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Probability Theory and Related Fields

Drift and diffusion for a mechanical system

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Summary. We consider a mechanical system in the plane, consisting of a vertical rod of length ℓ, with its center moving on the horizontal axis, subject to elastic collisions with the particles of a free gas, and to a constant force f. Assuming a suitable initial measure we show that the evolution of the system as seen from the rod is described by an exponentially ergodic irreducible Harris chain, implying convergence to a stationary invariant measure as $t \to \infty$. We deduce that in the proper scaling the motion of the rod is described as a drift plus a diffusion. We prove in conclusion that the diffusion is nondegenerate and that the drift is nonzero if $f \neq 0$ and has the same sign of f.

0 Introduction

In Physics the motion of a charged particle (c.p.) in a neutral gas in presence of a constant electric field is often described as a drift plus a brownian motion. Rigorous results in this sense are however difficult to obtain for any reasonable mechanical model, and they are not easy even if one simplifies the problem by eliminating the mutual interaction of the gas particles, by introducing additional rescalings, or even by resorting to stochastic evolution (see, e.g., [7, 9]). The main difficulty lies in the fact that one deals with a real nonequilibrium situation and there is no natural invariant measure given in advance. The invariant Gibbs measure for the coupled system cannot be normalized, for infinite volume, and it is not clear how it is related to the stationary probability distribution that is needed. We refer for this problem to the discussion in [9].

We study here a mechanical model in the plane, which consists of a rod of length ℓ and of mass M and a free gas of point-like particles of common mass m. The rod is supposed to move with its center on a line, called by convention

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"horizontal", keeping a position orthogonal to the line. The interaction of the rod with the free gas consists of elastic collisions, and a constant horizontal force f acts on the rod only. Under collision the particles preserve their vertical velocity, so that they eventually get out of the region of the plane accessible to the rod (a horizontal strip, denoted hereafter by $\mathscr{S}$), never to return.

The choice of the vertical velocity distribution is a delicate problem, as particles with small vertical velocity can stay for a long time in $\mathscr{S}$, preserving the "memory" of the past, and causing long time tails in the correlations. Such tails can be even nonintegrable, as it is apparently the case for the hard sphere tracer particle in the plane, for which numerical simulations show that the velocity correlations fall off as t^{-1}, so that diffusion, at least in the usual form, should not hold (see, e.g., [19]). For our model it is not clear whether the "fresh" particles coming from outside $\mathscr{S}$ provide enough stochasticity for the time correlations to be integrable. They should be integrable, however, for the equivalent model in dimension three or more. In view of the technical complexity of the problem, we remove all difficulties connected to small vertical velocities by assuming that the vertical velocity distribution has a "hole" in 0, i.e., denoting by v_2 the vertical velocity we assume that $|v_2| > u_0$, for some $u_0 > 0$ for all particles. This ensures that all particles which are at a given time inside $\mathscr{S}$, will get out of it after a "renewal time" $\tau = \ell/u_0$.

We assume as initial distribution μ an equilibrium Gibbs measure for the system "as seen from the rod" for $f = 0$, corresponding to some particle density ρ and some inverse temperature β. That is, particle positions are Poisson distributed in the plane $\mathbb{R}^2$, and the horizontal velocities of the particles and of the rod are Maxwell distributed. The vertical velocity distribution is denoted by $h(dv_2)$ and is assumed to have finite first moment, in order that the dynamics be well defined. One could consider other initial distributions, but absolute continuity and exponential falloff of the tails for the distribution of the horizontal velocity of the particles seem to be essential.

The existence of the dynamics for all times is not obvious for our model. In the Appendix A we prove that the dynamics exists for all times on a set of full measure.

By our choice of the initial measure the distribution of the particles coming into $\mathscr{S}$ is equal to the original Poisson distribution at all times, so that the evolution of the configuration in $\mathscr{S}$, "as seen from the rod" (which is given by positions and velocities of the particles in $\mathscr{S}$ and by the velocity V of the rod, and is denoted by X) is described by a Markov process with stationary (i.e., homogeneous in time) transition probabilities. It is convenient to consider the Markov chain obtained by observing the configuration X at discrete times $t_k = k\hat{\tau}$, with $\hat{\tau} > \tau$.

The main part of the paper is devoted to the proof that the chain is irreducible, aperiodic, and ergodic in the sense of Harris. The invariance of the (infinite) Gibbs equilibrium measure for the coupled system plays here an important role. As a consequence, we obtain the existence of a unique invariant measure ν_f for the system in the strip $\mathscr{S}$, "as seen from the rod", and we get in addition that ν_f is absolutely continuous with respect to the initial measure π, induced by μ. We prove in fact that the chain is geometrically ergodic [15], a result which implies that convergence to the invariant measure is exponentially fast, and hence that the time correlations fall off exponentially. The proof relies essentially on a suitable "relaxation condition" of the rod velocity. The argument is fairly general, and we present it separately in the Appendix B.

The properties of v_f imply the existence of a drift, and diffusion follows from exponential ergodicity [15]. We conclude by proving that the diffusion constant is positive, and that the drift is nonzero for $f \neq 0$ and has the sign of f.

To our knowledge there is no other rigorous result on the existence of drift and diffusion for a driven mechanical model in interaction with a free gas. Among the scanty results on related problems it is worth to mention here the ones in [2, 5].

The main technical point in the proof consists in the analysis of some events which occur with finite probability in a finite time and ensure the required randomness of the dynamics. The events should of course be simple enough, so that they may be fully controlled. Similar ideas have been applied in equilibrium situations by several authors in order to prove strong ergodic properties of some infinite particle systems with "local" interaction. We refer the interested reader to the papers [1, 4, 8, 10, 11, 17, 18].

The authors hope to provide in a future paper a proof of the validity of the Einstein relation between mobility and diffusion, which requires some additional uniform (in f) estimates.

For models with stochastic evolution an important result has been recently obtained by Lebowitz and Rost [14]. For a wide class of models of self diffusion they prove convergence, in some proper scaling, to a noncentered brownian motion, with drift and diffusion satisfying the Einstein relation.

Finally we would like to mention a very nice result [6] for a particular model of (deterministic) Lorentz gas, in which the absorption of energy by the medium is simulated by the so-called "gaussian dynamics". The result makes use of the powerful Markov partition techniques, and is remarkably complete, in that drift, diffusion, and the Einstein relation are rigorously proved. It is however not clear at the moment how gaussian dynamics can mimic the absorbtion of energy by the scatterers when they are not fixed.

1 Notation and formulation of the results

We consider a mechanical system in the plane $\mathbb{R}^2$, the points of which are denoted by $q = (q_1, q_2)$. We use the adjectives "horizontal" and "vertical" in reference to the q_1 and the q_2 axis, respectively. The system consists of a rod of mass M and of a gas of infinitely many point-like particles with common mass m. The rod, of length $\ell > 0$ and infinitely thin, is subject to the constraint of keeping a vertical position, with its center moving on the horizontal axis. The coordinate of the rod center is denoted by Q and its velocity by V.

We assume throughout that $M > m$. The case $M < m$ would require substantial changes in the proofs and is not considered here. The equal mass case is easier, but also requires some changes and is not considered.

The ideal gas is described by a locally (in q) finite subset Y of the one-particle phase space $\mathbf{M} \equiv \mathbb{R}^2 \times \mathbb{R}^2$. The collection of such subsets is denoted by $\mathcal{Y}$. As usual a point $(q, v) \in Y$ with $q = (q_1, q_2)$, $v = (v_1, v_2)$, stays for a particle with position q and velocity v. The topology of $\mathcal{Y}$ is the one for which a fundamental set of neighborhoods of the point $Y_0 \in \mathcal{Y}$ is provided by the sets $\mathcal{A}^n_{C,B} = \{Y \in \mathcal{Y}\colon\ |Y \cap C \times B| = n\}$, $n = |Y_0 \cap C \times B|$. Here by $|\cdot|$ we denote the cardinality of a set, n is a nonnegative integer number, and C and B are open subsets of $\mathbb{R}^2$ such that C is bounded and $Y_0 \cap \partial C \times B = \emptyset$. ($\partial\cdot$ denotes the boundary of a set.) The space $\mathcal{Y}$ with the given topology is a polish space (see, for instance [13]). We denote by $\mathfrak{M}$ the σ-algebra of the Borel subsets of $\mathcal{Y}$.

A configuration of the whole system is described by a point $\hat{\omega} = ((Q, V), Y)$ in the "extended phase space" $\hat{\Omega} = \mathbb{R}^2 \times \mathcal{Y}$. We also consider a reference frame moving with the rod, with the axes parallel to the fixed one, and the origin at the center of the rod. In the moving frame the system is described by a point $\omega = (V, Y)$ in the phase space $\Omega = \mathbb{R}^1 \times \mathcal{Y}$. The notation $V(\omega)$, $Q(\hat{\omega})$, $Y(\omega)$ and similar ones will be used for the projections.
For $B \subset \mathbf{M}$, and $a \in \mathbb{R}$ we denote by $B + a$ the horizontal shift

$$B + a = \{(q, v) \in \mathbf{M}:\ ((q_1 - a, q_2), v) \in B\}\,, \tag{1.1}$$

and the same notation will be used when $B \subset \mathbb{R}^2$ is a subset of the configuration space.

The topology on both $\hat{\Omega}$ and Ω is the product topology, and we denote by $\hat{\mathfrak{B}}$ and by $\mathfrak{B}$, respectively, the corresponding Borel σ-algebras. Ω can be seen as the quotient of $\hat{\Omega}$ with respect of the group of the horizontal shifts. One can identify Ω with the subset of $\hat{\Omega}$ corresponding to $Q = 0$, and define a projection $\Pi\colon\ \hat{\Omega} \to \Omega$ as follows

$$\Pi((Q, V), Y) = (V, Y - Q)\,. \tag{1.2}$$

Q and ω can be seen as the coordinates of $\hat{\omega} \in \hat{\Omega}$.

We introduce the measures μ on $(\Omega, \mathfrak{B})$ and $\hat{\mu}$ on $(\hat{\Omega}, \hat{\mathfrak{B}})$:

$$\mu(d\omega) = \sqrt{\frac{\beta M}{2\pi}} e^{-\beta M V^2/2} dV \mathcal{P}(dY), \quad \hat{\mu}(d\hat{\omega}) = \mu(d\omega) \times dQ\,. \tag{1.3a}$$

Here $\mathcal{P}$ is the Poisson field on the one-particle phase space $\mathbf{M}$ with the intensity measure

$$n(dq\, dv) = \rho \sqrt{\frac{\beta m}{2\pi}} e^{-\beta m v_1^2/2} h(dv_2) dv_1\, dq_1\, dq_2\,, \tag{1.3b}$$

ρ and β are positive constants, corresponding to the average particle density and to the inverse temperature of the system, $h(\cdot)$ is the distribution of the vertical velocity. We assume that the first moment exists,

$$m_2 = \int h(dv)|v| < \infty\,, \tag{1.4}$$

and that vertical velocities are separated from 0, i.e., for some $u_0 > 0$

$$h(\{v_2\colon\ |v_2| \leqq u_0\}) = 0\,. \tag{1.5}$$

The region of the plane accessible to the rod is

$$\mathscr{S} \equiv \{q \in \mathbb{R}^2 : \ |q_2| < \ell/2\} . \tag{1.6a}$$

We shall use a special notation for some subsets of $\mathbf{M}$, and for the corresponding configurations:

$$\begin{aligned} \mathbf{M}^s &= \mathscr{S} \times \mathbb{R}^2, & Y^s &= Y \cap \mathbf{M}^s \\ \mathbf{M}^+ &= \{(q, v) \in \mathbf{M} \backslash \mathbf{M}^s : \ q_2 v_2 < 0\}, & Y^{\text{in}} &= Y \cap \mathbf{M}^+ \\ \mathbf{M}^- &= \{(q, v) \in \mathbf{M} \backslash \mathbf{M}^s : \ q_2 v_2 > 0\}, & Y^{\text{out}} &= Y \cap \mathbf{M}^- . \end{aligned} \tag{1.6b}$$

$\mathbf{M}^+(\mathbf{M}^-)$ is the portion of the phase space $\mathbf{M}$ where the particles that visit $\mathscr{S}$ in the future (in the past) are located. Y^{in} and Y^{out} are called the "ingoing" and "outgoing" configurations respectively. The corresponding spaces are $\mathscr{Y}^{\text{in}} = \{Y^{\text{in}} : \ Y \in \mathscr{Y}\}$, and $\mathscr{Y}^{\text{out}} = \{Y^{\text{out}} : \ Y \in \mathscr{Y}\}$. The marginal distributions on $\mathscr{Y}^{\text{in}}$ or $\mathscr{Y}^{\text{out}}$ induced by $\mathscr{P}$ will as a rule be denoted by the same symbol $\mathscr{P}$.

The subsystem in $\mathscr{S}$ in the fixed reference frame and in the moving one is described by

$$\hat{X} = \hat{X}(\hat{\omega}) = ((Q, V), Y^s), \qquad X = X(\omega) = (V, Y^s) , \tag{1.7}$$

respectively. We also write $\hat{X} = (Q, X)$. The corresponding phase spaces and σ-algebras are denoted by $\hat{\mathscr{X}}$, $\hat{\mathfrak{B}}^s$ and $\mathscr{X}, \mathfrak{B}^s$, respectively. We use the notation $V(X)$, $Y^s(X), Q(\hat{X})$ and similar ones for the projections.

