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Probability Theory and Related Fields

Forward, backward and symmetric stochastic integration

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Summary. We define three types of non causal stochastic integrals: forward, backward and symmetric. Our approach consists in approximating the integrator. Two optics are considered: the first one is based on traditional usual stochastic calculus and the second one on Wiener distributions.

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0 Introduction and notations

Many authors have examined extensions of classical stochastic integrals to a certain class of anticipating integrands. A good list of references for this purpose is contained in [N]. Among the generalisations we find the classical backward stochastic integration [K1], the enlargement of filtrations [Je], the extension of Stratonovich integral [O], [Z], [NP], the Skorohod integration [N], and finally the forward integration [KR], [BM], [AP]. Let us remark that in the three last approaches the only integrator involved is essentially the Brownian motion.

In this paper we define forward, backward and symmetric integrals by a limit procedure. These integrals are respectively extensions of Ito, backward and Stratonovich integrals. We have focused our attention especially on the forward integration; the backward anticipating integral is defined in an analogous way and the symmetric integral is obtained by averaging. Our work has the following features:

i) through the definition, we make explicite the "forward" nature of the integral: in fact the integrator must operate with an infinitesimal anticipation with respect to the integrand,
ii) we introduce an effective non-causal stochastic integration with respect to more general integrators than the Brownian motion,

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iii) one of the straightforward consequence of the definition is an integration by parts formula,
iv) we relate the main anticipating stochastic integrals; in particular we connect Skorohod integral with the one defined through enlargement of filtration.

We introduce now some useful notations. Let $\varepsilon>0$. If $X=(X(t);\ 0\leq t\leq 1)$ and $Y=(Y(t);\ 0\leq t\leq 1)$ are two stochastic processes, continuous at 0 and 1, we set,

$$(1)_- \qquad I^-(\varepsilon, X, dY)=\int_0^1 X(t)\frac{Y((t+\varepsilon)\wedge 1)-Y(t)}{\varepsilon}\,dt,$$

$$(1)_+ \qquad I^+(\varepsilon, X, dY)=\int_0^1 X(t)\frac{Y(t)-Y((t-\varepsilon)\vee 0)}{\varepsilon}\,dt.$$

We also define by symmetry, $I^0=(I^+ + I^-)/2$. We denote by $\int_0^1 X d^- Y$ (resp. $\int_0^1 X d^+ Y$, $\int_0^1 X d^0 Y$) as the limit in probability of $I^-(\varepsilon, X, dY)$ (resp. $I^+(\varepsilon, X, dY)$, $I^0(\varepsilon, X, dY)$) when $\varepsilon\to 0+$; the first limit is called forward (resp. backward, symmetric) integral of X with respect to Y.

An obvious relation between $(1)_-$ and $(1)_+$ is given by,

$$(2) \quad I^-(\varepsilon, X, dY)+I^+(\varepsilon, Y, dX)=Y(1)\left[\frac{1}{\varepsilon}\int_{1-\varepsilon}^{1} X(s)\,ds\right]-X(0)\left[\frac{1}{\varepsilon}\int_0^{\varepsilon} Y(s)\,ds\right].$$

If we take the limit when ε goes to $0+$, we get the following integration by parts formula,

$$(3) \qquad \int_0^1 X d^- Y+\int_0^1 Y d^+ X=(XY)(1)-(XY)(0).$$

By symmetry we obtain,

$$(4) \qquad \int_0^1 X d^0 Y+\int_0^1 Y d^0 X=(XY)(1)-(XY)(0).$$

In our framework it is easy to see that the regularisation of the integrator with a mollifier naturally leads to the study of the objects I^- and I^+. [T] has approached the symmetric integral with respect to the Brownian sheet by making use of convolution procedure.

The article is organised in two sections. The first one is inspired by the tools of the classical stochastic calculus and the second one by the modern technics of the Wiener functionals analysis.

In Sect. 1, Theorem 1.1 gives a substitution formula for the forward integral with respect to a martingale; this result allows us easily to construct a solution to a system of stochastic differential equations with anticipating initial condition (Theorem 1.2). In the second part, we concentrate on the case where the integrator is a one-dimensional Brownian motion. Theorem 2.1 is the main result and relates our integrals to Skorohod integral and the trace of the Malliavin derivative of the integrand, the traces being defined in a weak sense.

1 A substitution theorem. Application to stochastic differential equations

We begin by showing that the forward integral of a previsible and bounded process with respect to a semimartingale equals the Ito integral.

Notations. 1°) $(\Omega, \mathscr{F}=(\mathscr{F}_t;\ 0\leqq t\leqq 1), P)$ will denote a classical filtered probability space, $\mathscr{F}$ satisfying the usual conditions. $\mathscr{P}$ is the σ-algebra generated by the previsible processes. Let $(B(t); 0\leqq t\leqq 1)$ be a usual $\mathscr{F}$-Brownian motion.
2°) If Y is a continuous $\mathscr{F}$-semimartingale and H a previsible process, such that $\int_0^1 H(s)^2 d\langle Y, Y\rangle(s)<\infty$, a.s., then $\int_0^1 H(s)\,dY(s)$ is usual stochastic integral with respect to Y.
3°) If f is a locally integrable function on $\mathbb{R}_+$, we denote by $\mathscr{S}(f)$ the set of all $t>0$ such that $\lim_{\varepsilon\to 0} \frac{1}{\varepsilon}\int_{t-\varepsilon}^{t} f(s)\,ds \neq f(t)$.

We recall the theorem of Stein ([S], Theorem 1, p5); for every $p>1$, there exists an universal constant c_p such that,

$$\int_0^1 M(s)^p\,ds \leqq c_p \int_0^1 |f(s)|^p\,ds,$$

where,

$$M(s)=\sup_{0<\varepsilon\leqq 1} \frac{1}{|V_\varepsilon|}\int_{s+V_\varepsilon} |f(t)|\,1_{[0,1]}\,dt, \qquad 0\leqq s\leqq 1,$$

and $V_\varepsilon=[-\varepsilon,\varepsilon]$ or $V_\varepsilon=[-\varepsilon,0]$ or $V_\varepsilon=[0,\varepsilon]$.
4°) Let $(X(t);\ 0\leqq t\leqq 1)$ be a stochastic process and $u\in[0,1]$. X^u is the process defined by: $X^u=X\,1_{[0,u]}$.

Lemma 1.1 ([RY], exercise 5.17, p165) *Assume M is a continuous square integrable $\mathscr{F}$-martingale, H a mapping defined on $\Omega\times\mathbb{R}_+\times\mathbb{R}_+$, bounded and $\mathscr{P}\otimes\mathscr{B}(\mathbb{R}_+)$-measurable. Then for every $s\geqq 0$ and $t\geqq 0$ we get,*

$$\int_0^s\left(\int_0^t H(u,v)\,dM(u)\right)dv=\int_0^t\left(\int_0^s H(u,v)\,dv\right)dM(u). \quad \square \tag{1.1}$$

Proposition 1.1 *Assume Y is a continuous $\mathscr{F}$-semimartingale, $Y=M+V$ is the canonical decomposition of Y, where M* (resp. V) *is a square integrable martingale* (resp. *V is a bounded variation process*) *defined on* $[0,1]$. *Suppose X is a $\mathscr{F}$-previsible and bounded process such that*:

$$\int_0^1 1_{\{s\in\mathscr{S}(X)\}}\left(d|V|(s)+d\langle M,M\rangle_s\right)=0 \qquad \text{a.s.} \tag{1.2}$$

Then for every $t \in [0,1]$, *we have a.s.*:

$$\int_0^1 X^t d^- Y = \int_0^t X(s)\, dY(s).$$

Remarks. 1°) The assumption (1.2) is realized if X is a left-continuous process, or V and $\langle M, M\rangle$ are two absolutely continuous processes with respect to the Lebesgue measure.

