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Sample path behaviour in connection with generalized arcsine laws

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Summary. Let $G = (G(t), t \geqq 0)$ be the process of last passage times at some fixed point of a Markov process. The Dynkin–Lamperti theorem provides a necessary and sufficient condition for $G(t)/t$ to converge in law as $t \to \infty$ to some non-degenerate limit (which is then a generalized arcsine law). Under this condition, we give a simple integral test that characterizes the lower-functions of G. We obtain a similar result for $A^+ = (A^+(t), t \geqq 0)$, the time spent in $[0, \infty)$ by a real-valued diffusion process, in connection with Watanabe's recent extension of Lévy's second arcsine law.

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1 Introduction

The celebrated first and second arcsine laws of P. Lévy [L] claim that the instant of the last zero on the time interval $[0, 1]$ of a real-valued Brownian motion B started at 0, and the total time spent in $[0, \infty)$ by B during the time-interval $[0, 1]$, are both distributed according to the arcsine law. That is their common distribution function is $(2/\pi)$ arcsin $\sqrt{x}$ $(0 \leqq x \leqq 1)$. We refer to Pitman–Yor [PY] for an elegant proof and to Bingham–Doney [BD] for further references.

It is well-known that Lévy's first arcsine law can be extended to a wide class of Markov processes. Specifically, let $X = (X_t, t \geqq 0)$ be a Borel right Markov process started from some recurrent point, say 0. When 0 is regular, we denote by $L = (L(t), t \geqq 0)$ a local time at 0, that is a continuous additive functional which increases only on the zero set of X. When 0 is irregular, we consider a sequence $\xi_1, \ldots, \xi_n, \ldots$ of independent exponential variables with parameter 1, which are independent of X. We then call local time at 0 the process $L(\cdot) = \xi_1 + \ldots \xi_{N(\cdot)}$, where $N(\cdot)$ is the number of zeros of X on the time interval $[0, \cdot]$ [This definition of local time in the irregular case may look awkward, but it is the most convenient for our purpose. Actually, this little artifice allows us to handle simultaneously the regular and irregular cases]. In both cases, the inverse local time $T(\cdot) = \inf\{s: L(s) > \cdot\}$ is a subordinator, that is T is a right-continuous increasing process with independent homogeneous increments. Denote its Laplace exponent by Φ,

$$\mathbb{E}(\exp\{-\lambda T(t)\}) = \exp\{-t\Phi(\lambda)\} \quad (\lambda, t \geqq 0).$$

Next, consider the process of last passage times at 0,

$$G(t) = \sup\{s < t : X_s = 0\} \quad (t \geqq 0),$$

and recall that $G(t)$ coincides with the inverse local time evaluated immediately before it exceeds t, i.e. $G(t) = T(L(t)-)$. Recall also that for every $\alpha \in (0,1)$, the so-called generalized arcsine law with parameter α is the probability measure on the unit interval with density $s^{\alpha-1}(1-s)^{-\alpha}[\Gamma(\alpha)\Gamma(1-\alpha)]^{-1}(0 < s < 1)$. Then the Dynkin–Lamperti theorem ensures that $G(t)/t$ converges in distribution as $t \to \infty$ to the generalized arcsine law with parameter $\alpha \in (0,1)$ if and only if Φ is regularly varying at $0+$ with index α, and this is also equivalent to $\lim_{t\to\infty} \mathbb{E}(G(t)/t) = \alpha$. Conversely, if $\mathbb{E}(G(t)/t)$ diverges as $t \to \infty$, then a fortiori $G(t)/t$ does not converge in distribution and Φ is not regularly varying at $0+$. Finally, the case when $\mathbb{E}(G(t)/t)$ converges to 0 (respectively, to 1) is somewhat degenerate, because then $G(t)/t$ converges in probability to 0 (respectively, to 1) and Φ is regularly varying at $0+$ with index 0 (respectively, with index 1). We refer to Sharpe [S] for material on Markov processes, local times ..., and to Bingham et al. [BGT] for a detailed account on regular variation and the Dynkin–Lamperti theorem.

