Probab. Theory Relat. Fields 92, 259–274 (1992)

Probability Theory and Related Fields

The ergodicity of a class of reversible reaction-diffusion processes

T.S. Mountford

Department of Mathematics, University of California, Los Angeles, CA 90024, USA

Received November 13, 1989; in revised form November 1, 1991

Summary. We build on recent results of Durrett, Ding and Liggett to establish ergodicity in a class of reversible reaction-diffusion processes.

This paper concerns a class of reaction-diffusion processes. The reaction-diffusion process has state space $X = \{\eta: Z^d \to Z^+\}$. We think of $\eta(x)$ as representing the number of particles at site x. The particles move as follows:

Particles are born at site x at rate $\beta(\eta(x))$. (Or alternatively, at rate $\beta(\eta(x))$, $\eta \to \eta + e_x$, where $e_x(y)$ equals 1 if $x = y$ and is 0 otherwise.)

Particles at site x die at rate $\delta(\eta(x))$. (Or alternatively, at rate $\delta(\eta(x))$, $\eta \to \eta - e_x$, where $e_x(y) = \delta_x(y)$). Necessarily $\delta(0) = 0$.

At rate $\eta(x)p(x, y)$, a particle jumps from x to y, where $p(x, y)$ is the transition matrix of an irreducible symmetric random walk on Z^d with $p(x, x) = 0$. (Alternatively, $\eta \to \eta - e_x + e_y$.)

Under suitable assumptions on β and δ (see Chen (1985)), there exists a continuous time Markov process on a subset of X as described above, which (formally) has generator

$$\Omega f(\eta) = \sum_x \beta(\eta(x))[f(\eta + e_x) - f(\eta)] + \sum_x \delta(\eta(x))[f(\eta - e_x) - f(\eta)]$$

$$+ \sum_x \sum_y \eta(x)p(x, y)[f(\eta - e_x + e_y) - f(\eta)] .$$

In a recent paper, Ding et al. (1990), hereafter referred to as DDL, dealt with reaction-diffusion processes where for some k,

$$\beta(x) = \sum_{j=0}^{k} b_j x^{(j)}$$

and

$$\delta(x) = \sum_{j=1}^{k+1} c_j x^{(j)}$$

Research supported by N.S.F. grant DMS-86-01800

and $b_j = \lambda c_{j+1} > 0$ for some $\lambda > 0$. In the above, $x^{(j)} = x(x-1)(x-2)\ldots(x-j+1)$, $x^{(0)} = 1$.

One of the reasons for the interest in this class of processes is that simple invariant measures are known. Janssen (1974) shows that in the above case v, the measure on Z^d where $\{\eta(x): x \in Z^d\}$ are independent Poisson random variables with parameter λ, is stationary and reversible for the process. In fact, DDL showed that if $b_0 > 0$ then v is the only stationary distribution and that it is the limit starting from any initial state. DDL's result is trivially not true if $b_0 = 0$, since then the birth rate at $\eta = 0$ is zero, so the point mass at $\eta(x) = 0$ is also stationary. In this case DDL conclude:

Theorem one. *In the case where* $b_0 = 0$, *the only translation invariant, stationary distributions are convex combinations of* v *and the point mass at* 0, δ_0.

However, for the class of reaction-diffusion processes under consideration, $b_0 = 0$ implies that $c_1 = 0$, so a process starting from a non-identically zero η_0 never dies out; that is, it never becomes identically zero. Of course this does not preclude the process tending to δ_0 in distribution. Shiga (1988) asked whether under these conditions the system beginning from $\eta_0 \neq 0$ must tend to v in distribution. We prove the following theorems:

Theorem two. *Let a reversible reaction diffusion process satisfy*
1 $b_0 = c_1 = 0$,
2 *for each* j, $b_j = \lambda c_{j+1}$,
3 $\Sigma |y| p(0, y) < \infty$,
then the process tends to v *in distribution, starting from any non-identically zero starting point.*

Theorem three. *For a reversible reaction diffusion process, under the above conditions,*
1 $b_0 = c_1 = 0$,
2 *for each* j, $b_j = \lambda c_{j+1}$,
3 *the probability distribution* $p(0, y)$ *lies in the domain of attraction of a stable law of index less than one,*
then the process tends to v *in distribution, starting from any non-identically zero starting point.*

The two results leave open the cases where the random variable corresponding to $p(0, y)$ is irregular or in the domain of attraction of the Cauchy law.

The two theorems have different proofs: Theorem two follows because of the "controllability" of random walks with first moment while Theorem three uses the fact that the overall process may be compared with auxiliary random walks which are transient. Despite this, the proofs have common elements, which are presented in Sect. 1. The proof of Theorem two is completed in Section Two while the proof of Theorem three is finished in the last section. In the remainder of this introduction we recall some definitions, recall some important facts and results from DDL, and make some simple observations.

Notation. Given a configuration η in X, the configuration η^x is given by $\eta^x(y) = \eta(y + x)$.

Given a subset A of X, the subset A^x is given by $\eta \in A$ if and only if $\eta^x \in A^x$.

Given a measure μ on X, the measure μ^x is given by $\mu^x(A^x) = \mu(A)$.

We denote the semi-group of the process by $S(t)$ $t \geqq 0$. For a configuration η, $S(t)\eta$ is the measure on X defined by $[S(t)\eta](A) = P^{\eta}[\eta_t \in A]$. For a subset A of X, $S(t)A$ denotes the function on X defined by $\eta \to [S(t)\eta](A)$.

Definitions. We use the standard partial ordering on $X = Z_+^{Z^d}$: $\eta \leqq \eta'$ if for every x in Z^d, $\eta(x) \leqq \eta'(x)$.
A function f on X is increasing if $\eta \leqq \eta'$ implies that $f(\eta) \leqq f(\eta')$. We use the partial ordering on measures on X given by: $\mu_1 \leqq \mu_2$ if $\int f \mathrm{d}\mu_1 \leqq \int f \mathrm{d}\mu_2$ for every increasing function f.
For an element η of X and a subset A of Z^d, the element $\eta \cap A$ of X is given by

$$\eta \cap A(x) = \eta(x) \quad \text{if } x \text{ is in } A \text{ and is zero otherwise.}$$

We say a site $z \in Z^d$ is occupied for a configuration η if $\eta(z) > 0$.

Some facts from DDL

A Attractiveness: Suppose that $\eta_0 \leqq \eta'_0$. Then two reaction-diffusion processes $\{\eta_t: t \geqq 0\}$ and $\{\eta'_t: t \geqq 0\}$ with $\eta_0 \leqq \eta'_0$ may be coupled so that for all times t, $\eta_t \leqq \eta'_t$.