2°) When $X(t,\omega) = \sum_{i=1}^n \alpha_i(t)\, G_i(\omega)$, where for every $1 \leqq i \leqq n$, α_i (resp. G_i) is a bounded function defined on $\mathbb{R}_+$ with compact support (resp. r.v.) and $\sum_{i=1}^n \int_0^1 1_{\{t \in \mathscr{S}(\alpha_i)\}}(d|V(s)| + d\langle M, M\rangle_s) = 0$ a.s., then

$$\int_0^1 X d^- Y = \sum_{i=1}^n \left(\int_0^1 \alpha_i(s)\, dY(s)\right) G_i.$$

3°) Assume X and Y are two continuous square integrable semimartingales, such that $X(0)=0$ or $Y(0)=0$. Let $\int_0^1 X(s) \circ dY(s)$ be the Stratonovich integral of X with respect to Y, then $\int_0^1 X(s) \circ dY(s) = \int_0^1 X d^0 Y$.

4°) If Z is a process, we denote by $\hat{Z}$ the process defined by: $\hat{Z}(t) = Z(1-t)$, $0 \leqq t \leqq 1$. From the identity, $I^+(\varepsilon, X, dY) = -I^-(\varepsilon, \hat{X}, d\hat{Y})$, we deduce that $\int_0^1 X d^+ B$ is the backward integral [K 1].

5°) Assume Y is a continuous square integrable $\mathscr{F}$-martingale, and $\mathscr{G} = (\mathscr{G}_t; 0 \leqq t \leqq 1)$ is a filtration containing $\mathscr{F}$ (i.e. $\mathscr{F}_t \subset \mathscr{G}_t\, \forall t \in [0,1]$). When X is only $\mathscr{G}$-adapted, $\int_0^1 X(t)\, dY(t)$ has no meaning. However if Y is a $\mathscr{G}$-continuous semimartingale, then the theory of enlargement of filtrations [Je], allows us to define $\int_0^1 X d^g Y$ as the usual $\mathscr{G}$-stochastic integral of X with respect to Y. Clearly we have:

$$\int_0^1 X^- dY = \int_0^1 X d^g Y.$$

Proof of Proposition 1.1 1°) Let $\varepsilon > 0$ and $t \in [0,1]$. We have,

$$I^-(\varepsilon, X^t, dY) = I^-(\varepsilon, X^t, dM) + I^-(\varepsilon, X^t, dV).$$

i) By Fubini theorem we get,

$$I^-(\varepsilon, X^t, dV) = \int_0^{t+\varepsilon} \left(\frac{1}{\varepsilon} \int_{s-\varepsilon}^{s} X^t(u)\, 1_{[0,1]}(u)\, du \right) 1_{[0,1]}(s)\, dV(s).$$

By (1.2), $d|V|$ almost surely, $\lim_{\varepsilon \to 0+} \frac{1}{\varepsilon} \int_{s-\varepsilon}^{s} X^t(u)\, du = X^t(s)$.

The process X^t is bounded, by the dominated convergence theorem we obtain,

$$\int_0^1 X^t d^- V = \int_0^t X(u)\, dV(u).$$

ii) Recall that,

$$I^-(\varepsilon, X^t, dM) = \frac{1}{\varepsilon} \int_0^t X(u)\, (M((u+\varepsilon) \wedge 1) - M(u))\, du.$$

We notice that $X(u)(M((u+\varepsilon) \wedge 1) - M(u))$ is the stochastic integral of the elementary process $(X(u)\, 1_{\{u < s \leqq u+\varepsilon\}};\ 0 \leqq s \leqq 1)$, with respect to the martingale $(M(t);\ 0 \leqq t \leqq 1)$.

By Lemma 1.1, we have,

$$I^-(\varepsilon, X^t, dM) = \int_0^{t+\varepsilon} \left(\frac{1}{\varepsilon} \int_{s-\varepsilon}^{s} X^t(u)\, 1_{[0,1]}(u)\, du \right) 1_{[0,1]}(s)\, dM(s).$$

But,

$$E\left[\left\{ \int_0^{t+\varepsilon} \left(\frac{1}{\varepsilon} \int_{s-\varepsilon}^{s} X^t(u)\, 1_{[0,1]}(u)\, du - X^t(s) \right) 1_{[0,1]}(s)\, dM(s) \right\}^2 \right]$$

$$\leqq E\left[\int_0^1 \left(\frac{1}{\varepsilon} \int_{s-\varepsilon}^{s} X^t(u)\, 1_{[0,1]}(u)\, du - X^t(s) \right)^2 d\langle M, M \rangle_s \right],$$

and $\lim_{\varepsilon \to 0+} \frac{1}{\varepsilon} \int_{s-\varepsilon}^{s} X^t(u)\, 1_{[0,1]}(u)\, du = X^t(s)$, $d\langle M, M \rangle_s$ a.s.

To process X^t is bounded; using again the dominated convergence theorem we show that $I^-(\varepsilon, X^t, dM)$ converges in L^2 to $\int_0^1 X^t(u)\, dM(u)$; then $\int_0^1 X^t d^- M = \int_0^t X(s)\, dM(s)$. □

We give now a partial reciprocity for Proposition 1.1. A "càdlàg" (resp. "càglàd") process is a process whose paths are a.s., right-continuous (resp. left-continuous) on [0, 1[(resp.]0, 1]), with limit from the left (resp. right) on]0, 1] (resp. [0, 1[).

Proposition 1.2 *Let Y be a $\mathscr{F}$-adapted càd làg process and suppose that $\int_0^1 X d^- Y$ exists for any bounded, càglàd and $\mathscr{F}$-previsible process X. Then Y is a $\mathscr{F}$-semimartingale.*

Remark (1.3) If Ω reduces to a single point, processes are deterministic functions defined on $[0, 1]$. In particular Proposition 1.2 tells us that if $\int_0^1 f d^- g$ exists for every bounded and càglàd function f, then g is a bounded variation function. The same conclusion holds if the $-$integral is replaced by $+$integral. We can prove that if $\int_0^1 f d^- g$ exists for every bounded and Borel function f, then g is absolutely continuous on $[0, 1]$.

Proof of Proposition 1.2 1°) We introduce two metric spaces $\mathscr{X}_1$ and $\mathscr{X}_2$ defined as follows:

(i) $\mathscr{X}_1$ is the set of bounded, $\mathscr{F}$-previsible and càdlàg processes defined on $[0, 1]$, and $d_1(X, Y) = \sup_{\substack{0 \leq t \leq 1 \\ \omega}} |X(t, \omega) - Y(t, \omega)|$, where X and Y belong to $\mathscr{X}_1$.

