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Exact convergence rates for the bounded law of the iterated logarithm in Hilbert space

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Summary. In a previous paper we obtained upper and lower class type results refining the bounded LIL for sums of iid Hilbert space valued mean zero random variables, whose covariance operators satisfy certain regularity assumptions. We now establish precise convergence rates for the bounded LIL in the "non-regular" case. It turns out that the almost sure behavior in this case is entirely different from the behavior in the previous situation.

1 Introduction

Let $X, X_1, X_2, \ldots$ be iid mean zero random variables taking values in a separable infinite-dimensional Hilbert space H with norm $\|\cdot\|$ and scalar product $(\cdot,\cdot)$. Set $S_n := \sum_{k=1}^{n} X_k$, $n \geqq 1$. Suppose that

$$(1.1) \qquad E[(X, y)^2] < \infty, \qquad y \in H,$$

and

$$(1.2) \qquad E[\|X\|^2 / L_2 \|X\|] < \infty,$$

where $Lt := \log(\max(t, e))$, $L_2 t := L(Lt)$, $t \geqq 0$.

Then it follows from a known result of DeAcosta and Kuelbs (1983) that with probability one,

$$(1.3) \qquad \limsup_{n \to \infty} \|S_n\| / \sqrt{2n L_2 n} = \sigma,$$

where $\sigma^2 := \sup\{E[(X, y)^2] : \|y\| \leqq 1\}$. In particular, one has for *any* $\varepsilon > 0$,

$$(1.4) \qquad P\{\|S_n\| \leqq \sigma((2+\varepsilon) n L_2 n)^{1/2} \text{ eventually}\} = 1.$$

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Fact (1.4), in turn, implies that there exists a null-sequence $\varepsilon_n \downarrow 0$ such that

$$P\{\|S_n\| \leqq \sigma((2+\varepsilon_n)\, n L_2\, n)^{1/2} \text{ eventually}\} = 1. \tag{1.5}$$

The above sequence $\{\varepsilon_n\}$ provides a kind of convergence rate for the bounded law of the iterated logarithm (LIL) (1.3), and it is therefore of interest to determine $\{\varepsilon_n\}$ in terms of appropriate characteristics of the distribution of X.

If X is *Gaussian*, then an application of the infinite-dimensional Kolmogorov-Erdös-Petrowski type integral test due to Kuelbs (1975) yields that one can choose $\{\varepsilon_n\}$ as

$$\varepsilon_n = a L_3\, n / L_2\, n, \qquad \text{if } a > d+2, \tag{1.6}$$

where d is the multiplicity of σ^2 which is an eigenvalue of the (compact) covariance operator of X, and $L_3\, t := L(L_2\, t)$, $t \geqq 0$.

Moreover, it follows from this result that the probability in (1.5) is zero whenever $a \leqq d+2$. We thus see that (1.6) gives the exact convergence rate for the Gaussian case.

Given this result, it is natural to ask what are the rates for non-Gaussian random variables. It has been shown by Einmahl (1989a) that the upper class part of the above integral test applies to arbitrary pregaussian mean zero random variables, i.e., to mean zero random variables satisfying

$$E[\|X\|^2] < \infty. \tag{1.7}$$

This clearly implies that, if we define $\{\varepsilon_n\}$ as in (1.6), then (1.5) remains valid for sums of iid random variables satisfying (1.7). The situation, however, becomes more complicated for random variables with infinite second strong moments. Contrary to the previous cases, it can happen that the covariance operator of X, which will be denoted by $\operatorname{cov}(X)$ in the sequel, is not compact. The quantity σ^2 is not necessarily an eigenvalue of $\operatorname{cov}(X)$, and even if this is the case, its multiplicity may be infinite. It will turn out that in order to establish convergence rates for the bounded LIL, one has to partition the class of random variables with infinite strong second moments into two subclasses:

The first class contains all random variables, whose covariance operators satisfy certain regularity assumptions. In particular, all random variables with *compact* covariance operators belong to this class. The second class contains among others all random variables whose covariance operators are multiples of the unit operator.

To be more specific, the regularity conditions determining the first class of random variables are as follows,

(1.8) σ^2 is an eigenvalue of $\operatorname{cov}(X)$,

(1.9) the corresponding eigenspace V of $\operatorname{cov}(X)$ is finite-dimensional,

and, for some $\rho < 1$,

$$\sup\{E[(X, y)^2] : y \in V^{\perp}, \|y\| \leqq 1\} \leqq \rho \sigma^2. \tag{1.10}$$

The second class consists of all random variables, where either of the conditions (1.8)–(1.10) fails; we call this the non-regular case.

Einmahl (1991) has recently obtained lower and upper class type results for random variables satisfying (1.8)–(1.10), from which one can easily get convergence rates in the first case (see Theorem A below). The main purpose of the present paper is to establish exact convergence rates in the non-regular case.

Theorem 1 *Let X be a mean zero random variable satisfying conditions* (1.1), (1.2), *and* $E[\|X\|^2]=\infty$. *If* $\varepsilon_n=a\beta_n L(L_2 n/\beta_n)/L_2 n$ *with* $a>1$, *where* $\beta_n:=E[\|X\|^2 1\{\|X\|\leqq\sqrt{n/(L_2 n)^5}\}]/\sigma^2$, *then*

$$P\{\|S_n\|\leqq\sigma((2+\varepsilon_n)nL_2 n)^{1/2} \textit{ eventually}\}=1.$$

Note that we have on account of (1.2),

$$\beta_n=o(L_2 n) \quad \text{as } n\to\infty. \tag{1.11}$$

In fact, one can even infer a somewhat stronger property from (1.2), namely,

$$\sum_{n=2}^{\infty}(\beta_n-\beta_{n-1})/L_2 n<\infty; \tag{1.12}$$

this implies (1.11) via the Kronecker lemma. It is now obvious that $\{\varepsilon_n\}$ is a null-sequence, but, since $E[\|X\|^2]=\infty$, we also have,

$$L_3 n/L_2 n=o(\varepsilon_n) \quad \text{as } n\to\infty. \tag{1.13}$$

This means that one cannot obtain the Gaussian convergence rate $O(L_3 n/L_2 n)$ from Theorem 1.

In order to formulate the corresponding results for random variables satisfying (1.8)–(1.10), we have to consider a sequence $\{\Sigma_n\}$ of positive semidefinite self-adjoint compact operators defined by

$$(x,\Sigma_n y)=E[(X'_n,x)(X'_n,y)], \qquad x,y\in H, \tag{1.14}$$

where $X'_n:=X_n 1\{\|X_n\|\leqq\sqrt{n/(L_2 n)^5}\}$, $n\geqq 1$.