Recently, S. Watanabe [W] extended Lévy's second arcsine law to one-dimensional diffusions. Specifically, suppose that X is a non-singular diffusion process on $(-\infty, \infty)$ and consider for every $t \geqq 0$

$$A^+(t) = \int_0^t \mathbf{1}_{\{X_s \geqq 0\}}\, ds, \qquad A^-(t) = t - A^+(t) = \int_0^t \mathbf{1}_{\{X_s < 0\}}\, ds.$$

As before, let L and T be, respectively, a local time at 0 and the inverse local time processes, and recall that the time-changed processes

$$T^+(t) = A^+ \circ T(t), \quad T^-(t) = A^- \circ T(t) \qquad (t \geqq 0)$$

are two independent subordinators. Let Φ^+ and Φ^- stand for their respective Laplace exponents, and note that $\Phi = \Phi^+ + \Phi^-$ is the Laplace exponent of T. Next, for $\alpha \in (0,1)$ and $p \in (0,1)$, consider the variable

$$Y_{p,\alpha} = p^{1/\alpha} S_\alpha (p^{1/\alpha} S_\alpha + (1-p)^{1/\alpha} S'_\alpha)^{-1},$$

where S_α and S'_α are two independent copies of the nonnegative stable variable with index α. Then $A^+(t)/t$ converges in distribution as $t \to \infty$ to $Y_{p,\alpha}$ if and only if

$$\Phi \text{ is regularly varying at } 0+ \text{ with index } \alpha \text{ and } \Phi^+ \sim p\Phi \text{ (at } 0+). \quad (1)$$

Conversely, if (1) fails, then either $A^+(t)/t$ does not converge in law as $t \to \infty$, or it converges to some degenerate distribution (the Dirac point mass at p, or a mixture of the Dirac point masses at 0 and 1). Finally, when the diffusion is in natural scale (there is of course no loss of generality in assuming this), (1) holds if and only if m^+ is regularly varying at ∞ with index $1/\alpha - 1$ and $m^-(x) \sim p^{1/\alpha}(1-p)^{1/\alpha} m^+(x)$ as $x \to \infty$, where $m^+(x) = m([0,x])$ and $m^-(x) = m([-x,0))$. In that case, the asymptotic behaviours of the Laplace exponents Φ^+ and Φ^- are determined by that of m^+ and m^-, by an application of a Tauberian theorem due to Kasahara [K] (*Warning*: there is a typographical error in the definition of the constant D_α on p. 70 of [K]), see also Kotani–Watanabe [KW]. The precise relation is

$$\Phi^+(\lambda) \sim (\alpha(1-\alpha))^\alpha \frac{\Gamma(1-\alpha)}{\Gamma(1+\alpha)} \lambda^\alpha \ell(1/\lambda) \quad (\lambda \to 0+),$$

where ℓ is a slowly varying function at ∞ such that an asymptotic inverse of $x \to xm^+(x)$ is $x \to x^\alpha/\ell(x)$, and a similar relation holds for Φ^- and m^-.

The purpose of this paper is to investigate the almost-sure asymptotic behaviour of the processes $G = (G(t), t \geqq 0)$ and $A^+ = (A^+(t), t \geqq 0)$ in the cases when non-degenerate extensions of Lévy's arcsine laws hold. Our interest in this problem arose from a recent paper of Hobson [H] who treats the Brownian case. His result is that if X is a real-valued Brownian motion and $h : (0,\infty) \to (0,\infty)$ a decreasing function, then a.s. for $V = A^+$ or G,

$$\liminf_{t\to\infty} V(t)/th(t) = 0 \text{ or } \infty$$

according as the integral

$$\int^\infty \sqrt{h(t)}\frac{dt}{t}$$

diverges or converges. See also Meyre–Werner [MW] for a multidimensional extension. We now state our results.

