\gamma(\omega))\big)}{\alpha_\gamma^{-1}\big(1-\hat{J}_\gamma(\omega)\big)}\,d\omega\,,\end{aligned}$$

which completes the proof by (4.15). □

Now we are in a position to investigate the Kac-Glauber process (1.6) by means of the coupling introduced in Section 3.

5 A priori bounds via coupling to the voter model

Having in mind (3.22), we are going to derive an energy inequality for the coupled process. By a direct calculation, see (3.19), we obtain

$$
\begin{aligned}
G_\gamma\big(h_\gamma(k,\sigma) - h_\gamma(k,\tilde\sigma)\big)^2 = \\
-4\big(h_\gamma(k,\sigma) - h_\gamma(k,\tilde\sigma)\big) \sum_{j\in\mathbb{Z}} J_\gamma(k-j)\big(c_\gamma(j,\sigma)\sigma(j) - \tilde c_\gamma(j,\tilde\sigma)\tilde\sigma(j)\big) \\
+4 \sum_{j:\sigma(j)\neq\tilde\sigma(j)} J_\gamma^2(k-j)(c_\gamma(j,\sigma) + \tilde c_\gamma(j,\tilde\sigma)) \\
+4 \sum_{j:\sigma(j)=\tilde\sigma(j)} J_\gamma^2(k-j)|c_\gamma(j,\sigma) - \tilde c_\gamma(j,\tilde\sigma)| \,.
\end{aligned}
\tag{5.1}
$$

The critical terms containing $c_\gamma + \tilde c_\gamma$ can be estimated by means of (3.18). For brevity set $\tilde h_\gamma := h_\gamma(\cdot,\tilde\sigma)$, we get

$$
\begin{aligned}
& G_\gamma\,(h_\gamma - \tilde h_\gamma)^2 + J_\gamma^2 * G_\gamma\,(\sigma - \tilde\sigma)^2 - 2(h_\gamma - \tilde h_\gamma)(\Delta_\gamma h_\gamma - \Delta_\gamma \tilde h_\gamma) \\
& \le 2(h_\gamma - \tilde h_\gamma)\big((1-\tilde\beta)J_\gamma * \tilde h_\gamma - J_\gamma * (h_\gamma - \tanh(\beta h_\gamma)\big) + 2J_\gamma^2 * |c_\gamma - \tilde c_\gamma|(\sigma + \tilde\sigma)^2 \\
& \le 2(h_\gamma - \tilde h_\gamma)\big((1-\tilde\beta)J_\gamma * \tilde h_\gamma - (1-\beta)J_\gamma * h_\gamma - \Omega(\beta h_\gamma)\big) \\
& - 2(h_\gamma - \tilde h_\gamma)\Delta_\gamma \Omega(\beta h_\gamma) + 4J_\gamma^2 * \big(\beta|h_\gamma| + \tilde\beta|\tilde h_\gamma|\big)
\end{aligned}
\tag{5.2}
$$

as $|\tanh x| \le |x|$.

In the forthcoming calculations it will frequently be exploited that Δ_γ is a negative operator in $\ell^2(\mathbb{Z})$. Indeed, we have

$$
\begin{aligned}
\langle\varphi,\psi\rangle_+ &:= -\langle\varphi, \Delta_\gamma\psi\rangle \\
&= \sum_{\ell>0} J_\gamma(\ell) \sum_{k\in\mathbb{Z}} \varphi(k)\big(2\psi(k) - \psi(k+\ell) - \psi(k-\ell)\big) \\
&= \sum_{\ell>0} J_\gamma(\ell) \sum_{j=0}^{\ell-1} \sum_{k\in\ell\mathbb{Z}+j} \big(\varphi(k+\ell) - \varphi(k)\big)\big(\psi(k+\ell) - \psi(k)\big) \\
&= \sum_{\ell>0} J_\gamma(\ell) \sum_{k\in\mathbb{Z}} \big(\varphi(k+\ell) - \varphi(k)\big)\big(\psi(k+\ell) - \psi(k)\big)\,,
\end{aligned}
\tag{5.3}
$$

thus $\langle\cdot,\cdot\rangle_+$ is a new scalar product in $\ell^2(\mathbb{Z})$; set $\|\varphi\|_+ := \langle\varphi,\varphi\rangle_+^{1/2}$ and remember that it does depend on γ. In view of (3.9) and (3.22) it is not surprising that our energy inequality will be formulated in terms of $\|\cdot\|$ and $\|\cdot\|_+$.