(ii) $\mathscr{X}_2$ is equal to the set of $\mathscr{F}_1$-measurable r.v., and $d_2(X, Y) = E(|X - Y)| \wedge 1)$ where X and Y are two elements of $\mathscr{X}_2$.

$(\mathscr{X}_1, d_1)$ and $(\mathscr{X}_2, d_2)$ are two complete metric spaces which moreover satisfy,

(a) $d_i(X, Y) = d_i(X - Y, 0)$, $i = 1$ or 2.
(b) for every X in $\mathscr{X}_i$ (resp. α in $\mathbb{R}$), $t \to tX$ (resp. $Y \to \alpha Y$) is a continuous function from $\mathbb{R}$ (resp. $(\mathscr{X}_i, d_i)$) to $(\mathscr{X}_i, d_i)$, $i = 1$ or 2.

According to ([DS], Chap. II), $(\mathscr{X}_1, d_1)$ and $(\mathscr{X}_2, d_2)$ are two F-spaces. $(\mathscr{X}_2, d_2)$ is equal to $\mathscr{X}_2$ equipped with the topology defined by the convergence in probability.

We introduce a family $\{\mu_\varepsilon; \varepsilon > 0\}$ of linear operators from $(\mathscr{X}_1, d_1)$ to $(\mathscr{X}_2, d_2)$. Let $\varepsilon > 0$. We set $\mu_\varepsilon(Z) = I^-(\varepsilon, Z, dY)$. We have,

$$|\mu_\varepsilon(Z)| \leq \sup_{0 \leq t \leq 1} |Z(t, \omega)| \frac{1}{\varepsilon} \int_0^1 |Y((t + \varepsilon) \wedge 1) - Y(t)| \, dt.$$

Then μ_ε is continuous. According to our assumptions, $\mu_\varepsilon(Z)$ converges in $(\mathscr{X}_2, d_2)$ to $\int_0^1 Z d^- Y$.

In particular for every fixed Z in $(\mathscr{X}_1, d_1)$, the family $\{\mu_\varepsilon(Z), \varepsilon > 0\}$ is bounded in $(\mathscr{X}_2, d_2)$. Theorem 18 p. 55 of [DS] proves that $Z \to \int_0^1 Z d^- Y$ is a linear and continuous map from $(\mathscr{X}_1, d_1)$ to $(\mathscr{X}_2, d_2)$.

2°) Let $\mathscr{X}'_1$ be the set of elementary previsible processes Z of the type: $Z = \sum_{i=1}^n Z_i 1_{]t_i, t_{i+1}]}$ where $0 \leq t_1 < \ldots < t_n < t_{n+1} = 1$, and Z_i is bounded and

$\mathscr{F}_{t_i}$-measurable, for every $i \in \{1, \ldots, n\}$. $\mathscr{X}'_1$ is a subspace of $\mathscr{X}_1$ and $Z \to \int_0^1 Z d^- Y$ is continuous map from $(\mathscr{X}'_1, d_1)$ to $(\mathscr{X}_2, d_2)$.

We fix an element Z of the previous type. We have,

$$I^-(\varepsilon, Z, dY) = \sum_{i=1}^n I^-(\varepsilon, Z_i 1_{]t_i, t_{i+1}]}, dY) = \sum_{i=1}^n Z_i I^-(\varepsilon, 1_{]t_i, t_{i+1}]}, dY).$$

Since

$$I^-(\varepsilon, 1_{]t_i, t_{i+1}]}, dY) = \frac{1}{\varepsilon} \int_{t_{i+1}}^{t_{i+1}+\varepsilon} Y(t \wedge 1)\, dt - \frac{1}{\varepsilon} \int_{t_i}^{t_i+\varepsilon} Y(t \wedge 1)\, dt,$$

and Y is a right-continuous process, we get,

$$\int_0^1 Z d^- Y = \sum_{i=1}^n Z_i (Y(t_{i+1}) - Y(t_i)).$$

From the Bichteler-Jacod theorem ([DM], Chap. VIII, p. 400), Y is a $\mathscr{F}$-semimartingale. □

Theorem 1.1 *Assume G is $\mathbb{R}^d$-valued* r.v., $\alpha > 1$, $q > 2\alpha/(\alpha - 1)$, $\delta > d(2\alpha + q)/2\alpha$, *and X a $\mathscr{B}(\mathbb{R}^d) \otimes \mathscr{P}$-measurable map from $\mathbb{R}^d \times \Omega \times [0, 1]$ to $\mathbb{R}$ such that for every positive N, we have:*

$$(1.4) \qquad \begin{cases} \text{(i)} \quad E\left[\int_0^1 |X(0, s)|^q\, ds\right] < \infty, \\ \text{(ii)} \quad E\left[\int_0^1 |X(a, s) - X(b, s)|^q\, ds\right] \leqq c_N |a - b|^\delta;\ \forall |a| \leqq N, \forall |b| \leqq N. \end{cases}$$

Let $(M(t); 0 \leqq t \leqq 1)$ be a continuous local martingale such that $d\langle M \rangle(t) = h(t)\, dt$ and

$$(1.5) \qquad E\left[\int_0^1 h(t)^\alpha\, dt\right] < \infty.$$

Then a.s., *for every $t \in [0, 1]$, $\int_0^1 X^t(G, \cdot)\, d^- M$ exists and*

$$(1.6) \qquad \int_0^1 X^t(G, \cdot)\, d^- M = \left(\int_0^1 X^t(a, u)\, dM(u)\right)_{a=G}.$$

Remarks (1.7). 1°) By (1.5) we have $E[\langle M, M \rangle (1)^\alpha] < \infty$, then M is a $L^{2\alpha}$-martingale.

2°) In the Brownian case, if X satisfies (1.4) with $q > 2$ and $\delta > d$ then (1.6) holds (choose $\alpha > 1$ large enough such that $\delta > d(2\alpha + q)/2\alpha$ and $q > 2\alpha/(\alpha - 1)$).

Proof of Theorem 1.1 1) Let E be the space $\mathscr{C}([0, 1])$ equipped with the uniform norm: $\|f\| = \sup_{0 \leqq u \leqq 1} |f(u)|$. $(E; \|.\|)$ is a Banach space. Let $((U(a, t);\ 0 \leqq t \leqq 1);\ a \in \mathbb{R}^d)$

and $((V(\varepsilon, a, t);\ 0 \leqq t \leqq 1);\ a \in \mathbb{R}^d \varepsilon > 0)$ be two families of E-valued random processes defined by,

$$U(a, t) = \int_0^1 X^t(a, s)\, dM(s) = \int_0^t X(a, s)\, dM(s),$$

$$V(\varepsilon, a, t) = I^-(\varepsilon, X^t(a, \cdot), dM) = \int_0^t \left(\frac{1}{\varepsilon} \int_{s-\varepsilon}^{s} X(a, u)\, 1_{\{u \geqq 0\}}\, du\right) dM(s),$$

for every $t \in [0, 1]$.