For $L > 0$ we define

$$\mathscr{S}_L = \{q \in \mathscr{S} : \ |q_1| < L\}, \qquad \mathbf{M}^s_L = \mathscr{S}_L \times \mathbb{R}^2 . \tag{1.8}$$

We denote by π the measure induced by μ on $\mathscr{X}$:

$$\pi(A) \equiv \mu(\{\omega : \ X(\omega) \in A\}), \quad A \in \mathfrak{B}^s . \tag{1.9}$$

The dynamics on $\hat{\Omega}$ is the usual dynamics corresponding to elastic collisions. All particles keep their velocity until they collide with the rod. Upon collision, the vertical velocities do not change, and the horizontal velocities change according to the following formulas, in which v_1 and V denote the incoming velocities of the particle and of the rod, respectively, and v_1' and V' the corresponding outgoing velocities

$$\begin{aligned} V' &= \alpha V + (1 - \alpha) v_1 , \\ v_1' &= (1 + \alpha) V - \alpha v_1 . \end{aligned} \tag{1.10}$$

Here $\alpha = (M - m)/(M + m) > 0$, as $M > m$. Between collisions the rod moves with a constant acceleration f/M. The dynamics is completed by prescribing velocities at collision times to be the outgoing ones.

The flow on $\hat{\Omega}$ corresponding to the dynamics just described is denoted by $\{\hat{T}^t\colon\ t \in \mathbb{R}\}$. The corresponding flow $\{T^t\colon\ t \in \mathbb{R}\}$ on Ω is given by the relation

$$T^t \Pi \hat{\omega} = \Pi \hat{T}^t \hat{\omega}\,,$$

where Π is the projection defined in (1.2). The definition is correct, since the flow $\{\hat{T}^t\colon\ t \in \mathbb{R}\}$ commutes with the group of the horizontal shifts

$$\hat{T}^t(\hat{\omega} + a) = \hat{T}^t\hat{\omega} + a\,.$$

The above prescriptions define the dynamics only for those initial points $\omega \in \Omega$ such that:

(i) There are no multiple collisions, i.e., the particles collide with the rod one at a time.

(ii) The rod undergoes only a finite number of collisions in a finite time.

(iii) The configuration $Y(T^t\omega) \cap (B \times \mathbb{R}^2)$, where $B \subset \mathbb{R}^2$ is any measurable subset of finite Lebesgue measure, is finite for all $t \in \mathbb{R}$.

As usual for infinite particle systems, one has to prove the existence of a subset $\Omega' \subset \Omega$, for which the dynamics is defined, invariant with respect to the dynamics, and of full measure with respect to the initial measure.

The following result is proved in the Appendix A.

Theorem 1.1 *There is a measurable subset $\Omega' \subset \Omega$, invariant with respect to the dynamics, such that the dynamics exists for $\omega \in \Omega'$, and $\mu(\Omega') = 1$ for all choices of ρ, β and of the vertical velocity distribution h satisfying condition* (1.4).

We set

$$\hat{\Omega}' = \{(Q, \omega)\colon\ \omega \in \Omega'\}, \quad \mathscr{X}' = \{X(\omega)\colon\ \omega \in \Omega'\}\,. \tag{1.11}$$

By Theorem 1.1, the dynamics $\{\hat{T}^t\colon\ t \in \mathbb{R}\}$ exists for $\hat{\omega} \in \hat{\Omega}'$, and we have $\hat{\mu}(\hat{\Omega}\backslash\hat{\Omega}') = 0$, $\pi(\mathscr{X}') = 1$. For $f = 0$ μ is an equilibrium measure, invariant under $\{T^t\colon\ t \in \mathbb{R}\}$.

The "free dynamics" $\{T_0^t\colon\ t \in \mathbb{R}\}$ is defined for any subset $C \subset \mathbf{M}$ as

$$T_0^t C = \{(q, v) \in \mathbf{M}\colon\ (q - tv, v) \in C\}\,. \tag{1.12}$$

The free dynamics is well defined if we have for all times only a finite number of particles in finite volumes. It is a consequence of our proof of Theorem 1.1 (Appendix A) that the free dynamics is defined for all times for all $Y(\omega)$, $\omega \in \Omega'$.

The evolution of the system in the strip $\mathscr{S}$, denoted by

$$X_t(\omega) \equiv X(T^t\omega), \quad t > 0, \ \ \omega \in \Omega' \tag{1.13}$$

is described by a Markov process the transition probabilities of which are determined by the Poisson measure $\mathscr{P}$ as follows

$$P^t(X, A) = \mathscr{P}(\{Y \in \mathscr{Y}\colon\ T^t(X \cup Y^{\text{in}}) \in A\}), \quad X \in \mathscr{X}', \ \ A \in \mathfrak{B}^s\,. \tag{1.14}$$

The restrictions of $\mathfrak{B}^s$ and π to $\mathscr{X}'$ are denoted by the same symbol. The transition kernel $P^t(X, \cdot\,)$ is then defined for $X \in \mathscr{X}'$, as a probability measure on the measurable space $(\mathscr{X}', \mathfrak{B}^s)$.

We now state the main results of the present paper. The proofs are given in Sect. 2.

Theorem I (Convergence to the invariant measure) *There is a probability measure* ν_f *on* $\mathfrak{B}^s$, *invariant with respect to* $\{P^t: t \geqq 0\}$, *equivalent to* π, *and such that for all* $X \in \mathscr{X}'$

$$\|P^t(X, \cdot) - \nu_f\| \leqq g(X)e^{-\kappa t},$$

where $\|\cdot\|$ *denotes the variation distance between measures,* κ *is a positive constant and* $g \in L^1(\nu_f)$.

We denote the displacement of the rod up to time t by

$$Q(t, \omega) = \int_0^t V(T^s\omega)ds. \tag{1.15}$$

Theorem II (Drift) *The limit*

$$d_f \equiv \lim_{t\to\infty} \frac{1}{t} Q(t, \omega) = \mathbb{E}_{\nu_f} V$$

exists, is finite, and does not depend on ω, *for* μ-*a.a.* $\omega \in \Omega'$. *Moreover* $f d_f > 0$ *for* $f \neq 0$.

Theorem III (Diffusion) *The process*

$$\xi_t(s) = \frac{Q(st, \omega) - d_f st}{\sqrt{t}}$$

tends weakly, as $t \to \infty$, *in the space of the continuous functions of* s, *to the Wiener process* $W_{\sigma_f}(s)$ *with nondegenerate diffusion constant* $\sigma_f > 0$.

2 Proofs

The dynamics $\{\hat{T}^t: t \in \mathbb{R}^1\}$ in $\hat{\Omega}'$ can be expressed as

$$\hat{T}^t(Q, \omega) = \left(Q + \int_0^t V(T^s\omega)ds, \quad T^t\omega\right), \tag{2.1}$$

and the Gibbs measure

$$\hat{\mu}_f(d\hat{\omega}) = e^{\beta f Q}\mu(d\omega)dQ = e^{\beta f Q}\hat{\mu}(d\hat{\omega}) \tag{2.2}$$

is invariant for $\{\hat{T}^t\}$. Consider the function $\psi(\hat{\omega}) = \phi(\omega)\mathbb{I}_J(Q)$, where $\phi(\cdot)$ is a bounded measurable function, $J = (a, b)$ is an interval and $\mathbb{I}$. denotes the indicator function. ψ is summable with respect to $\hat{\mu}_f$, and the invariance condition implies, for all $t \in \mathbb{R}$,

$$\int_{\hat{\Omega}} \hat{\mu}_f(d\hat{\omega})\psi(\hat{\omega}) = \int_{\hat{\Omega}} \hat{\mu}_f(d\hat{\omega})\psi(\hat{T}^t\hat{\omega}).$$

By manipulating the right-hand side we get

$$\int_\Omega \mu(d\omega)\phi(\omega)\int_a^b e^{\beta f Q}dQ = \int_\Omega \mu(d\omega)\phi(T^t\omega)e^{-\beta f \int_0^t V(T^s\omega)ds}\int_a^b e^{\beta f Q}dQ\,,$$

which implies, since a and b are arbitrary,

$$\int_\Omega \mu(d\omega)\phi(\omega) = \int_\Omega \mu(d\omega)\phi(T^t\omega)e^{-\beta f \int_0^t V(T^s\omega)ds}\,. \tag{2.3}$$

As a consequence we have the following result.

Lemma 2.1 *Let* $\{\mu_t\colon\ t \in \mathbb{R}\}$, $\mu_0 = \mu$, *denote the family of the measures generated by the dynamics* $\{T^t\}$ *on* Ω'. *Then for all* $t \in \mathbb{R}$ *the measure* μ_t *is equivalent to the measure* μ, *and the Radon Nikodým derivative is equal to*

$$\frac{d\mu_t}{d\mu}(\omega) = e^{\beta f \int_0^t V(T^{-t+s}\omega)ds}\,. \tag{2.4}$$

Proof. It is enough to replace ω by $T^{-t}\omega$ in Eq. (2.3). □
In what follows $X \cup Y$ denotes the point $\omega = (V(X), Y^s(X) \cup Y)$.

Lemma 2.2 *For any* $t \geqq 0$ *the measures* πP^t *and* π *are equivalent on* $\mathfrak{B}^s$, *and the Radon Nikodým derivative of* πP^t *with respect to* π *is given by the formula*

$$\frac{d(\pi P^t)}{d\pi}(X) \equiv \rho_t(X) = \int \mathscr{P}(dY)e^{\beta f \int_0^t V(T^{-s}(X\cup Y^{\text{out}}))ds}\,. \tag{2.5}$$

Proof. For $A \in \mathfrak{B}^s$ we have, using Lemma 2.1,

$$\begin{aligned}\pi P^t(A) &= \mu_t(\{\omega\colon\ X(\omega) \in A\})\\ &= \int_A \pi(dX)\int_{\mathscr{Y}} \mathscr{P}(dY)e^{\beta f \int_0^t V(T^{-s}(X\cup Y^{\text{out}}))ds} = \int_A \rho_t(X)\pi(dX)\,,\end{aligned}$$

which proves the lemma. □

For $(q, v) \in \mathbf{M}$ we consider the parameters

$$\begin{aligned}&t^e(q, v) = \inf\ \{t\colon\ q + vt \in \mathscr{S}\} \quad && t^o(q, v) = t^o(q_2, v_2) = \sup\ \{t\colon\ q + vt \in \mathscr{S}\}\,,\\ &q_1^e(q, v) = q_1 + t^e(q, v)v_1, && q_1^o(q, v) = q_1 + t^o(q, v)v_1\,,\end{aligned} \tag{2.6}$$

which are the entrance and exit times in $\mathscr{S}$ of the particle (q, v), and its horizontal coordinates at those times. For all particles, by (1.5), we have $0 < t^o(q, v) - t^e(q, v) < \tau$ where

$$\tau = \frac{\ell}{u_0}\,. \tag{2.7}$$

Furthermore, for $t_1 < t_2$, and for any measurable $B \subset \mathbf{M}$, we denote by

$$\mathbf{N}(t_1, t_2; B) = \bigcup_{t \in (t_1, t_2)} T_0^{-t} B\,, \tag{2.8}$$

the set of the points that pass through B by the free dynamics in the time interval (t_1, t_2). Finally, for $B \subset \mathbf{M}$, $X \in \mathscr{X}'$ we introduce the quantities

$$W(B) = \sup\left\{|v_1|:\ (q, v) \in B:\ q_1 v_1 < 0, \tau|v_1| \geqq \frac{1}{4}|q_1|\right\}\,,$$
$$W(X) = \max\{|V(X)|, W(Y^s(X))\}\,. \tag{2.9}$$

In what follows we assume throughout that $f > 0$. The case $f = 0$ is actually simpler, but some of the formulas have to be changed. For the sake of brevity we leave to the reader the fairly simple extension of the results to the case $f = 0$.

Remark. 2.1 The condition $q_1 v_1 < 0$ identifies the particles the velocity of which is directed towards the rod, hereafter denoted as "incoming particles". (The particles with $q_1 v_1 > 0$ are called "outgoing".) Only incoming particles can increase the velocity of the rod (in absolute value) by collision. Let $X = (V, Y^s)$. The condition $W(X) < U$ is equivalent to $|V| < U$ plus the condition $\mathscr{A}_U \cap Y^s = \emptyset$, where

$$\mathscr{A}_U = \{(q, v) \in \mathbf{M}:\ q_1 v_1 < 0,\ |q_1| \leqq 4\tau|v_1|,\ |v_1| \geqq U\}\,.$$

Let $W(X) < U$ and consider the dynamics $\{T^s X:\ s \in \mathbb{R}\}$, i.e., the dynamics of the point $\omega = (V, Y^s)$. If $(q, v) \in Y^s$ is an incoming particle with $|v_1| \geqq U$, and no collision occurs, then $|q_1| > 4|v_1|\tau$ and the absolute value of the horizontal position at time τ is $||q_1| - |v_1|\tau|$, so that its distance from the origin is larger than $3U\tau$ for all $s \in [0, \tau]$. Moreover, if the rod does not collide with incoming particles with velocity larger than U in absolute value, it cannot travel more than a distance $U\tau + (f/2M)\tau^2$. If U is so large that $f\tau/MU < 1$, then $U\tau + (f/2M)\tau^2 < \frac{3}{2}U\tau$. Hence $W(X) < U$ implies that in the dynamics $\{T^s X\}$ the rod does not collide in the time interval $[0, \tau]$ with incoming particles with velocity larger in absolute value than U.