B We can start the process with $\eta_0 = \eta^{\infty}$, that is the state where the number of particles at each site is infinite. In this case, for every $m \in Z^+$ and every strictly positive t, $E[(\eta_t^{\infty}(0))^m] < \infty$.

It should be noted that Theorem One and facts A and B imply that $S(t)\eta^{\infty}$ tends to ν in distribution as t tends to infinity. Fact A ensures that we may couple a reaction-diffusion process $\{\eta_t: t \geqq 0\}$ with the process $\{\eta_t^{\infty} : t \geqq 0\}$ so that for all t, $\eta_t \leqq \eta_t^{\infty}$. It follows from the above observation that for any η_0 and for any increasing function f

$$\limsup_{t \to \infty} \int f(\eta)\mathrm{d}[S(t)\eta_0](\eta) \leqq \int f(\eta)\mathrm{d}\nu(\eta)\,.$$

Suppose now that we knew $S(t)e_0$ tends to ν in distribution. It immediately follows from the translation invariant nature of the process and the translation invariance of ν that $S(t)e_x$ tends to ν as well. If η_0 is non-zero, there is an x with $e_x \leqq \eta_0$. It follows from Fact A (again) that for increasing, continuous, and bounded f

$$\int f(\eta)\mathrm{d}\nu(\eta) = \liminf \int f(\eta)\mathrm{d}[S(t)e_x](\eta) \leqq \liminf_{t \to \infty} \int f(\eta)\mathrm{d}[S(t)\eta_0](\eta)\,.$$

We could then conclude that for any non-zero η_0 and an increasing, continuous, and bounded f,

$$\lim_{t \to \infty} \int f(\eta)\mathrm{d}[S(t)\eta_0](\eta) = \int f(\eta)\mathrm{d}\nu(\eta)\,.$$

Such functions constitute a convergence-determining class. Thus, the problem of showing that for any starting configuration η, $S(t)\eta$ tends to ν in distribution as t tends to infinity is reduced to showing that $S(t)\eta$ tends to ν in distribution for $\eta = e_0$.

Consequently, in this paper every reaction-diffusion process $\{\eta_t: t \geqq 0\}$ has $\eta_0 = e_0$ unless stated to the contrary. All systems other than η_t^{∞} are finite systems in the sense that the initial configuration only has a finite number of occupied sites and these sites are occupied by finitely many particles.

The author is pleased to thank Tom Liggett for introducing and explaining the problem and Tokuzo Shiga for showing that Theorem Two could be proved in greater generality than the author originally thought. The author is also grateful for their helpful comments and encouragement.

Section one

In this section we show that convergence to the upper invariant measure follows if two related conditions are satisfied:

Proposition 1.1 *Suppose that*
i. *For every $\varepsilon > 0$ there exists a K so that*

$$\liminf_{t\to\infty} \frac{1}{t}\int_0^t [S(t)\eta_0] \quad [\text{there exists } |x| \leqq K \text{ with } \eta(x) > 0]\,dt > 1-\varepsilon$$

and
ii. *For every site x in Z^d, there exists P^{η_0} a.s. a (random) integer n so that $\eta_n(x) > 0$.*
Then the measures $S(t)\eta_0$ converge to ν in distribution.

The proposition is proved via the following lemmas.

Lemma 1.2 *Under conditions* (i) *and* (ii) *above, the measures*

$$\mu_t = \frac{1}{t}\int_0^t S(u)\eta_0\,du$$

converge to ν as t tends to infinity.

Proof. We first note that while the space X is not compact, we have for every $t \geqq 1$ that $S(t)\eta_0 \leqq S(1)\eta^\infty$, so the collection of measures $\frac{1}{t}\int_0^t S(u)\eta_0\,du$ is tight. Therefore, convergence to ν will follow if every convergent sequence of measures $\frac{1}{t_n}\int_0^{t_n} S(u)\eta_0\,du$ with t_n tending to infinity has ν as its limit.

Let us take t_n to be any sequence of times tending to infinity for which $\mu = \lim \mu_{t_n}$ exists. We first show that μ must be translation invariant.

Note that since the process is translation invariant, we must have

$$\lim_{n\to\infty} \frac{1}{t_n}\int_0^{t_n} S(u)e_x\,du = \mu^x .$$

The attractiveness of the system ensures that for any η_n with $\eta_n(x) > 0$, we must have for t larger than n

$$S(t-n)e_x \leqq S(t-n)\eta_n .$$

Thus it follows that if A_n is the event $\{n = \inf\{m\colon \eta_m(x) > 0\}\}$, then for t greater than N

$$S(t)\eta_0 \geqq \sum_{r=1}^{N} P^{\eta_0}[A_r]S(t-r)e_x + \left(1 - \sum_{r=1}^{N} P^{\eta_0}[A_n]\right)\delta_0 .$$

Integrating this inequality, we obtain for large n

$$\frac{1}{t_n}\int_N^{t_n} S(t)\eta_0 \mathrm{d}t \geqq \sum_{r=1}^{N} \frac{1}{t_n}\int_N^{t_n} P^{\eta_0}[A_r]S(t-r)e_x \mathrm{d}t + \left(1-\frac{N}{t_n}\right)\left(1-\sum_{r=1}^{N} P^{\eta_0}[A_r]\right)\delta_0 .$$

Assumption (ii) of Proposition 1.1 guarantees that as N becomes large, $\sum_{r=1}^{N} P^{\eta_0}[A_r]$ tends to 1. Thus we let n tend to infinity, then let N tend to infinity, and use assumption (ii) of Proposition 1.1 to obtain $\mu \geqq \mu^x$. But we may interchange the roles of 0 and x and obtain $\mu^x \geqq \mu$ in a similar fashion. Thus $\mu = \mu^x$ and the measure is translation invariant.

It follows from the main theorem in DDL, quoted in this paper as Theorem One, that μ must be a convex combination of δ_0 and ν.

Let B_K be the set $\{\eta : \exists |x| \leqq K \text{ with } \eta(x) > 0\}$. Assumption (i) of Proposition 1.1 can be rephrased as

$$\text{for each } \varepsilon > 0 \text{ there exists } K \text{ s.t. } \liminf_{t\to\infty} \frac{1}{t}\int_0^t S(t)\eta_0[B_K]\,\mathrm{d}t > 1-\varepsilon .$$

But for our given sequence t_n, this must imply that $\mu(B_K) > 1-\varepsilon$. Since ε may be taken as small as desired and $\delta_0(B_K) = 0$, we must have that μ equals ν. □

The above lemma shows that the Cesaro means of the measures $S(t)\eta_0$ converge to the upper invariant measure ν. It remains to show that $S(t)\eta_0$ converges to ν as t tends to infinity. This is a common problem in particle systems. The following is a simple general lemma.