We set: $p = 2\delta\alpha/(2\alpha + q)$, $\gamma = q\alpha/(2\alpha + q)$; then $p > d$ and $1 < \gamma < \alpha$. We use successively Doob and Burkholder-Davis-Gundy inequalities,

$$E[\|U(a, .)\|^{2\gamma}] \leqq CE\left[\left(\int_0^1 X(a, s)^2\, h(s)\, ds\right)^\gamma\right] \leqq CE\left[\int_0^1 X(a, s)^{2\gamma}\, h(s)^\gamma\, ds\right].$$

Let $p' = \alpha/\gamma$; $q' = (2\alpha + q)/2\alpha$ is the conjugate exponent of p', then,

$$E[\|U(a, .)\|^{2\gamma}] \leqq C' \left\{E\left[\int_0^1 X(a, s)^q\, ds\right]\right\}^{1/q'},$$

where $C' = C\left\{E\left[\int_0^1 h(s)^\alpha\, ds\right]\right\}^{1/p'}$.

Using (1.4) and also the maximal inequality in $L^q[0, 1]$ ([S], Theorem 1, p. 5), we get,

$$E[\|U(a, .)\|^{2\gamma}] < \infty, \qquad E[\|V(\varepsilon, a, .)\|^{2\gamma}] < \infty. \tag{1.8}$$

$$E[\|V(\varepsilon, a, .) - V(\varepsilon, b, .)\|^{2\gamma}] + E[\|U(a, .) - U(b, .)\|^{2\gamma}] \leqq k_N |a - b|^p. \tag{1.9}$$

where k_N is a constant.

2) By (1.9) and Kolmogorov lemma, U admits a continuous version U_0, and U_0 also verifies:

$$E[\|U_0(a, .) - U_0(b, .)\|^{2\gamma}] \leqq k_N |a - b|^p; \quad \forall |a| \leqq N \quad \text{and} \quad |b| \leqq N. \tag{1.10}$$

3) We recall Garsia, Rodemich and Rumsey lemma (G.R.R.) stated by Barlow and Yor ([BY]).

Fix $p > d$, $\gamma > 0$, $0 < m < p - d$, $N > 0$, $K_N = \{a \in \mathbb{R}^d;\ |a| \leqq N\}$ and $\{V(a), a \in K_N\}$ a family of E-valued random variables such that,

$$E[\|V(a) - V(b)\|^{2\gamma}] \leqq k|a - b|^p, \qquad \forall a \in K_N, \qquad b \in K_N.$$

Then, there exist two constants C_1 and C_2, independent of the process V and a r.v. Γ such that,

(i) $\|V(a) - V(b)\|^{2\gamma} \leqq C_1 |a - b|^m \Gamma\ \forall a \in K_N,\ b \in K_N$

(ii) $E(\Gamma) \leqq C_2 k$.

4) There exist two constants C_1 and C_2 and a r.v. $\Gamma(\varepsilon)$ such that,

$$\|V(\varepsilon, a,.)-V(\varepsilon, b,.)\|^{2\gamma} \leqq C_1 |a-b|^m \Gamma(\varepsilon), \tag{1.11}$$

$$E(\Gamma(\varepsilon)) \leqq C_3, \tag{1.12}$$

with $C_3 = C_2 k_N$.
5) Let $\tau > 0$ and $G^N = G\, 1_{\{|G| \leqq N\}}$.

$$P(\|V(\varepsilon, G,.)-U_0(G,.)\| > \tau) \leqq P(|G| > N) + P(\|V(\varepsilon, G^N,.)-U_0(G^N,.)\| > \tau).$$

Even if it means replacing G by G^N, we can assume that $|G|$ being bounded by N. Fix $n > N$ and G_n a discrete r.v. (with finite values) such that $G_n \in K_N$ and $\|G - G_n\|_\infty \leqq 1/n$. We deduce from (1.11) and (1.12) the inequalities,

$$E[\|V(\varepsilon, G,.)-V(\varepsilon, G_n,.)\|^{2\gamma}] \leqq C_1\, E[|G-G_n|^m \Gamma(\varepsilon)] \leqq C_1\, C_3 (1/n)^m. \tag{1.13}$$

A similar proof using (1.10) shows that,

$$E[\|U_0(G,.)-U_0(G_n,.)\|^{2\gamma}] \leqq C_4 (1/n)^m. \tag{1.14}$$

For every fixed $a > 0$, we deduce from (1.8) and the maximal inequality that the limit in the $L^{2\gamma}(E)$-sense of $V(\varepsilon, a,.)$ is equal to $U_0(a,.) = U(a,.)$. Moreover it is clear that if G is a discrete r.v. which takes its values in K_N, $V(\varepsilon, G,.)$ converges in $L^{2\gamma}(E)$ to $U_0(G,.)$.

It is sufficient to use now the two uniform inequalities (1.13) and (1.14). □

Let us discuss now the application of Theorem 1.1 to the systems of stochastic differential equations (S.D.E.'s) with an initial non-adapted value. We work in the general outline defined by Jacod ([Ja], Chap. XIV). We introduce $\tilde{\Omega} = \mathscr{C}([0,1], \mathbb{R})$, $\tilde{\mathscr{F}} = (\tilde{\mathscr{F}}_t; 0 \leqq t \leqq 1)$ the natural filtration on $\tilde{\Omega}$, $\Omega' = \Omega \times \tilde{\Omega}$, $\mathscr{F}' = (\bigcap_{s \leqq t} (\mathscr{F}_s \otimes \tilde{\mathscr{F}}_s);\ 0 \leqq t \leqq 1)$ and $\mathscr{P}'$ the σ algebra generated by the previsible processes defined on Ω'. Let $(B_1, \ldots, B_n)$ be a n-dimensional Brownian motion defined on $(\Omega, \mathscr{F} = (\mathscr{F}_t;\ 0 \leqq t \leqq 1), P)$. We assume that $\sigma = (\sigma_1, \ldots, \sigma_n)$ and b to satisfy,

(1.15) b and σ_i for every $1 \leqq i \leqq n$, maps $[0,1] \times \Omega'$ to $\mathbb{R}^d$ and $\mathscr{P}'$-measurable.

There exists two constants $K_1 > 0$ and $K_2 > 0$ such that,

$$\sum_{i=1}^{n} |\sigma_i(s, \omega, \tilde{\omega}) - \sigma_i(s, \omega, \tilde{\omega}_1)| + |b(s, \omega, \tilde{\omega}) - b(s, \omega, \tilde{\omega}_1)| \leqq K_1 \|\tilde{\omega} - \tilde{\omega}_1\|_s, \tag{1.16}$$

$$\sum_{i=1}^{n} |\sigma_i(s, \omega, \tilde{\omega})|^2 + |b(s, \omega, \tilde{\omega})|^2 \leqq K_2 (1 + \|\tilde{\omega}\|_s^2), \tag{1.17}$$

for every $0 \leqq s \leqq 1$, $\omega \in \Omega$, $\tilde{\omega}$ and $\tilde{\omega}_1$ in $\mathscr{C}([0, s])$, where $\|\tilde{\omega}\|_s = \sup_{0 \leqq u \leqq s} |\tilde{\omega}(u)|$.

When assumptions (1.15), (1.16) and (1.17) are realized, we know ([Ja], (14.50)) that the systems of S.D.E.'s:

$$X(t) = x + \sum_{i=1}^{n} \int_0^t \sigma_i(s, \omega, X_.)\, dB_i(s) + \int_0^t b(s, \omega, X_.)\, ds, \tag{1.18}$$

has a unique solution, where $x \in \mathbb{R}^d$, and $X_.$ is the element of $\tilde{\Omega}$ defined by, $X_.(s) = X(s, \omega)$. We denote by $X(x, \cdot)$ the unique solution of (1.18).