Lemma 2.3 *There is a constant C such that for all U large enough*

$$\pi(\{X:\ W(X) \geqq U\}) < e^{-CU^2}\,. \tag{2.10}$$

Proof. Setting $d_k = \max\{1, (|k| - 1)\}$ and

$$A_k = \{\max\{|v_1|:\ (q, v) \in Y^s,\ q_1 \in (4k\tau U, 4(k+1)\tau U)\} > d_k U\}\,,$$

we have

$$\{W(X) \geqq U\} \subset \bigcup_{k=-\infty}^{\infty} A_k \cup \{|V| \geqq U\}\,.$$

Since $\pi(A_k) \leqq c_1 U e^{-c_2 d_k^2 U^2}$, for some $c_1, c_2 > 0$, we get the result by summing over k. □

Remark. 2.2 As a consequence of Lemma 2.3 $W(X) < \infty$ π-a.e. . Let $\mathbf{P}$ denote the velocity reversal, i.e. the operation that changes Y into $\mathbf{P}Y = \{(q, v) : (q, -v) \in Y\}$, and $X = (V, Y^s) \in \mathscr{X}$ into $\mathbf{P}X = (-V, \mathbf{P}Y^s)$. The measure π is left invariant by $\mathbf{P}$, hence $W(\mathbf{P}X) < \infty$ π-a.e. . It is convenient to include in the definition of $\mathscr{X}'$ the condition $W(X) + W(\mathbf{P}X) < \infty$. Note that one can define $\mathbf{P}$ in Ω and $\hat{\Omega}$ in an obvious way, and clearly $\mathbf{P}$ leaves the measures μ and $\hat{\mu}$ invariant.

Lemma 2.4 *For any $X_0 \in \mathscr{X}'$ and $t > 3\tau$ the measure π is absolutely continuous with respect to $P^t(X_0, \cdot)$.*

Proof. Though based on simple ideas, the proof is technically involved. We begin with a brief intuitive discussion, which may serve as a guideline to the reader.

The proof relies on a simple mechanism that brings the initial point X_0 into any other fixed, but arbitrary, point X at time t, and is easily described as follows. Consider the image $T^{-s_1}X$ of X under the backward dynamics in the absence of particles outside $\mathscr{S}$, as described in Remark 2.1, for $s_1 > \tau$. $T^{-s_1}X = (V(T^{-s_1}X), Y(T^{-s_1}X))$, and the particles are outside $\mathscr{S}$ in the region $T_0^{-s_1}\mathbf{M}^s$. Let now Y^{in} be such that in the dynamics $T^s(X_0 \cup Y^{\text{in}})$ the following happens: (i) no particles of Y^{in} enter the strip $\mathscr{S}_L$ up to time $s_0 > \tau$, for some $L > 2s_0 U$ and $U > W(X_0)$ large enough, except one particle (q^*, v^*) which is such that it collides with the rod at time s_0 and the velocity of the rod jumps from $V(T^{s_0}X_0)$ to $V(T^{-s_1}X)$; (ii) the configuration of the particles of Y^{in} that at time $t = s_0 + s_1$ are in $\mathscr{S}$, shifted forward by s_0 in time, and seen from the rod position at that time (i.e., the configuration $T_0^{s_0} Y^{\text{in}} \cap T_0^{-s_1}\mathbf{M}^s - Q(T^{s_0}X_0)$) coincides with $Y(T^{-s_1}X)$. If no recollision with the particle (q^*, v^*) takes place, it is easy to see that $X(T^t(X_0 \cup Y^{\text{in}})) = X$, since the particle (q^*, v^*) is out of $\mathscr{S}$ by time t, and the dynamics T^{s_1} applied to $(V(T^{-s_1}X), Y(T^{-s_1}X))$ will restore X.

The technical part of the proof consists in showing that the configurations Y^{in} constructed in this way and such that $X(T^t(X_0 \cup Y^{\text{in}})) \in A$ with $\pi(A) > 0$ have positive measure $\mathscr{P}$. Condition (i) can be changed into a positive measure condition for each X by requiring that the colliding particle is in some small neighborhood $\mathscr{U}$ of (q^*, v^*). To accomplish the proof we then use the fact that the velocity of the rod after collision is a C^1 map of full rank on $\mathscr{U}$, and the absolute continuity of the time shifts of μ with respect to μ.

Let $X_0 \in \mathscr{X}'$ be fixed, and the positive number U be such that $W(X_0) < U$. We set

$$A_U \equiv \{X \in \mathscr{X}' : (W(X) + W(\mathbf{P}X)) < U\}, \tag{2.11}$$

and fix some positive numbers τ', τ'' and L so that

$$2\tau + \tau'' < t - \tau' < t - \tau, \quad L > 2(U + (f/M)t)t. \tag{2.12}$$

We split the incoming configuration $Y \in \mathscr{Y}^{\text{in}}$ as $Y = \hat{Y}_1 \cup \hat{Y}_2 \cup Y_3$, with

$$\hat{Y}_1 = Y \cap T_0^{-t+\tau'}\mathbf{M}^-, \quad \hat{Y}_2 = Y \cap T_0^{-t+\tau'}(\mathbf{M}\backslash\mathbf{M}^-) \cap T_0^{-t}(\mathbf{M}\backslash\mathbf{M}^+). \quad (2.13)$$

In words, the particles of $\hat{Y}_1$ cross $\mathscr{S}$ and are out of $\mathscr{S}$ at time $t-\tau'$, the particles of $\hat{Y}_2$ are in $\mathbf{M}^s \cup \mathbf{M}^+$ at time $t-\tau'$, and by time t have entered $\mathscr{S}$, while the particles of the complement Y_3 enter the strip only after time t. We further set

$$\begin{aligned} \hat{Y}_1 &= Y_1 \cup \bar{Y}_1, \quad Y_1 = \hat{Y}_1 \cap \mathbf{N}(0, t-\tau'; \mathbf{M}_L^s), \qquad \bar{Y}_1 = \hat{Y}_1 \backslash Y_1\,, \\ \hat{Y}_2 &= Y_2 \cup \bar{Y}_2, \quad Y_2 = \hat{Y}_2 \cap T_0^{-t}\mathbf{M}^s = Y \cap T_0^{-t}\mathbf{M}^s, \quad \bar{Y}_2 = \hat{Y}_2 \backslash Y_2\,. \end{aligned} \quad (2.14)$$

We shall construct a subset $\mathscr{Y}_U \subset \mathscr{Y}^{\text{in}}$ by giving Y_1, Y_2 and $\bar{Y}_2 = \hat{Y}_2 \cap T_0^{-t}\mathbf{M}^-$ a special form, so that at some time $s \in (\tau, t-\tau)$, the velocity of the rod is brought close to the value $V(T^{-t+s}X)$ for $X \in A_U$.

We first observe that the difference

$$\Delta = \Delta(X) \equiv V(T^{-t+s}X) - V(T^s X_0) \quad (2.15)$$

is constant for $s \in (\tau, t-\tau)$, as no collision takes place in the dynamics $T^u X$, for $|u| > \tau$ (all particles are out of $\mathscr{S}$). We fix $v_2^* \in$ supp $h(dv_2)$, and denote by $\tau^* = l/|v_2^*| < \tau$ the corresponding "crossing time". Let $X \in A_U$ be fixed and such that

$$|\Delta| > (1-\alpha)\tau^* f/2M \equiv \Delta_0\,. \quad (2.16)$$

We choose $Y_1 = (q^*, v^*)$ with the value of v_2^* fixed above, and q_2^* such that $t_*^e = t^e(q_2^*, v_2^*) \in (\tau, \tau+\tau'')$. This ensures, by the first inequality (2.12), that $t_*^o = t_*^e + \tau^* < t_*^e + \tau < t - \tau'$, i.e., the particle is out of $\mathscr{S}$ at time $t-\tau'$. q_1^*, v_1^* are chosen as functions of the quantities

$$\bar{V}_0 = V(T^{t_*^e}X_0), \quad V^* = V(T^{t-t_*^e}X), \quad \bar{Q}_0 = \int_0^{t_*^e} V(T^s X_0)ds\,.$$

Suppose that $V^* - \bar{V}_0 = \Delta(X) > 0$. We should have $\hat{q}_1^* = q_1^* + v_1^* t_*^e - \bar{Q}_0 < 0$, since the velocity of the rod has to increase, and the collision is on the left. We fix the collision time $t_* \in (t_*^e, t_*^o)$, and determine q_1^* and v_1^* by solving the equations

$$(v_1^* - \bar{V}_0)^2 + 2\hat{q}_1^* \frac{f}{M} = \frac{\Delta^2}{(1-\alpha)^2}, \qquad t_* - t_*^e = \frac{M}{f}\left(v_1^* - V_0 - \frac{\Delta}{1-\alpha}\right). \quad (2.17)$$

The dynamics $T^s(X_0 \cup Y_1)$ is then such that a single collision takes place at the time t_*, and the velocity of the rod jumps by Δ. For $\Delta < 0$ the collision is on the right, i.e., $\hat{q}_1^* > 0$. We can again determine (q_1^*, v_1^*) by the equations (2.17), but we have to discuss possible recollisions. Inequality (2.16) shows however that no recollision occurs for the configuration $X_0 \cup (q^*, v^*)$, since the recollision time is

$$t_* + 2\frac{M}{f}(V(T^{t_*}X_0) - v_1^*) > t_*^e + 2\frac{M}{f}(1-\alpha)|\Delta| > t_*^e + \tau^* .$$

For $|\Delta| \leqq \Delta_0$ recollisions can occur for negative Δ. We then construct Y_1 as a configuration of two particles. The first one (q^+, v^+) is chosen exactly as (q^*, v^*) above (with vertical velocity $v_2^+ = v_2^*$), for a velocity jump $\Delta' = -2\Delta_0$. We denote by t_+^e, and t_+, the entrance and collision times of (q^+, v^+). The second particle $(q^*, v^*) \in Y_1$ is now fixed in such a way that the entrance time t_*^e and the collision time t_* satisfy the inequalities $t_*^e \in (\tau, \tau + \tau'')$ and $t_* \in (t_+, t_+^e + \tau^*)$. q_1^* and v_1^* are then determined by equations analogue to (2.17), in such a way as to provide a further jump of the velocity of the rod by $2\Delta_0 + \Delta > \Delta_0$. The collision with (q^*, v^*) is on the left, since the velocity of the rod has to increase. Possible recollisions with (q^+, v^+) are avoided by choosing t_* so that $t_* - t_*^e \in (\delta^*, \tau^*)$, where δ^* is some constant depending only on Δ_0 and on the parameters of the model.

We have shown that for all $X \in A_U$ we can construct a configuration Y_1, depending on X, such that in the dynamics of $X_0 \cup Y_1$ the rod collides with a particle $(q^*, v^*) \in Y_1$ at some time $t_* \in (\tau, 2\tau + \tau'')$, and the outgoing velocity of the rod for $s > t_*$ is given by $V(T^{-t+s}X)$. From now on q_2^*, v_2^* and t_* are assumed to be fixed and independent of $X \in A_U$.

Let $\mathcal{O}_\delta(x,y)$, $(x,y) \in \mathbb{R}^2$ denote the open sphere with radius δ, and $\mathcal{U}_\delta(q,v) = \mathcal{O}_\delta(q_1, v_1) \times \mathcal{O}_\delta(q_2, v_2)$. For $X \in A_U$, and denoting symmetrization by $[\cdot]_\Sigma$, we set

$$\mathcal{Y}_X^{(1)} = \begin{cases} \mathcal{U}_\delta(q^*(X), v^*(X)) & \text{if } |\Delta| > \Delta_0, \\ [\mathcal{U}_\delta(q^+(X), v^+(X)) \times \mathcal{U}_\delta(q^*(X), v^*(X))]_\Sigma & \text{if } |\Delta| \leqq \Delta_0. \end{cases} \tag{2.18a}$$

Let $Y_1^* = (q^*(X), v^*(X))$ if $|\Delta| > \Delta_0$, and $Y_1^* = (q^+(X), v^+(X)) \cup (q^*(X), v^*(X))$ if $|\Delta| \leqq \Delta_0$. For any $X \in A_U$ and $Y \in \mathcal{Y}_X^{(1)}$ the dynamics $\{T^s(X_0 \cup Y_1)\}$ and $\{T^s(X_0 \cup Y_1^*)\}$ are close for $s \in [0, t - \tau')$ if δ is small. In particular, the particles of Y_1 collide only once, before time $t - \tau'$, and produce a jump in the velocity which is continuous function of $Y_1 \in \mathcal{Y}_X^{(1)}$, and close to Δ. We also set

$$\mathcal{Y}^{(1)} = \bigcup_{X \in A_U} \mathcal{Y}_X^{(1)} . \tag{2.18b}$$

For $Y_1 \in \mathcal{Y}^{(1)}$ we denote by

$$Q_0(Y_1) = \int_0^{t-\tau'} V(T^s(X_0 \cup Y_1))\,ds, \quad V_0(Y_1) = V(T^{t-\tau'}(X_0 \cup Y_1)) \tag{2.19}$$

the displacement and the velocity of the rod at time $t - \tau'$, and introduce the set

$$\mathcal{Y}_U = \{Y \in \mathcal{Y}^{\text{in}}:\ Y_1 \in \mathcal{Y}^{(1)},\ (\bar{Y}_2 + Q_0(Y_1)) \cap \mathbf{N}(t - \tau', t; \mathbf{M}_L^s) = \emptyset\} .$$

(2.20)

Let $\mathfrak{M}_1$ be the σ-algebra generated by Y_1, and $\mathscr{P}^{(1)}$ the measure induced on $\mathscr{Y}^{(1)}$ by $\mathscr{P}$. We have

$$\begin{aligned}\mathscr{P}(\mathscr{Y}_U) &= \int_{\mathscr{Y}\text{in}} \mathscr{P}(dY)\mathscr{P}(\mathscr{Y}_U|\mathfrak{M}_1) \\ &= \int_{\mathscr{Y}^{(1)}} \mathscr{P}^{(1)}(dY)\mathscr{P}(\{Y:\ (\bar{Y}_2+Q_0(Y_1))\cap \mathbf{N}(t-\tau',t;\mathbf{M}_L^s)=\emptyset\}) \\ &\doteq \mathscr{P}^{(1)}(\mathscr{Y}^{(1)})p_2,\end{aligned} \tag{2.21a}$$

where, setting $\bar{Y}_2^* = Y\cap(\mathbf{M}\backslash\mathbf{M}^-)\cap T_0^{-\tau'}\mathbf{M}^-$,

$$\begin{aligned}p_2 &= \mathscr{P}(\{Y:\ (\bar{Y}_2+Q_0(Y_1))\cap \mathbf{N}(t-\tau',t;\mathbf{M}_L^s)=\emptyset\}) \\ &= \mathscr{P}(\{Y:\ \bar{Y}_2^*\cap \mathbf{N}(0,\tau';\mathbf{M}_L^s)=\emptyset\}) > 0\,.\end{aligned} \tag{2.21b}$$

Here we have used the independence of the distributions of Y_1 and $\hat{Y}_2$, as configurations in nonintersecting regions of $\mathbf{M}^+$, and the invariance of $\mathscr{P}$ with respect to horizontal translations and to the free dynamics.