Lemma 1.3 *Consider a measurable bounded function $b(t)$ on $t \geqq 0$. If $\limsup_{t\to\infty} b(t) = \lim_{t\to\infty} \frac{1}{t}\int_0^t b(s)\,\mathrm{d}s = B$, then for each $\varepsilon > 0$ we have*

$$\lim_{t\to\infty} \frac{1}{t}\int_0^t I_{\{b(s) < B-\varepsilon\}}\,\mathrm{d}s = 0 .$$

Proof. Given $\delta > 0$, there exists T so that $b(t) < B + \delta$ for all t larger than T.

$$\begin{aligned} B &= \lim_{t\to\infty} \frac{1}{t}\int_0^t b(s)\mathrm{d}s \\ &\leqq \liminf_{t\to\infty}\left(\frac{1}{t}\int_0^t (B-\varepsilon)I_{\{b(s)<B-\varepsilon\}}\mathrm{d}s + \frac{1}{t}\int_T^t (B+\delta)I_{\{b(s)\geqq B-\varepsilon\}}\mathrm{d}s\right). \end{aligned}$$

This entails that

$$\limsup_{t\to\infty} \frac{1}{t}\int_0^t I_{\{b(s)<B-\varepsilon\}}\mathrm{d}s \leqq \frac{\delta}{\delta+\varepsilon} .$$

Since δ can be made arbitrarily small we are done. □

Let A be any increasing event (that is, the indicator function I_A is an increasing function). Then $S(t)\eta^\infty(A)$ tends to $\nu(A)$ and by attractiveness $S(t)\eta^\infty(A) \geqq S(t)\eta_0(A)$. Thus, applying Lemma 1.3 to the bounded function $b(t) = S(t)\eta_0(A)$ with $B = \nu(A)$, we obtain

Corollary 1.4 *Given an increasing event A and any $\varepsilon > 0$,*

$$\lim_{t\to\infty} \frac{\lambda[\{u \in [0,t]:\ S(u)\eta_0(A) < \nu(A) - \varepsilon\}]}{t} = 0\,.$$

Before completing the proof of Proposition 1.1 we require one last lemma. This lemma is really just a reformulation of Theorem B on page 68 of Halmost (1950).

Lemma 1.5 *Suppose B is a measurable subset of $[0, T]$ with $\lambda(B) > 9/10T$ (λ denotes Lebesgue measure). Then for every z in $[T, 3/2T]$, there exist x and y in B with $z = x + y$.*

Proof. Suppose not. In that case, there exists a z in the interval $[T, 3/2T]$ which cannot be expressed in the desired form. This property of z is equivalent to the sets B and $z - B$ being disjoint. However $\lambda([0, T] \cap z - B)$ must exceed $4/10T$ and so $\lambda(B \cap z - B) \geqq 9/10T + 4/10T - T \geqq 3/10T$. This contradiction establishes the lemma. □

Proof of Proposition 1.1

To show that $S(t)\eta_0$ converges in distribution to ν it will suffice to show that

$$S(t)\eta_0(A) \quad \text{tends to } \nu(A) \text{ in distribution}$$

for every cylinder set A. In turn, to show the above it will be sufficient to show this for every increasing cylinder set A.

Recall that B_n is the subset of X consisting of configurations which have an occupied site within n of the origin. Fix ε positive but otherwise arbitrarily small. We can find an n so big that $\nu(B_n) > 1 - \varepsilon/2$. Corollary 1.4 implies that

$$\liminf_{t\to\infty} \frac{\lambda[\{u \in [0,t]:\ S(u)\eta_0(B_n) > 1 - \varepsilon\}]}{t} = 1\,. \tag{$**$}$$

For this n let $i_1, i_2, \ldots i_M$ be an (unimportant but fixed) ordering of the elements of the lattice within distance n of the origin. Using Corollary 1.4 once more we find that for a fixed increasing set A we must have

$$\liminf_{t\to\infty} \frac{\lambda[\{u \in [0,t]:\ S(u)\eta_0(A^{i_k}) > \nu(A) - \varepsilon\}]}{t} = 1$$

for each k.

Now let V be the set of times u for which

(a) $S(u)\eta_0(B_n) > 1 - \varepsilon$

(b) $S(u)\eta_0(A^{i_k}) > \nu(A) - \varepsilon$ for all k. It follows from the preceding statements that there is a T such that for all $t > T$, $\lambda(V \cap [0, t]) > 9/10t$. It follows from Lemma 1.5 that for all $t > T$ we can find s and u in V with $s + u = t$. We fix such a triple. Now for $k = 1, 2, \ldots, M$, let subsets of X be defined by $C_k = \{\eta: k \text{ is the smallest } j \text{ with } \eta(i_j) > 0\}$. So $\bigcup_k C_k = B_n$ and by condition (a) in the definition of the time set B we have $S(s)\eta_0(\bigcup_{k=1}^M C_k) > 1 - \varepsilon$. Also by attractiveness of the system and the fact that A is an increasing event, we have

$$\text{for } \eta \in C_k,\quad [S(u)\eta](A) \geqq [S(u)e_{i_k}](A) = [S(u)e_0](A^{-i_k})\,.$$

Since u is a member of V, this last expression is greater than $v(A) - \varepsilon$. Thus on the set B_n we have

$$[S(u)\eta](A) \geqq v(A) - \varepsilon .$$

Therefore, using the semigroup property and the fact that $s \in V$, we have

$$\begin{aligned}[S(t)\eta_0](A) &= [S(s)\eta_0](S(u)A) \geqq [S(s)\eta_0](I_{B_n}S(u)A) \\ &\geqq (v(A) - \varepsilon)[S(s)\eta_0](I_{B_n}) \geqq (1 - \varepsilon)[v(A) - \varepsilon] .\end{aligned}$$

Since A is an arbitrary increasing set and ε is arbitrarily small, the proof is complete. □

Section two

The object of this section is to prove Theorem Two. Given the results of Section One, we have reduced the problem to verifying conditions (i) and (ii) of Proposition 1.1.

Throughout this section we take the transition probabilities $p(x, y) = p(0, x - y) = p(0, y - x)$ to satisfy

$$\sum_y |y| p(0, y) < \infty ,$$

and to be the transition probabilities of an irreducible random walk. We give a simple consequence of these assumptions.