A discrete version of the energy inequality (3.22) follows now from (5.2) by summing for $k \in \mathbb{Z}$ and rearranging the sums. Since we are working in an infinite volume, the weight functions θ_r play again an important role at this step.

5.4 Lemma. *If* $0 < \gamma < r$ *then for* $\varphi, \psi \in \ell^\infty(\mathbb{Z})$ *we have*

$$
\langle\theta_r^2\varphi, J_\gamma * \psi\rangle \le \langle\theta_r\varphi, \theta_r\psi\rangle + \frac{r}{\gamma}\|\theta_r\varphi\|\,\|\theta_r\psi\| \tag{i}
$$

and

$$
\begin{aligned}
\langle\theta_r^2\varphi, -\Delta_\gamma\psi\rangle &\le \langle\theta_r\varphi, \theta_r\psi\rangle_+ + \frac{r^2}{\gamma^2}\|\theta_r\varphi\|\,\|\theta_r\psi\| \\
&+ \frac{r}{\gamma}\big(\|\theta_r\varphi\|_+\|\theta_r\psi\| + \|\theta_r\varphi\|\,\|\theta_r\psi\|_+\big)\,.
\end{aligned}
\tag{ii}
$$

Proof. The first inequality is a direct consequence of $|\theta_r(k \pm \ell) - \theta_r(k)| \leq r\gamma^{-1}\theta_r(k)$ for $r\ell \leq 1$. To prove the second one we use an elementary identity, see (5.3),

$$\begin{aligned}\langle \theta_r^2 \varphi, -\Delta_\gamma \psi\rangle - \langle \theta_r \varphi, \theta_r \psi\rangle_+ &= \sum_{\ell>0} J_\gamma(\ell) \sum_{k\in\mathbb{Z}} \theta_r(k)\varphi(k) \\ &\times \Big(\big(\theta_r(k+\ell) - \theta_r(k)\big)\psi(k+\ell) + \big(\theta_r(k-\ell) - \theta_r(k)\big)\psi(k-\ell)\Big) \\ = -\sum_{\ell>0} J_\gamma(\ell) &\sum_{k\in\mathbb{Z}} \big(\theta_r(k+\ell)\varphi(k+\ell) - \theta_r(k)\varphi(k)\big)\big((\theta_r(k+\ell) - \theta_r(k)\big)\psi(k) \\ + \sum_{\ell>0} J_\gamma(\ell) &\sum_{k\in\mathbb{Z}} \theta_r(k)\varphi(k)\big(\theta_r(k+\ell) - \theta_r(k)\big)\big(\psi(k+\ell) - \psi(k)\big)\,.\end{aligned}$$

On the other hand, $\theta_r(k)\psi(k+\ell) = \theta_r(k+\ell)\psi(k+\ell) + \big(\theta_r(k+\ell) - \theta_r(k)\big)\psi(k+\ell)$, thus the statement follows directly by the Cauchy inequality. □

Now we are in a position to summarize the above calculations. Let $h_{\gamma,t} := h_\gamma(\cdot,\sigma_t)$, $\tilde h_\gamma := h_\gamma(\cdot,\tilde\sigma_t)$, and remember that $\tilde\beta \leq 1$ and $\sigma_0 = \tilde\sigma_0$ by assumption. We have

5.5 Proposition. *Under condition (2.2) there exist some constants $\beta' > 1$, $\gamma', \delta' > 0$ and $K < +\infty$ such that*

$$\begin{aligned}\mathbf{E}\|\theta_r h_{\gamma,t} - \theta_r \tilde h_{\gamma,t}\|^2 &+ \frac{1}{9}\int_0^t \mathbf{E}\|\theta_r h_{\gamma,s}^2\|^2 ds + \frac{1}{4}\int_0^t \mathbf{E}\|\theta_r h_{\gamma,s} - \theta_r \tilde h_{\gamma,s}\|_+^2\, ds \\ &\leq Kt\, \exp\Big(\frac{Kr^2t}{\gamma^2}\Big)\Big(\frac{\gamma^{4/3}}{r} + \frac{r^3}{\gamma^4} + \frac{(\beta-\tilde\beta)^4}{r(1-\tilde\beta)^2} + \frac{1}{t}\int_0^t \mathbf{E}\|\theta_r \tilde h_{\gamma,s}^2\|^2\, ds\Big)\end{aligned}$$

for all $t, r > 0$ whenever $\gamma < \gamma'$, $r < \delta'\gamma$ and $|1-\beta| \leq 1-\tilde\beta \leq \beta' - 1$.