Let now A be fixed, and such that $\pi(A)>0$. We can assume that for, some $U<\infty, A\subset A_U$. Note that if $Y\in\mathscr{Y}_U$, then

$$X_t = X(T^t(X_0\cup Y)) = X(T^{\tau'}(V_0(Y_1), T_0^{t-\tau'}Y_2+Q_0(Y_1))\ )\,. \tag{2.22}$$

In fact, in the time interval $(t-\tau',t]$ the rod can only collide with the particles of $T_0^{t-\tau'}Y_2$, since, by the second inequality (2.12), and the definition of A_U, it cannot get out of $\mathscr{S}_L$, so that it cannot collide with the particles of $T_0^{t-\tau'}\bar{Y}_2$. Hence X_t is a function of Y_1 and $\tilde{Y}_2 := T_0^{t-\tau'}Y_2+Q_0(Y_1)$. We perform the change of variables $Y_2\to\tilde{Y}_2$, and, by the independence of Y_1 and Y_2, and the invariance of $\mathscr{P}$ with respect to horizontal translations and to the free dynamics, we see that Y_1 and $\tilde{Y}_2$ are independent, and the marginal distribution of $\tilde{Y}_2$ is the same as the distributioin of $Y_2^* = Y\cap T_0^{-\tau'}\mathbf{M}^s$ induced by $\mathscr{P}$, which is denoted by $\mathscr{P}^{(2)}$.

Suppose now that $|\Delta(X)|>\Delta_0$ for all $X\in A$. Then $\mathscr{Y}_A^{(1)} = \bigcup_{X\in A}\mathscr{Y}_X^{(1)}\subset\mathscr{Y}^{(1)}$ is a set of one particle configurations, and setting, $\mathscr{U}=\{(q_1,v_1):\ (q,v)\in\mathscr{Y}_A^{(1)}\}$, we have

$$\begin{aligned}P^t(X_0,A) &\geqq \int_{\mathscr{Y}_U}\mathscr{P}(dY)\mathbb{I}_A(X(T^t(X_0\cup Y))) \\ &\geqq \int_{\mathscr{Y}_A^{(1)}}\mathscr{P}^{(1)}(dY_1)\mathscr{P}(\{Y:\ (V_0(Y_1),\tilde{Y}_2)\in T^{-\tau'}A\}|\mathfrak{M}_1) \\ &= \int_{\mathscr{U}} m^{(1)}(dq_1\ dv_1)\mathscr{P}^{(2)}(\{Y_2^*:\ (V_0(Y_1),Y_2^*)\in T^{-\tau'}A\})\,,\end{aligned}$$

where $m^{(1)}$ is the measure induced on $\mathscr{U}$ by the restriction of $\mathscr{P}^{(1)}$ to $\mathscr{Y}_A^{(1)}$.

Q_0 and V_0 are functions of (q_1,v_1) defined on $\mathscr{U}$, which we denote as

$$Q_0(Y_1)=\psi(q_1,v_1)\qquad V_0(Y_1)=\varphi(q_1,v_1)\,, \tag{2.23}$$

and $m^{(1)}$ is equivalent to the Lebesgue measure $dq_1\, dv_1$ on the open set $\mathscr{U}$, since $\int_{\mathscr{C}_\delta(q_2^*,v_2^*)} dq_2\, h(dv_2) > 0$. Hence we prove the lemma if we show that $\mu(A) > 0$ implies that the set

$$\mathscr{U}' = \{(q_1, v_1) \in \mathscr{U}:\ \mathscr{P}^{(2)}(\mathscr{Y}^*_{\varphi(q_1,v_1)}) > 0\} \subset \mathscr{U}\,,$$

where $\mathscr{Y}_V^* = \{Y_2^*:\ (V, Y_2^*) \in T^{-\tau'}A\}$, is of positive Lebesgue measure. To do this, let $Y_2' = Y \cap \mathbf{M}^- \cap T^{\tau'}(\mathbf{M}\backslash\mathbf{M}^-)$, and consider the set

$$\mathscr{A} = \{\omega \in \Omega':\ X(\omega) \in A:\ (Y_2'(\omega) + Q'(X(\omega))) \cap \mathbf{N}(-\tau', 0; \mathbf{M}_L^s) = \emptyset\}\,,$$

where

$$Q'(X) = -\int_0^{\tau'} V(T^{-s}X)ds = \int_0^{\tau'} V(T^s\mathbf{P}X)ds\,.$$

Q' is $\mathfrak{B}^s$-measurable, and, by translation invariance of the measure $\mathscr{P}$, we have

$$\mu(\mathscr{A}) = \int_A \pi(dX)\mu(\mathscr{A}|\mathfrak{B}^s) = \pi(A)\mathscr{P}(\{Y:\ Y_2' \cap \mathbf{N}(-\tau', 0; \mathbf{M}_L^s) = \emptyset\})\,.$$

Since the last factor on the right is positive, $\pi(A) > 0$ implies $\mu(\mathscr{A}) > 0$ and, by Lemma 2.2 $\mu(T^{-\tau'}\mathscr{A}) = \mu_{\tau'}(\mathscr{A}) > 0$. Using the independence of Y_2^* and $\bar{Y}_2^*$ and (2.21b) we find

$$\begin{aligned}\mu(T^{-\tau'}\mathscr{A}) &= \mu(\{\omega:\ (V(\omega), Y_2^*(\omega)) \in T^{-\tau'}A, \bar{Y}_2^*(\omega) \cap \mathbf{N}(0, \tau'; \mathbf{M}_L^s) = \emptyset\}) \\ &= p_2\mu(\{\omega:\ (V(\omega), Y_2^*(\omega)) \in T^{-\tau'}A\}) = p_2\int_{\mathscr{V}} g(dV)\mathscr{P}^{(2)}(\mathscr{Y}_V^*)\,.\end{aligned}$$

Here g denotes the gaussian density of the distribution of V induced by μ, and $\mathscr{V} = \{V:\ (V, Y_2^*) \in T^{-\tau'}A$ for some $Y_2^*\}$. Hence $\pi(A) > 0$ implies that the set $\mathscr{V}' = \{V \in \mathscr{V}:\ \mathscr{P}^{(2)}(\mathscr{Y}_V^*) > 0\}$ is of positive Lebesgue measure.

By our construction $\varphi^{-1}(\mathscr{V}') \subset \mathscr{U}'$. The function $\varphi(q_1, v_1)$ is differentiable with derivatives

$$\begin{aligned}\frac{\partial\varphi}{\partial q_1} &= -(1-\alpha)\frac{f}{M\ (V(T^{t_c}X_0) - v_1)}\,, \\ \frac{\partial\varphi}{\partial v_1} &= (1-\alpha)\left(1 - \frac{ft_c}{M\ (V(T^{t_c}X_0) - v_1)}\right)\,,\end{aligned} \tag{2.24}$$

where t_c is the collision time. If δ is small t_c is close to t_* and the derivatives (2.24) are nonzero for all $(q_1, v_1) \in \mathscr{U}$, so that φ is a C^1 map of full rank. Hence the Lebesgue measure $dq_1\, dv_1$ of $\varphi^{-1}(\mathscr{V}')$ is positive, and the Lebesgue measure of $\mathscr{U}'$ is also positive.

If A is such that $|\Delta(X)| \leqq \Delta_0$ on some subset of A of positive μ measure the proof is somewhat more complicated, but follows the same lines. We omit the details. The lemma is proved. □

We prove next a property of nondegeneracy of the distribution of the displacement $Q_t(\omega)$, defined by Eq. (1.15), which will ensure nondegeneracy of the limiting brownian motion. In what follows we denote by P_λ the distribution of the Markov process $\{X_t\colon\ 0 \leqq t < \infty\}$ with initial measure λ, and by $\mathbb{E}(\,\cdot\,|X_0, X_t)$ the conditional distribution with respect to the σ-algebra generated by X_0 and X_t.

Corollary 2.5 *Let t be as in the previous lemma. Then the conditional dispersion*

$$\mathbb{D}(Q_t|X_0, X_t) = \mathbb{E}(\ (Q_t - \mathbb{E}(Q_t|X_0X_t))^2|X_0, X_t)$$

is P_π-almost everywhere positive.

Proof. The result is intuitively obvious, since $\mathbb{D}(Q_t|X_0, X_t) = 0$ on a set of positive measure would imply that on that set Q_t is a function of X_0 and X_t only, which is impossible since X_0 and X_t are relative to the rod position and cannot determine the absolute shift Q_t.

For the proof it is enough to show that for any pair $A_0, A \in \mathfrak{B}^s$ such that $\int_{A_0} \pi(dX_0)P^t(X_0, A) > 0$ we have

$$\int_{A_0} \pi(dX_0) \int_{A} P^t(X_0, dX_t)\mathbb{D}(Q_t|X_0, X_t) > 0\,.$$

The hypothesis implies that $\pi_t(A) > 0$ and hence, by Lemma 2.2, that $\pi(A) > 0$. We shall prove that for all $X_0 \in \mathcal{X}'$ and $A \in \mathfrak{B}^s$ such that $\pi(A) > 0$ we have

$$I_{X_0, A} \equiv \int_{A} P^t(X_0, dX_t)\mathbb{D}(Q_t|X_0X_t) > 0\,.$$

Let X_0 be fixed. It is not restrictive to assume that for some U large enough we have both $W(X_0) < U$ and $A \subset A_U$, where A_U is defined by Eq. (2.11). Going back to the mechanical system we express $X_s, 0 \leqq s \leqq t$ as a function of $Y \in \mathcal{Y}^{\text{in}}\colon\ X_s = X(T^s(X_0 \cup Y))$. We denote by $\mathfrak{M}_0^t$ the σ-algebra of subsets of $\mathcal{Y}^{\text{in}}$ generated by the functions $X_s, 0 \leqq s \leqq t$ (for X_0 fixed), by $\mathfrak{M}_t$ the subalgebra generated by X_t, and by $\mathbb{E}_{X_0}$ the expectation with respect to the measure induced by $\mathcal{P}$ on $\mathfrak{M}_0^t$. Then clearly

$$I_{X_0, A} = \int_{\{X_t \in A\}} \mathbb{E}_{X_0}((Q_t - \mathbb{E}_{X_0}(Q_t|\mathfrak{M}_t)\ )^2|\mathfrak{M}_t)\mathcal{P}(dY)\,.$$

Assume that A is such that $|\Delta(X)| > \Delta_0$ for all $X \in A$, and set $\tilde{\mathcal{Y}} = \{Y \in \mathcal{Y}_U\colon Y_1 \in \mathcal{Y}_A^{(1)}\}$, $\mathcal{Y}_U$ and $\mathcal{Y}_A^{(1)}$ being defined in the course of the previous proof. We again go over to the variables $Y_1, \tilde{Y}_2$, and by Eq. (2.22) the restriction of $\mathfrak{M}_t$ to $\tilde{\mathcal{Y}}$ coincides with the σ algebra generated by $V_0(Y_1) = \varphi(q_1, v_1), (q_1, v_1) \in \mathcal{U}$, and by $\tilde{Y}_2$. We denote it by $\mathfrak{M}_t^*$. Observe that

$$Q_t = Q_0(Y_1) + Q_t^*, \qquad Q_t^* = \int_{t-\tau'}^{t} V(T^s(V_0(Y_1), \tilde{Y}_2))\ ds\,.$$

Denoting by $\mathbb{D}_{X_0}(\,\cdot\,|\mathfrak{N}\,)$ the conditional dispersion with respect to a subalgebra $\mathfrak{N} \subset \mathfrak{M}$, i.e., $\mathbb{D}_{X_0}(\,\cdot\,|\mathfrak{N}\,) = \mathbb{E}_{X_0}(\ (\,\cdot\ - \mathbb{E}(\,\cdot\,|\mathfrak{N}\,))^2|\mathfrak{N}\,)$, and observing that

Q_t^* is $\mathfrak{M}_t^*$-measurable, we have,

$$\mathbb{D}_{X_0}(Q_t|\mathfrak{M}_t^*) = \mathbb{D}_{X_0}(Q_0|\mathfrak{M}_t^*) = \mathbb{D}_{X_0}(Q_0|\mathfrak{B}_1),$$

where $\mathfrak{B}_1$ is the σ algebra of subsets of $\mathscr{U}$ generated by φ. The equality $\mathbb{D}_{X_0}(Q_0|\mathfrak{M}_t^*) = \mathbb{D}_{X_0}(Q_0|\mathfrak{B}_1)$ comes from the independence of Y_1 and $\tilde{Y}_2$.