Lemma 2.1 *There exists a constant $c > 0$ such that for all $\theta \in R^d$ of Euclidean norm equal to unity, $\sum_y |\langle y, \theta\rangle| p(0, y) > c$.*

Proof. The map on the d-dimensional unit sphere $\theta\colon \to \langle y, \theta\rangle$ is continuous for each y. It follows, therefore, from the condition $\sum_y |y| p(0, y) < \infty$ and the Dominated Convergence Theorem that the map $\theta\colon \to \sum_y |\langle y, \theta\rangle| p(0, y)$ is continuous. Since the underlying random walk is irreducible, $\sum_y |\langle y, \theta\rangle| p(0, y)$ must be strictly positive for each non-zero θ. The result now follows from the compactness of the unit sphere. □

We introduce a Markov process $\{(Y_t, Z_t)\colon t \geqq 0\}$ on $Z^d \times \{1, 2\}$ with generator

$$\begin{aligned}\Omega f(x, v) = {} & I_{\{v=1\}} \sum_y p(0, y)(f(x + y, 1) - f(x, 1)) \\ & + I_{\{v=2\}} \sum_{\langle y, x\rangle \leqq 0} p(0, y)(f(x + y, 1) - f(x, 2)) \\ & + I_{\{v=2\}} \sum_{\langle y, x\rangle > 0} p(0, y)(f(x, 1) - f(x, 2)) \\ & + I_{\{v=1\}} \beta(1)(f(x, 2) - f(x, 1)) \\ & + I_{\{v=2\}} \delta(2)(f(x, 1) - f(x, 2)) .\end{aligned}$$

The process Z_t jumps from 1 to 2 and back in a way that mimics the way particles are born and die in the reaction-diffusion process. Y_t jumps as a random walk with jump probabilities $p(0, y)$ if $Z_t = 1$, but if $Z_t = 2$, jumps away from the origin are suppressed; instead Z_t will jump to state 1. We also assume that $(Y_0, Z_0) = (0, 1)$.

We can regard (Y_t, Z_t) as a process on X by identifying $(y, z) \in Z^d \times \{1, 2\}$ with $\alpha \in X$ where $\alpha(y) = z$ and $\alpha(x) = 0$ for $x \neq y$. It follows from Corollary A2 of Shiga and Uchiyama (1986) that we can couple the processes (Y_t, Z_t) and η_t (recall that $\eta_0 = e_0$) so that for all times t

$$\eta_t(Y_t) \geqq Z_t \, .$$

That is, the state Y_t is always occupied by the configuration η_t and if Z_t equals 2, then at least two particles are present at the site Y_t. Since the transition probabilities are symmetric, $\sum_{\langle y, x \rangle \leqq 0} p(0, y)$ is at least 1/2 for all sites x. Consequently, the jump rate for Y_t is always between 1 and 2.

We define the stopping times T_n by

$$T_0 = 0; \quad \text{for } n \geqq 1 \; T_n = \inf\{t > T_{n-1} \colon Y_t \neq Y_{T_{n-1}}\} \, .$$

The strong Markov property ensures that the discrete time process $\{W_n = Y_{T_n} \colon n \geqq 0\}$ is a Markov chain. The following lemma follows easily from the definition of the transition rates for the process (Y_t, Z_t).

Lemma 2.2 *Let $Q(x, z)$ be the transition probabilities for the Markov chain W. There exists a strictly positive α so that for all sites x*

$$2\, p(0, y) \geqq Q(x, y) \geqq (1 + \alpha) p(0, y) \quad \textit{for} \quad \langle x, y \rangle \leqq 0$$

$$\textit{and } Q(x, y) \leqq (1 - \alpha) p(0, y) \quad \textit{for} \quad \langle x, y \rangle > 0 \, .$$

Lemma 2.3 *The Markov chain $\{W_n \colon n \geqq 1\}$ is positive recurrent and irreducible.*

Proof. The irreducibility of W follows easily from the irreducibility of $p(\,,)$.

Theorem 9.1 of Tweedie (1976) ensures that to prove positive recurrence it suffices to show that for some ε and N

$$E[|W_{n+1}| \,|\, W_n = y] \leqq |y| - \varepsilon$$

whenever $|y| > N$. Showing this constitutes the remainder of the proof. We pick K so large that $\sum_{|y| > K} |y| p(0, y) < \alpha c/10$, where α and c are the constants of lemmas 2.2 and 2.1 respectively. Now consider $E[|W_{n+1}| \,|\, W_n = x]$ for x of large magnitude. Write the random quantity $W_{n+1} - x$ as Y_n^x. Then

$$E[|W_{n+1}| \,|\, W_n = x] = E[|Y_n^x + x|] \leqq E[|Y_n^x| + |x| I_{\{|Y_n^x| > K\}}]$$

$$+ E[|Y_n^x + x| I_{\{|Y_n^x| \leqq K\}}] \, .$$

From our choice of K, the first term on the right hand side of the inequality is less than $|x| P[|Y_n^x| > K] + c\alpha/5$. Since the random quantity $|Y_n^x|$ is bounded by K, we can use the binomial expansion to write

$$E[|Y_n^x + x| I_{\{|Y_n^x| \leqq K\}}] = E[(|Y_n|^2 + 2\langle x, Y_n \rangle + |x|^2)^{1/2} I_{\{|Y_n^x| \leqq K\}}]$$

$$= E\left[\left(|x| + \left\langle Y_n^x, \frac{x}{|x|} \right\rangle + O\left(\frac{1}{|x|}\right)\right) I_{\{|Y_n^x| \leqq K\}}\right] .$$

It follows from Lemma 2.2 that

$$E\left[\left\langle Y_n^x, \frac{x}{|x|}\right\rangle I_{\{|Y_n^x| \leqq K\}}\right] < \sum_{|y| \leqq K, \langle x,y\rangle \leqq 0)} (1+\alpha)p(0,y)\left\langle \frac{x}{|x|}, y\right\rangle$$

$$+ \sum_{|y| \leqq K, \langle x,y\rangle > 0)} (1-\alpha)p(0,y)\left\langle \frac{x}{|x|}, y\right\rangle$$

$$\leqq -\alpha \sum_{|y| \leqq K} p(0,y)\left|\left\langle \frac{x}{|x|}, y\right\rangle\right| .$$

From our choices of K and c, it follows that the last term is less than $-\alpha(c - c\alpha/10)$, which is less than $-\alpha c/2$. Collecting all our work together, we find that

$$E[|W_{n+1}||W_n = x] \leqq P[|Y_n^x| > K]|x| + c\alpha/5 + P[|Y_n^x| \leqq K]|x|$$

$$- c\alpha/2 + O\left(\frac{1}{|x|}\right) \leqq |x| - c\alpha/5$$

for $|x|$ large enough and the lemma is proven. □

Proof of Theorem Two. We are now ready to verify conditions (i) and (ii) of Proposition 1.1. Lemma 2.3 states that the Markov chain W is positive recurrent and irreducible. Therefore, for each x in Z^d, there will be a time t at which the process Y_t hits x and stays there for at least unit time. This ensures that (ii) holds for the process η_t. It remains to establish (i). The Markov chain W has an invariant probability measure π. By Lemma 2.2, irrespective of the state Y_{n-1}, $\{T_n - T_{n-1}\}$ are stochastically greater than exponential random variables with mean 1/2 and stochastically less than exponential random variables of mean 1. It follows from the strong law of large numbers that for any subset A of Z^d we have with probability 1

$$\limsup_{t\to\infty} \frac{\lambda\{s \leqq t : Y_s \in A\}}{t} \leqq 2\pi(A) .$$