Theorem 1. *Suppose that X is a Borel right Markov process started from 0 and let Φ stand for the Laplace exponent of the inverse local time. Suppose that Φ is regularly varying at $0+$ with index $\alpha \in (0,1)$ and put $\varphi(x) = 1/\Phi(1/x)$. The asymptotic rate of growth of the last passage process G is determined as follows. If $f : (0,\infty) \to (0,\infty)$ is an increasing function, then a.s.*

$$\liminf_{t\to\infty} f(G(t))/t = 0 \textit{ or } \infty$$

according as the integral

$$\int^\infty \Phi(1/f(t))\, d\varphi(t)$$

diverges or converges.

Theorem 2. *Suppose that X is a non-singular diffusion on $(-\infty,\infty)$, denote by T the inverse local time at 0 and by Φ its Laplace exponent. Consider A^+, the process of the time spent by X in $[0,\infty)$, and let Φ^+ stand for the Laplace exponent of $T^+ = A^+ \circ T$. Suppose that Φ is regularly varying at*

$0+$ *with index* $\alpha \in (0,1)$ *and that* $\Phi^+ \sim p\Phi$ *at* $0+$ *for some* $p \in (0,1)$. *Put* $\varphi(x) = 1/\Phi(1/x)$. *Then, if* $f : (0,\infty) \to (0,\infty)$ *is an increasing function, we have a.s.*

$$\liminf_{t\to\infty} f(A^+(t))/t = 0 \text{ or } \infty$$

according as the integral

$$\int^{\infty} \Phi(1/f(t))\, d\varphi(t)$$

diverges or converges.

It should be clear that Theorem 1 is in fact a result about subordinators, in the sense that it involves the Markov process X only through the inverse local time. But any subordinator can be thought of as the inverse local time of some Markov process, and the present formulation does not induce a loss of generality.

It is interesting to observe that the integral test of Theorem 2 is the same as that of Theorem 1, and in particular, does not depend on the constant $p \in (0,1)$ which represents the limiting proportion of long excursions which are positive. When the hypotheses of Theorem 2 are fulfilled, the lower-functions of G and of A^+ thus coincide, but their upper-functions are different (plainly, $G(t) = t$ for infinitely many t's whereas $A(t) < t$ for all $t > 0$). Note also that information on the upper-functions of A^+ follows from Theorem 2 applied to the diffusion-X and the identity $A^+(t) = t - A^-(t)$.

Of course, $\Phi(\lambda) = \sqrt{2\lambda}$ in the Brownian case, and it is immediate to check that Theorems 1–2 agree with the results of Hobson. Note also that when X is a so-called skew Bessel process of dimension $2 - 2\alpha$ $(0 < \alpha < 1)$, then the integral test in Theorem 1 and 2 reduces to

$$\int^{\infty} f(t)^{-\alpha} t^{\alpha-1} dt = \infty \text{ or } < \infty .$$

This was obtained also by Hobson in his Ph.D. dissertation. We refer to Barlow et al. [BPY] for the arcsine law for skew Bessel processes.

Our approach relies on some sample path properties of subordinators, which are developed in Sect. 2. This material is used in Sect. 3 to establish Theorems 1 and 2. Finally we discuss some extensions of these results in Sect. 3.

2 Some sample path properties of subordinators

Throughout this section, $T = (T(t), t \geqq 0)$ stands for a subordinator with Laplace exponent Φ. Denote its potential measure by U,

$$U(dx) = \int_0^{\infty} \mathbb{P}(T(t) \in dx)\, dt ,$$

and recall that the Laplace transform of U is $1/\Phi$. We first establish an easy result on the convergence of certain integrals of T.