Theorem 1.2 *Assume* ξ *is a* r.v. $\mathscr{F}_1$*-measurable*, $\tau \in [0, 1]$, *and* Y *is the process*: $Y(t) = X(\xi, t)$; $0 \leqq t \leqq 1$. *Then* a.s.,

$$Y(\tau) = \xi + \sum_{i=1}^{n} \int_0^1 \sigma_i(s, \omega, Y(.))\, 1_{[0,\tau]}\, d^- B_i(s) + \int_0^\tau b(s, \omega, Y(.))\, ds. \tag{1.19}$$

Proof. We set $U_i(t, x, \omega) = \sigma_i(t, \omega, X(x, \cdot, \omega))\, 1_{[0,\tau]}(t)$, $0 \leqq i \leqq n$. It is clear from (1.16) that:

$$\sum_{i=1}^{n} |U_i(t, x, \omega) - U_i(t, y, \omega)| \leqq K_1 \|X(x,., \omega) - X(y,., \omega)\|_t, \tag{1.20}$$

for every $t \in [0, 1]$.
According to an easy adaptation of [K2], Theorem 2.1, p. 211 there exist two constants C_1 and C_2 such that,

$$E[\|X(x,.) - X(y,.)\|^q] \leqq C_1 |x - y|^q, \tag{1.21}$$

$$E[\|X(0,.)\|^q] \leqq C_2, \tag{1.22}$$

where $q > 0$. We chose $q > \max(2, d)$. By (1.16), (1.17) a straightforward calculation shows that U_i satisfies (1.4) for every $i \in \{1, \ldots, n\}$; we can apply Theorem 1.1, and (1.19) follows immediately. □

2 Wiener analysis approach (Malliavin calculus)

In this section, we would like to examine the convergence of the ε-integrals defined in the Introduction through a Wiener interpretation; [W] and [BH] will be the basic references on this subject.

All along this section, r will be a fixed real number and $1 < q \leqq 2$.

Let $(\Omega = \mathscr{C}[0, 1]$, $H = L^2[0, 1)]$, $P)$ be the canonical Wiener space with the usual Brownian motion $(B(t);\ 0 \leqq t \leqq 1)$. E will be a separable Hilbert space, with inner product $(.,.)$. $\mathscr{D}_\infty(E)$ will be the algebra of E-valued Wiener functionals and the space of E-valued Wiener distributions will be the dual space $\mathscr{D}_{-\infty}(E)$. We recall that

$$\mathscr{D}_{-\infty}(E) = \bigcup_{\substack{p>1 \\ s \in \mathbb{R}}} \mathscr{D}_{p,s}(E), \qquad \mathscr{D}_\infty(E) = \bigcap_{\substack{p>1 \\ s \in \mathbb{R}}} \mathscr{D}_{p,s}(E),$$

where $(\mathscr{D}_{p,s}(E); p>1, s\in\mathbb{R})$ is the family of Sobolev-Watanabe-Krée spaces. $\langle .,. \rangle$ will stand for the duality between $\mathscr{D}_{-\infty}(E)$ and $\mathscr{D}_{\infty}(E)$. If $E=\mathbb{R}$, we will simply drop E. $L^q(E)$ stands for $L^q(\Omega; E)$; we recall that $\mathscr{D}_{q,0}(E)=L^q(E)$. The gradient operator D maps continuously $\mathscr{D}_{q,s}$ into $\mathscr{D}_{q,s-1}(H)$. The divergence operator δ (Skorohod integral) maps continuously $\mathscr{D}_{q,s}(H)$ into $\mathscr{D}_{q,s-1}$. D and δ are dual operators.

Let denote $H_n=H\otimes\ldots\otimes H$ (n times). If $T\in\mathscr{D}_{q,r}(H_n)$, $\alpha\in H_n$, we denote by T_α the Wiener distribution in $\mathscr{D}_{q,r}$ defined by, $\langle T_\alpha, Y\rangle=\langle T, \alpha\otimes Y\rangle$. If $T\in\mathscr{D}_{q,r}(H_n)$ there exists a unique $\tilde{T}\in L^q([0,1]; \mathscr{D}_{q,r})$ such that,

$$T_\alpha=\int_{[0,1]^n}\alpha(t)\,\tilde{T}(t)\,dt, \tag{2.1}$$

where the integral is understood in Pettis sense; this follows from a slight extension of [BH], Proposition III 1.1.8. $\tilde{T}$ will be the $\mathscr{D}_{q,r}$-decomposition of T. Clearly, we can replace $\mathscr{D}_{q,r}$ by $\mathscr{D}_{-\infty}$. If $T\in\mathscr{D}_{-\infty}$, then $DT\in\mathscr{D}_{-\infty}(H)$, $(D_s T; 0\leq s\leq 1)$ will denote its $\mathscr{D}_{-\infty}$-decomposition, and $D_\alpha T$ stands for $(DT)_\alpha$. Since $\mathscr{D}_\infty\otimes H_n$ is dense in $\mathscr{D}_\infty(H_n)$, then (2.1) can be extended to,

$$\langle T, Z\rangle=\int_{[0,1]^n}\langle Z(t), \tilde{T}(t)\rangle\,dt, \tag{2.2}$$

for every Z in $\mathscr{D}_\infty(H_n)$.

The Wick product of two Wiener distributions is defined in [BH], exercise 1.6 p. 133, with the help of the notion of characteristic functionals and will be symbolized by: However if $U=I_n(f_n)$, $V=I_m(g_m)$ are Wiener iterated Ito integrals where $f_n\in L^2([0,1]^n)$ and $g_m\in L^2([0,1]^m)$, then $U\colon V=I_{n+m}(f_n\otimes g_m)$. If $h\in H$, $Y\in L^q$ then the definition of the Wick product implies that $\delta h\colon Y\in\mathscr{D}_{q,-1}$ and

$$\delta h\colon\; Y=\delta(h\otimes Y). \tag{2.3}$$

From now on Δ will be an element of $\{-,+,0\}$. For simplicity we will write $I^\Delta(\varepsilon, X)$ instead of $I^\Delta(\varepsilon, X, dB)$. Let $X=(X(t);\ 0\leq t\leq 1)$ be a process in $L^q(H)$. By standard arguments it is easy to check that $I^\Delta(\varepsilon, X)\in L^{p'}(H)$, for some $p'>1$.

The following three functions which are elements of H_2 will play an important role in the sequel,

$$\alpha_{\varepsilon,-}=\frac{1}{\varepsilon}1_{\{(s,t)\in[0,1]^2\mid t\leq s\leq t+\varepsilon\}},\qquad \alpha_{\varepsilon,+}=\frac{1}{\varepsilon}1_{\{(s,t)\in[0,1]^2\mid t-\varepsilon\leq s\leq t\}},$$

$$\alpha_{\varepsilon,0}=\frac{1}{2\varepsilon}1_{\{(s,t)\in[0,1]^2\mid t-\varepsilon\leq s\leq t+\varepsilon\}}.$$

$X_{\varepsilon,\Delta}$ will be the process defined by

$$X_{\varepsilon,\Delta}(s)=\int_0^1 X(t)\,\alpha_{\varepsilon,\Delta}(s,t)\,dt,\qquad s\in[0,1].$$

In the definition of $I_w^\Delta(\varepsilon, X)$ we make use of the Wick product instead of the ordinary product of random variables, more precisely,

$$I_w^-(\varepsilon, X) = \int_0^1 X(t) \colon \left[\frac{B((t+\varepsilon)\wedge 1) - B(t)}{\varepsilon}\right] dt. \tag{2.4}$$

Similarly to the definition of $I^+(\varepsilon, X)$ and $I^0(\varepsilon, X)$, we can define $I_w^+(\varepsilon, X)$ and $I_w^0(\varepsilon, X)$.