Proof. Multiplying (5.2) by $\theta_r^2(k)$ and summing for $k \in \mathbb{Z}$, by a direct calculation it follows that

$$\mathbf{E}\|\theta_r f_{\gamma,t}\|^2 + 2\int_0^t \mathbf{E}\|\theta_r f_{\gamma,s}\|_+^2\, ds \leq \int_0^t \mathbf{E}R(s)\, ds\,, \tag{5.6}$$

where $f_{\gamma,t} = h_{\gamma,t} - \tilde h_{\gamma,t}$ and $R(t) = R_1 + R_2 + R_3 + R_4 + R_5$. The remainders R_i, $1 \leq i \leq 5$ are defined and estimated as follows. From (ii) of Lemma 5.4

$$\begin{aligned}R_1 := 2\langle \theta_r^2 f_\gamma, \Delta_\gamma f_\gamma\rangle + 2\|\theta_r f_\gamma\|_+^2 &\leq \frac{2r^2}{\gamma^2}\|\theta_r f_\gamma\|^2 + \frac{4r}{\gamma}\|\theta_r f_\gamma\|\, \|\theta_r f_\gamma\|_+ \\ &\leq \frac{26r^2}{\gamma^2}\|\theta_r f_\gamma\|^2 + \frac{1}{6}\|\theta_r f_\gamma\|_+^2\end{aligned}$$

as $2uv \leq zu^2 + v^2/z$ for any $z > 0$, while from (i) of Lemma 5.4 by the Schwarz inequality

$$\begin{aligned} R_2 &:= 2\langle \theta_r^2 f_\gamma, (1-\tilde\beta) J_\gamma * \tilde h_\gamma - (1-\beta) J_\gamma * h_\gamma \rangle \\ &\le -(2-2\delta')(1-\tilde\beta)\|\theta_r f_\gamma\|^2 + 4|\beta - \tilde\beta|\, \|\theta_r f_\gamma\|\, \|\theta_r h_\gamma\| \\ &\le 2(\beta-\tilde\beta)^2 (1-\delta')^{-1}(1-\tilde\beta)^{-1}\|\theta_r h_\gamma\|^2 . \end{aligned}$$

To handle the next term we need some facts on hyperbolic tangent. It is an odd function such that $0 \le x - \tanh x \le x^3/3$ and $0 \le \tanh x - x + x^3/3 \le 2x^5/15$ for all $x \ge 0$, moreover $x - \tanh x \ge x^3/5$ if $0 \le x \le \beta''$, where $\beta'' > 1$ is so small that $(\beta'')^5 + 5\tanh \beta'' \le 5\beta''$. Using an elementary inequality, $2uv^3/3 \le 75u^4 + v^4/15$, we get

$$\begin{aligned} R_3 := -2\langle \theta_r^2 f_\gamma, \Omega(\beta h_\gamma)\rangle &\le \frac{2\beta^3}{3}\langle \theta_r^2, |\tilde h_\gamma h_\gamma^3|\rangle - \frac{2\beta^3}{5}\|\theta_r h_\gamma^2\|^2 \\ &\le 75\beta^3 \|\theta_r \tilde h_\gamma^2\|^2 - \frac{\beta^3}{3}\|\theta_r h_\gamma^2\|^2 ; \end{aligned}$$

Since $\|\varphi\|_+^2 \le 2\|\varphi\|^2$ and $3\|\theta_r \Omega(\beta h_\gamma)\| \le \beta^3 \|\theta_r h_\gamma^2\|$ as $|h_\gamma| \le 1$, by the Schwarz inequality

$$\begin{aligned} R_4 := 2\langle \theta_r^2 f_\gamma, \Omega(\beta h_\gamma)\rangle_+ &\le \frac{\sqrt{8}\beta^3}{3}\|\theta_r f_\gamma\|_+ \|\theta_r h_\gamma^2\| + \frac{3r\beta^3}{\gamma}\|\theta_r f_\gamma\|\, \|\theta_r h_\gamma^2\| \\ &\le \frac{4}{3}\|\theta_r f_\gamma\|_+^2 + \frac{2\beta^6}{9}\|\theta_r h_\gamma^2\|^2 + \frac{81 r^2}{2\gamma^2}\|\theta_r f_\gamma\|^2 . \end{aligned}$$