Let $\sigma^2_{1,n}\geqq\sigma^2_{2,n}\geqq\dots$ be the positive eigenvalues of Σ_n arranged in a non-increasing order and taking into account the multiplicities. If there are only finitely many positive eigenvalues, set $\sigma^2_{i,n}=0$ for large i. Put

$$\gamma'_n:=\sum_{i=d+1}^{\infty}\log(\sigma^2/(\sigma^2-\sigma^2_{i,n})),$$

where $d:=\dim(V)$, which is finite on account of (1.9). Note that by (1.10),

$$\gamma'_n\leqq\left(\sum_{i=d+1}^{\infty}\sigma^2_{i,n}\right)\Big/\sigma^2(1-\rho)\leqq\operatorname{trace}(\Sigma_n)/\sigma^2(1-\rho)=\beta_n/(1-\rho). \tag{1.15}$$

Applying Theorem 3, Einmahl (1991), we then obtain:

Theorem A *Let X be a mean zero random variable satisfying conditions* (1.1), (1.2), (1.8)–(1.10).

i) *If* $\varepsilon'_n = a((d+2) L_3 n + \gamma'_n)/L_2 n$ *and* $a > 1$, *then*

$$P\{\|S_n\| \leqq \sigma((2+\varepsilon'_n) n L_2 n)^{1/2} \text{ eventually}\} = 1.$$

ii) *Under the additional assumption*

(1.16) $$E[(X, y)^2 1\{|(X, y)| \geqq t\}] = O((L_2 t)^{-1}) \quad \text{as } t \to \infty, y \in V$$

we have if $\varepsilon'_n = a((d+2) L_3 n + \gamma'_n)/L_2 n$ *and* $a \leqq 1$,

$$P\{\|S_n\| \leqq \sigma((2+\varepsilon'_n) n L_2 n)^{1/2} \text{ eventually}\} = 0.$$

Recalling (1.13) and (1.15), we see that the convergence rate in the "regular" case is of order $O((\beta_n \vee L_3 n)/L_2 n)$, which is better than the rate in Theorem 1. Our next result, however, shows that the latter rate is sharp if either of the conditions (1.8)–(1.10) fails.

Theorem 2 *Let* $\tilde{\beta}_n \uparrow \infty$ *be a sequence of positive real numbers satisfying* (1.12). *If the underlying p-space* $(\Omega, \mathscr{F}, P)$ *is non-atomic, one can construct random variables* $X: \Omega \to H$ *satisfying* (1.1), (1.2),

(1.17) $$E[\|X\|^2 1\{\|X\| \leqq \sqrt{n}/(L_2 n)^5\}] \leqq \tilde{\beta}_n \sigma^2,$$

and one of the following three conditions,

(1.18a) σ^2 *is not an eigenvalue of* $\operatorname{cov}(X)$,

(1.18b) $\operatorname{cov}(X) = \sigma^2 I$, *where* I *is the unit operator,*

(1.18c) $\begin{cases} \sigma^2 \text{ is an eigenvalue of } \operatorname{cov} X; \\ \text{the corresponding eigenspace } V \text{ of } \operatorname{cov}(X) \text{ is finite-dimensional}; \\ \text{and } \sup\{E[(X, y)^2]: \|y\| \leqq 1, y \in V^{\perp}\} = \sigma^2, \end{cases}$

such that, if $\varepsilon_n = a \tilde{\beta}_n L(L_2 n/\tilde{\beta}_n)/L_2 n$ *and* $a < 1$ *we have:*

(1.19) $$P\{\|S_n\| \leqq \sigma((2+\varepsilon_n) n L_2 n)^{1/2} \text{ eventually}\} = 0.$$

It seems worthwhile to mention that in one case our Theorem 1 also provides exact convergence rates for the clustering in the LIL. This problem has been extensively studied in the literature (see Bolthausen (1978); Grill (1987); Goodman and Kuelbs (1991a, 1991b)). To formulate our result, let us recall that one has under assumptions (1.1) and (1.2) with probability one,

(1.20) $$C(\{S_n/(2n L_2 n)^{1/2}\}) = K,$$

where $K := \{\operatorname{cov}(X)^{1/2} \cdot y: \|y\| \leqq 1\}$, and $C(\{z_n\})$ denotes the set of limit points of the sequence $\{z_n\} \subset H$ (see Corollary 4 and Lemma 6.1, De Acosta and Kuelbs (1983)).

Similarly as in (1.5), one can infer from (1.20) that there exists a null-sequence $\delta_n \downarrow 0$ such that

(1.21) $$P\{S_n/(2n L_2 n)^{1/2} \in K^{\delta_n} \text{ eventually}\} = 1,$$

where $K^{\delta} := \{y \in H: \inf_{x \in K} \|x - y\| \leqq \delta\}$, $\delta > 0$.

If we now consider mean zero random variables X: $\Omega \to H$ such that $\operatorname{cov}(X) = I$, it follows from Theorem 1 that (1.21) holds true if

$$\delta_n = a\beta_n L(L_2 n/\beta_n)/L_2 n, \qquad a > 1/4. \tag{1.22}$$

Moreover, it follows from Theorem 2 that there exist mean zero random variables with $\operatorname{cov}(X) = I$, where the probability in (1.21) is equal to zero if $a < 1/4$.

Goodman and Kuelbs (1991 b) have shown that the convergence rate for the Brownian motion (considered as a Gaussian random variable in the Hilbert space $\mathscr{L}_2[0, 1]$) is of order $O((L_3 n)^{1/3}/(L_2 n)^{2/3})$. It is also known that in this case the rate in (1.21) cannot be better than $O((L_2 n)^{-2/3})$ (see Goodman and Kuelbs (1991 a)). Comparing these results with (1.22), it is easy to see that the above rates will be better if we assume,

$$E[\|X\|^2/(L_2 \|X\|)^\alpha] < \infty \qquad \text{for some } \alpha < 1/3. \tag{1.23}$$

This might be somewhat surprising since we are now dealing with random variables having infinite second strong moments. But this only demonstrates that, unlike the convergence rates for the bounded LIL, the rates for the clustering are highly dependent on the geometric structure of the cluster set K.

Returning to the bounded LIL, one can also infer from (1.3) that for some null-sequence $\tilde{\varepsilon}_n \downarrow 0$,

$$P\{\|S_n\| > \sigma((2 - \tilde{\varepsilon}_n) n L_2 n)^{1/2} \text{ infinitely often}\} = 1. \tag{1.24}$$

If X satisfies conditions (1.1), (1.2), (1.8)–(1.10) and (1.16), one can choose $\tilde{\varepsilon}_n = 0$, and, in fact, a much more precise result can be obtained from Theorem A(ii). But this is no longer true if condition (1.16) fails. In the classical setting of real-valued random variables, the following somewhat surprising result can be inferred from the work of Feller (1946).

