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Probability Theory and Related Fields

Completeness of location families, translated moments, and uniqueness of charges

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Summary. A sufficient condition for statistical completeness of location families generated by a probability density in euclidean space is given. As an application, completeness of families generated by a symmetric stable law is proved. Our criterion, complementing a classical result of Wiener and recent work of Isenbeck and Rüschendorf, is in terms of regularity of the generating density and zerofreeness of its characteristic function. Its proof rests on a local version of the convolution theorem for Fourier transforms of tempered distributions. A more general version of the criterion is applicable to apparently different problems, as is illustrated by giving a simultaneous proof of a theorem on translated moments by P. Hall and a uniqueness result of M. Riesz in potential theory.

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1 Introduction and main result

In view of the importance of the statistical notion of completeness (see e.g. [9] and [10]) of families of probability measures (laws), it is not surprising that some work has been done in order to study this and related concepts for their own sake (recent examples are [1], [7], [13], and [14]), partly with the object of deciding whether given families are complete. The latter seems to be quite difficult in general.

Before supporting this view in the case of location families, let us recall definitions and fix some notation. If f is a probability density with respect to Lebesgue measure λ^d in d-dimensional euclidean space $\mathbf{R}^d$, let $f_\theta(x) := f(\theta - x)$ be the reflected and translated density and call

$$\mathscr{F} := \{f_\theta \lambda^d : \theta \in \mathbf{R}^d\}$$

the location family generated by f. Let $L^1(f_\theta \lambda^d)$ denote, as usual, the space of integrable functions with respect to the parenthetical measure and put

$$L^1(\mathscr{F}) := \bigcap_{\theta \in \mathbf{R}^d} L^1(f_\theta \lambda^d).$$

For $g \in L^1(\mathscr{F})$ write

$$h(\theta) := \int g(x) f_\theta(x) \lambda^d(\mathrm{d}x) \tag{1}$$
$$= \int g(x) f(\theta - x) \, \mathrm{d}x.$$

Then completeness of $\mathscr{F}$ means:

$$g \in L^1(\mathscr{F}), \quad h(x) = 0 \quad (x \in \mathbf{R}^d) \Rightarrow g(x) = 0 \quad [\lambda^d]. \tag{2}$$

$[\lambda^d]$ indicates, as usual, that a statement holds outside some null set for λ^d.

Concerning the special case of location families on the line, completeness of $\mathscr{F}$ is known from the literature in the following cases via the arguments indicated below.

(i) $f\lambda^1 = N(0, 1)$, the standard normal law,
(ii) $f(x) = \Gamma(\alpha)^{-1} \exp(\alpha x - e^x)$, i.e. $f\lambda^1$ is the law of $\log Y$ where Y is a gamma distributed random variable with fixed shape parameter $\alpha > 0$ and scale parameter 1,
(iii) $f(x) = \frac{1}{\pi}(1 + x^2)^{-1}$, the Cauchy density,
(iv) $f(x) = \frac{1}{2} e^{-|x|}$,
(v) $f\lambda^1 = \chi_n^2(\delta^2)$ with fixed positive integer n and $\delta^2 \geqq 0$,
(vi) $f(x) = \frac{2}{\pi} e^x (1 + e^{2x})^{-1}$.

In cases (i) and (ii), $\mathscr{F}$ is an exponential family of full rank so completeness of $\mathscr{F}$ follows from the completeness theorem for exponential families [9, p. 46]. This method of proving completeness is not as general as it appears, since, by a theorem of Dynkin and Ferguson [9, p. 35], every exponential location family is, apart from a scale parameter, given by (i) or (ii).

In case (iii), completeness of $\mathscr{F}$ was proved by Pollard [15] by exhibiting a rather complicated inversion formula for the so-called Poisson transformation, which transforms any $g \in L^1(\mathscr{F})$ into the function h defined by (1). The existence of any inversion formula implies in particular injectivity of the transform which is equivalent to statistical completeness. Actually Pollard treated a more general problem, with $L^1(\mathscr{F})$ replaced by some larger space.

Case (iv) is handled by Oosterhoff and Schriever [14] in a special, elementary way.

Case (v), also due to Oosterhoff and Schriever [14], is based on a theorem producing new complete families out of given ones.

The final case (vi) is implicit in [11, p. 314], where completeness of the scale family generated by the absolute value of a Cauchy distributed random variable with zero median is proved, which yields, by taking logarithms, the complete location family with f as above.

Apart from the trivial possibility of introducing a scale parameter in each case, there do not seem to be any (or at least not many) further examples of complete location families in the literature. This small number of complete location families and the variety of methods used for obtaining them naturally results in the demand of a general completeness criterion for location families.