Finally, as $\theta_r(k \pm \ell) \le 2\theta_r(k)$ if $|\gamma\ell| \le 1$, and $\|\theta_r\|^2 = O(1/r)$, from (2.2) and the Schwarz inequality

$$\begin{aligned} R_5 &:= 4\beta' \langle J_\gamma^2 * \theta_r^2, |h_\gamma| + |\tilde h_\gamma|\rangle \le 16\beta' \sum_{k\in\mathbb{Z}} J_\gamma^2(k)\, \langle \theta_r^2, |h_\gamma| + |\tilde h_\gamma|\rangle \\ &\le K_1' \gamma \langle \theta_r^2, |h_\gamma| + |\tilde h_\gamma|\rangle \le K_2' \Big(\frac{\gamma^{4/3}}{r} + \gamma^{2/3}\big(\|\theta_r h_\gamma\|^2 + \|\theta_r \tilde h_\gamma\|^2\big)\Big) . \end{aligned}$$

Observe now that $2\|\theta_r h_\gamma\|^2 \le z^{-1}\|\theta_r\|^2 + z\|\theta_r h_\gamma^2\|^2$ for each $z > 0$, thus the estimation of R_2, R_4 and R_5 can be completed by choosing z small. Summarizing the calculations above we obtain

$$\begin{aligned} &R(t) + \frac{1}{9}\|\theta_r h_\gamma^2\|^2 + \frac{1}{4}\|\theta_r h_\gamma\|_+^2 \\ &\quad \le K\Big(\frac{r^2}{\gamma^2}\|\theta_r f_\gamma\|^2 + \frac{\gamma^{4/3}}{r} + \frac{r^3}{\gamma^4} + \frac{(\beta-\tilde\beta)^4}{r(1-\tilde\beta)^2} + \|\theta_r \tilde h_\gamma^2\|^2 \Big) \end{aligned} \tag{5.7}$$

provided that $\beta' - 1 > 0$ is small enough, which completes the proof of Proposition 5.5 by the Gronwall inequality. □

Let us remark that the formulation of Proposition 5.5 does not rely on the scaling rule (1.7).

6 Completion of the proofs

Now we are in a position to materialize the ideas outlined in Section 3. Adopt the scaling rule (1.7) and let $\tilde{\beta} = 1 - \tilde{\rho}\alpha_\gamma$, where $\tilde{\rho} \geq 0$ and $|\rho| \leq \tilde{\rho}$ for convenience. The moment estimates of the previous sections can be reformulated at the macroscopic level as follows. Under conditions (2.2)-(2.4) the coupled process satisfies

$$\limsup_{\gamma \to 0} \int_0^\tau d\vartheta \int_{-\infty}^{\infty} \theta_q(x)\big(\big(\mathbf{E}\xi_\gamma^4(x,\vartheta) + \mathbf{E}\tilde{\xi}_\gamma^4(x,\vartheta)\big)\, dx < +\infty \tag{6.1}$$

for all $q, \tau > 0$. Let $\mathbb{P}_\gamma$ denote the joint distribution induced by the coupled process $\zeta_\gamma := (\xi_\gamma, \tilde{\xi}_\gamma)$ on $\mathbf{D}[\mathbb{R}_+, \mathscr{D}'(\mathbb{R}) \times \mathscr{D}'(\mathbb{R})]$.

In view of a general theorem of [Fo] the proof of the tightness of the family $\{\mathbb{P}_\gamma : \gamma > 0\}$ amounts to proving the tightness of the projections $X_\gamma(\varphi, \tau)$ and $\tilde{X}_\gamma(\varphi, \tau)$ on $\mathbf{D}[\mathbb{R}_+, \mathbb{R}]$ for each $\varphi \in \mathscr{D}(\mathbb{R})$. From (3.10)

$$X_\gamma(\varphi,\tau) = X_\gamma(\varphi,0) + \int_0^\tau \mathscr{L}_\gamma X_\gamma(\varphi,\vartheta)\, d\vartheta + W_\gamma(\varphi,\tau); \tag{6.2}$$

$$\mathscr{L}_\gamma X_\gamma(\varphi,\vartheta) = \int_{-\infty}^{\infty} \xi_\gamma(x,\vartheta)\big(\Delta_\gamma - \rho\mathscr{J}_\gamma^* - \beta\delta_\gamma^{-2}\bar{\Omega}(\beta\delta_\gamma\xi_\gamma(x,\vartheta))\mathscr{J}_\gamma^*\big)\varphi(x),$$