Let us take A to be the set $\{y : |y| > n\}$ where n is so large that $\pi(A) < \varepsilon/4$. Then we have

$$P\left[\frac{\lambda\{s \leqq t : |Y_s| > n\}}{t} < \frac{3\varepsilon}{4}\right] \to 1 \quad \text{as } t \to \infty .$$

Condition (i) follows via an application of Fubini's Theorem. □

Section three

In this section we prove Theorem Three. Henceforth, we assume that the probability law $p(0, y)$ is in the domain of attraction of a (necessarily symmetric) stable random variable of index $\alpha < 1$. As with Theorem Two, it only remains to show that under the conditions of this section conditions (i) and (ii) of Proposition 1.1 are satisfied. As will be evident from the proof, no generality will be lost by assuming that the dimension of the state space for the underlying random walk is one. To minimize notational complexity, we make this assumption. Throughout this section any random walk is assumed to be a continuous time random walk with jump rates (and probabilities) $p(x, y)$.

We use block arguments based on ideas found in Bramson (1989) and Bramson et al. (1989). For a given M and K, we consider random variables $I_{z,n}$ defined by

$$I_{z,n} = 1 \quad \text{if } \eta_{nK}(y) > 0 \quad \text{for some } y \text{ in } ((z-1)M, (z+1)M)$$
$$= 0 \quad \text{otherwise .}$$

We compare the above variables with a 1-dependent oriented percolation system. An oriented percolation system is a directed graph with vertex set equal to $Z \times Z^+$ and directed edges from vertices (z, n) to vertices $(z+1, n+1)$ or $(z-1, n+1)$. These edges are open with (bond) probability p and closed with probability $1-p$. The system is 1-dependent if edges between disjoint vertex pairs are independent. We write ψ_n^A for the set of z so that (z, n) is connected to some point in $A(\subset Z \times \{0\})$ by a path of open edges. If $A = (0, 0)$, the superscript is suppressed.

The majority of work in this section is to the end of proving

Proposition 3.1 *Consider the reaction-diffusion process $\{\eta_t : t \geqq 0\}$ with $\eta_0 = e_0$. For a given M and K we define $I_{z,n}$ by*

$$I_{z,n} = 1 \quad \textit{if } \eta_{nK}(y) > 0 \quad \textit{for some } y \textit{ in } ((z-1)M, (z+1)M)$$
$$= 0 \quad \textit{otherwise .}$$

Given $\varepsilon > 0$, we may choose M and K so that the process η can be coupled wth a 1-dependent oriented percolation system with the bond probability equal to $1-\varepsilon$, so that if $z \in \psi_n$, then $I_{z,n} = 1$.

The general results of Durrett (1984) can then be used to complete the proof of Theorem Three.

We record some facts to be used later.

Fact 3.1
Let W_α be a symmetric stable process of index α (see e.g. Ethier and Kurtz (1985), Chap. 3, Sects. 6–8). For a continuous time random walk γ_t with jumps distributed as $p(x,)$, there are constants $t(M)$ so that the continuous time process $V^M(s) = \gamma_{t(M)s}/M$ converges to W_α in the Skorohod topology on $D[0, 1]$ (see e.g. Ethier and Kurtz (1985), Chap. 3, Sects. 6–8).

Fact 3.2
A continuous time random walk γ_t with transition probabilities $p(x, y)$ is transient (see e.g. Feller (1971), Theorem 3, page 580).

Fact 3.3
Given $\varepsilon > 0$, we can find $M(\varepsilon)$ so that $|z| > M$ implies $P^0[\gamma_t \text{ hits } z] < \varepsilon$. Here $P^y[\ ]$ denotes the probability of an event for a random walk starting at y. (See for example Spitzer (1965), P3 page 293.)
The following corollary is self evident:

Corollary 3.4 *Given M and $\varepsilon > 0$, we can find $R(\varepsilon, M)$ so that $|z| > R$ implies $P^z[|\gamma_t| < M$ for some $t] < \varepsilon$.*

Lemmas 3.5–3.9 below establish couplings between our reaction diffusion process and systems of random walks. These latter processes are easier to deal with in establishing that the number of occupied sites grows large as t becomes large.

The following lemma is just an application of Corollary A2 of Shiga and Uchiyama (1986, page 114). (A system of k random walks can be considered as a finite process on $X = Z_+^{Z^d}$.)

Lemma 3.5 *Let X_t^i, $i = 1, 2, \ldots, k$, be independent random walks until $T = \inf\{t > 0: \exists\, i \neq j$ so that $X_t^i = X_t^j\}$. At time T let each process be taken to a graveyard state Δ. If η_t is a reaction-diffusion process with $\eta_0(X_0^i) > 0$ for each i, then there is a coupling of the processes so that for all t*

$$\eta_t(X_t^i) > 0 \quad \textit{for each } t\ .$$

By convention we write $\eta_t(\Delta) > 0$ *for each t.*

Remark. If for each different i and j, X_0^i and X_0^j are at least $M(\varepsilon)$ apart, then it follows from Fact 3.3 that $P[T < \infty] < k^2\varepsilon$. When Lemma 3.5 is used, we will usually be dealing with starting positions so that $P[T < \infty]$ is negligible.

For $L < N$ in Z^+, let us define an (L, N) shooting process $Y_t^{L,N}$ as a continuous time Markov process on $S_1 \cup Z^2 \cup \Delta$ where S_1 equals $Z \times \{A, B\}$ and Δ is a graveyard state that the process can never leave. We will always have $Y_0^{L,N}$ in S_1. The process $Y_t^{L,N}$ has generator

$$\text{on } S_1 \ (\Omega f)(x, A) = \sum_y p(x, y)(f(y, A) - f(x, A)) + \beta(1)(f(x, B) - f(x, A))\ ,$$

$$(\Omega f)(x, B) = \sum_y p(x, y)(f(y, A) - f(x, B)) + \delta(1)(f(x, A) - f(x, B))$$

$$+ \sum_{L < |x-y| < N} p(x, y)(f(x, y) - f(x, B))$$

on Z^2

$$(\Omega f)(x, y) = \sum_{y \neq z} p(x, z)(f(z, y) - f(x, y)) + \sum_{z \neq x} p(y, z)(f(x, z) - f(x, y))$$

$$+ (p(x, y) + p(y, x))(f(\Delta) - f(x, y))\ .$$

As is evident from the generator on Z^2, after the process jumps from S_1 to Z^2 it can be thought of as a pair of particles performing independent random walks which disappear on meeting. Let $T^{L,N} = \inf\{t: Y_t^{L,N} \in Z^2\}$. On the stochastic interval $[0, T^{L,N})$, $Y_t^{L,N}$ can be written as (X_t^1, N_t) where X^1 is a random walk and $N_t \in \{A, B\}$. When $N_t = B$, the process "shoots out" or creates a second particle at rate $\sum_{L < |y| < N} p(0, y) > 0$. Let $V^{L,N} = \inf\{t: Y_t^{L,N} \in \Delta\}$. On the stochastic interval $[T^{L,N}, V^{L,N})$, $Y_t^{L,N}$ can be written as (X_t^a, X_t^b) where the Xs are independent random walks which disappear on meeting at $V^{L,N}$.