Lemma 1. *Let* $g : [0, \infty) \to [0, \infty)$ *be a decreasing function. The following assertions are equivalent*:

(i)
$$\int_0^\infty g(x) U(dx) < \infty$$

(ii)
$$\mathbb{P}\left(\int_0^\infty g(T(t))\, dt < \infty \right) = 1$$

(iii)
$$\mathbb{P}\left(\int_0^\infty g(T(t))\, dt < \infty \right) > 0$$

Proof. The derivations (i)$\Rightarrow$(ii)$\Rightarrow$(iii) are obvious. Suppose that (iii) holds and pick $\varepsilon > 0$ and $k > 0$ such that

$$\mathbb{P}\left(\int_0^\infty g(T(t))\, dt < k \right) > \varepsilon\,.$$

Next, consider for every integer $n > 0$ the stopping time

$$\tau_n = \inf \left\{ t : \int_0^t g(T(s))\, ds \geqq kn \right\}\,.$$

By the Markov property, conditionally on $\tau_n < \infty$, the process $T'(\,\cdot\,) = T(\tau_n + \,\cdot\,) - T(\tau_n)$ is a subordinator distributed as T. Using the hypothesis that g decreases, we get

$$\begin{aligned} \mathbb{P}(\tau_{n+1} = \infty \mid \tau_n < \infty) &= \mathbb{P}\left(\int_0^\infty g(T'(t) + T(\tau_n))\, dt \leqq k \mid \tau_n < \infty \right) \\ &\geqq \mathbb{P}\left(\int_0^\infty g(T'(t))\, dt < k \right) > \varepsilon\,. \end{aligned}$$

This shows that $\int_0^\infty g(T(t))\, dt$ is bounded from above by kV, where V is a geometric variable. This ensures that its expectation is finite and (i) follows. □

The next lemma provides a relation between on the one hand, the relative size of the subordinator and its jumps, and on the other hand, certain integrals of T. Denote the jump of T at time $t \geqq 0$ by $\Delta_t = T(t) - T(t-)$, the Lévy measure by Π, and the tail of the Lévy measure by $\overline{\Pi}$, that is $\overline{\Pi}(x) = \Pi(x, \infty)$.

Lemma 2. *For every Borel function* $b : [0, \infty) \to [1, \infty)$, *the events*

$$\{\Delta_t > b(T(t-)) \text{ infinitely often as } t \to \infty\}$$

and

$$\left\{ \int_0^\infty \overline{\Pi} \circ b(T(t))\, dt = \infty \right\}$$

coincide up to a null set.

Proof. The process of the jumps Δ is a Poisson point process with intensity Π, and hence the compensated sum

$$\sum_{s \leqq t} \mathbf{1}_{\{\Delta_s > b(T(s-))\}} - \int_0^t \overline{\Pi} \circ b(T(s))\,ds \quad (t \geqq 0)$$

is a martingale. The assertion follows from the theorem of convergence of martingales by an argument similar to that of Proposition 3.1 in Hobson [H]. □

An immediate combination of Lemmas 1 and 2 yields the following.

Lemma 3. *Let* $f : [0, \infty) \to [0, \infty)$ *be an increasing function. Then*

$$\mathbb{P}(\Delta_t > f(T(t-)) \text{ infinitely often as } t \to \infty) = 0 \text{ or } 1$$

according as the integral

$$\int^{\infty} \overline{\Pi} \circ f(x) U(dx)$$

converges or diverges.

Alternatively, Lemma 3 can be deduced from a recent result of Erickson (Theorem 1(ii) in [E]) and the compensation formula for Poisson point processes.

Our final lemma concerns the rate of growth of T when its Laplace exponent is regularly varying.