where $W_\gamma(\varphi, \tau)$ is a martingale, $\bar{\Omega}(u) := u^{-1}\Omega(u)$ and an abbreviation $\mathscr{J}_\gamma * \varphi \equiv \mathscr{J}_\gamma^* \varphi$ is used. A similar expression is obtained for $\tilde{X}_\gamma$ by omitting $\bar{\Omega}$, see (3.12). The associated quadratic form is just

$$\begin{aligned} \mathscr{D}_\gamma(\varphi,\tau) &:= \mathscr{L}_\gamma X_\gamma^2(\varphi,\tau) - 2X_\gamma(\varphi,\tau)\mathscr{L}_\gamma X_\gamma(\varphi,\tau) \\ &= 2\int_{-\infty}^{\infty} (\mathscr{J}_\gamma * \varphi)^2(x)\big(1 - \mathbf{s}_\gamma(x,\tau)\tanh(\beta\delta_\gamma\xi_\gamma(x,\tau))\big)\, dx, \end{aligned} \tag{6.3}$$

see (3.14) for the case of the scaled voter process. Of course,

$$Z_\gamma(\varphi,\tau) := W_\gamma^2(\phi,\tau) - \int_0^\tau \mathscr{D}_\gamma(\varphi,\vartheta)\, d\vartheta$$

and the analogously defined processes $\tilde{Z}_\gamma$ are also martingales. Moreover, (6.1) implies in both cases for all $\varphi \in \mathscr{D}(\mathbb{R})$ and $\tau \geq 0$ that $W_\gamma^2(\varphi,\tau) - Z_\gamma(\varphi,\tau) \to 2\tau(\varphi,\varphi)$ in probability as $\gamma \to 0$. Since the jumps of any of the above processes go to zero uniformly as $\gamma \to 0$, it is easy to verify a general criterion of tightness in the Skorohod space, see Theorem 9.4 of Chapter 3 in [EK]. Technical details of this routine calculation and some further references can be found in [HS2], see Theorem 1.11 there, or Section 3 of [BPRS]. We could also apply the martingale CLT, see version (b) of Theorem 1.4 in Chapter 7 of [EK], because the processes $X_\gamma(\varphi,\tau) - W_\gamma(\varphi,\tau)$ are locally equicontinuous and bounded in time with large probability, therefore it is sufficient to control the martingale components, W_γ. This latter approach identifies the limits of W_γ and $\tilde{W}_\gamma$ as a Wiener process of variance $2(\varphi,\varphi)$.

Proof of Proposition 3.15. Since the family $\tilde{\mathscr{P}}_\gamma$ is tight, and $B(u) \equiv \tilde{\rho}$ in this case, the statement is an immediate consequence of Definition 2.7, Theorem 2.8 and the uniqueness of the martingale problem for the generalized Ornstein-Uhlenbeck process (3.5), see Theorem 1.15 in [HS2] or Section 3 of [BPRS]. Let $\tilde{X}(\varphi,\tau)$ denote the limiting process, then

$$W(\varphi,\tau) := \tilde{X}(\varphi,\tau) - \tilde{X}(\varphi,0) - \int_0^\tau \tilde{X}(a\varphi'' - \tilde{\rho}\varphi,\vartheta)\,d\vartheta$$

is a Wiener process for each $\varphi \in \mathscr{D}(\mathbb{R})$, thus the underlying white noise $W(\phi)$ can be constructed by expanding $\phi \in \mathscr{D}'(\mathbb{R}^2)$ as $\phi(x,\tau) = \Sigma\,\psi_k(\tau)\varphi_k(x)$. Remember that there is a one to one correspondence between $\tilde{\xi}$ and the white noise. □

The crucial step towards proving Theorems 2.10 and 2.12 consists in the evaluation of the nonlinear term of (3.21). Let Ψ_ℓ be the mollifier introduced before Proposition 4.13, we have

Proposition 6.4. *Under conditions (2.2)-(2.4)*

$$\lim_{\ell\to\infty}\limsup_{\gamma\to 0}\int_0^\tau d\vartheta\int_{-\infty}^{\infty}\theta_q(x)\mathbf{E}|\gamma^{-1}\Omega(\beta\delta_\gamma\xi_\gamma(x,\vartheta)) - 3^{-1}(\Psi_\ell*\xi_\gamma(x,\vartheta))^3|\,dx = 0$$

for all $\tau, q > 0$.