Theorem B *Let $\tilde{\varepsilon}_n \downarrow 0$ be a null-sequence. One can find a random variable $X: \Omega \to \mathbb{R}$ with mean zero and variance one such that*

$$P\{|S_n| \leqq ((2 - \tilde{\varepsilon}_n) n L_2 n)^{1/2} \textit{ eventually}\} = 1. \tag{1.25}$$

This means that sums of iid mean zero random variables with variance one can be much *smaller* than sums of independent standard normal random variables. Note also that (1.25) implies for any $\delta > 0$,

$$E[|X|^{2+\delta}] = \infty. \tag{1.26}$$

To see (1.26), observe that if $E[|X|^{2+\delta}]$ were finite for some $\delta > 0$, the Kolmogorov-Erdös-Petrowski integral test would apply, and the probability in (1.25) would be zero.

Our last theorem is to show that this phenomenon occurs in Hilbert space even in a more extreme form.

Theorem 3 *Let $\tilde{\varepsilon}_n \downarrow 0$ be a null-sequence, and let g: $[0, \infty) \to (0, \infty)$ be a non-decreasing function such that*

$$g(t)/L_2 t \to 0 \qquad \textit{as } t \to \infty. \tag{1.27}$$

If the underlying p-space $(\Omega, \mathscr{F}, P)$ is non-atomic, one can find a mean zero random variable $X: \Omega \to H$ satisfying

(1.28) $$E[\|X\|^2/g(\|X\|)] = \infty$$

and

(1.29) $$\operatorname{cov}(X) = I,$$

such that

(1.30) $$P\{\|S_n\| \leqq ((2-\tilde{\varepsilon}_n)\, n L_2\, n)^{1/2} \textit{ eventually}\} = 1.$$

The proofs of Theorems 1, 2 and 3 will be presented in Sects. 2, 3, and 4, respectively.

2 Proof of Theorem 1

The proof is based on two auxiliary results which might be of independent interest.

Lemma 1 *Let $\eta_1, \ldots, \eta_d$ be independent standard normal random variables, where $d \geqq 3$. Then we have for $t \geqq 2d$,*

$$P\{\eta_1^2 + \ldots + \eta_d^2 \geqq t\} \leqq A_1\, d^{-1/2} \exp\left(-\frac{t}{2} + \left(\frac{d}{2} - 1\right) \log\left(\frac{et}{d-2}\right)\right),$$

where $A_1 > 0$ is a universal constant.

Proof. Since $\eta_1^2 + \ldots \eta_d^2$ has a gamma distribution, an elementary argument based on integration by parts yields,

$$P\{\eta_1^2 + \ldots + \eta_d^2 \geqq t\} = \left(\sum_{i=0}^{j-1} y^{\alpha-1-i}\Gamma(\alpha-i)^{-1}\right) \exp(-y)$$
$$+ \Gamma(\alpha-j)^{-1} \int_y^\infty u^{\alpha-j-1} \exp(-u)\, du,$$

where $\alpha = d/2$, $y = t/2$, and $j = \alpha - 1$ or $= \alpha - 1/2$ according as d is even or odd. Since in both cases $\alpha - j - 1 \leqq 0$, we have:

$$\int_y^\infty u^{\alpha-j-1} \exp(-u)\, du \leqq y^{\alpha-j-1} \exp(-y).$$

Combining this with the inequality $\Gamma(\alpha)/\Gamma(\alpha-i) \leqq \alpha^i$, $0 \leqq i \leqq j$, we find that the above probability is less than or equal to

$$\Gamma(\alpha)^{-1}\, y^{\alpha-1} \exp(-y) \sum_{i=0}^{j} (\alpha/y)^i \leqq 2\Gamma(\alpha)^{-1}\, y^{\alpha-1} \exp(-y) \qquad \text{if } y \geqq 2\alpha.$$

Using similar arguments as in the proof for Stirling's formula (see, for instance, Feller (1957), p. 51), one easily finds that

$$\Gamma(\alpha) \geqq c \cdot ((\alpha-1)/e)^{\alpha-1/2} \qquad \text{for } \alpha \geqq 3/2,$$

where $c>0$ is a universal constant.

Combining the last two inequalities, we obtain the assertion of the lemma. □

Using Lemma 1, Einmahl (1991), we can infer from the above lemma:

Lemma 2 *Let Y be a Gaussian mean zero random variable in Hilbert space. Let $\sigma_1^2 \geqq \sigma_2^2 \geqq \dots$ be the eigenvalues of* $\operatorname{cov}(Y)$. *Then we have for $b \geqq \sigma_1^2$, $t \geqq \sqrt{2d}$ and $d \geqq 3$:*

$$P\{\|Y\| \geqq t\} \leqq A_1 d^{-1/2} \exp\left(-\frac{t^2}{2b} + \left(\frac{d}{2}-1\right) \log\left(\frac{et^2}{b(d-2)}\right) + \alpha_Y\right),$$

where $\alpha_Y := \frac{1}{2} \sum_{j=d+1}^{\infty} \log(b/(b-\sigma_j^2))$.

We now have all tools which are required for the proof of Theorem 1. W.l.o.g. we assume $\sigma^2=1$. Further let for any $a>1$, ψ_a be an increasing function such that

$$\psi_a(n) = (2L_2 n + a\beta_n L(L_2 n/\beta_n))^{1/2}. \tag{2.1}$$

By (1.11) we have for large enough n,

$$(2L_2 n)^{1/2} \leqq \psi_a(n) \leqq 2(L_2 n)^{1/2}. \tag{2.2}$$

Using the same arguments as in the proof of Theorem 3(A), Einmahl (1991), Theorem 1 reduces to verifying that for any $K>0$,

$$\sum_{j=1}^{\infty} P\{\|Y_{n_{j+1}}\| \geqq \psi_a(n_j) - K/\psi_a(n_j)\} < \infty, \tag{2.3}$$

where $\{Y_n\}$ is a sequence of Gaussian mean zero random variables with $\operatorname{cov}(Y_n) = \Sigma_n$, which is defined by (1.14), and $n_j := [\exp(j/Lj)]$, $j \geqq 1$.