Notation. Given an L, N shooting process $Y_t^{L,N}$, X_t^1 is a random walk so that $Y_t^{L,N} = (X_t^1, N_t)$ for $t < T^{L,N}$. For all $t \geqq T^{L,N}$ (X_t^a, X_t^b) denotes a pair of random walks (killed upon meeting) so that $Y_t^{L,N} = (X_t^a, X_t^b)$ for $t \geqq T^{L,N}$.

Lemma 3.6 *Suppose that $Y_0^{L,N} \in S_1$. For any L and N with $\sum_{L<|y|<N} p(0, y) > 0$ we have $T^{L,N} < \infty$ a.s. but $\lim_{L\to\infty} P[V^{L,N} < \infty] = 0$.*

Remark. It should be noted that the distributions of the stopping times $T^{L,N}$ and $T^{L,N}$ are independent of the initial point of $Y^{L,N}$ in S_1.

Proof. Let $k = \sum_{L<|y|<N} p(0, y)$. The process N_t jumps from A to B at constant rate $\beta(1)$ and when the process N_t is equal to B, $Y_T^{L,N}$ will jump from S_1 to Z^2 at rate k. Since the process is time homogeneous, it must jump eventually from S_1 to Z^2.

On the other hand during the time interval $[T^{L,N}, V^{L,N}]$ the process $Y^{L,N}$ can be written as two independent random walks X^a and X^b which are distance more than L apart at time $T^{L,N}$ and for which $V^{L,N} = \inf\{t > T^{L,N}: X^a = X^b\}$. The result now follows from Fact 3.3, the Markov property, and the translation invariance properties of random walks. □

A k-tuple of processes $(Y_t^{L,N}, X_t^2, \ldots, X_t^k)$ is a k-dimensional L, N shooting process if

(1) For $t < T = \inf\{s\colon Y_s^{L,N} \in Z^2$ or $\exists 1 \leqq i < j \leqq k$ with $X_s^i = X_s^j\}$, the process $Y^{L,N}$ behaves like an L, N shooting process and the $X^j\, j > 1$ behave as random walks. All processes behave independently.

(2) If T, of (1) above, $= T^{L,N} = \inf\{s\colon Y_s^{L,N} \in Z^2\}$, the on $[T, \infty)$ the $k+1$ processes $X_t^a, X_t^b, X_t^2, \ldots, X_t^k$ behave as system of independent random walks which disappears when any two meet.

(3) If $T = \inf\{s\colon \exists 1 \leqq i < j \leqq k$ with $X_s^i = X_s^j\}$, then all processes are instantly sent to Δ a graveyard state where they remain.

Lemma 3.7 *Let $(Y_t^{L,N}, X_t^2, X_t^3, \ldots, X_t^k)$ be a k-dimensional L, N shooting process. Let $Y_0^{L,N} = (x_1, n) \in S_1$ and $X_0^j = x_j$ for $j > 1$.*
Let η_t be a reaction-diffusion process with
For each j $\eta_0(x_j) > 0$ and $\eta_0(x_1) \geqq 2$ if $n = B$.
Then the above processes can be coupled together so that for all t.
i. *For $j \in [2, r]$ $\eta_t(X_t^j) > 0$.*
ii. *For $t < T^{L,N}$, if $Y_t^{L,N} = (x, n)$ then $\eta_t(x) > 0$ and $\eta_t(x) \geqq 2$ if $n = B$.*
and
iii. *For $t \geqq T^{L,N}$, $\eta_t(X_t^a), \eta_t(X_t^b) > 0$.*

Proof. The Lemma is proved by simply comparing relevant jump rates and applying Corollary A.2 of Shiga and Uchiyama (1986). □

Lemma 3.8 *Given integers K and k and an $\varepsilon < 0$, there exists $T(\varepsilon, k, N)$ so that for $t > T$*

$$P\{\text{there exist } x_1, x_2, \ldots x_k \text{ each distance } K \text{ apart from the others,}$$
$$\text{so that } \eta_t(x_i) > 0 \forall i] > 1 - \varepsilon\,.$$

We use induction on k. The result is trivially true for $k = 1$ and any K and ε. Suppose now that the result has been proved for $k-1$ and any K and $\varepsilon > 0$. Fix ε and K. Choose M so large that for $|z| > M$,

$$P^z[\exists t > 0\colon |X_t| < 2K] < \frac{\varepsilon}{10k^2}\,.$$

Then pick $N > M$ with $\sum_{M<|z|<N} p(0, z) > 0$. Finally we choose K_1 so large that $|z| > K_1$ implies $P^z[|y_t| < 2N$ for some $t] < \varepsilon/10k^2$.

By induction there exists $T_1 = T(\varepsilon/4, k-1, K_1)$ so that

$$S(T_1)\eta_0(\text{there exists } x_1, x_2, x_{k-1} \text{ (each pair } K_1 \text{ apart) } s.t.\ \eta(x_i) > 0\ \forall i) > \varepsilon/4\,.$$

Let $(Y^{L,N}, X^2, X^3, \ldots X^{k-1})$ be a $k-1$ dimensional L, N shooting process. By Lemma 3.7 and the above definition of T_1, outside of a set of probability $\varepsilon/4$, we can couple η_{s+T_1} and independent processes $Y_s^{L,N}, X_s^2, \ldots, X_s^{k-1}$ so that

$$\eta_{T_1+t}(X_t^i) > 0 \quad \text{for } i = 1, 2, 3, \ldots k-1\,,$$

$$\eta_{T_1+t}(X_t^1) \geqq 2 \quad \text{for } N_t = B \text{ and } t < T^{L,N}\,.$$

$$\eta_{T_1+t}(X_t^a), \eta_{T_1+t}(X_t^b) \geqq 1 \quad \text{for } t \geqq T^{L,N}\,,$$

and

$$Y_0^{L,N} = (x_1, 1), X_0^j = x_j \quad \text{for } j > 1 \text{ with } |x_i - x_j| > K_1 \text{ for } i \neq j\,.$$

Our choice of K_1 ensures that outside of a set of probability $\varepsilon/4 + (k-1)^2 \dfrac{\varepsilon}{10k^2} \leqq 3\varepsilon/8$ at time $T_1 + T^{L,N}$ the shooting process has not hit Δ, and we have

a $Y_{T^{L,N}}^{L,N} = (y_1, y_2)$

b $X^j = y_{j+1}$ for $j > 1$, with $|y_j - y_i| > N$ for $i \neq j$.