By (2.3), we get

$$\left[\frac{B((t+\varepsilon)\wedge 1) - B(t)}{\varepsilon}\right] \colon X(t) = \delta(\alpha_{\varepsilon,-}(.,t)) \colon X(t) = \delta[\alpha_{\varepsilon,-}(.,t)\, X(t)],$$

$$I_w^\Delta(\varepsilon, X) = \delta(X_{\varepsilon,\Delta}). \tag{2.5}$$

Lemma 2.1 *Let $X \in L^q(H)$. Then $X_{\varepsilon,\Delta} \in L^q(H)$ and*

$$I^\Delta(\varepsilon, X) = I_w^\Delta(\varepsilon, X) + D_{\alpha_{\varepsilon,\Delta}} X. \tag{2.6}$$

Proof. For simplicity α will stand for $\alpha_{\varepsilon,\Delta}$; since α is bounded by $1/\varepsilon$, the first statement is obvious. Therefore $I_w^\Delta(\varepsilon, X) \in \mathscr{D}_{q,-1}$. We set,

$$a(\varepsilon) = \langle I_w^\Delta(\varepsilon, X), Y\rangle, \qquad Y \in \mathscr{D}_\infty.$$

Using duality we get,

$$a(\varepsilon) = \langle X_{\varepsilon,\Delta}, DY\rangle = E\left[\int_0^1 D_r Y\left(\int_0^1 \alpha(r,t)\, X(t)\, dt\right) dr\right]$$

$$= E\left[\int_0^1 X(t)\left(\int_0^1 \alpha(r,t)\, D_r Y dr\right) dt\right] = E\left[\int_0^1 X(t)\, D_{\alpha(\cdot,t)} Y dt\right].$$

According to [W], Proposition 1.11, p. 51, we have,

$$D_{\alpha(\cdot,t)} Y = Y\delta(\alpha(\cdot,t)) - \delta(\alpha(\cdot,t)\, Y).$$

Therefore, we can write

$$a(\varepsilon) = E(I^\Delta(\varepsilon, X)\, Y) - a_1(\varepsilon), \tag{2.7}$$

where,

$$a_1(\varepsilon) = E\left[\int_0^1 X(t)\, \delta(\alpha(\cdot,t)\, Y)\, dt\right].$$

Using once more duality we obtain,

$$a_1(\varepsilon) = E\left[\int_0^1 \int_0^1 \alpha(r,t)\, Y D_r X(t)\, dr\, dt\right] = E(Y D_\alpha X).$$

This and (2.7) show (2.6). □

At this stage, we need a suitable notion of trace for $T\in\mathcal{D}_{-\infty}(H_2)$. Let $T\in\mathcal{D}_{q,r}(H_2)$; we will say that T has a $\mathcal{D}_{q,r}$ Δ-trace if $\lim_{\varepsilon\to 0+} T_{\alpha_{\varepsilon,\Delta}}$ weakly exists in $\mathcal{D}_{q,r}$. Obviously $\mathcal{D}_{q,r}$ can be replaced by $\mathcal{D}_{-\infty}$; in this case, we will speak about $\mathcal{D}_{-\infty}$ trace.

Let F be a topological vector space, and $f\colon [0,1]^2\to F$, Pettis-integrable. We will say that f has a $F\Delta$-trace or simply a Δ-trace if $\lim_{\varepsilon\to 0+} \int_{[0,1]^2} \alpha_{\varepsilon,\Delta}(u)\,f(u)\,du$ weakly exists in F. This limit is denoted by $\mathrm{Tr}^{\Delta}(f)$. This definition has been performed by [RV]. It is clear that $T\in\mathcal{D}_{q,r}(H_2)$ (resp. $T\in\mathcal{D}_{-\infty}(H_2)$) has a $\mathcal{D}_{q,r}$ (resp. $\mathcal{D}_{-\infty}$) Δ-trace iff $\tilde{T}$ has a $\mathcal{D}_{q,r}$ (resp. $\mathcal{D}_{-\infty}$) Δ-trace.

Theorem 2.1 *Let us suppose $X\in L^q(H)$. Then*

a) $\lim_{\varepsilon\to 0+} I_w^{\Delta}(\varepsilon, X)=\delta X$ *in* $\mathcal{D}_{q,-1}$.

b) $\lim_{\varepsilon\to 0+} I^{\Delta}(\varepsilon, X)$ *weakly exists in* $\mathcal{D}_{q,-1}$ (resp. $\mathcal{D}_{-\infty}$) *iff DX has a* $\mathcal{D}_{q,-1}$ (resp. $\mathcal{D}_{-\infty}$) *Δ-trace.*

In this case $\lim_{\varepsilon\to 0+} I^{\Delta}(\varepsilon, X)$ *is equal to the sum of δX and the Δ-trace of DX.*

Remark 2.1 The theorem is still true if we replace $L^q(H)$ by $\mathcal{D}_{q,1}(H)$ and $\mathcal{D}_{q,-1}$ by L^q.

Proof. Using part a) and (2.6) we establish b).

In order to prove a) we observe that it is enough to check that,

$$\lim_{\varepsilon\to 0} X_{\varepsilon,\Delta}=X \quad \text{in } L^q(H). \tag{2.8}$$

After this we can use the continuity of δ from $L^q(H)$ into $\mathcal{D}_{q,-1}$. As for the proof of (2.8) we need the following lemma.

Lemma 2.2 *Let Y belonging to $L^q(H)$, $V_\varepsilon=[-\varepsilon,\varepsilon]$* (resp. *$V_\varepsilon=[-\varepsilon,0]$ or $V_\varepsilon=[0,\varepsilon]$*). *Then* $\frac{1}{|V_\varepsilon|}\int_{s+V_\varepsilon} Y(t)\,1_{[0,1]}(t)\,dt$ *converges in $L^q(H)$* to Y, *when ε goes to* 0.

Proof. We set,

$$M(s)=\sup_{0<\varepsilon\leqq 1}\frac{1}{|V_\varepsilon|}\int_{s+V_\varepsilon} |Y(t)|\,1_{[0,1]}(t)\,dt, \qquad 0\leqq s\leqq 1.$$

Using Stein inequality we obtain,

$$E\left[\left(\int_0^1 M(s)^2\,ds\right)^{q/2}\right]\leqq C^{q/2}\,E\left[\left(\int_0^1 Y(s)^2\,ds\right)^{q/2}\right]$$

By the dominated convergence theorem we can conclude. □

Remark 2.2 We would like now to answer the following question: are the two approaches of Sects. 1 and 2 compatible? In other words, let X be a process in $L^q(H)$; let us suppose that $I^\Delta(\varepsilon, X)$ converges in probability to an integrable v.a. Z and also weakly converges in $\mathscr{D}_{-\infty}$ to an integrable r.v. Z'. Do we have $Z=Z'$ a.s.? The answer is not known; however, if we replace the convergence in probability with the L^1 convergence, it is positive. [Hint: Let $\mathscr{H}$ be the set of bounded r.v. φ such that $\varphi=f(B(t_1), \ldots, B(t_n))$, where $0 \leqq t_1 < \ldots < t_n \leqq 1$ and f is a C^∞ bounded function such that any derivative is also bounded. If $\varphi \in \mathscr{H}$, we have $E[\varphi Z]=E[\varphi Z']$. This equality can be extended to $\varphi \in L^\infty \cap \mathscr{D}_\infty$ by density of $\mathscr{H}$ into $L^\infty \cap \mathscr{D}_\infty$].