Proof. Since $|\Omega(u) - 3^{-1}u^3| \le 2|u^5|/15$, from (6.1)

$$\limsup_{\gamma\to 0}\mathbf{E}\int_0^\tau d\vartheta\int_{-\infty}^{\infty}\theta_q(x)|\gamma^{-1}\Omega(\beta\delta_\gamma\xi_\gamma(x,\vartheta)) - 3^{-1}\xi_\gamma^3(x,\vartheta)|\,dx = 0\,.$$

On the other hand, $|\xi_\gamma^3 - (\Psi_\ell*\xi_\gamma)^3| \le 2|\xi_\gamma - \Psi_\ell*\xi_\gamma|\,|\xi_\gamma^2 + (\Psi_\ell*\xi_\gamma)^2|$ and $\xi_\gamma = \tilde{\xi}_\gamma + \eta_\gamma$, thus taking into account (6.1) and Proposition 4.13, the statement reduces to

$$\lim_{\ell\to\infty}\limsup_{\gamma\to 0}\int_0^\tau d\vartheta\int_{-\infty}^{\infty}\theta_q(x)\mathbf{E}\big(\eta_\gamma(x,\vartheta)) - \Psi_\ell*\eta_\gamma(x,\vartheta)\big)^2\,dx = 0 \qquad (6.5)$$

by means of the Schwarz inequality. Indeed, as

$$\|\theta_r h_\gamma - \theta_r\tilde{h}_\gamma\|_+^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi}(1 - \hat{J}_\gamma(\omega))|\hat{g}_{\gamma,r}|^2\,d\omega\,,$$

where $g_{\gamma,r} := \theta_r h_\gamma - \theta_r\tilde{h}_\gamma$, following (4.14)-(4.16) we obtain (6.4) directly from the explicit bound of Proposition 5.5. for $\|\cdot\|_+$. □

To complete the proof of (2.10), let us now return to (3.19)-(3.21).

Proof of Theorem 2.10. Let $Y_\gamma(\phi)$ denote the stochastic integral on the left hand side of (3.21), from (3.19)

$$\mathbf{E}Y_\gamma^2(\phi) \le 4\delta_\gamma\int_0^\infty d\vartheta\int_{-\infty}^{\infty}\big(\mathscr{J}_\gamma*\phi(x,\vartheta)\big)^2\big(\beta\mathbf{E}|\xi_\gamma(x,\vartheta)| + \tilde{\beta}\mathbf{E}|\tilde{\xi}_\gamma(x,\vartheta)|\big)\,dx\,,$$

which vanishes as $\gamma \to 0$ in view of (6.1). The nonlinear term of (3.21) is controlled by Proposition 6.4, and from (6.1) we know also that all limit distributions of ζ_γ are concentrated on measurable functions. Therefore as $(\Psi_\ell * \xi_\gamma)^3$ is a continuous functional on $\mathscr{D}(\mathbb{R})$, we have (3.16) with probability one with respect to any limit distribution, which completes the proof by Theorem 2.8. Indeed, the Gaussian component $\tilde{\xi}$ is uniquely determined, and the proof of Theorem 2.8 yields a pathwise uniqueness of ξ for each given trajectory of $\tilde{\xi}$. Therefore $\mathbb{P}_\gamma \Longrightarrow \mathbb{P}$ as $\gamma \to 0$, where $\mathbb{P}$ is a uniquely specified joint distribution of ξ and $\tilde{\xi}$ such that ξ solves (1.9). □

The study of the equilibrium states is based on a representation $\bar{\mu}^t_\gamma \Longrightarrow \lambda^\beta_\gamma$ as $t \to +\infty$ allowing us to get information on λ^β_γ from the a priori bounds of the previous two sections, see also (3.23).

Proof of Theorem 2.12. Let $\beta = 1 - \rho\gamma^{2/3}$ and $\tilde{\beta} = 1 - \gamma^{2/3}$ in the construction of the coupled process. Combining Proposition 5.5 and Proposition 4.11, and exploiting the translation invariance of the initial distribution we get a uniform bound

$$\int h^4_\gamma(k,\sigma)\,\bar{\mu}^t_\gamma(d\sigma) + \sum_{\ell>0} J_\gamma(\ell) \int \left(h_\gamma(k+\ell,\sigma) - h_\gamma(k,\sigma)\right)^2 \bar{\mu}^t_\gamma(d\sigma) \leq K(\rho)\gamma^{4/3} \tag{6.6}$$

for all $k \in \mathbb{Z}$ and $\gamma, t > 0$. Indeed, multiplying both sides of the inequality (5.5) by rt^{-1}, using (5.3),(4.11), and sending $r \to 0$, we obtain (6.6) by a direct computation. Since (6.6) extends by continuity to the family λ^β_γ, we see that the initial distributions $\mu_\gamma = \lambda^\beta_\gamma$ satisfy the moment condition (2.4), therefore the family $\mathscr{P}^\beta_\gamma$ of equilibrium processes is tight in $\mathbf{D}[\mathbb{R}_+, \mathscr{D}'(\mathbb{R})]$. The starting point of the identification of the limiting process is the KMS condition (3.24).