We now couple the process $\eta_{T_1+T^{L,N}+s}$ with a system of independent random walks $Y_s^j(Y_0^j = y_j)$ as in Lemma 3.5. Removing a further set of probability $\varepsilon/10$ we have that for all times s the $\{Y_s^j\}$ are all at least N apart. Thus we have shown that outside of a set of probability at most $3\varepsilon/8 + \varepsilon/10 < \varepsilon/2$, we have for $t > T_1 + T^{L,N}$ that there exist sites z_i $i = 1, 2, \ldots, k$ each $> K$ apart from the others so that $\eta_t(z_i) > 0$. The result now follows by taking $T = T(\varepsilon, k, K)$ so large that $P[T < T_1 + T^{L,N}]$ is less than $\varepsilon/2$. □

Definition. For an interval I, $\{\eta_t^I : t \geqq 0\}$ denotes the modified reaction-diffusion process for which no sites are occupied outside I and for which particles attempting to jump outside I are destroyed. Formally η_t^I has generator

$$\begin{aligned}\Omega^I f(\eta) = &\sum_{x\in I} \beta(\eta(x))[f(\eta + e_x) - f(\eta)] + \sum_{x\in I} \delta(\eta(x))[f(\eta - e_x) - f(\eta)]\\ &+ \sum_{x\in I}\left(\sum_{y\in I} \eta(x)p(x, y)[f(\eta - e_x + e_y) - f(\eta)]\right.\\ &\left.+ \sum_{y\in I^c} \eta(x)p(x, y)[f(\eta - e_x) - f(\eta)]\right).\end{aligned}$$

The following is a direct consequence of Corollary A2 of Shiga and Uchiyama (1986).

Lemma 3.9 *Let $\eta_0^I(x_i) > 0$ for $i = 1, 2, \ldots k$. Consider independent random walks $X^i(X_0^i = x_i)$ which are killed outside I, and which all simultaneously die the first time two of the walks meet. We may couple η_t^I with the random walks so that for all $t > 0$*

$$\eta_t^I(X_t^i) > 0\,.$$

An analogue of Lemma 3.7 also holds. The following corollary follows simply from the proof of Lemma 3.8 above.

Corollary 3.10 Given integers V, K and k and an $\varepsilon > 0$, there exists $T(\varepsilon, k, N)$ and $R(\varepsilon, k, N)$ so that for any $\eta_0^{(-R,R)}$ with $\eta_0^{(-R,R)}(z) > 0$, for some z in $[-V, V]$

$$P\,[\text{there exist } x_1, x_2, \dots x_k \text{ each distance } K \text{ apart from the others, so that } \eta_T^{(-R,R)}(x_i) > 0 \forall i] > 1 - \varepsilon\,.$$

We are ready to begin our renormalization process. For the time being let k and ε be fixed. Choose K to be greater than the $M(\varepsilon)$ of Fact 3.3. Then pick $T(=T(\varepsilon, k, K))$ and $R(=R(\varepsilon, k, K))$ as in Corollary 3.10.

It follows from Fact 3.2 and simple properties of symmetric stable processes that there exists a $c > 0$ (not depending on k, K, ε) etc.) so that for M sufficiently large we will have

$$\inf_{|z| < M/2} P[\gamma_{t(M)}^M \in (3M/4, M)] > c > 0$$

where γ^M is a random walk killed outside $(-M, M)$ and $t(M)$ is the number given in Fact 3.2. We may choose such an M and will assume that $M > R/4$.

For $(z, n) \in Z^1 \times Z_+$, $z + n = 0 (\bmod 2)$, we will write $I^{z,n} = 1$ if $\eta_{(t(M)+T)n}(x) > 0$ for some $x \in ((z-1)M, (z+1)M)$; $I^{z,n} = 0$ otherwise. In the next few paragraphs we will develop some theory which will enable us to compare the set of (z, n) for which $I^{z,n} = 1$ with the set of points connected to (0, 0) in a supercritical, 1-dependent, oriented percolation. The lemma underneath follows from the attractiveness of our reaction-diffusion processes (modified or otherwise) and Corollary A2 of Shiga and Uchiyama (1986).

Lemma 3.11 *Consider modified reaction-diffusion processes η_t^I and independent processes $\eta_t^{I_j}$ for $j = 1, 2, \dots k$ where*
1 The I^j are disjoint sub intervals of I
2 $\eta_0^{I^j}(x) \leqq \eta_0^I(x)$. We may couple the processes so that for all t

$$\text{For } x \text{ in } I^j, \eta_t^{I^j}(x) \leqq \eta_t^I(x)\,.$$

Lemma 3.12 *Consider the process $\eta_t^{(-M,M)}$ with $\eta_0^{(-M,M)}(x) > 0$ for $|x| < M/4$. Outside a set of probability $\varepsilon(k^2 + 1) + (1-c)^k$ we have $\eta_{t(M)+T}^{(-M,M)}(y) > 0$ for some y in $(3M/4, M)$.*
(Similarly for some y in $(-M, -3M/4)$.)

Proof. From Corollary 3.10 it follows that (outside of a set of probability ε), $\eta_T^{(-M,M)}(x_i) > 0$ for k distinct points $x_1, x_2, \dots x_k$ each K apart from the rest and in the interval $(-M/2, M/2)$. (Recall that $R < M/4$). We now use our coupling of Lemma 3.9 and assume that there are k independent random walks X^j, killed on leaving $(-M, M)$, so that $\eta_{T+t}^{(-M,M)}(X_t^j) > 0$. We recall how M was selected and deduce that if there does not exist a site y in $(3M/4, M)$ which has a particle at time $t(M) + T$ then either
(i) Two of the random walks X_t^i must have met
or
(ii) For each k, $X_{t(M)}^k$ is not in $(3M/4, M)$.
The first event has probability bounded by $k^2\varepsilon$ while the second has probability less than $(1-c)^k$. □

Lemma 3.11 allows us to assume the existence of processes (for $z+n$ even) ${}^n\eta_t^{((z-1)M,(z+1)M)}$ with the properties

1 For $s \in [0, t(M)+T]$ ${}^n\eta_s^{((z-1)M,(z+1)M)}(x) \leqq \eta_{n(t(M)+T)+s}(x)$

2 ${}^n\eta_0^{((z-1)M,(z+1)M)}(x) = \eta_{n(t(M)+T)}(x)$ for $x \in ((z-1)M, (z+1)M)$.