By the properties of the conditional distribution and of the dispersion we have

$$\begin{aligned} I_{X_0A} &\geqq \int_{\tilde{\mathscr{Y}}\cap\{X_t\in A\}} \mathbb{D}_{X_0}(Q_t|\mathfrak{M}_t)\mathscr{P}(dY) \\ &= \mathscr{P}(\tilde{\mathscr{Y}}) \int_{\{X_t\in A\}} \mathbb{E}((Q_t - \mathbb{E}(Q_t|\mathfrak{M}_t))^2|\mathfrak{M}_t^*)\hat{\mathscr{P}}(dY) \\ &\geqq \mathscr{P}(\tilde{\mathscr{Y}}) \int_{\{X_t\in A\}} \hat{\mathscr{P}}(dY)\mathbb{D}_{X_0}(Q_0|\mathfrak{B}_1), \end{aligned}$$

where $\hat{\mathscr{P}}$ denotes the normalized restriction of $\mathscr{P}$ to $\tilde{\mathscr{Y}}$. $\mathbb{D}(Q_0|\mathfrak{B}_1)$ is everywhere positive, since, as it would not be hard to work out, the jacobian of the transformation $(q_1, v_1) \to (\psi(q_1, v_1), \varphi(q_1, v_1))$ is constant, and equal to $(1-\alpha)^2$ on $\mathscr{U}$. Hence $I_{X_0,A}$ is positive if $\mathscr{P}(\tilde{\mathscr{Y}} \cap \{X_t \in A\}) > 0$. But this property is, as shown in the course of the previous proof, a consequence of the condition $\pi(A) > 0$.

The proof when A is such that $|\Delta(X)| \leqq \Delta_0$ on a subset of A of positive π measure follows the same lines. □

Corollary 2.6 *There is a positive measure λ such that for $t > 3\tau$*

$$P^t(X_0, dX) \geqq \mathbb{I}_{A_U}(X_0)\lambda(dX). \tag{2.25}$$

Proof. The proof again makes use of the constructions of Lemma 2.4. Let $X_0 \in A_U$ be fixed, and $\Delta = -V(T^t X_0)$. Suppose for the moment that $|\Delta| > \Delta_0$. We determine (q^*, v^*) as a function of Δ and $t^* < t - \tau'$ by Eqs. (2.17), and consider the dynamics $\{T^s(X_0 \cup Y_1)\}$ for $Y_1 \in \mathscr{U}_\delta(q^*, v^*)$.

Let $V_t = V(T^t(X_0 \cup Y_1))$ and $Q_t = \int_0^t ds\ V(T^s(X_0 \cup Y_1))$ denote the velocity and the displacement at time t. Clearly $V_t = \varphi(q_1, v_1) + (f/M)\tau' \equiv \varphi_1(q_1, v_1)$, and the image under V_t of $\mathscr{O}_\delta(q_1^*, v_1^*)$ is an open interval, $\mathscr{I}_{X_0}$, which contains the origin, by our choice of Δ. Since the derivatives (2.24) are bounded away from 0, the intersection of the intervals $\mathscr{I}_{X_0}$ for all $X_0 \in A_U$ such that $|\Delta| > \Delta_0$ contains a nondegenerate open interval $\mathscr{I}$, and $0 \in \mathscr{I}$. Moreover for any such X_0 the measure induced on $\mathscr{I}$ by the Lebgesgue measure on $\mathscr{O}_\delta(q_1^*, v_1^*)$, via the map φ_1, is absolutely continuous with respect to dV_1, with some density $f_{X_0}(V_1)$ which has the property that for all $V_1 \in \mathscr{I}$

$$0 < \beta_1 \leqq f_{X_0}(V_1) \leqq \beta_2 < \infty.$$

Let

$$\tilde{\mathscr{Y}}^{(X_0)} = \{Y:\ Y_1 \in \mathscr{U}_\delta(q^*, v^*), (\hat{Y}_2 + Q_1(Y_1)) \cap \mathbf{N}(t-\tau', t; \mathbf{M}_L^s) = \emptyset\}.$$

Then, by repeating the arguments in the proof of Lemma 2.4, we see that

$$P^t(X_0,A) \geqq \int_{\tilde{\mathscr{Y}}(X_0)} \mathscr{P}(dY)\mathbb{I}_A(X_t) \geqq \text{ const } \hat{\lambda}(A),$$

$$\hat{\lambda}(dV\, dY^s) = dV\ \mathbb{I}_{\mathscr{I}}(V)\mathscr{P}(dY^s)\mathbb{I}_{\mathscr{N}_L}(Y^s),$$

where $\mathscr{N}_L = \{Y:\ Y \cap \mathbf{N}(-\tau', 0; \mathbf{M}_L^s) = \emptyset)\}$. Hence the result follows.

On the set $\{X_0 \in A_U:\ |\Delta| \leqq \Delta_0\}$ we can repeat the same arguments, by taking Y_1 in a set of two-particle configurations, as in the proof of Lemma 2.4. The procedure is more complicated, but straightforward. We omit the details. □

The following lemma shows that large velocities slow down with large probability.

Lemma 2.7 *Let* $2\tau < t < 5\tau$ *and* $\kappa \in (0, (t-2\tau)\rho\ell \log(1/\alpha))$ *be a fixed constant. Then there is a positive number* U_0 *such that the following inequalities hold for* $U > U_0$, *and some positive constants* c_1, c_2:

$$P^t(X, \{W \geqq U\}) < e^{-c_1 U^2} \quad \text{if } W(X) < U - \frac{f}{M}t \tag{2.26a}$$

$$P^t(X, \{W \geqq U\}) < e^{-c_2 U}, \quad \text{if } W(X) < Ue^{\kappa U}. \tag{2.26b}$$

Proof. The proof is somewhat long, though straightforward, and could be simplified if one is content with a weaker result than the exponential estimate (2.26b). As for Lemma 2.4 we begin with a short description of the main idea, that may serve as a guideline for the reader.

The relaxation of velocities is brought about by the gaussian distribution of incoming velocities. Suppose that the rod keeps a high velocity $\bar{U}$ for some time $\bar{t}$. Then typically it will collide with particles with velocity not exceeding $\log(\bar{U}\bar{t})$ in absolute value, except for a set with measure of the order $e^{-\text{const} \log \bar{U}}$. If no particles with velocity larger than $\log(\bar{U}\bar{t})$ can collide, the rod can keep a velocity larger than $\mathcal{O}(\log \bar{U})$ only if it collides with a small number of slow particles, since, roughly speaking, each time that it collides with a slow particle its velocity drops by a factor α. The probability of this event turns to be again of the order $\mathcal{O}(e^{-\text{const} \log \bar{U}})$.

Passing to the proof, we set $\tilde{U} = U - (f/M)t$ and $\hat{U} = \gamma\tilde{U}$, for some $\gamma \in (0, 1)$. Let $\bar{Y} = Y^{\text{in}} \cap T_0^{-t}(\mathbf{M}\backslash\mathbf{M}^+)$, and $Y^i = \bar{Y} \cap \mathbf{N}(0, t; \mathbf{M}_L^s)$, for some $L > 0$, $Y^e = \bar{Y}\backslash Y^i$, and consider the set

$$B_{U,L} = \{\ Y \in \mathscr{Y}^{\text{in}} : \sup\{|v_1|:\ (q_1, v_1) \in Y^i\} < \hat{U}\}\,.$$

Probability estimates as the ones in the proof of Lemma 2.3 show that for some constant $\bar{c}_1$ and U large enough

$$P(B_{U,L}) > 1 - e^{-\bar{c}_1 U^2}\,. \tag{2.27}$$

A short analysis of the proof shows that for any choice of the constant $b < \beta m/2$ we can choose $\bar{c}_1$ in such a way that the estimate (2.27) holds for all $L < e^{bU^2}$.

Take $L = 2tU$ and consider the dynamics $T^s(X \cup Y)$ for $W(X) < U$ and $Y \in B_{U,L}$. A slight modification of the argument in Remark 2.1 shows that the rod keeps a velocity less than U in absolute value up to time t and travels less than a distance $L/2$. The velocity of the particles of Y^i, i.e., of these that enter the strip $\mathscr{S}_L$ up to time t, is bounded by $\hat{U}$ if they did not collide. By collision they may be accelerated to higher velocities, but then they become outgoing. Hence the condition $W(X_t) < U$ can be violated only by the particles of Y^e. Again we observe that the distributions of Y^i and Y^e are independent, and, for $Y \in B_{U,L}$ the displacement is a function of Y^i only:

$$Q_t = \int_0^t ds\ V(T^s(X \cap Y^i)) .$$

Hence, for $Y \in B_{U,L}$, $W(X_t) \geqq \hat{U}$ implies $W(Y^* + Q_t) \geqq \hat{U}$, where $Y^* = T_0^t Y^e \cap \mathbf{M}^s$. Let $\mathfrak{M}_i$ denote the σ algebra generated by Y^i. Then, taking into account the inequality

$$\mathscr{P}(\{W(Y^* + Q_t) \geqq \hat{U}\}|\mathfrak{M}_i) \leqq \mathscr{P}(\{W(Y^s) \geqq \hat{U}\})$$

we get

$$\begin{aligned} &\mathscr{P}(\{Y \in \mathscr{Y}^{\text{in}}\colon\ W(X(T^t(X \cup Y)) \geqq \hat{U}\}) \\ &\quad \leqq \mathscr{P}(B^c_{U,L}) + \int_{B_{U,L}} \mathscr{P}(dY)\mathscr{P}(\{W(Y^* + Q_t) \geqq \hat{U}\}|\mathfrak{M}_i) \\ &\quad \leqq \mathscr{P}(B^c_{U,L}) + \mathscr{P}(B_{U,L})\mathscr{P}(\{W(Y^s) \geqq \hat{U}\}) . \end{aligned} \tag{2.28}$$

Hence inequality (2.26a) follows from the estimate (2.27) and Lemma 2.3.

To prove inequality (2.26b), let $L = 2tUe^{\kappa U} + (f/M)t^2$, assuming that U is so large that $L < e^{(\beta m/4)U^2}$. By repeating the analysis of $T^s(X \cup Y^i)$, for $Y \in B_{U,L}$, we see that the rod cannot travel by the time t more than a distance $L/2$. We set for brevity $X_s = X(T^s(X \cup Y))$, $\tilde{X}_s = X(T^s(X \cup Y^i))$, for $Y \in B_{U,L}$, and $V_s = V(X_s) = V(\tilde{X}_s)$, $s \in [0,t]$, so that $X_t = \tilde{X}_t \cup (Y^* + Q_t)$, the notation being as above. Let

$$D_{U,X} = \{Y \in B_{U,L}\colon\ \inf_{s \in (\tau,t)} |V_s| < \tilde{U}\} .$$

If $Y \in D_{U,X}$, then $W(\tilde{X}_t) < U$. In fact all particles that collide in the time interval (τ,t) come from Y^i and have velocity less than $\hat{U}$ in absolute value. Once the absolute value of the velocity of the rod has fallen under $\tilde{U}$, it can only increase by acceleration, not exceeding U. The particles of Y^i that have been accelerated to velocities larger than U are outgoing. Hence the conditional probability $\mathscr{P}(W(X_t) \geqq U|\mathfrak{M}_i)$, for $Y \in D_{U,X}$ is bounded by $\mathscr{P}(W(Y^* + Q_t) \geqq U|\mathfrak{M}_i)$, which is estimated as before.

We set $\bar{D}_{U,X} = B_{U,L} \backslash D_{U,X}$. The collision rules (1.10) imply that if V' and V have opposite signs, then $|V'| < (1-\alpha)|v_1|$. Hence in the dynamics $T^s(X \cup Y)$, $Y \in B_{U,L}$, the velocity of the rod cannot change sign in the time interval $(\tau,t]$ without dropping below $\hat{U} < \tilde{U}$, so that $\bar{D}_{U,X}$ splits into two nonintersecting subsets $\bar{D}^{\pm}$, according to the sign of the rod velocity.

Another consequence of the collision rules (1.10) is that for $Y \in B_{U,L}$, a particle that collides at some time $s \in (\tau, t)$ for $|V^-(X_s)| \geqq \tilde{U}$, where $V^-(X_s) = \lim_{\varepsilon \downarrow 0} V(X_{s-\varepsilon})$ is the incoming velocity, can recollide only after time t. In fact, supposing $V^-(X_s) \geqq \tilde{U}$, as $|v_1| < \hat{U}$, the outgoing velocity of the particle is

$$v_1' = V^- + \alpha(V^- - v_1) \geqq V^- + \alpha(1-\gamma)\tilde{U} ,$$

so that no recollision can occur for U so large that $\alpha(1-\gamma)\tilde{U} > (t-\tau)f/M$.

Let $\mathscr{P}_{U,L} = \mathscr{P}(\cdot | B_{U,L})$ and $\mathfrak{M}_0^r$ denote the σ-algebra of subsets of $B_{U,L}$ generated by the trajectory $\{V_s\colon\ 0 \leqq s \leqq r\}$. If $r \in [2\tau, t)$, then the distribution of the trajectory $\{V_s\colon\ r < s \leqq t^*\}$, where $t^* = \min\{s > r\colon\ |V_s| < \tilde{U}\}$ induced by the conditional measure $\mathscr{P}_{U,L}(\,\cdot\,|\mathfrak{M}_0^r)$, computed at a point $Y \in B_{U,L}$ such that $|V_r| \geqq \tilde{U}$, depends only on V_r. In fact, by the observations above, the condition $|V_r| \geqq \tilde{U}$ implies that all particles that collide in the interval (τ, t^*) cannot recollide, while those that collided in the interval $(0,\tau]$ are out of the game at time r. Assuming, for definiteness, positive velocities, i.e., $V_r \geqq \tilde{U} > 0$, collisions in the time interval (r, t^*) can occur only with "fresh" particles on the right.