Lemma 4. *Suppose that* Φ *is regularly varying at* $0+$ *with index* $\alpha \in (0, 1)$ *and let* $f : (0, \infty) \to (0, \infty)$ *be an increasing function. Then*

$$\lim_{t \to \infty} T(t)/f(t) = 0 \quad a.s. \tag{2}$$

if and only if

$$\int^{\infty} \Phi(1/f(x))\,dx < \infty . \tag{3}$$

Moreover, if the preceding assertions fail, then

$$\limsup_{t \to \infty} T(t)/f(t) = \infty \quad a.s.$$

Proof. Assume first that (3) holds. We deduce from the obvious inequality

$$\begin{aligned} \mathbb{P}(T(t) \geqq a) &\leqq (1 - 1/e)^{-1} \mathbb{E}(1 - \exp\{-T(t)/a\}) \\ &= (1 - 1/e)^{-1} (1 - \exp\{-t\Phi(1/a)\}) \end{aligned}$$

applied for $t = 2^{n+1}$ and $a = f(2^n)$ that

$$\mathbb{P}(T(2^{n+1}) \geqq f(2^n)) \leqq 2(1 - 1/e)^{-1} 2^n \Phi(1/f(2^n)) .$$

Since $t \to \Phi(1/f(t))$ decreases, we deduce from (3) that the series $\sum 2^n \Phi(1/f(2^n))$ converges, and by the Borel–Cantelli lemma,

$$T(2^{n+1}) < f(2^n) \text{ for all integers } n \text{ large enough, a.s.}$$

An immediate argument of monotonicity shows that the latter implies that $T(t) < f(t)$ for all real number t large enough, a.s., and since (3) still holds when one replaces f by εf for some $\varepsilon \in (0,1)$ (because Φ is concave), this proves (2).

Now assume that (3) fails. The Lévy–Khintchine formula gives

$$\Phi(\lambda)/\lambda = \delta + \int_0^\infty e^{-\lambda t}\overline{\Pi}(t)\,dt\,,$$

where $\delta > 0$ is the drift coefficient. An application of a Tauberian theorem and the monotone density theorem yields

$$\Gamma(1-\alpha)\overline{\Pi}(x) \sim \Phi(1/x) \quad (x \to \infty) \tag{4}$$

and it follows that

$$\int^\infty \overline{\Pi} \circ f(x)\,dx = \infty\,. \tag{5}$$

Recall that the Lévy measure $\overline{\Pi}$ is the intensity of the Poisson point process of the jumps Δ, so (5) implies

$$\limsup_{t\to\infty} \Delta_t/f(t) \geqq 1 \quad \text{a.s.}$$

A fortiori $\limsup_{t\to\infty} T(t)/f(t) \geqq 1$ a.s., and since (3) fails as well when f is replaced by f/ε for some $\varepsilon \in (0,1)$ (because Φ is concave), we have

$$\limsup_{t\to\infty} T(t)/f(t) = \infty \qquad \text{a.s.} \qquad \square$$

Remarks. 1. Under the additional condition that $t \to f(t)/t$ increases, Theorem 6.1 in Fristedt [Fr] provides an integral test to decide whether f is an upper-function of a general subordinator, which is equivalent to that of Lemma 4. See also Feller [Fe]. On the other hand, Lemma 4 is well-known in the stable case, see e.g. Theorem 11.2 in [Fr].

2. We point out that Lemma 4 holds actually under the weaker condition that the integrated tail of the Lévy measure $t \to \int_0^t \overline{\Pi}(x)\,dx$ has positive increase in the sense of de Haan and Resnick, see [BGT] on page 71 for a precise definition. Specifically, Φ is concave and increasing, and hence it is O-regularly varying. By the Tauberian Theorem 2.10.2 in [BGT] and the Lévy–Khintchine formula, we see that

$$\int_0^t \overline{\Pi}(x)\,dx \overset{\cup}{\underset{\cap}{}} t\Phi(1/t) \quad (t \to \infty)\,,$$

where the notation $a \overset{\cup}{\underset{\cap}{}} b$ means $a = O(b)$ and $b = O(a)$. The integrated tail has plainly bounded increase and so does $\overline{\Pi}$. An application of the O-version of the monotone density theorem (Proposition 2.10.3 in [BGT]) then implies that

$\overline{\Pi}(x) \overset{\cup}{\cap} \Phi(1/x)$. This shows that (5) holds whenever (3) fails, and we conclude that in that case, $\limsup_{t\to\infty} T(t)/f(t) = \infty$ a.s. The derivation (3) ⇒ (2) does not use any extra assumption on Φ. We also recall that some condition on the growth of the tail of the Lévy measure is indeed needed for the validity of the conclusions of Lemma 4. For instance, if the subordinator has finite expectation $\mathbb{E}(T(1)) = \mu < \infty$, then the law of large numbers gives $\lim_{t\to\infty} T(t)/t = \mu$ a.s.