Set d_n: $\max\{k: \sigma_{k,n}^2 > 1-\delta\} \vee 3$, where $\delta := (a-1)/6 \wedge 1/2$, and $\{\sigma_{k,n}^2 : k \geqq 1\}$ is the eigenvalue sequence of the operator Σ_n. Applying Lemma 2 with $b=1$ and $d = d_{n_{j+1}}$, we obtain for $t \geqq \sqrt{2d_{n_{j+1}}}$:

$$P\{\|Y_{n_{j+1}}\| \geqq t\} \leqq A_1 \exp\left(-\frac{t^2}{2} + \alpha_j(t)\right), \tag{2.4}$$

where $\alpha_j(t) := (\frac{1}{2} d_{n_{j+1}} - 1) \quad L(et^2/(d_{n_{j+1}} - 2)) + \tilde{\alpha}_j$, and $\tilde{\alpha}_j := \frac{1}{2} \sum_{i=d_{n_{j+1}}+1}^{\infty} \log(1/(1-\sigma_{i,n_{j+1}}^2))$. By the definition of d_n we have,

$$\tilde{\alpha}_j \leqq \frac{1}{2\delta} \sum_{i=1}^{\infty} \sigma_{i,n_{j+1}}^2 = \frac{1}{2\delta} \beta_{n_{j+1}}. \tag{2.5}$$

Also note that

$$\beta_{n_{j+1}} \geqq \sum_{i=1}^{d_{n_{j+1}}} \sigma_{i,n_{j+1}}^2 \geqq (1-\delta)\, d_{n_{j+1}}.$$

Recalling that $\delta < 1/2$, we find that

$$d_{n_{j+1}} \leqq (1+2\delta)\, \beta_{n_{j+1}}. \tag{2.6}$$

Combining (2.6) and (1.11), we see that $d_{n_{j+1}}/L_2\, n_{j+1} \to 0$ as $j \to \infty$. We now can infer from (2.5) and (2.6) for $t \leqq 2(L_2\, n_j)^{1/2}$ and large j,

$$2\alpha_j(t) \leqq (1+3\delta)\, \beta_{n_{j+1}}\, L(L_2\, n_{j+1}/\beta_{n_{j+1}}). \tag{2.7}$$

Setting $\mathbb{N}_1 := \{j: \beta_{n_{j+1}} \leqq (1+\delta/3)\, \beta_{n_j}\}$, we obtain for $t \leqq 2(L_2\, n_j)^{1/2}$ and large $j \in \mathbb{N}_1$,

$$2\alpha_j(t) \leqq (1+4\delta)\, \beta_{n_j}\, L(L_2\, n_j/\beta_{n_j}). \tag{2.8}$$

Because of (2.6) and (1.11), we can now infer from (2.4) that for large enough $j \in \mathbb{N}_1$,

$$P\{\|Y_{n_{j+1}}\| \geqq \psi_a(n_j) - K/\psi_a(n_j)\} \leqq A_1\, e^K \exp(-L_2\, n_j - \delta\, \beta_{n_j}\, L(L_2\, n_j/\beta_{n_j})).$$

Since we assume that $E[\|X\|^2] = \infty$, it follows that $\beta_n \uparrow \infty$. We thus have for large j,

$$\beta_{n_j}\, L(L_2\, n_j/\beta_{n_j}) \geqq 3\delta^{-1} L_3\, n_j,$$

whence the last term is

$$O((Ln_j)^{-1} (L_2\, n_j)^{-3}).$$

Recalling the definition of the sequence $\{n_j\}$, we readily find that

$$\sum_{j \in \mathbb{N}_1} P\{\|Y_{n_{j+1}}\| \geqq \psi_a(n_j) - K/\psi_a(n_j)\} < \infty.$$

Thus, it remains to show that

$$\sum_{j \in \mathbb{N}_2} P\{\|Y_{n_{j+1}}\| \geqq \psi_a(n_j) - K/\psi_a(n_j)\} < \infty, \tag{2.9}$$

where

$$\mathbb{N}_2 := \{j: \beta_{n_{j+1}} > (1+\delta/3)\, \beta_{n_j}\}.$$

Assumption (1.2) implies that

(2.10) $$\sum_{j=1}^{\infty} (\beta_{n_{j+1}} - \beta_{n_j})/L_2 n_j < \infty.$$

Since we have for $j \in \mathbb{N}_2$,

$$\beta_{n_{j+1}} - \beta_{n_j} > (1-(1+\delta/3)^{-1})\, \beta_{n_{j+1}} > \delta\, \beta_{n_{j+1}}/4,$$

we can infer from (2.10),

(2.11) $$\sum_{j \in \mathbb{N}_2} \beta_{n_{j+1}}/L_2 n_j < \infty.$$

Using Markov's inequality and recalling (2.2), we find that for large enough j,

$$P\{\| Y_{n_{j+1}} \| \geqq \psi_a(n_j) - K/\psi_a(n_j)\} \leqq E[\| Y_{n_{j+1}} \|^2]/L_2 n_j = \beta_{n_{j+1}}/L_2 n_j.$$

Combining this inequality with (2.11), we obtain (2.9), thereby completing the proof of Theorem 1. □

3 Proof of Theorem 2

The proof is based on the following three lemmas.

Lemma 3 *Let $\eta_1, \ldots, \eta_d$ be independent standard normal random variables, where $d \geqq 3$. Then we have for $t \geqq 1$,*

$$P\{\eta_1^2 + \ldots + \eta_d^2 \geqq t\} \geqq A_2\, d^{-1/2} \exp\left(-\frac{t}{2} + \left(\frac{d}{2} - 1\right) \log\left(\frac{et}{d-2}\right)\right),$$

where A_2 is a universal constant.

The proof of Lemma 3 is very similar to that of Lemma 1, and it is therefore omitted. Using Lemma 3, we obtain the following result for Gaussian random variables.

Lemma 4 *Let Y be a Gaussian mean zero random variable in Hilbert space. Let $\sigma_1^2 \geqq \sigma_2^2 \geqq \ldots$ be the eigenvalues of* cov(Y). *Then we have for $0 < b \leqq \sigma_d^2$, $t > b^{1/2}$, and $d \geqq 3$,*

$$P\{\| Y \| \geqq t\} \geqq A_2\, d^{-1/2} \exp\left(-\frac{t^2}{2b} + \left(\frac{d}{2} - 1\right) \log\left(\frac{et^2}{b(d-2)}\right)\right).$$

Proof. First recall that

$$\| Y \|^2 \overset{\mathcal{D}}{=} \sum_{j=1}^{\infty} \sigma_j^2 \eta_j^2,$$

where η_j, $j \geqq 1$ are independent standard normal variables. This enables us to conclude that

$$P\{\|Y\| \geqq t\} = P\left\{\sum_{j=1}^{\infty} \sigma_j^2 \eta_j^2 \geqq t^2\right\}$$
$$\geqq P\left\{\sum_{j=1}^{d} \eta_j^2 \geqq t^2/b\right\}.$$

Applying Lemma 3, we obtain the assertion. □

We finally need,

Lemma 5 *Let Σ: $H \to H$ be a positive semidefinite, selfadjoint compact operator with eigenvalues $\sigma_1^2 \geqq \sigma_2^2 \geqq \ldots$. Let V be a d-dimensional subspace of H such that*

$$(y, \Sigma y) \geqq b \|y\|^2, \qquad y \in V.$$

Then we have: $\sigma_d^2 \geqq b$.