If $\mathbf{M}_a^{s+} = \{(q,v) \in \mathbf{M}\backslash\mathbf{M}^-\colon\ q_1 \in (0,a)\}$, $a > 0$, then at time r the particles that can collide up to time t^* are a subset of

$$Y_r = T_0^r Y^i \cap (\mathbf{M}_{L/2}^{s+} + Q_r) .$$

(To check this observe that at time r no particle with $q_1 > Q_r$, and $|v_1| < \hat{U}$ could collide with the rod in the past.) Now the distribution of Y_r induced by $\mathscr{P}_{U,L}$ is Poisson with intensity measure $n(dq\,dv)\mathbb{I}_{\mathscr{A}_r}$, $\mathscr{A}_r = \{(q,v) \in \mathbf{M}_{L/2}^{s+} + Q_r\colon\ |v_1| < \hat{U}\}$. This follows by observing that $\mathscr{P}_{U,L}$ is the Poisson measure on $\mathbf{M}^+$ with intensity measure $n(dq\,dv)\mathbb{I}_{\mathscr{A}_{U,L}}(q,v)$, where $\mathscr{A}_{U,L} = \mathbf{M}^+\backslash\{(q,v) \in \mathbf{N}(0,t;\mathbf{M}_L^s)\colon\ |v_1| \geqq \hat{U}\}$, and using the obvious properties of the evolution of $\mathscr{P}$ under free dynamics.

In conclusion the trajectory $\{V_s\colon\ r < s \leqq t^*\}$ depends only on V_r and $\hat{Y}_r = Y_r - Q_r$. As the distribution of $\hat{Y}_r$ is independent of $\mathfrak{M}_0^r$, the conditional distribution of $\{V_s\colon\ r < s \leqq t^*\}$ depends only on V_r.

We proceed to estimate the probability $\mathscr{P}_{U,L}(\bar{D}^+|\mathfrak{M}_0^{2\tau})$. Let $V_{2\tau} = \bar{V}$, and assume, of course, $\bar{V} \in [\tilde{U}, Ue^{\kappa U} + (f/M)2\tau)$. By what we said above the process $\{V_s\colon\ r < s < t^*\}$ corresponds to a collision process with Poisson distributed particles, no recollision, and instantaneous collision rate

$$r(V_s) = \rho\ell\sqrt{\frac{\beta m}{2\pi}} \int_{|v_1|<\hat{U}} (V_s - v_1)e^{-\beta m v^2/2}\,dv_1 .$$

We divide the interval $(2\tau, t]$ into subintervals $I_k = (t_k, t_{k+1}]$, $t_k = 2\tau + k\delta$, $k = 1,2,\ldots,N$, where N is some large integer and $\delta = (t-2\tau)/N$, and consider the sequence $\{V_{t_k}\colon\ k = 0,1,\ldots,k^*\}$, where $k^* = \min\{k\colon\ V_{t_k} < \tilde{U}\}$ if $V_t < \tilde{U}$, and $k^* = N$ otherwise. We introduce a reference jump process $\{\hat{V}_k\colon\ k = 0,1,\ldots\}$

with $\hat{V}_0 = \bar{V}$ and the following transition probabilities

$$P(\hat{V}_{k+1} = v|\hat{V}_k = u) = \begin{cases} p_\delta(\tilde{U}) & \text{for } v = \alpha u + (1-\alpha)\hat{U}\,, \\ 1 - p_\delta(\tilde{U}) & \text{for } v = u\,, \\ 0 & \text{otherwise}\,. \end{cases} \tag{2.29}$$

Here $p_\delta(V)$ is the probability that the rod with initial velocity V and collision rate r undergoes at least one collision in the following time interval of length δ:

$$p_\delta(V) = 1 - e^{-\int_0^\delta r(V+(f/M)s)ds} = \delta r(V)(1 + \mathcal{O}(\delta^2))\,.$$

A coupling between the dynamics of the rod and the sequence $\{\hat{V}_k\colon\ 0 \leqq k \leqq k^*\}$ can be chosen in the following way. We introduce random variables with values in $\{0,1\}$, $\{\xi_k\colon\ 0 \leqq k < k^*\}$, which are mutually independent for any choice of the sequence $\{V_{t_k}\}$ and such that $P(\xi_k = 1) = p_\delta(\tilde{U})/p_\delta(V_{t_k})$, $k = 0, 1, \ldots, k^* - 1$. If the rod collides in the interval I_k we set $\hat{V}_{k+1} = \hat{V}_k$ if $\xi_k = 0$, and $\hat{V}_{k+1} = \alpha\hat{V}_k + (1-\alpha)\hat{U}$ if $\xi_k = 1$. If it does not collide we set $\hat{V}_{k+1} = \hat{V}_k$. It is easy to see that

$$V_{t_k} \leqq \hat{V}_k + \delta k \frac{f}{M}, \quad k = 0, 1, \ldots, k^*\,. \tag{2.30}$$

In fact the inequality holds for $k = 0$, and can be proved by induction, noting that if no collision takes place in the interval I_k, then $V_{t_{k+1}} = V_{t_k} + \delta(f/M)$, and if at least one collision takes place, then the largest possible value of $V_{t_{k+1}}$ does not exceed $\alpha V_{t_k} + (1-\alpha)\hat{U} + (f/M)\delta$, corresponding to one single collision taking place at time t_k, with a particle of velocity $\hat{U}$.

As a consequence we have, by (2.30),

$$\mathscr{P}_{U,L}(\bar{D}^+|V_{2\tau} = \bar{V}) \leqq P(\hat{V}_N + (t-\tau)\frac{f}{M} \geq \tilde{U}|\hat{V}_0 = \bar{V})\,.$$

If m jumps take place for the process $\{\hat{V}_k\colon\ k = 1, \ldots, N\}$ then $\hat{V}_N = \alpha^m\bar{V} + (1-\alpha^m)\hat{U}$, so that the condition $\hat{V}_N + (t-2\tau)f/M \geqq \tilde{U}$ implies, recalling that $\bar{V} < Ue^{\kappa U} + (f/M)2\tau$,

$$m \leqq m_0 = \left[\left(\log\frac{1}{\alpha}\right)^{-1} \log\frac{\bar{V} - \hat{U}}{\tilde{U}\left(\gamma - \frac{(t-2\tau)f}{M\tilde{U}}\right)}\right] < \frac{\kappa}{\log\frac{1}{\alpha}} U(1 + o(1))\,,$$

where $[\cdot]$ denotes the integer part, and the asymptotics is for large U. On the other hand the average value of m, which has a binomial distribution (N trials) with probability $p_\delta(\tilde{U})$, is

$$\bar{m} = \mathbb{E}m = Np_\delta(\tilde{U}) \asymp (t-2\tau)r(\tilde{U}) = (t-2\tau)\rho\ell U(1 + o(1))\,.$$

By applying the standard exponential Chebyshev inequality and taking N large (i.e., δ small) we see that there is a constant $c' > 0$ such that $P(m \leqq m_0|\hat{V}_0 = \bar{V}) \leqq e^{-c'U}$, which implies $\mathscr{P}_{U,L}(\bar{D}^+|\mathfrak{M}_0^{2\tau}) \leqq e^{-c'U}$.

For negative $V_{2\tau}$ we obtain, in a similar way, $\mathscr{P}_{U,L}(\bar{D}^-|\mathfrak{M}_0^{2\tau}) \leqq e^{-c'U}$. The conclusion now follows by an estimate analogous to (2.28). □

Remark. 2.3 For $t \in (3\tau, 5\tau)$ and $\kappa \in (0, \tau\rho\ell \log(1/\alpha))$ the left-hand side of inequalities (2.26a, b) can be replaced by $P^t(X, A_U^c)$, where A_U^c is the complement of the set A_U defined by formula (2.11). The proof follows by applying the previous lemma for some $t' \in (2\tau, 3\tau)$, and by observing that after a time τ all outgoing particles with velocity larger than U (which could be there because of collisions), get out of the game. Hence in order to ensure $W(\mathbf{P}X_t) < U$ it is enough to control the velocities of the incoming particles in $Y^{\mathrm{in}} \cap T_0^{-1}\mathbf{M}^s$, which is done as in Lemma 2.3.

We now discretize time and consider the Markov chain with state space $\mathscr{X}$ and transition probability $\mathbb{P} = P^{4\tau}$. The results above imply some important properties for the chain $(\mathscr{X}, \mathbb{P})$, which we state in the form of a lemma.

Lemma 2.8 *For the chain* $(\mathscr{X}, \mathbb{P})$ *the set* $\mathscr{X}'$ *is absorbing, and has full* π *measure*: $\pi(\mathscr{X}') = 1$. *Moreover the following properties hold*:
(i) *The restriction of the chain to* $\mathscr{X}'$ *is* π*-irreducible and aperiodic*;
(ii) $\pi\mathbb{P}$ *and* π *are equivalent and* π *is a maximal irreducibility measure*;
(iii) *For all* $X \in \mathscr{X}'$ π *is absolutely continuous with respect to* $\mathbb{P}(X, \cdot)$;
(iv) *For* U *large enough the set* A_U, *defined by* (2.11), *is a minorant* (*or* "*small*") *set for the chain* $(\mathscr{X}, \mathbb{P})$.

Proof. $\mathscr{X}'$ is absorbing since Ω' is invariant, and obviously has full π measure. Assertions (i) and (iii) come from Lemma 2.4. Assertion (ii) comes from Lemma 2.1, recalling that the equivalence of π and $\pi\mathbb{P}$ implies that π is a maximal irreducibility measure. Assertion (iv) comes from Corollary 2.6, in fact the condition that A_U is a minorant set is equivalent to inequality (2.25) [5]. □

From now on we consider the chain $(\mathscr{X}', \mathbb{P})$, and write for simplicity $\mathscr{X}$ instead of $\mathscr{X}'$.

Proof of Theorem I. It is easy to see that the chain $(\mathscr{X}, \mathscr{P})$ satisfies a relaxation condition in the sense of Definition B.1 of Appendix B, by taking the function W as $\max\{W(X), W(\mathbf{P}X)\}$ and $\phi(U) = Ue^{\kappa U}$. In fact, Condition (i) of Definition B.1 is obviously satisfied, Condition (ii) follows by Lemma 2.4, and Condition (iii) follows by Corollary 2.6. The proof of Theorem I then follows by extending the results of Theorem B.1 to continuous time.

By Theorem B.1 we have, for all $f \neq 0$, the existence of an absorbing measurable set $\mathscr{H} \subset \mathscr{X}$ such that the chain $(\mathscr{H}, \mathbb{P})$ is Harris recurrent. Moreover there is a unique $\mathscr{P}$-invariant measure v_f on $\mathfrak{B}^s$, concentrated on $\mathscr{H}$, and the chain is geometrically ergodic, i.e., there are a constant $\chi > 0$, and a function $\hat{g}(X) \geqq 0$, $\hat{g} \in L^1(\mathscr{X}, v_f)$ such that

$$\|\mathbb{P}^n(X, \cdot) - v_f\| < \hat{g}(X)e^{-\chi^n}, \tag{2.31}$$

for all $X \in \mathscr{H}$, where $\| \cdot \|$ denotes the variation distance.

We first prove that v_f is invariant with respect to P^t, $t \in \mathbb{R}_+$. Note that $v_f P^t \sim v_f$ (where $\sim$ denotes equivalence). In fact $v_f \sim \pi$, which implies

$\nu_f P^t \sim \pi P^t$, and, by Lemma 2.2, $\nu_f P^t \sim \pi \sim \nu_f$. Hence $\nu_f P^t = \nu_f \mathbb{P}^n P^t = \nu_f P^t \mathbb{P}^n \to \nu_f$ as $n \to \infty$.

Taking now $m(t) = [t/4\tau]$, we have, for any $X \in \mathscr{H}$

$$\|P^t(X, \cdot) - \nu_f\| = \|\mathbb{P}^{m(t)} P^{t-4\tau m(t)}(X, \cdot) - \nu_f P^{t-4\tau m(t)}(X, \cdot)\|$$

$$\leqq \hat{g}(X) e^{-\chi m(t)} \leqq e^{\chi} \hat{g}(X) e^{-(\chi/4\tau)t} .$$

Theorem I is proved.

Before proving Theorems II and III we need two more auxiliary results.

Proposition 2.9 *There are finite constants d_f and $\sigma_f \geqq 0$ such that for any initial distribution λ of the Markov process $\{X_t\colon t \geqq 0\}$ for which* $\lim_{t\to\infty} \|\lambda P^t - \nu_f\| = 0$ *the following assertions hold*:

(i) (*Existence of the drift*)

$$\lim_{t\to\infty} \frac{1}{t} \int_0^t V(X_s)\, ds = d_f = \nu_f(V) \quad P_\lambda \ a.e. \ ; \tag{2.32}$$

(ii) (*Diffusion*) *The distribution of the process*

$$\xi_t^\varepsilon = \sqrt{\varepsilon} \int_0^{t/\varepsilon} (V(X_s) - d_f)\, ds$$

induced by P_λ converges as $\varepsilon \to 0$ to that of the Wiener process $W_{\sigma_f^2 t}$.

Proof. It is enough to prove the assertion for $\lambda = \nu_f$.