Theorem 2.1 generalizes in some sense the results of [Z] and [SU] concerning the relation between Ogawa symmetric integral and traces of derivatives. [Z] introduces the traces in a functional analysis framework and [SU] through a limiting technique. The different traces are not really comparable, see [HM].

Let us relate our concept of trace with the classical traces coming out from functional analysis. Let g be a E-valued function defined on $[0,1]$. We say that g has a E classical trace if g is a square integrable Bochner function and there exists an element denoted by $\mathscr{T}_{\text{cl}}(g) \in E$, such that for any complete orhonormal sequence $(\varphi_n, n \geqq 0)$ in H,

$$\sum_{n=0}^{\infty} \int_{[0,1]^2} \varphi_n(u)\, \varphi_n(v)\, g(u,v)\, du\, dv, \tag{2.9}$$

converges, in E, to $\mathscr{T}_{\text{cl}}(g)$.

Remarks 2.3 1°) Let denote $\tilde{g}(u,v)=(g(u,v)+g(v,u))/2$. In the definition, g does not need to be symmetric; however obviously, g has a E classical trace if and only if $\tilde{g}$ has one.
2°) This concept of trace appears in [R]. Let $e \in F$ and $K_e\, L^2[0,1] \to L^2[0,1]$, the linear operator defined by

$$(K_e\, \varphi)(u) = \int_0^1 \varphi(v)\, (\tilde{g}(u,v), e)\, dv.$$

Rosinski ([R] Proposition 2.1 and Corollary 2.2) has proved, that the sum (2.9) converges for every $(\varphi_n, n \geqq 0)$ iff K_e is a nuclear operator. In this case (2.9) does not depend on $(\varphi_n, n \geqq 0)$.

We begin with the scalar case, i.e. $E=\mathbb{R}$.

Proposition 2.1 *Let g be a symmetric function of H_2. Assume g has a $\mathbb{R}$ classical trace then for any $\Delta \in \{-, +, 0\}$ g has a Δ-trace and* $\operatorname{Tr}^\Delta(g)=\mathscr{T}_{\text{cl}}(g)$.

Remark 2.4 It is clear that if g is continuous on $\{(s,t); 0 \leqq t \leqq s \leqq 1\}$ (resp. $\{(s,t); 0 \leqq s \leqq t \leqq 1\}$) then g has a $-$trace (resp. $+$trace). Recall that Balakrishnan ([B], p. 126) gives an example of a real valued function g such that g is continuous and has no classical trace, this shows that Proposition 2.1 admits no converse.

Proof. Let consider the operator $G: H \to H$, defined by

$$G\psi(t) = \int_0^1 g(t,u)\, \psi(u)\, du, \qquad t \in [0,1].$$

Since G is a symmetric operator. So by ([B], Theorem 3.4.3, p. 115), G is nuclear iff g has a classical trace. Moreover G is diagonalisable; let consider a complete orthonormal basis $(\varphi_n, n \geqq 0)$ of H, such that for every n, φ_n is an eigenvector of G associated with the eigenvalue λ_n. $(\varphi_n \otimes \varphi_m; n \geqq 0, m \geqq 0))$ is an orthonormal basis of H_2. Then

$$g = \sum_{n,m} a_{n,m} \varphi_n \otimes \varphi_m,$$

$$\text{where } a_{n,m} = \int_{[0,1]^2} \varphi_n(u)\, \varphi_m(v)\, g(u,v)\, du\, dv.$$

We have

$$a_{n,m} = \int_0^1 \varphi_n(u)\, G(\varphi_m)(u)\, du = \lambda_n\, \delta_{n,m},$$

where $\delta_{n,m}$ is the Kronecker symbol. Consequently

$$g = \sum_n \lambda_n\, \varphi_n \otimes \varphi_n.$$

Since G is nuclear then

$$\sum_n |\lambda_n| < \infty \quad \text{and} \quad \mathcal{T}_{\text{cl}}(g) = \sum_n \lambda_n.$$

$$\int_{u-\varepsilon}^{u} g(u,v)\, 1_{[0,1]}(v)\, dv = \sum_n \lambda_n\, \varphi_n(u) \int_{u-\varepsilon}^{u} \varphi_n(v)\, 1_{[0,1]}(v)\, dv, \qquad u \in [0,1].$$

We introduce

$$S_n(u) = \sup_{0<\varepsilon<1} \left(\frac{1}{\varepsilon} \left| \int_{u-\varepsilon}^{u} \varphi_n(v)\, 1_{[0,1]}(v)\, dv \right| \right), \qquad u \in [0,1].$$

It is clear that

$$\left| \lambda_n\, \varphi_n(u) \frac{1}{\varepsilon} \int_{u-\varepsilon}^{u} \varphi_n(v)\, 1_{[0,1]}(v)\, dv \right| \leqq |\lambda_n|\, |\varphi_n(u)|\, S_n(u), \qquad u \in [0,1].$$

By Stein theorem we get,

$$\lim_{\varepsilon \to 0+} \left(\frac{1}{\varepsilon} \int_{u-\varepsilon}^{u} \varphi_n(v)\, 1_{[0,1]}(v)\, dv \right) = \varphi_n(u), \qquad \text{for a.e. } u \in [0,1],$$

$$\int_0^1 (S_n(v))^2\, dv \leqq C \int_0^1 \varphi_n(v)^2\, dv.$$

Using Cauchy-Schwarz inequality, we have

$$\int_0^1 \Big(\sum_n |\lambda_n|\, |\varphi_n(u)|\, S_n(u)\Big)\, du \leqq \sqrt{C} \sum_n |\lambda_n| < \infty.$$

We can apply the dominated convergence theorem,

$$\mathrm{Tr}^{-}(g)=\lim_{\varepsilon\to 0+}\int_{[0,1]^2} g(u,v)\,\alpha_{\varepsilon,-}(u,v)\,du\,dv=\sum_n \lambda_n=\mathcal{T}_{\mathrm{cl}}(g).$$

Since g is symmetric, it is obvious that all the Δ-traces are equal. □

Suppose now that f is a F-valued function defined on $[0,1]^2$. We introduce f^{Δ} a F-valued function defined on $[0,1]^2$ by

$$f^{\Delta}(s,t)=\begin{cases} f(s\wedge t, s\vee t) & \text{if } \Delta=+,\\ f(s\vee t, s\wedge t) & \text{if } \Delta=-,\\ \tilde{f}(s,t) & \text{if } \Delta=0.\end{cases}$$

The function f^{Δ} is symmetric and $f^0=\tilde{f}=(f^{+}+f^{-})/2$.