Proof. Let $\{e_i\}$ be a sequence of orthonormal eigenvectors corresponding to the eigenvalues $\{\sigma_i^2\}$. Set $\tilde{V} := \operatorname{span}\{e_1, \ldots, e_{d-1}\}$, and let π be the orthogonal projection from V onto $\tilde{V}$. Using a well-known result from linear algebra, we find that

$$\dim \pi^{-1}\{0\} \geqq \dim(V) - \dim(\tilde{V}) = 1.$$

This implies that there exists a vector $y \in V \cap \tilde{V}^{\perp} = \pi^{-1}\{0\}$ with $\|y\| = 1$. Since $y \in \tilde{V}^{\perp}$, we have,

(3.1) $$(y, \Sigma y) \leqq \sigma_d^2.$$

On the other hand, since $y \in V$, we also have:

(3.2) $$(y, \Sigma y) \geqq b.$$

Combining (3.1) and (3.2), we obtain the assertion of Lemma 5. □

We now start the proof of Theorem 2. Set $\delta_k := 1/2 L_2 n$, $\tilde{\beta}_{n-1}^{1/2} < k \leqq \tilde{\beta}_n^{1/2}$, and $a_k := \sqrt{n}/(L_2 n)^5 =: c_n$, $\tilde{\beta}_{n-1} < k \leqq \tilde{\beta}_n$, $n \geqq 1$, where $\tilde{\beta}_0 := 0$. Let $\{\delta_k'\}$ be a sequence satisfying

(3.3) $$0 \leqq \delta_k' \leqq \delta_k, \qquad k \geqq 1.$$

Since the p-space $(\Omega, \mathscr{F}, P)$ is non-atomic, there exists a sequence of independent random variables $\{\xi_k\}$ satisfying

(3.4) $$P\{\xi_k = \pm a_k\} = (1 - \delta_k')/2 a_k^2, \qquad P\{\xi_k = 0\} = 1 - (1 - \delta_k')/a_k^2$$

(use Theorem 5.2 and Problem 5.2, Billingsley (1986)). Let $\{e_k\}$ be a complete orthonormal basis of H, and set

(3.5) $$X = \sum_{k=1}^{\infty} \xi_k e_k.$$

Using the independence of the ξ_k's, it is easily checked that

$$E[(X, y)^2] \leqq \|y\|^2, \qquad y \in H \tag{3.6}$$

Moreover, it follows that

$$\begin{aligned} E[\|X\|^2/L_2\|X\|] &\leqq \sum_{k=1}^{\infty} E[\xi_k^2/L_2|\xi_k|] \\ &\leqq \sum_{k=1}^{\infty} (L_2 a_k)^{-1} \\ &\leqq \sum_{n=1}^{\infty} ([\tilde{\beta}_n] - [\tilde{\beta}_{n-1}])/L_2 c_n, \end{aligned}$$

where the last series is finite since $\{\tilde{\beta}_n\}$ satisfies condition (1.12).

We thus see that any random variable as above satisfies conditions (1.1) and (1.2). In order to verify (1.17), note that

$$\begin{aligned} E[\|X\|^2 1\{\|X\| \leqq c_n\}] &= \sum_{j=1}^{\infty} E[(X, e_j)^2 1\{\|X\| \leqq c_n\}] \\ &= \sum_{j=1}^{\infty} E[\xi_j^2 1\{\|X\| \leqq c_n\}] \\ &\leqq \sum_{j=1}^{\infty} E[\xi_j^2 1\{|\xi_j| \leqq c_n\}] \\ &\leqq [\tilde{\beta}_n]. \end{aligned}$$

To see the last inequality, observe that

$$E[\xi_j^2 1\{|\xi_j| \leqq c_n\}] = 0 \qquad \text{if } a_j > c_n.$$

We next show (1.19). For that purpose we need lower bounds for the eigenvalues $\sigma_{k,n}^2$ of the operator Σ_n, which is defined by (1.14).

Using the independence of the ξ_k's, we have for large enough n,

$$\begin{aligned} E[\xi_k^2 1\{\|X\| \leqq c_n\}] &\geqq E\left[\xi_k^2 1\left\{\xi_k^2 \leqq \frac{3}{4} c_n^2, \sum_{j \neq k} \xi_j^2 \leqq \frac{c_n^2}{4}\right\}\right] \\ &\geqq E[\xi_1^2 1\{|\xi_k| \leqq c_{[n/2]}\}] \, P\{\|X\| \leqq c_n/2\}. \\ &\geqq E[\xi_k^2 1\{|\xi_k| \leqq c_{[n/2]}\}] - P\{\|X\| > c_n/2\}. \end{aligned}$$

If we have $k_{n,1} := [\tilde{\beta}_{[n/2]}^{1/2}] \leqq k \leqq [\tilde{\beta}_{[n/2]}] =: k_{n,2}$, it follows from (3.3) for large enough n,

$$E[\xi_k^2 1\{\|X\| \leqq c_n\}] \geqq 1 - (L_2 n)^{-1}. \tag{3.7}$$

Set $V_n := \text{span}\{e_k : k_{n,1} \leqq k \leqq k_{n,2}\}$, and let $y = \sum_{k_{n,1}}^{k_{n,2}} y_k e_k$ be a vector in V_n. Since the random variables ξ_k are symmetric and independent, it follows that the random variables $\xi_k 1\{\|X\| \leqq x_n\}$ are uncorrelated. This enables us to infer from (3.7) that

$$\begin{aligned}(y, \Sigma_n y) &= E[(X, y)^2 1\{\|X\| \leqq c_n\}] \\ &= E\left[\left(\sum_{k_{n,1}}^{k_{n,2}} y_k \xi_k 1\{\|X\| \leqq c_n\}\right)^2\right] \\ &= \sum_{k_{n,1}}^{k_{n,2}} y_k^2 E[\xi_k^2 1\{\|X\| \leqq c_n\}] \\ &\geqq (1 - (L_2 n)^{-1}) \|y\|^2.\end{aligned}$$

Recalling Lemma 5, we find that for large n,

$$\sigma^2_{k_n, n} \geqq 1 - (L_2 n)^{-1}, \tag{3.8}$$

where $k_n := k_{n,2} - k_{n,1} + 1$. To simplify our notation, we set for $a < 1$,

$$\phi_a(n) := (2 L_2 n + a \tilde{\beta}_n L(L_2 n/\tilde{\beta}_n))^{1/2}, \qquad n \geqq 1.$$

Using Lemma 13, Einmahl (1991), and arguing as in Sect. 5, Einmahl (1989b), one can reduce the proof of (1.19) to showing that

$$P\{\|S'_n\|/\sqrt{n} \geqq \phi_a(n) + 1/(L_2 n)^{1/2} \text{ infinitely often}\} = 1, \tag{3.9}$$

where $S'_n := \sum_{j=1}^{n} X_j 1\{\|X_j\| \leqq c_j\}$, $n \geqq 1$.