Let P denote any limit point of P^β_γ as $\gamma \to 0$, in view of the first part of (6.6), P can be considered as a Borel probability on $\mathscr{C}(\mathbb{R})$. Although P is a limit distribution with respect to the weak topology of $\mathbb{L}^4_{\text{loc}}(\mathbb{R})$, we know also that it is a stationary measure of (1.9). Since the solutions to (1.9) are continuous with probability one even if the initial value is not so, see [Iwa1] or [Fun1], we really have $P[\mathscr{C}(\mathbb{R})] = 1$. Now following the lines of the derivation of (3.16) from (3.21) via Proposition 6.4, we obtain (3.25) as a consequence of (3.24) and (6.6), consequently every limit distribution P of the Gibbs states P^β_γ is a reversible measure of the macroscopic equation. In view of the main result of [Iwa2] every reversible measure is a $P(\phi^4)_1$ Gibbs random field, see (2.11)

We also know that there is a diffusion process χ of law P^* such that $P[d\chi|\mathscr{A}^c_r] = P^*[d\chi|\mathscr{A}^c_r]$. This process is given by

$$d\chi_t = -V'(\chi_t)\,dt + \sqrt{2a}\,dw_t\,, \tag{6.7}$$

where $V \geq 0$ is smooth and $\lim V(x) = +\infty$ as $x \to \pm\infty$; w is a standard Wiener process. In fact, V is determined in terms of the ground state of a related Schrödinger equation, see page 172 of [Sim] with further references on the regularity properties of V. Let $q = q_t(x,y)$ denote the transition density of this process, and consider an $\mathscr{A}^c_r$-measurable, continuous and bounded function Φ; for all $t > 0$ we have

$$\int \Phi\, dP = \int P^*[\Psi|\mathscr{A}^c_{r+t}]\, dP \tag{6.8}$$
$$= \int dP \int\int_{-\infty}^{\infty} \frac{\Psi(y_1,y_2)q_t(\chi_{-r-t},y_1)q_{2r}(y_1,y_2)q_t(y_2,\chi_{r+t})}{q_{2r+2t}(\chi_{-r-t},\chi_{r+t})}\, dy_1\, dy_2$$

where $\Psi(x_1,x_2) = P^*[\Phi|\mathscr{A}^c_r]$ if $\chi_{-r} = x_1$ and $\chi_r = x_2$. The evaluation of (6.8) is now based on the ergodic properties of P^*.

It is plain that (6.7) defines a reversible process: $v(x)q_t(x,y) = v(y)q_t(y,x)$ is an identity if $v(x) := \exp(-aV(x))$. By a direct calculation we obtain an energy inequality,

$$\int_{-\infty}^{\infty} \left(\phi(t,x) - \bar{\varphi}\right)^2 v(x)\, dx + 2a \int_0^t ds \int_{-\infty}^{\infty} \left(\partial_x \psi(s,x)\right)^2 v(x)\, dx$$
$$\leq \int (\varphi(x) - \bar{\varphi})^2 v(x)\, dx$$

where $\varphi : \mathbb{R} \mapsto \mathbb{R}$ is continuous and bounded,

$$\bar{\varphi} := \frac{\int_{-\infty}^{\infty} \varphi(x)v(x)\, dx}{\int_{-\infty}^{\infty} v(x)\, dx}, \quad \psi(t,x) := \int_{-\infty}^{\infty} q_t(x,y)\varphi(y)\, dy .$$

In the same way as above we see that the space integral of $(\partial_x \psi)^2 v$ is again a decreasing function of time, consequently

$$\int_{-\infty}^{\infty} (\phi(t,x) - \bar{\varphi})^2 v(x)\, dx + 2at \int_{-\infty}^{\infty} \left(\partial_x \psi(t,x)\right)^2 v(x)\, dx$$
$$\leq \int_{-\infty}^{\infty} (\varphi(x) - \bar{\varphi})^2 v(x)\, dx . \tag{6.9}$$

This means that for any $\varepsilon > 0$ and $\ell < +\infty$ we have some $T = T_\varepsilon(\ell,\varphi)$ such that $|\psi(t,x) - \bar{\varphi}| \leq \varepsilon$ whenever $t > T$. Moreover, T depends only on the bound of φ. Therefore (6.8) turns into $P(\Phi) = P^*(\Phi)$ by sending $t \to +\infty$, which completes the proof. □

Acknowledgements. We are indebted to P. Bleher, J.L. Lebowitz and E. Presutti for stimulating discussions and useful comments. This work was supported in parts by Hungarian National Foundation for Scientific Research, grant No. 1902, by a European Community Grant, CIPA-CT92-4016, and by CNR, Italy. We enjoyed very kind hospitality at the University of Rome II. Tor Vergata, and at the Eötvös Loránd University of Budapest, respectively.