3. Proof of the theorems

Theorems 1 and 2 now follow readily from Lemmas 3 and 4 and Tauberian theorems. In the sequel, T stands for the inverse of the local time L, its Laplace exponent is Φ.

Proof of Theorem 1. Recall that the Laplace transform of the potential measure U is $1/\Phi$, so by a Tauberian theorem,

$$\Gamma(1+\alpha)U(x) \sim 1/\Phi(1/x) = \varphi(x) \quad (x \to \infty)\,.$$

We then deduce from (4) and Lemma 3 that

$$\mathbb{P}(\Delta_t > f(T(t-)) \text{ infinitely often as } t \to \infty) = 0 \text{ or } 1\,.$$

according as

$$\int^{\infty} \Phi(1/f(x))\, d\varphi(x) \tag{6}$$

converges or diverges.

Next, recall that $G(t) = T(L(t)-)$ for all $t \geqq 0$ a.s., so $\Delta_t > f(T(t-))$ infinitely often as $t \to \infty$ if and only if $t - G(t) > f(G(t))$ infinitely often. We deduce that a.s.,

$$\liminf_{t\to\infty} \{f(G(t)) + G(t)\}/t \geqq 1 \text{ or } \liminf_{t\to\infty} f(G(t))/t \leqq 1$$

according as (6) converges or diverges.

First, assume that (6) diverges. Then the same holds when f is replaced by f/ε for some $\varepsilon \in (0,1)$ (because Φ is concave), and it follows that $\liminf_{t\to\infty} f(G(t))/t = 0$ a.s. Finally, assume that (6) converges. Then

$$\lim_{x\to\infty} \Phi(1/f(x))\varphi(x) = \lim_{x\to\infty} \Phi(1/f(x))/\Phi(1/x) = 0$$

(since $\Phi(1/f)$ decreases), and this forces

$$\lim_{x\to\infty} f(x)/x = \infty\,, \tag{7}$$

because Φ is concave. As a consequence, we have

$$\liminf_{t\to\infty} f(G(t))/t \geqq 1 \quad \text{a.s.}$$

and since (6) converges as well when f is replaced by εf for some $\varepsilon \in (0,1)$, we conclude that $\lim_{t\to\infty} f(G(t))/t = \infty$ a.s. □

Proof of Theorem 2. Recall that $T^+ = A^+ \circ T$ and $T^- = A^- \circ T$ are two independent subordinators. Lemmas 1 and 4 yield

$$\liminf_{t\to\infty} f(T^+(t))/T^-(t) = \infty \text{ or } 0 \quad \text{a.s.}$$

according as the integral

$$\int^{\infty} \Phi^-(1/f(x))U^+(dx) \tag{8}$$

converges or diverges, where Φ^- is the Laplace exponent of T^- and U^+ the potential measure of T^+. An argument based on a Tauberian theorem shows that one can replace (8) by (6).

Assume first that (6) diverges, so $\liminf_{t\to\infty} f(A^+ \circ T(t))/A^- \circ T(t) = 0$ a.s., and since $A^- \circ T(t) \leqq T(t)$, a fortiori $\liminf_{t\to\infty} f(A^+(t))/t = 0$ a.s.