Let the subsequence $\{m_n\}$ be defined by $m_n := [\exp(3 n L_2 n)]$, $n \geqq 1$. We first show for $0 < a < 1$ and $K > 0$,

$$\sum_{j=1}^{\infty} P\{\|S'_{m_j} - S'_{m_{j-1}}\| \geqq \sqrt{m_j}(\phi_a(m_j) + K(L_2 m_j)^{-1/2})\} = \infty. \tag{3.10}$$

Let $\{Y_n\}$ be a sequence of independent Gaussian mean zero random variables with $\text{cov}(Y_n) = \Sigma_n = \text{cov}(X'_n)$, $n \geqq 1$. Then we can infer from Lemma 5, Einmahl (1991):

$$\begin{aligned}&P\{\|S'_{m_j} - S'_{m_{j-1}}\| \geqq \sqrt{m_j}(\phi_a(m_j) + K(L_2 m_j)^{-1/2})\} \\ &\geqq P\left\{\left\|\sum_{m_{j-1}+1}^{m_j} Y_k\right\| \geqq \sqrt{m_j}(\phi_a(m_j) + 2K(L_2 m_j)^{-1/2})\right\} \\ &\quad - A K^{-3} (L_2 m_j)^{3/2} m_j^{-1/2} E[\|X'_{m_j}\|^3],\end{aligned}$$

where $A > 0$ is a universal constant. We now claim that

$$\sum_{j=1}^{\infty} (L_2 m_j)^{3/2} m_j^{-1/2} E[\|X'_{m_j}\|^3] < \infty. \tag{3.11}$$

Relation (3.11) implies that in order to prove (3.10) it is enough to check that

$$(3.12)\qquad \sum_{j=1}^{\infty} P\left\{\left\|\sum_{m_{j-1}+1}^{m_j} Y_k\right\| \geqq \sqrt{m_j}(\phi_a(m_j)+2K(L_2 m_j)^{-1/2})\right\}=\infty.$$

To see (3.11), set $n_l:=2^l$, $l\geqq 1$, $l_j:=\min\{l: 2^l \geqq m_j\}$, $l\geqq 1$. Then we have $l_1<l_2<l_3<\dots$, and it is clear that

$$\sum_{j=1}^{\infty}(L_2 m_j)^{3/2} m_j^{-1/2} E[\|X'_{m_j}\|^3] \leqq \sqrt{2}\sum_{j=1}^{\infty}(Ll_j)^{3/2} n_{l_j}^{-1/2} E[\|X'_{n_{l_j}}\|^3].$$

Relation (3.11) now follows from Lemma 8, Einmahl (1991).

As to (3.12), observe that $\operatorname{cov}(Y_k)-\operatorname{cov}(Y_{m_{j-1}})$ is positive semidefinite if $k \geqq m_{j-1}$. This enables us to conclude that

$$(3.13)\qquad P\left\{\left\|\sum_{m_{j-1}+1}^{m_j} Y_k\right\| \geqq \sqrt{m_j}(\phi_a(m_j)+2K(L_2 m_j)^{-1/2})\right\}$$
$$\geqq P\{\|Y_{m_{j-1}}\| \geqq (1-m_{j-1}/m_j)^{-1/2}(\phi_a(m_j)+2K(L_2 m_j)^{-1/2})\}$$

Since $m_{j-1}/m_j \leqq (L_2 m_j)^{-1}$ for large j, we find that the last probability is

$$\geqq P\{\|Y_{m_{j-1}}\| \geqq \phi_a(m_j)+K_1(L_2 m_j)^{-1/2}\}$$

for some constant $K_1>2K$. This probability, in turn, is for large enough j,

$$\geqq K_2 d_j^{-1/2} \exp(\tfrac{1}{2}(d_j-2) L(L_2 m_j/(d_j-2))-\phi_a^2(m_j)/2b_j),$$

where $d_j:=k_{m_{j-1}}$, $b_j=1-(Lj)^{-1}$, and K_2 is a positive constant.

Here we have used Lemma 4 in conjunction with (3.8). Setting $\delta:=(1-a)/2$, we obtain after some calculation that the last term is

$$\geqq K_3(Lm_j)^{-1}(L_2 m_j)^{-1/2} \exp(\tfrac{1}{2}(\tilde\beta_{m_{j-2}}-(a+\delta)\tilde\beta_{m_j}) L(L_2 m_j/\tilde\beta_{m_j}))$$

provided j is large enough. To verify the last inequality, notice that $m_{j-2} \leqq [m_{j-1}/2]$, whence we have for large j,

$$\tilde\beta_{m_{j-2}}-\delta\tilde\beta_{m_j} \leqq d_j-2 \leqq \tilde\beta_{m_j}=o(L_2 m_j).$$

Set

$$\mathbb{N}_1:=\{j: \tilde\beta_{m_j}(a+\delta)<\tilde\beta_{m_{j-2}}\}$$
$$\mathbb{N}_2:=\mathbb{N}-\mathbb{N}_1.$$

Using (1.12), we readily obtain,

$$(3.14)\qquad \sum_{j=3}^{\infty}(\tilde\beta_{m_j}-\tilde\beta_{m_{j-2}})/L_2 m_j<\infty.$$

If $j\in\mathbb{N}_2$, we have

$$\tilde\beta_{m_j}-\tilde\beta_{m_{j-2}} \geqq \delta\tilde\beta_{m_j}.$$

Since $\tilde{\beta}_n \uparrow \infty$, we can infer from (3.14),

$$\sum_{j \in \mathbb{N}_2} (L_2 m_j)^{-1} < \infty. \tag{3.15}$$

But from the definition of $\{m_j\}$ it also follows that

$$\sum_{j=1}^{\infty} (L m_j)^{-1} (L_2 m_j)^{-1/2} = \infty. \tag{3.16}$$

This implies in combination with (3.15),

$$\sum_{j \in \mathbb{N}_1} (L m_j)^{-1} (L_2 m_j)^{-1/2} = \infty. \tag{3.17}$$

Since we have according to the previous considerations for large $j \in \mathbb{N}_1$,

$$P\left\{\left\| \sum_{m_{j-1}}^{m_j} Y_k \right\| \geqq \sqrt{m_j}(\phi_a(m_j) + 2K(L_2 m_j)^{-1/2})\right\} \geqq K_3 (L m_j)^{-1} (L_2 m_j)^{-1/2},$$

(3.12), and, consequently, (3.10) follows from (3.17).