The first statement comes from Theorem I and the fact that $|V(\cdot)| \leqq W(\cdot) \in L^p(\mathscr{X}, \nu_f)$, for any $p \geqq 1$, which follows by Theorem B.1. As for Assertion (ii), let $\mathfrak{B}^n$ and $\mathfrak{B}_n$ be the σ algebras generated by the variables $\{X_t\colon t \geqq n\}$, and $\{X_t\colon t \leqq n\}$, respectively, for $n = 0, 1, \ldots$. Consider the operator

$$\Pi^n_{p,r} : \ L^p(\Omega, \mathfrak{B}^n, P_{\nu_f}) \to L^r(\Omega, \mathfrak{B}_0, P_{\nu_f}), \quad p, r \geqq 1 ,$$

defined by setting $\Pi^n_{p,r}\xi = \mathbb{E}(\xi|\mathfrak{B}_0) - \mathbb{E}\xi$, where $\mathbb{E}$ refers to the measure P_{ν_f}. A simple check shows that, by Theorem I,

$$\|\Pi^n_{\infty,1}\| \leqq 2e^{-\kappa n} \int \nu_f(dX) g(X), \qquad \|\Pi^n_{p,p}\| \leqq 2, \quad p > 1 ,$$

where $\|\cdot\|$ denotes the operator norm. Taking $p = 4$ and noting that $\frac{1}{3} + (1 - \frac{1}{3})\frac{1}{4} = \frac{1}{2}$, we get, by the Riesz–Thorin Interpolation Theorem (see, e.g., [16]), that $\|\Pi^n_{6,2}\| \leqq \text{const}\, e^{-(\chi/3)n}$. Hence for $t \geqq n \geqq 1$

$$\|\mathbb{E}((V(X_n) - d_f)|\mathfrak{B}_0)\|_2 \leqq \|V\|_6 \|\Pi^n_{6,2}\| \leqq \text{const}\, e^{-(\kappa/3)n} . \tag{2.33}$$

We can then apply, for example, Theorem 3.79 of [12], and obtain the result. □

Remark. 2.4 It follows from the results of [12, VIII Sect. 5e] that we also have "mixing convergence", i.e., for any $\eta \in L^\infty(\Omega, \mathfrak{B}_0, P_{\nu_f})$ and any bounded functional ψ on the space of the continuous function on $\mathbb{R}_+$ we have, as $\varepsilon \to 0$,

$$\mathbb{E}(\psi(\xi^\varepsilon.)\eta) \to \mathbb{E}\eta\, \mathbb{E}_{\sigma f w}(\psi),$$

where $\mathbb{E}_{\sigma w}$ denotes expectation with respect to the Wiener process with dispersion σ^2.

The following Proposition concludes the proofs of Theorems II and III.

Proposition 2.10 *The following assertions hold.*
(i) $\sigma_f > 0$ *for all* $f \in \mathbb{R}_1$.
(ii) $f d_f > 0$ *if* $f \neq 0$.

Proof. We denote by

$$Q_n = \int_{4\tau n}^{4\tau(n+1)} V(X(T^s\omega))\, ds$$

the displacement between times $4\tau n$ and $4\tau(n+1)$, and by $S_m = \sum_{k=0}^{m-1} Q_k$ the total displacement up to time $4m\tau$. Let $\mathfrak{J}$ denote the σ-algebra generated by the variables $X_0, X_1, \ldots$, where we write X_k for $X(T^{4\tau k}\omega)$. Note that the displacements $\{Q_n\colon\ n = 0, 1, \ldots\}$ are conditionally independent, with respect to $\mathfrak{J}$, i.e., they are independent once we fix the sequence $X_0, X_1, \ldots$. Hence, denoting by $\mathbb{D}(\,\cdot\,|\mathfrak{J})$ the conditional variance, we get

$$\mathbb{D}(S_m|\mathfrak{J}) = \sum_{k=0}^{m-1} \mathbb{D}(Q_k|\mathfrak{J}) = \sum_{k=0}^{m-1} \mathbb{D}(Q_k|X_{k-1}X_k).$$

By Corollary 2.5, we have $\mathbb{E}\mathbb{D}(Q_n|X_{n-1}X_n) = a > 0$, so that

$$\frac{1}{m}\mathbb{D}S_m \geqq \frac{1}{m}\mathbb{E}\mathbb{D}(S_m|\mathfrak{J}) = \frac{1}{m}\sum_{k=1}^{m-1} \mathbb{E}\mathbb{D}(Q_k|X_{k-1}X_k) = a,$$

where $\mathbb{D}$ and $\mathbb{E}$ denote variance and expectation with respect to $P_{\nu f}$. Assertion (i) is proved.

As for assertion (ii), assume $f > 0$ and take $\phi = 1$ in Eq. (2.3). We get

$$\int \mu(d\omega) e^{-\beta f \int_0^t V(T^s\omega)\, ds} = 1.$$

By the Chebyshev inequality, for any $\varepsilon > 0$,

$$\mu(\{\omega\colon\ S_m \leqq -\varepsilon\sigma_f\sqrt{4m\tau}\}) \leqq e^{-\beta f \varepsilon \sigma_f \sqrt{4m\tau}}. \tag{2.34}$$

On the other hand, by the central limit theorem and assertion (i) we have

$$\mu(\{\omega\colon\ S_m - 4m\tau d_f \leqq -\varepsilon\sigma_f\sqrt{4m\tau}\}) \to \frac{1}{\sqrt{2\pi\sigma_f}} \int_\varepsilon^\infty e^{-s^2/2\sigma_f^2}\, ds. \tag{2.35}$$

The right-hand side of (2.35) is bounded away from 0, while the right-hand side of (2.34) tends to 0 as $m \to \infty$, so that they are incompatible if $f > 0$ and $d_f \leqq 0$.

Appendix A: Existence of dynamics

Proof of Theorem 1.1 It is well known (see for example [3]) that there is a subset $\mathcal{Y}' \subset \mathcal{Y}$, invariant with respect to the free dynamics $\{T_0^s: s \in \mathbb{R}\}$, such that $\mathcal{P}(\mathcal{Y}') = 1$, and for $Y \in \mathcal{Y}'$ and any measurable bounded set $\mathbf{C} \subset \mathbb{R}^2$ the cardinality of $T_0^s Y \cap (\mathbf{C} \times \mathbb{R}^2)$ is finite for all $s \in \mathbb{R}$, i.e., on $\mathcal{Y}'$ the free dynamics is defined for all times.

We fix some positive t and L, and set

$$Y_L^t = Y \cap \mathbf{N}(0, t; \mathbf{M}_L^s) . \tag{A.1}$$

It is easy to check that $\mathbf{N}(0, t; \mathbf{M}_L^s)$ is contained in the region $\mathbf{B}_{L,t} = \mathbf{B}_{L,t}^+ \cup \mathbf{B}_{L,t}^-$, where

$$\begin{aligned}\mathbf{B}_{L,t}^+ = &\left\{(q,v):\ v_1 > 0,\ v_2 > 0,\ -L - v_1 t < q_1 < L,\ -\frac{\ell}{2} - v_2 t < q_2 < \frac{\ell}{2}\right\} \\ &\cup \left\{(q,v):\ v_1 > 0,\ v_2 < 0,\ -L - v_1 t < q_1 < L,\ -\frac{\ell}{2} < q_2 < \frac{\ell}{2} - v_2 t\right\} ,\end{aligned} \tag{A.2}$$

and $\mathbf{B}_{L,t}^-$ is obtained from $\mathbf{B}_{L,t}^+$ by reversing the sign of q_1 and v_1. Since $n(\mathbf{B}_{L,t}) < \infty$, Y_L^t is finite for $\mathcal{P}$-almost all Y.

Let $V_L = 2C\sqrt{\log L/m\beta}$ and

$$\mathcal{A}_L^t = \{Y \in \mathcal{Y}':\ \max\{|v_1|:\ (q_1, v_1) \in Y_L^t\} \leqq V_L\} . \tag{A.3}$$

As usual we denote by $(\mathcal{A}_L^t)^c$ the complement of $\mathcal{A}_L^t$. The following lemma holds.

Lemma A.1 *If $C > 1$, then there is a positive constant C_t such that for L large enough*

$$\mathcal{P}((\mathcal{A}_L^t)^c) \leqq \frac{C_t}{L^{1+r}\sqrt{\log L}}, \quad r = C^2 - 1 . \tag{A.4}$$

Proof. The probability that Y_L^t contains no particle with $|v_1| \geqq V_L$ is not smaller than the probability that the set $\mathbf{B}_{L,t}^0 = \mathbf{B}_{L,t} \cap \{|v_1| \geqq V_L\}$ is empty, i.e., $\mathcal{P}((\mathcal{A}_L^t)^c) \leqq 1 - e^{-b_{L,t}}$, where $b_{L,t} = n(\mathbf{B}_{L,t}^0)$. The proof follows by the following asymptotics, which holds for large L:

$$\begin{aligned} b_{L,t} &= \rho(\ell + m_2 t)\sqrt{\frac{2\beta m}{\pi}} \int_{V_L}^{\infty} (2L + v_1 t) e^{-(\beta m/2) v_1^2} dv_1 \\ &\asymp \sqrt{\frac{2}{\pi}} \rho(\ell + m_2 t) \frac{1}{L^{2C^2-1}} \left(\frac{1}{C\sqrt{\log L}} + \frac{t}{L\sqrt{\beta m}} \right) < \frac{C_t}{L^{1+r}\sqrt{\log L}} . \end{aligned}$$

Here $m_2 = \int |v_2| h(dv_2)$, and C_t is a constant independent of L. The proof follows by the inequality $1 - e^{-x} \leqq x$ which holds for all $x \geqq 0$. $\square$

Form now on we fix $C > 1$ and $t > 0$. Passing to integer L, the previous lemma gives that $\mathscr{P}$-a.a. $Y \in \mathscr{Y}$ belong to $\mathscr{A}_L^t$ for all L large enough. Let $\mathscr{A}_{t,L} = \{\omega\colon\ Y(\omega) \in \mathscr{A}_L^t,\ |V(\omega)| < V_L\}$. Then the set

$$\Omega_t = \bigcup_{L_0=2}^{\infty} \bigcap_{L=L_0}^{\infty} \mathscr{A}_{t,L} \tag{A.5}$$

has full μ measure. We set furthermore

$$\mathscr{A}_{t,L}^N = \{\omega \in \mathscr{A}_{t,L}\colon\ |Y_L^t| = N\}, \quad N = 0,1,2\dots,$$

where $|\cdot|$ denotes the cardinality of the configuration. $\mathscr{A}_{t,L}^N$ can be identified with a subset of the product $\mathbb{R} \times ((\mathbf{N}(0,t;\mathbf{M}_L^s)^N)_\Sigma$, where $(\cdot)_\Sigma$ denotes symmetrization. By labeling the particles, we can identify $\mathscr{A}_{t,L}^N$ with a subset $\tilde{A}_{t,L}^N$ of $\mathbb{R} \times (\mathbf{N}(0,t;\mathbf{M}_L^s))^N$, and the measure induced by μ on $\tilde{A}_{t,L}^N$ is absolutely continuous with respect to $d\bar{m}_N = dV \prod_{i=1}^N dq_1^{(i)}\, dv_1^{(i)}\, dq_2^{(i)}\, h(dv_2^{(i)})$.

The collision rules (1.10) imply the Liouville theorem for our dynamics, i.e., that the measure $\bar{m}_N$ is preserved: $\bar{m}_N(A) = \bar{m}_N(T^tA)$. Let $\mathscr{I}_{t,L}^N \subset \tilde{\mathscr{A}}_{t,L}^N$ denote the set of the particle configurations that have a multiple collision by time t. We write $\mathscr{I}_{t,L}^N = \bigcup_\Gamma I_\Gamma^t$, where Γ is the set of the labels of the particles that enter the multiple collision, and runs over all possible subsets of $\{1,2,\dots,N\}$ with cardinality $|\Gamma| > 1$. We denote by X the generic point in $\tilde{A}_{t,L}^N$, by $t^*(X)$ the time of the multiple collision and by $\bar{t}(X)$ the last collision time before t^* (with the proviso that $\bar{t}(X) = 0$ if t^* is the first collision time). For any positive integer n, and $h = t/2^n$, we set

$$I_\Gamma^{k,n} = \{X \in I_\Gamma^t\colon\ \bar{t}(X) \leqq kh,\ t^*(X) > kh\}, \quad k = 0,\dots,2^n - 1, \quad I_\Gamma^{(n)} = \bigcup_{k=2}^{2^n} I_\Gamma^{k,n}.$$

Clearly the sets $I_\Gamma^{(n)}$ are nondecreasing and $\bigcup_{n=1}^\infty I_\Gamma^{(n)} = I_\Gamma^t$.

It is easy to see that the Lebesgue measure of those X's for which two or more particles and the rod are at the same time at the same given position corresponds to a subvariety of 0 Lebesgue measure of $\mathbb{R}^{4N+1}$. Hence, by the Liouville theorem $\bar{m}_N(I_\Gamma^{k,n}) = \bar{m}_N(T^{kh}I_\Gamma^{k,n}) = 0$, which implies $\bar{m}_N(\mathscr{I}_{t,L}^N) = 0$.

To exclude infinitely many collisions in a finite time we use a similar argument. Let $\mathscr{J}_{t,L}^N \subset \tilde{\mathscr{A}}_{t,L}^N \backslash \mathscr{I}_{t,L}^N$ denote the subset of $\tilde{\mathscr{A}}_{t,L}^N$ for which infinitely many collisions take place in the time interval $(0,t)$. As before, we write $\mathscr{J}_{t,L}^N = \bigcup_\Gamma J_\Gamma^t$, where Γ now denotes the set of labels of the particles that undergo infinitely many collisions, and again runs over all possible subsets of $\{1,2,\dots,N\}$ with cardinality $|\Gamma| > 1$, since infinitely many collisions of the rod with only one particle in a finite time are impossible. $t^*(X)$ now denotes the accumulation point of the monotonic sequence of the collision times with the particles of Γ, and $\bar{t}(X)$ denotes the last collision time before t^* with the particles that are not in Γ (and we set $\bar{t}(X) = 0$ if there is no such collision).

Again we set

$$J_\Gamma^{k,n} = \{X \in J_\Gamma^t \colon \bar{t}(X) \leqq kh,\ t^*(X) > kh\}, \quad k = 0, \ldots, 2^n - 1,\ J_\Gamma^{(n)} = \bigcup_{k=0}^{2^n-1} J_\Gamma^{k,n},$$

for h as before. The sets $J_\Gamma^{(n)}$ are nondecreasing and $\bigcup_{n=1}^{\infty} J_\Gamma^{(n)} = J_\Gamma^t$.