Proposition 2.2 *Let g be an element of $L^2([0,1]^2,E)$. If g^{Δ}* (resp. g) *has a E classical trace then g has a Δ-trace and $\mathcal{T}_{\mathrm{cl}}(g^{\Delta})=\mathrm{Tr}^{\Delta}(g)$* (resp. $\mathcal{T}_{\mathrm{cl}}(g)=\mathcal{T}_{\mathrm{cl}}(g^0)$ $=\mathrm{Tr}^{\Delta}(g^0)$ *for every* $\Delta\in\{+,-,0\}$).

Proof. Assume g is a symmetric function and g admits a E classical trace. Let $y\in E$. We denote by g_0 the element of H_2 defined by

$$g_0(s,t)=(g(s,t),y),\qquad (s,t)\in[0,1]^2.$$

g_0 is a symmetric function, admits a $\mathbb{R}$ classical trace and

$$\mathcal{T}_{\mathrm{cl}}(g_0)=(\mathcal{T}_{\mathrm{cl}}(g),y).$$

Therefore by Proposition 2.1, g_0 has a Δ-trace. We set

$$\mathrm{Tr}^{\varepsilon,\Delta}(f)=\int_{[0,1]^2} f(u,v)\,\alpha_{\varepsilon,\Delta}(u,v)\,du\,dv.$$

Then,

$$(\mathrm{Tr}^{\varepsilon,\Delta}(g),y)=\mathrm{Tr}^{\varepsilon,\Delta}(g_0).$$

Consequently, $(\mathrm{Tr}^{\varepsilon,\Delta}(g),y)$ converges to $(\mathcal{T}_{\mathrm{cl}}(g),y)$; this means that g has a Δ-trace and $\mathcal{T}_{\mathrm{cl}}(g)=\mathrm{Tr}^{\Delta}(g)$.

We study now the general case. We suppose that g^{Δ} has a E classical trace. By the previous step, g^{Δ} has a Δ-trace and $\mathcal{T}_{\mathrm{cl}}(g^{\Delta})=\mathrm{Tr}^{\Delta}(g^{\Delta})$. An easy calculation shows that $\mathrm{Tr}^{\varepsilon,\Delta}(g^{\Delta})=\mathrm{Tr}^{\varepsilon,\Delta}(g)$. In particular, g has a Δ-trace and

$$\mathrm{Tr}^{\Delta}(g^{\Delta})=\mathrm{Tr}^{\Delta}(g)=\mathcal{T}_{\mathrm{cl}}(g).\quad \square$$

Let us write now some consequences for stochastic processes. Let X be an element of $L^2(H)$ (resp. $\mathcal{D}_{2,1}(H)$), $\tilde{D}X=(D_s X(t);\,(s,t)\in[0,1]^2)$ the $\mathcal{D}_{2,-1}$ (resp. L^2)-decomposition of DX and

$$D^{\Delta}X=(\tilde{D}X)^{\Delta}.$$

Theorem 2.2 *If $D^{\Delta}X$ has a L^2* (resp. $\mathcal{D}_{2,-1}$) *classical trace then $\tilde{D}X$* admits *a L^2* (resp. $\mathcal{D}_{2,-1}$) *Δ-trace and*

(i) $\mathscr{T}_{cl}(D^\Delta X) = \mathrm{Tr}^\Delta(DX)$,
(ii) *$I^\Delta(\varepsilon, X)$ weakly converges to $\delta X + \mathscr{T}_{cl}(D^\Delta X)$.*

Remarks 2.5 1°) Let X be an element of $\mathscr{D}_{2,1}(H)$ and assume that a.s. DX admits a Δ-trace, then $\int_0^1 X\,d^\Delta B$ exists and $\int_0^1 X\,d^\Delta B = \delta X + \mathrm{Tr}^\Delta DX$, a.s. This is an easy consequence of (2.6) and Remark 2.1.
2°) Let Φ be the family of all complete orthonormal basis of H. Then the weak limit of $I^\Delta(\varepsilon, X)$ is equal to the symmetric type integral introduced by Zakai ([Z]). Some authors call it also, Ogawa integral ([Nu], [NZ]).
3°) Previous result extends some considerations of [NP] Sect. 7, (see also [N], Sect. 6). The authors define a class Γ_c^2 of stochastic processes $X \in \mathscr{D}_{2,1}(H)$ such that $(\tilde{D}X)$ is "continuous". If X belongs to Γ_c^2, then the forward (resp. backward, symmetric) integral of X is equal to the sum of δX and the $-$trace (resp. $+$trace, 0trace) of DX.

Moreover the weak limit of $I^-(\varepsilon, X)$ (resp. $I^+(\varepsilon, X)$; $I^0(\varepsilon, X)$) is equal to the forward (resp. backward; symmetric) integral defined by Nualart.

The final result relates the Skorohod integral and the integral defined through the enlargement of filtration. Assume $\mathscr{G} = (\mathscr{G}_t; 0 \leqq t \leqq 1)$ is a filtration on $(\Omega, \mathscr{F} = (\mathscr{F}_t; 0 \leqq t \leqq 1), P)$, satisfying the usual conditions, and "bigger" than $\mathscr{F}$ (i.e. $\mathscr{F}_t \subset \mathscr{G}_t$ for every $t \in [0,1]$). We suppose that

(i) B is a $\mathscr{G}$-semimartingale, $B = \tilde{B} + V$ is the $\mathscr{G}$-canonical decomposition of B, where $V_t = \int_0^t H(s)\,ds$, $0 \leqq t \leqq 1$,
(ii) there exists $p > 1$ such that $(H(s); 0 \leqq s \leqq 1)$ belongs to $L^p([0,1] \times \Omega, \mathbb{R})$.
We know that $\tilde{B}$ is a $\mathscr{G}$ Brownian motion (see for instance [Je]).

Proposition 2.3 *Assume X is a $\mathscr{G}$-previsible process, belonging to $L^q([0,1] \times \Omega, \mathbb{R}) \cap L^q(H)$ where q is the conjugate exponent of p, $\mathrm{Tr}^-(DX)$ exists in $\mathscr{D}_{-\infty}$ and belongs to L^1. Then*

$$\int_0^1 X\,d^- B = \int_0^1 X\,d^g B = \delta X + \mathrm{Tr}^-(DX) \quad \text{a.s.} \tag{2.10}$$

Proof. Through a slight modification in the proof of Proposition 1.1, it is easy to check that $\int_0^1 X\,d^- B$ is the limit in L^1 of $I^-(\varepsilon, X)$. We have already noticed that $\int_0^1 X\,d^- B = \int_0^1 X\,d^g B$. Theorem 2.1 and Remark 2.2 implie the second equality in (2.10). □

Example. Let $\mathscr{G} = (\mathscr{G}_t; 0 \leqq t \leqq 1)$ be the smallest filtration such that, $\mathscr{F}_t \subset \mathscr{G}_t$ for every $0 \leqq t \leqq 1$, $B(1)$ is $\mathscr{G}_0$-measurable. By ([Je], Theorem 3.23 p. 46) we have

$$H(s) = \frac{B(1) - B(s)}{1 - s}, \quad 0 \leqq s < 1.$$

Since $E(|H(s)|^p)=(1-s)^{-p/2}E(|B(1)|^p)$, we can apply Proposition 2.3 with $1<p<2$.

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