In view of Borel-Cantelli, it is now clear that for any $K>0$ with probability one,

$$\|S'_{m_j} - S'_{m_{j-1}}\| \geqq \sqrt{m_j}(\phi_a(m_j) + K(L_2 m_j)^{-1/2}) \text{ i.o.} \tag{3.18}$$

On the other hand, it follows from Lemma 11, Einmahl (1991) that with probability one,

$$\|S_n - S'_n\| = o(n L_2 n) \quad \text{as } n \to \infty. \tag{3.19}$$

This means in view of (1.4) and (3.6), that we have with probability one,

$$\|S'_{m_{j-1}}\| \leqq (3 m_{j-1} L_2 m_j)^{1/2} \text{ eventually.} \tag{3.20}$$

Using the fact that for large enough j,

$$m_{j-1}^{1/2} / m_j^{1/2} \leqq (L_2 m_j)^{-1},$$

we obtain from (3.18) (applied with $K=3$) and (3.20) that with probability one,

$$\begin{aligned} \|S'_{m_j}\| &\geqq \|S'_{m_j} - S'_{m_{j-1}}\| - \|S'_{m_{j-1}}\| \\ &\geqq \sqrt{m_j}(\phi_a(m_j) + (L_2 m_j)^{-1/2}) \text{ i.o.} \end{aligned}$$

This completes the proof of (3.9), thereby establishing (1.19). To finish the proof of Theorem 2, consider the three sequences,

(3.21 a) $\qquad \delta'_n = \delta_n, \quad n \geqq 1$

(3.21 b) $\qquad \delta'_n = 0, \quad n \geqq 1,$

(3.21 c) $\qquad \delta'_1 = 0, \quad \delta'_n = \delta_n, \quad n \geqq 2.$

It is straightforward to verify that $\sigma^2=1$ and that $\operatorname{cov}(X)$ satisfies (1.18a), (1.18b) or (1.18c), according as we define $\{\delta'_n\}$ by (3.21a), (3.21b) or (3.21c). (Note that in the last case, $V=\operatorname{span}\{e_1\}$ is 1-dimensional.) Theorem 2 has been proved. □

4 Proof of Theorem 3

W.l.o.g. we assume that the function g satisfies the condition

$$g(t)/L_2\, t \text{ is decreasing} \tag{4.1}$$

If (4.1) is not satisfied, we can find a non-decreasing function $\tilde{g}(t)\geqq g(t)$ such that $\tilde{g}(t)/L_2\, t$ is decreasing and still $\tilde{g}(t)/L_2\, t\to 0$.

Since $E[\|X\|^2/\tilde{g}(\|X\|)]=\infty$ implies (1.28), it is clear that it is enough to prove Theorem 3 for functions with the property (4.1).

Set $\rho_n:=g(e^e)\,n/g(\exp(\exp(n))$, $n\geqq 1$, $\rho_0:=0$, $a_n:=\exp(\exp(\rho_n^2/(\rho_n-\rho_{n-1})))$, $n\geqq 1$. From the definition of ρ_n it immediately follows that

$$(\rho_n-\rho_{n-1})/\rho_n\leqq 1/n, \qquad n\geqq 1, \tag{4.2}$$

which in turn implies

$$a_n\geqq\exp(\exp(n\,\rho_n))\geqq\exp(\exp(n)), \qquad n\geqq 1. \tag{4.3}$$

Set $\tilde{\rho}_n:=\rho_{[(L_2 n)^{1/2}]}$, $n\geqq 1$, and let $\delta_n\downarrow 0$ be a sequence such that for large enough j, if $n_{j-1}<n\leqq n_j$,

$$\delta_n:=(\tfrac{1}{2}\tilde{\varepsilon}_{n_{j-1}}+\tfrac{3}{2}(\log\tilde{\rho}_{n_j})/\tilde{\rho}_{n_j}+3L_3\, n_{j-1}/L_2\, n_{j-1})\wedge\tfrac{1}{2},$$

where $n_j:=[\exp(j/Lj)]$, $j\geqq 1$. Let $\{\xi_k\}$ be a sequence of independent random variables satisfying

$$P\{\xi_k=\pm a_k\}=\tfrac{1}{2}(1-\delta_{m_k})\,a_k^{-2}, \tag{4.4}$$

$$P\{\xi_k=\pm c_m\}=\tfrac{1}{2}(\delta_{m-1}-\delta_m)\,c_m^{-2}, \qquad m>m_k, \tag{4.5}$$

$$P\{\xi_k=0\}=1-(1-\delta_{m_k})\,a_k^{-2}-\sum_{m=m_k+1}^{\infty}(\delta_{m-1}-\delta_m)\,c_m^{-2}, \tag{4.6}$$

where $c_m:=\sqrt{m}/(L_2\, m)^5$, $m_k:=\min\{m: c_m\geqq a_k\}$, $k\geqq 1$. Let $\{e_k\}$ be a complete orthonormal basis of H, and set,

$$X=\sum_{k=1}^{\infty}\xi_k\, e_k.$$

It is easily checked that

$$E[\xi_k^2]=1, \qquad k\geqq 1. \tag{4.7}$$

Using the fact that the ξ_k's are independent and that $\{e_k\}$ is complete in H, we readily obtain,

(4.8) $$E[(X,y)^2] = \|y\|^2, \quad y \in H.$$

Relation (4.8) immediately implies (1.29). Further note that

$$\begin{aligned} E[\|X\|^2/L_2\|X\|] &\leqq \sum_{k=1}^{\infty} E[\xi_k^2]/L_2 a_k \\ &\leqq \sum_{k=1}^{\infty} \frac{\rho_k - \rho_{k-1}}{\rho_k^2} \\ &\leqq 1 + \int_1^{\infty} x^{-2}\,\mathrm{d}x = 2. \end{aligned}$$

Thus, we have

(4.9) $$E[\|X\|^2/L_2\|X\|] < \infty.$$

Next observe that on account of (4.1) and (4.3),

$$g(e^e)/g(a_k) \geqq \rho_k (L_2 a_k)^{-1} \geqq (\rho_k - \rho_{k-1})/\rho_k, \quad k \geqq 1.$$

Using (4.2) which implies that $\rho_k \leqq 2\rho_{k-1}$, $k \geqq 2$, we get:

(4.10) $$\sum_{k=1}^{\infty} g(a_k)^{-1} = \infty.$$

Consider now the events $B_k := \{\xi_k = a_k\} \cap \bigcap_{j \neq k} \{\xi_j = 0\}$, $k \geqq 1$. Then we have

$$\begin{aligned} E[\|X\|^2/g(\|X\|)] &= \sum_{k=1}^{\infty} E[\xi_k^2/g(\|X\|)] \\ &\geqq \sum_{k=1}^{\infty} E[\xi_k^2 I_{B_k}/g(\|X\|)] \\ &= \sum_{k=1}^{\infty} P(B_k)\, a_k^2/g(a_k) \\ &\geqq \frac{1}{4} \sum_{k=1}^{\infty} \Big(\prod_{j \neq k} P\{\xi_j = 0\}\Big)\, g(a_k)^{-1} \\ &\geqq \frac{1}{4} \prod_{j=1}^{\infty} (1 - a_j^{-2}) \sum_{k=1}^{\infty} g(a_k)^{-1} \\ &\geqq \frac{1}{4} \exp\left(-2 \sum_{j=1}^{\infty} a_j^{-2}\right) \sum_{k=1}^{\infty} g(a_k)^{-1}, \end{aligned}$$

where we use the inequalities

$$P\{\xi_j=0\}\geqq 1-a_j^{-2}\geqq \exp(-2a_j^{-2}), \quad j\geqq 1.$$

It is clear that $\sum_{j=1}^{\infty} a_j^{-2}<\infty$, which in conjunction with the above lower bounds implies (1.28).