Then assume that (6) converges, so $\lim_{t\to\infty} f(A^+ \circ T(t))/A^- \circ T(t) = \infty$ a.s. We deduce from (7) and the identity $A^+(t) + A^-(t) = t$ that

$$\lim_{t\to\infty} f(A^+ \circ T(t))/T(t) = \infty \quad \text{a.s.} \tag{9}$$

Next, take any sample path for which (9) holds, pick M arbitrarily large and choose t_M such that

$$f(A^+ \circ T(t)) \geqq 2MT(t) \quad \text{for all } t > t_M$$

and [recall (7)]

$$f(t) \geqq 2Mt \quad \text{for all } t > T(t_M-)\,.$$

Then take any $t > t_M$. If $T(t) \leqq 2T(t-)$, then $f(A^+ \circ T(t-)) \geqq MT(t)$ and an immediate argument of monotonicity shows that

$$f(A^+(s)) \geqq Ms \quad \text{for all } s \in [T(t-),\, T(t)]\,. \tag{10}$$

If $T(t) > 2T(t-)$ and the excursion of the diffusion on $(T(t-), T(t))$ is negative, then $A^+(s) = A^+ \circ T(t)$ for all $s \in [T(t-), T(t)]$ and (10) obviously holds. Finally, suppose that $T(t) > 2T(t-)$ and the excursion of the diffusion on $(T(t-), T(t))$ is positive. Then, by monotonicity, $f(A^+(s)) \geqq Ms$ for all $s \in [T(t-), 2T(t-)]$, whereas for $s \in (2T(t-), T(t)]$, $A^+(s) \geqq s - T(t-) > T(t_M-)$ and thus

$$f(A^+(s)) \geqq f(s - T(t-)) \geqq 2M(s - T(t-)) \geqq Ms\,.$$

Hence (10) holds in all cases, and since M can be chosen arbitrarily large, the proof is now complete. □

3. Some extensions

A perusal of the proofs shows that Theorems 1 and 2 can be extended using the notion of positive increase. The argument is similar to that in the second remark after the proof of Lemma 4, so we omit the details. Specifically, assume that $\int_0^{\cdot} \overline{\Pi}(x)\,dx$ has positive increase. Then, the conclusions of Theorem 1 are valid. Next, let $\overline{\Pi}^+$ and $\overline{\Pi}^-$ be the tails of the Lévy measures of T^+ and T^-, respectively, and assume that $\int_0^{\cdot} \overline{\Pi}^+(x)\,dx$ and $\int_0^{\cdot} \overline{\Pi}^-(x)\,dx$ both have positive increase. Suppose moreover that $\Phi^+ \overset{\cup}{\cap} \Phi^-$ in the neighbourhood of 0+. Then the conclusions of Theorem 2 are valid. We also point out that some growth condition on the tail(s) of the Lévy measure(s) is needed in order to ensure the conclusions of Theorems 1 (2). For instance, if $T(1)$ has finite expectation, that is if $\int^{\infty} \overline{\Pi}(x)\,dx < \infty$, then the law of large numbers implies that $\lim_{t\to\infty} G(t)/t = 1$ a.s.

In the Brownian case, Hobson [H] also characterized the sample path behaviour of $G(t)$ and $A^+(t)$ for as $t \to 0+$. The methods of the present paper can be easily adapted to deal with small times, the precise statements and proofs corresponding to Theorems 1–2 are left to the interested reader. The methods also apply to give a discrete time analogue of Theorem 2, which is connected to Watanabe's arcsine law for non-homogeneous birth and death processes [W].

Finally, we mention that there exists another extension of Lévy's second arcsine law, namely for real-valued Lévy processes that fulfill Spitzer's condition, see Getoor–Sharpe [GS]. It does not seem that the method used in this work can be applied to investigate the almost-sure asymptotic behaviour of the time spent in $[0,\infty)$ by a Lévy process (difficulties are related to the jumps of the Lévy process across 0). Nonetheless, Shi–Werner [SW] solved very recently this problem for stable Lévy processes, using an argument based on the scaling property and ergodic theory.

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