3 The process ${}^n\eta_s^{((z-1)M,(z+1)M)}$ is conditionally independent of the processes ${}^m\eta_s^{((y-1)M,(z+1)M)}$ for $m \leqq n((y,m) \neq (z,n))$ given $\eta_{n(t(M)+T)}$.

Proof of Proposition 3.1 We shall say that (z,m) connects with $(y, m+1)$ $(y = z \pm 1)$ if ${}^m\eta_{t(M)+T}^{((z-1)M,(z+1)M)}(x) > 0$ for some x, $|x-y| < M/4$. It follows from Lemma 3.11 (and the translation invariant properties of the process) that:

(i) The event $\{(z,m)$ connects with $(y,m+1)\}$ has conditional probability $> 1 - \varepsilon(k^2+1) - (1-c)^k$ given the σ field $F_m (= \sigma\{\eta_s : s \leqq m(t(M)+T)\})$, on $I^{z,m} = 1$.

(ii) The events $\{(z,m)$ connects with $(y,m+1)\}$ is conditionally independent of $\sigma(\{(z',m)$ connects with $y', m+1)\}, z' \neq z)$ given F_m.

Given these observations Proposition 3.1 follows easily since ε can be arbitrarily small and k arbitrarily large.

Proof of Theorem Three. To prove Theorem Three it will suffice to verify that conditions (i) and (ii) of Proposition 1.1 hold. To this end we recall some facts about oriented percolation found in Durrett (1984).

Fact 3.13

For oriented percolation with the connection probability sufficiently close to 1, there exists a $\gamma >$ so that for $A \subset Z^1 \times \{n\}$,

$$P[A \text{ is not connected to the infinite cluster}] < e^{-\gamma|A|}.$$

See Durrett (1984) pages 1026–1029.

From this easily follows

Fact 3.14

As p tends towards 1, $P[(0,0)$ is connected to the infinite cluster $C]$ tends to 1.

Fact 3.15

If $l_n = \inf\{z\colon (z,n)$ is connected to $(0,0)\}$ and $r_n = \sup\{z\colon (z,n)$ is connected to $(0,0)\}$. Then on $\Omega^{(0,0)} = \{(0,0)$ is connected to the infinite cluster $C\}$ we have for supercritical percolation that

$$\frac{r_n}{n} \to \alpha(>0), \frac{l_n}{n} \to -\alpha \quad \text{as } n \text{ tends to } \infty.$$

See Durrett (1984), pages 1005, 1024–1025, and in the interval $[l_n, r_n] \times \{n\}$ the points in C coincide with the points connected to $Z^1 \times \{0\}$. See Durrett (1984), page 1021.

Using the second part of Fact 3.19 and time reversal it follows that

Fact 3.16

$\inf_{n \text{ even}} P[(0,0)$ is connected to $(0,n)]$ tends to 1 as p, the connection probability tends to 1. See Durrett (1984), pages 1021–1023.

The following also follows easily from time reversal ideas.

Fact 3.17
On the event $\Omega^{(0,0)}$ there exist infinitely many points $(0, n)$ which are connected to $(0, 0)$.
We proceed to proving that property (i) holds. Fact 3.16 states that for any $\delta > 0$ there exists a p_δ so that for connection probabilities $p \geqq p_\delta$ $\inf_{n \text{ even}} P[(0, 0)$ is connected to $(0, n)] > 1 - \delta$. We may choose ε so small and k so large that $(k^2 + 1)\varepsilon + (1 - c)^k < \delta$. By Proposition 3.1 we can then find M so large that the random set of vertices $\{(z, n)\colon I^{(z,n)} = 1\}$ contains the vertices of a 1-dependent oriented percolation with connection probability $> (1 - \delta)$ connected to $(0, 0)$.

It follows that at times $2n(t(M) + T)$ the probability that there exists an x in $(-M, M)$ with $\eta_{2n(t(M)+T)}(x) > 0$ exceeds $(1 - \delta)$. We now choose D so large that for each t in $[0, t(M) + T]$ $P[|\gamma_t| > D] < \delta$. It follows from the Markov property of the process η_t and Lemma 3.5 with $k = 1$, that for all t the probability that there exists an x in $(-(M + D), (D + M))$ with $\eta_t(x) > 0$ exceeds $(1 - 2\delta)$. Since δ is arbitrary we have proved that condition (i) holds.

To prove (ii) we first note that if $I^{0,n}$ is equal to 1 for infinitely many n, then (outside of a set of probability zero), the coupling of Lemma 3.5 entails that for each x there must exist infinitely many integers m with $\eta_m(x) > 0$. So given $\delta > 0$ we can take M as in the preceding paragraph. Fact 3.16 entails that for this renormalization the probability that $I^{0,n}$ is 1 for infinitely many n exceeds $1 - 2\delta$. Property (ii) follows from the arbitrariness of δ.

References

Bramson, M.: Survival of nearest particle systems with low birth rates. Ann. Probab. **17**, 433–443 (1989)

Bramson, M., Durrett, R., Swindle, G.: Statistical mechanics of crabgrass. Ann. Probab. **17**, 444–481 (1989)

Durrett, R.: Oriented percolation in two dimensions. Ann. Probab. **12**, 999–1040 (1984)

Chen, M.: Infinite dimensional reaction diffusion processes. Acta Math. Sin. New Ser. **1**, 261–273 (1985)

Ding, W., Durrett, R., Liggett, T.: Ergodicity of reversible reaction diffusion processes. Probab. Theory Relat. Fields **85**, 13–26 (1990)

Ethier, S., Kurtz, T.: Markov processes. New York: Wiley 1986

Feller, W.: An introduction to probability theory and its application II. New York: Wiley

Halmos, P.: Measure theory. New York: Van Nostrand 1950

Janssen, H.: Stochastisches Reaktionsmodell fur einem Nichtgleichgewichts-plasinubergang. Z. Phys. **270**, 67–73 (1974)

Shiga, T.: Stepping stone models in population genetics and population dynamics. Stochastic Processes Phys. Eng. 345–355 (1988) Dordrecht: D. Reidel 1988

Shiga, T., Uchiyama, K.: Stationary states and their stability of the stepping stone model involving mutation and selection. Probab. Theory Relat. Fields **73**, 87–118 (1986)

Spitzer, F.: Principles of random walk. New York: Van Nostrand 1964

Tweedie, R.: Criteria for classifying general Markov chains. Adv. Appl. Probab. **8**, 737–771 (1976)