It remains to verify (1.30). Let $\phi:[0,\infty)\to(0,\infty)$ be an increasing function such that $\phi(n)=((2-\tilde{\varepsilon}_n)L_2 n)^{1/2}$, $n\geqq 1$.

Recalling (4.8) and (4.9), we can see that, as in the proof of Theorem 1, it is enough to show for any $K>0$,

$$\sum_{j=1}^{\infty} P\{\|Y_{n_{j+1}}\|\geqq\phi(n_j)-K/\phi(n_j)\}<\infty, \tag{4.11}$$

where Y_n, $n\geqq 1$ are Gaussian mean zero random variables with $\mathrm{cov}(Y_n)=\Sigma_n$, $n\geqq 1$, and Σ_n is defined as in (1.14).

We now want to apply Lemma 2, Sect. 2, for which reason we need an upper bound for the largest eigenvalue $\sigma_{1,n}^2$ of Σ_n. Note that if $m_k\leqq n$,

$$\begin{aligned} E[\xi_k^2 1\{\|X\|\leqq c_n\}]&\leqq E[\xi_k^2 1\{|\xi_k|\leqq c_n\}]\\ &=(1-\delta_{m_k})+\sum_{m=m_k+1}^{n}(\delta_{m-1}-\delta_m)\\ &=1-\delta_n. \end{aligned}$$

If $m_k>n$, we have,

$$E[\xi_k^2 1\{\|X\|\leqq c_n\}]=0. \tag{4.12}$$

Using the symmetry and the independence of the ξ_k's, we obtain for $y=\sum_{k=1}^{\infty} y_k e_k\in H$,

$$\begin{aligned} E[(X,y)^2 1\{\|X\|\leqq c_n\}]&=\sum_{k=1}^{\infty} y_k^2 E[\xi_k^2 1\{\|X\|\leqq c_n\}]\\ &\leqq\|y\|^2(1-\delta_n), \end{aligned}$$

whence we have:

$$\sigma_{1,n}^2\leqq 1-\delta_n, \quad n\geqq 1. \tag{4.13}$$

Set $V_n=\mathrm{span}\{e_1,\ldots,e_{k_n}\}$, where $k_n:=\max\{k: m_k\leqq n\}$. Then it easily follows from (4.12) that

$$E[(X,y)^2 1\{\|X\|\leqq c_n\}]=0, \quad y\in V_n^{\perp}. \tag{4.14}$$

Relation (4.14) in turn implies that

$$\sigma_{i,n}^2=0, \quad i>k_n. \tag{4.15}$$

We now need an upper bound for k_n. On account of (4.3) we have,

(4.16) $$k_n \leqq \tilde{k}_n,$$

where $\tilde{k}_n := \max\{k: \tilde{m}_k \leqq n\}$, $n \leqq 1$, and $\tilde{m}_k := \min\{m: c_m \geqq \exp(\exp(k\rho_k))\}$. It is obvious that

(4.17) $$\tilde{k}_n \rho_{\tilde{k}_n} \leqq L_2 c_n \leqq (\tilde{k}_n + 1) \rho_{\tilde{k}_n+1} \leqq (\tilde{k}_n + 1)^2.$$

In particular, $(\tilde{k}_n + 1) \geqq (L_2 c_n)^{1/2}$, which in conjunction with the first inequality in (4.17) and (4.1) implies for large enough n,

(4.18) $$k_n \leqq \tilde{k}_n \leqq 2 L_2 n/\tilde{\rho}_n := d_n.$$

Applying Lemma 2 with $b = 1 - \delta_{n_{j+1}}$, and $d = d_{n_{j+1}}$, we find that for large enough j,

$$\begin{aligned} P\{\|Y_{n_{j+1}}\| \geqq \phi(n_j) - K/\phi(n_j)\} &\leqq K_1 \exp\left(-\frac{1-\tilde{\varepsilon}_{n_j}/2}{1-\delta_{n_{j+1}}} L_2 n_j + L_2 n_{j+1} \log(2e\tilde{\rho}_{n_{j+1}})/\tilde{\rho}_{n_{j+1}}\right) \\ &\leqq K_1 \exp\left(-\left[\frac{1-\tilde{\varepsilon}_{n_j}/2}{1-\delta_{n_{j+1}}} - \frac{3}{2}(\log \tilde{\rho}_{n_{j+1}})/\tilde{\rho}_{n_{j+1}}\right] L_2 n_j\right) \\ &\leqq K_1 (Ln_j)^{-1} \exp\left(-\frac{1}{2}(2\delta_{n_{j+1}} - \tilde{\varepsilon}_{n_j} - 3(\log \tilde{\rho}_{n_{j+1}})/\tilde{\rho}_{n_{j+1}}) L_2 n_j\right), \end{aligned}$$

where $K_1 > 0$ is a constant.

Recalling the definition of δ_n, we readily obtain that the last term is

$$O((Ln_j)^{-1}(L_2 n_j)^{-3}) = O(j^{-1}(Lj)^{-2}).$$

This shows that (4.11) is true, and Theorem 3 has been proved. □

References

Billingsley, P.: Probability and measure. 2nd edn. New York: Wiley 1986

Bolthausen, E.: On the speed of convergence in Strassen's law of the iterated logarithm. Ann. Probab. **6**, 668–672 (1978)

De Acosta, A., Kuelbs, J.: Some results on the cluster set $C(\{S_n/a_n\})$ and the LIL. Ann. Probab. **11**, 102–122 (1983)

Einmahl, U.: A note on the law of the iterated logarithm in Hilbert space. Probab. Theory Relat. Fields **82**, 213–223 (1989a)

Einmahl U.: The Darling-Erdös theorem for sums of i.i.d. random variables. Probab. Theory Relat. Fields **82**, 241–257 (1989b)

Einmahl, U.: On the almost sure behavior of sums of i.i.d. random variables in Hilbert space. Ann. Probab. **19**, 1227–1263 (1991)

Feller, W.: The law of the iterated logarithm for identically distributed random variables. Ann. Math. **47**, 631–638 (1946)

Feller, W.: An introduction to probability theory and its applications, I. 2nd edn. New York: Wiley 1957

Goodman, V., Kuelbs, J.: Rates of clustering in Strassen's LIL for Brownian motion. J. Theor. Probab. **4**, 285–310 (1991a)

Goodman, V., Kuelbs, J.: Rates of clustering for some Gaussian self-similar processes. Probab. Theory Relat. Fields **88**, 47–75 (1991b)

Grill, K.: On the rate of convergence in Strassen's law of the iterated logarithm. Probab. Theory Relat. Fields **74**, 585–589 (1987)

Kuelbs, J.: Sample path behavior for Brownian motion in Banach spaces. Ann. Probab. **3**, 247–261 (1975)