References

[BPRS] Bertini L., Presutti E., Rüdiger B., Saada, E., Dynamical fluctuations at the critical point, Th. Probab. Appl. **38** (1993), 689–741

[COP] Cassandro M., Orlandi E., Presutti E., Interface and typical Gibbs configurations for one-dimensional Kac potentials, Probab. Th. Rel. Fields **96** (1993), 57–96

[DOPT] De Masi A., Orlandi E., Presutti E., Triolo L., Glauber evolution with Kac potentials: I. Macroscopic equations and fluctuation theory. CARR Report No. 9/92

[DP] De Masi A., Presutti E., Mathematical Methods of Hydrodynamical Limits, Lecture Notes in Mathematics No. 1501, Springer-Verlag, Berlin-Heidelberg, 1991 Berlin-Heidelberg

[EK] Ethier S.N., Kurtz T.G., Markov Processes, Characterization and Convergence, Wiley, New York, 1986

[Fo] Fouque J.P., La convergence en loi pour les processus à veleurs dans un espace nucléaire, Ann. Inst. Henri Poincaré – Probabilités et Statistiques **20** (1984), 225–245

[Frie] Friedman A., Partial Differential Equations of Parabolic Type, Prentice-Hall, Englewood Cliffs, New Jersey, 1964

[Frit] Fritz J., On the hydrodynamic limit of a Ginzburg-Landau lattice model. The law of large numbers in arbitrary dimensions, Probab. Th. Rel. Fields **81** (1989), 291–318

[FR] Fritz J., Rüdiger B., Approximation of a one-dimensional stochastic PDE by local mean field type lattice systems, IMA Volumes in Mathematics, to appear

[Fun1] Funaki T., Regularity properties for stochastic partial differential equations of parabolic type, Osaka J. Math. **28** (1991), 495–516

[Fun2] Funaki T., Reversible measures of multi-dimensional Ginzburg-Landau type continuum model, Osaka J. Math. **28** (1991) 463–494

[HH] Hochenberg P.C., Halperin P.I., Theory of dynamical critical phenomena, Rev. Mod. Phys. **49** (1977), 435–479

[HS1] Holley R., Stroock D.W., Generalized Orstein-Uhlenbeck processes and infinite particle branching Brownian motions, Publ. RIMS, Kyoto Univ. A **14** (1978), 741–814

[HS2] Holley R., Stroock D., Central limit phenomena of various interacting systems, Ann. Math. **110** (1979), 333–393

[Iwa1] Iwata K., An infinite dimensional stochastic differential equation with state space C(R), Probab. Th. Rel. Fields **74** (1987), 141–159

[Iwa2] Iwata K., Reversible measures of a $P(\phi)_1$ time evolution, Probabilistic Methods in Mathematical Physics (Ito, K., Ikeda, N., eds.) Proceedings of Taniguchi Symposium, Katata, 1985, pp. 195–209

[KUH] Kac M., Uhlenbeck G., Hemmer P.C., On the van der Waals theory of vapour-liquid equilibrium. I. Discussion of a one dimensional model, J. Math. Phys. **4** (1963), 216–228

[Lig] Liggett T.M., Interacting Particle Systems, Springer-Verlag, New York, 1985

[LP] Lebowitz J., Penrose O., Rigorous treatment of the van der WaalsMaxwell theory of the liquid-vapour phase transition, J. Math. Phys **7** (1966), 98–113

[MT] Mueller, C., Tribe R., A stochastic PDE arising from the long range voter model, Probab. Th. Rel. Fields **100** (1994), 131–142

[Sim] Simon B., Functional Integration and Quantum Physics, Academic Press, New York, 1979

[Wal] Walsh J.B., An introduction to stochastic partial differential equations, , Springer-Verlag, Berlin Heidelberg New York, Lecture Notes in Mathematics **1180** (1986), 265–439