To show that $J_\Gamma^{k,n}$ has zero measure we need to analyse the dynamics in the interval (kh, t^*). Only the particles of Γ collide with the rod, and we can neglect vertical velocities. Clearly if $\{u_k\}_{k=0}^{\infty}$ is the sequence of the velocities assumed by one of the particles on the left (right), then it is strictly decreasing (increasing). Hence $\lim_{k\to\infty} u_k = V^*$, and V^* is the common limiting velocity as $s \uparrow t^*$ for all particles of Γ and for the rod. At time kh we go over to the barycenter frame of reference, so that, denoting by $V_G(s)$ the velocity of the barycenter and by $G(s)$ its position, we have $V_G(kh) = G(kh) = 0$. Let Q_t denote the position of the rod at time t, and $\mu = M + pm$ the total mass, p being the number of elements of Γ. Conservation of energy gives

$$T_G(s) + \frac{1}{2}\mu V_G(s)^2 - fQ_s = T_G(0) - fQ_0, \quad kh \leqq s \leqq t^*, \tag{A.6}$$

where $T_G(s)$ is the kinetic energy in the center of mass reference system:

$$T_G(s) = \frac{m}{2}\sum_{i=1}^{p}(v_i(s) - V_G(s))^2 + \frac{M}{2}(V(s) - V_G(s))^2,$$

v_i being the velocity at time s of the i-th particle and $V(s)$ the velocity of the rod. Taking into account that the motion of G is uniformly accelerated we have

$$\lim_{s\to t^*} T_G(s) = 0, \quad \lim_{s\to t^*} V_G(s) = V^* = \frac{f}{\mu}t^*$$

$$\lim_{s\to t^*} G(s) = \lim_{s\to t^*} Q_s = Q^* = \frac{f}{2\mu}(t^*)^2.$$

Substituting into (A.6), and going back to the frame of reference with the origin at the rod, we find that in order that infinitely many collisions take place we must have

$$T_G(0) + fG(0) = 0. \tag{A.7}$$

This is impossible if the barycenter is on the right of the rod, $G(0) > 0$, and corresponds to a subvariety of zero Lebesgue measure otherwise.

Taking countable unions over N and then over L, we see that the subset $\mathcal{N}_t^+ \subset \Omega_t$ for which either a multiple collision or infinitely many collisions take place up to time t has μ measure 0. For the reverse dynamics (negative time) we find a similar subset $\mathcal{N}_t^-$ of zero μ measure. Taking integer t, and intersections, we find a subset $\Omega' \subset \Omega$ on which the dynamics is defined for all times. Ω' is clearly invariant with respect to the dynamics.

Theorem 1.1 is proved.

Appendix B

Definition B.1 We say that the chain $(\mathscr{E}, \mathfrak{B}, P)$ satisfies a relaxation condition if there are positive functions $W: \mathscr{E} \to \mathbb{R}_+$ and $\phi: \mathbb{R}_+ \to \mathbb{R}_+$, and positive numbers a, U_0 such that following conditions hold:
(i)

$$\phi^{(n)}(U) > (n+1)U, \quad U > U_0\,, \tag{B.1}$$

where $\phi^{(n)} = \phi^{(n-1)} \circ \phi$, $n = 1, \ldots$ denotes the n-th iterate of the function ϕ (with $\phi^{(0)}(x) = x$);
(ii) If $A_U = \{X : W(X) < U\}$, then for all $X \in \mathscr{E}$ such that $W(X) < \phi(U)$

$$P(X, A_U^c) < e^{-aU}, \quad U > U_0\,; \tag{B.2}$$

(iii) for any $U > U_0$ one can find a positive measure λ and an integer $n_0 > 0$, which may depend on U and are such that

$$P^{n_0}(X_0, dX) \geqq \mathbb{I}_{A_U}(X_0)\lambda(dX)\,. \tag{B.3}$$

Theorem B.1 *Suppose that $(\mathscr{E}, \mathfrak{B}, P)$ is an irreducible aperiodic chain satisfying a relaxation condition in the sense of Definition B.1, and let π be a maximal irreducibility measure for it. Then the following assertions hold:*
(i) *There is an absorbing set $\mathscr{H} \subset \mathscr{E}$ such that the chain restricted to $\mathscr{H}$ is Harris recurrent.*
(ii) *There is a unique P-invariant probability measure ν concentrated on $\mathscr{H}$, and it is equivalent to π.*
(iii) *$W \in L^p(\mathscr{E}, \nu)$ for all $p \in (0, \infty)$.*
(iv) *The chain is geometrically ergodic, i.e., one can find a function $g(X) \in L^1(\mathscr{E}, \nu)$ and a constant χ such that for all $X \in \mathscr{H}$*

$$\|P^n(X, \cdot) - \nu(\cdot)\| < g(X)e^{-\chi \cdot n}\,, \tag{B.4}$$

where $\| \cdot \|$ denotes the variation norm.
(v) *For any function $F : \mathscr{E} \to \mathbb{R}$ such that $\nu(F) = 0$, and $|F(X)| < C_F$ $(1 + W(X))$ for some constant $C_F < \infty$, and for any $X_0 \in \mathscr{H}$, as $N \to \infty$*

$$\frac{1}{\sqrt{N}} \sum_{j=1}^{N-1} F(X_j) \Rightarrow \mathscr{N}(0, \sigma^2)\,,$$

where $\Rightarrow$ denotes weak convergence, $\mathscr{N}(0, \sigma^2)$ is the centered normal distribution with dispersion $\sigma^2 \geqq 0$, and the distribution of the sum is the one induced by P_{X_0}.

Proof. The proof is based on Propositions B.2 and B.3 below.

Proposition B.2 *For $t > 0$ let $k_U(t)$ denote the smallest integer $n \geqq 1$ such that $t < \phi^{(n)}(U)$. Then for U large enough and for all $n \geqq k_U(W(X))$*

$$P^n(X, A_U^c) < \frac{1}{1 - e^{-aU}} e^{-aU}\,. \tag{B.5}$$

Proof. Let

$$a_U^{(n)} = \sup_{W(X)<\phi^{(n)}(U)} P^n(X, A_U^c) .$$

For $U > U_0$ and U_0 large enough $a_U^{(1)} < e^{-aU}$, by inequality (B.2). If $n \geqq 2$ we have, for $W(X) < \phi^{(n)}(U)$

$$\begin{aligned} P^n(X, A_U^c) &= \int_{A^c_{\phi^{(n-1)}(U)}} P(X, dX_1) P^{n-1}(X_1, A_U^c) \\ &\quad + \int_{A_{\phi^{(n-1)}(U)}} P(X, dX_1) P^{n-1}(X_1, A_U^c) \\ &\leqq e^{-a\phi^{(n-1)}(U)} + a_U^{(n-1)} . \end{aligned}$$

Inequality (B.5) then follows from inequality (B.1):

$$\sum_{k=1}^{\infty} e^{-a\phi^{(k)}(U)} \leqq e^{-aU} \sum_{k=0}^{\infty} e^{-kaU} = \frac{e^{-aU}}{1 - e^{-aU}} .$$

Proposition B.1 is proved. □

Consider the potential kernel

$$G(X, \cdot) = \sum_{j=0}^{\infty} P^j(X, \cdot) .$$

By inequality (B.5) we have $G(X, A_U) = \infty$ for all X, which implies (see [15]) that the chain is recurrent. Hence one can find an absorbing set $\mathscr{H} \subset \mathscr{E}$ and a P-invariant σ-finite measure ν such that π is absolutely continuous with respect to ν, and $\nu(B) > 0$ implies

$$P_X \left(\sum_{j=1}^{\infty} \mathbb{I}_B(X_j) = \infty \right) = 1 . \tag{B.6}$$

From now on we identify $\mathscr{E}$ and $\mathscr{H}$. According to the standard terminology $(\mathscr{E}, P)$ is a Harris recurrent chain.

We next show that ν is a finite measure, and hence can be normalized to a probability measure. Since π is a maximal irreducibility measure, λ has to be absolutely continuous with respect to π: $\lambda(dX) = \rho(X)\pi(dX)$. We can find a positive γ and a set $D \subset \{X : \rho(X) > \gamma\}$ such that $\pi(D) > 0$ and $\nu(D) < \infty$. Then by inequalities (B.3) and (B.5), for any choice of X, we can find n large enough so that

$$P^n(X, D) \geqq \int_{A_U} P^{n-n_0}(X, dX_1) P^{n_0}(X_1, D) \geqq \gamma \pi(D) P^{n-n_0}(X, A_U) > \bar{\gamma} > 0 .$$

Then $\nu(D) < \infty$ implies that $\nu(\mathscr{E})$ is finite, since

$$\nu(D) = \lim_{n\to\infty} \int_{\mathscr{E}} \nu(dX) P^n(X, D) \geqq \int_{\mathscr{E}} \nu(dX) \liminf_{n\to\infty} P^n(X, D) \geqq \bar{\gamma}\nu(\mathscr{E}) .$$

As the chain is Harris recurrent, we have, by Orey's theorem [15], that for all $X \in \mathscr{E}$

$$\lim_{n\to\infty} \|\nu(\cdot) - P^n(X,\cdot)\| = 0\,. \tag{B.7}$$

Relation (B.6) implies that ν is an irreducibility measure, hence it is equivalent to π. Uniqueness of the invariant measure ν is a standard fact. Assertions (i) and (ii) of the theorem are proved.

Assertion (iii) follows by observing that if M is a positive integer such that $U = M^{1/p} > U_0$, then, by Proposition B.2 we have

$$\nu((W(X))^p > M) = \lim_{n\to\infty} P^n(X_0, A_U^c) \leqq Ce^{-aM^{1/p}}\,.$$

In what follows for any measurable set $B \subset \mathscr{E}$ we set

$$\tau_B(X) = \min\{j \geqq 1 :\ X_j \in B\}\,.$$

Proposition B.3 *There are positive constants $\bar{a} > 0$ and $C_1(U)$ such that for U large enough and all X for which $k_U(W(X)) \leqq n$*

$$P_X(\tau_{A_U} > n+k) < C_1(U)e^{-\bar{a}kU}, \quad k \geqq 1\,. \tag{B.8}$$

Proof. For $k = 1$ the result follows from Proposition B.2. We fix $\bar{a} \in (0, a)$ and proceed by induction, setting $\gamma = a - \bar{a}$ and assuming that $e^{-\gamma U_0} < \frac{1}{2}$. Suppose that inequality (B.8) holds for all $k < m$, and let X be fixed in such a way that $W(X) < \phi^{(n)}(U)$. We have

$$P_X(\tau_{A_U} > n+m) = \int_{A_U^c} P^n(X, dX_n) \int_{A_U^c} P(X_n, dX_{n+1}) \cdots \int_{A_U^c} P(X_{n+m-2}, dX_{n+m-1}) P(X_{n+m-1}, A_U^c)\,. \tag{B.9}$$

We set

$$a_U(n,m) = \sup_{W(X)<\phi^{(n)}(U)} P_X(\tau_{A_U} > n+m)\,,$$

and split the integration over X_{n+m-k} into the regions $A^c_{\phi^{(k)}(U)}$, and $A_U^c \backslash A^c_{\phi^{(k)}(U)}$, for $k = 1, \ldots, m$. By splitting the integral over X_{n+m-1} we get by (B.9)

$$a_U(n,m) \leqq e^{-aU} a_U(n, m-1) + a_U^{(1)}(n, m-1)\,,$$

$$a_U^{(1)}(n,m-1) = \sup_{W(X)<\phi^{(n)}(U)} \int_{A_U^c} P^n(X, dX_n) \cdots \int_{A_U^c} P(X_{n+m-3}, dX_{n+m-2}) P(X_{n+m-2}, A^c_{\phi(U)})\,.$$

By splitting the integral over X_{n+m-2} we get

$$a_U^{(1)}(n, m-1) \leqq a_U(n, m-2)e^{-a\phi(U)} + a_U^{(2)}(n, m-2)\,, \tag{B.10}$$

where $a_U^{(2)}(n, m-2)$ corresponds to integrating the first $m-2$ variables over A_U^c, with the function $P(X_{n+m-3}, A^c_{\phi^{(2)}(U)})$. We go on until we are left with the integral over X_n, for which, by inequality (B.5), we get

$$\sup_{W(X)<\phi^{(n)}(U)} P^n(X, A^c_{\phi^{(m)}(U)}) \leqq \sup_{W(X)\leqq\phi^{(n+m)}(U)} P^n(X, A^c_{\phi^{(m)}(U)}) \leqq C(U)e^{-a\phi^{(m)}(U)},$$

where $C(U) = (1 - e^{-aU})^{-1}$. In conclusion we get, using once again inequality (B.1),

$$\begin{aligned} a_u(n,m) & \\ &\leqq e^{-aU}a_U(n,m-1) + e^{-a\phi(U)}a_U(n,m-2) + \cdots + e^{-a\phi^{(m-1)}(U)}a_U(n,1) \\ &\quad + C(U)e^{-a\phi^{(m)}(U)} \\ &\leqq C(U)e^{-\bar{a}mU}\sum_{k=1}(e^{-\gamma U})^k \leqq C_1(U)e^{-\bar{a}mU}. \quad \square \end{aligned}$$

As a consequence of Proposition B.3 we have

$$\sup_{X\in A_U} \mathbb{E}_X e^{\tau A_U} < \infty. \tag{B.11}$$

Relation (B.11) implies that the chain is geometrically ergodic, i.e., assertion (iv) [15].

As for assertion (v), note that by Harris ergodicity (B.7) we need to prove the result only for the invariant distribution ν. By assertion (iii) we have $F \in L^p(\mathscr{E}, \nu)$ for all p. By geometric ergodicity (B.4) we are dealing with a stationary process with exponential decay of the strong (or α-) mixing coefficient. Hence the result follows by standard results (see, for instance, [15]).

Theorem B.1 is proved.

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