Heuristically, it is clear what such a general criterion should look like (compare [5, p. 1673]): Since the rightmost member in (1) is an integral of convolution type, completeness of $\mathscr{F}$ may be written as

$$h = g * f = 0 \;\Rightarrow\; g = 0, \tag{3}$$

which, by taking Fourier transforms, should be equivalent to

$$\hat{h} = \hat{g}\hat{f} = 0 \;\Rightarrow\; \hat{g} = 0, \tag{4}$$

which in turn should be the case iff

$$\hat{f} \neq 0. \tag{5}$$

The problem with these heuristics is to assign a meaning to $\hat{g}$ and to explain what (5) precisely means.

The present paper gives a solution of this problem by imposing suitable restrictions on f (and thus indirectly, via the definition of $L^1(\mathscr{F})$, also on g). This enables us to prove the following completeness criterion.

Theorem 1.1 *Assume that the generating density f of the location family $\mathscr{F}$ satisfies the following conditions.*

(i) $f(x+y) \leqq C(1+|x|^2)^k f(y)$ *for some finite C and positiv integer k and all x, $y \in \mathbf{R}^d$,*

(ii) $\hat{f}$, *the characteristic function of f, is infinitely often differentiable in $\mathbf{R}^d \setminus A$ for some finite set A.*

Then $\mathscr{F}$ is complete iff $\hat{f}$ has no zeros.

Note that, for $\mathscr{F}$ to be complete, necessity of the zerofreeness of $\hat{f}$ is trivial (and well known), regardless of whether (i) and (ii) are assumed or not: If t is a zero of $\hat{f}$, take $g(x) = \cos tx$, the real part of $\exp(-itx)$, to show that (2) is not valid. It is however not clear, to what extent the conditions (i) and (ii) may be relaxed without affecting the sufficiency. See Sect. 4 below.

The following simple proposition compares condition (i) above with two more easily checked conditions.

Proposition 1.2 *Condition* (i) *in Theorem 1.1 is implied by*

(i′) $0 < c \leqq f(x)(1+|x|)^{\alpha} \leqq C < \infty$, $x \in \mathbf{R}^d$, *for some real c, C and $\alpha > 0$,*

and implies

(i″) $c(1+|x|^2)^{-k} \leqq f(x) \leqq C(1+|x|^2)^k$, $x \in \mathbf{R}^d$, *for some positive c and finite C and the same k as in* (i).

It turns out that our approach to Theorem 1.1 leads in fact to the considerably stronger Theorem 1.3 below. We need some more notation in order to state it.

By a complex measure G on $\mathbf{R}^d$ we mean a complex valued countably additive set function defined on the class of bounded Borel sets. We denote integration with respect to G and its total variation measure by $\mathrm{d}G(x)$ and $|\mathrm{d}G(x)|$, respectively. If f is any complex or extended real valued Borel function, then $M(\mathscr{F})$ denotes the set of those complex measures G for which

$$\int |f(x-y)|\,|\mathrm{d}G(y)| < \infty \quad [\lambda^d] \tag{6}$$

holds. Clearly, the space $L^1(\mathscr{F})$, defined above in the special case where f is a probability density, may be identified with a subset of the set of nonnegative and absolutely continuous elements of $M(\mathscr{F})$.

Theorem 1.3 *Let f be any complex or extended real Borel function on $\mathbf{R}^d$. Assume that f is locally integrable and satisfies the following conditions.*

(i) *The inequality*

$$\sum_{i=1}^{m} |f(x-b_i)| \geqq \frac{c}{(1+|x|^2)^k} \tag{7}$$

and the implication

$$|x-y| \geqq R \;\Rightarrow\; |f(x-y)| \leqq C(1+|x|^2)^k \sum_{i=1}^{m} |f(-y-b_i)| \tag{8}$$

hold for some positive integer m, $b_1, \ldots, b_m \in \mathbf{R}^d$, $c>0$, and R, C, $k<\infty$.

(ii) *$\hat{f}$, the Fourier transform of f in the sense of tempered distributions, is outside some finite set A given by an infinitely often differentiable function.*

Then the implication

$$G \in M(\mathscr{F}), \quad \int f(x-y)\,\mathrm{d}G(y) = 0 \quad [\lambda^d] \;\Rightarrow\; G=0 \tag{9}$$

is true if either f is integrable and $\hat{f}$ is zerofree in R^d or f is not integrable and $\hat{f}$ is zerofree in $\mathbf{R}^d \setminus A$.

Theorem 1.3 contains Theorem 1.1 as a special case, since the usual characteristic function of a probability density f is also the Fourier transform of f in the sense of tempered distributions.

The following is an analogue of Proposition 1.2.

Proposition 1.4 *Condition* (i) *in Theorem 1.3 is fulfilled if the implication*

$$|x| \geqq R \;\Rightarrow\; 0 < c \leqq f(x)(1+|x|)^{\alpha} \leqq C < \infty$$

holds for some real R, c, C, and α.

The rest of this paper is organized as follows. Section 2 gives proofs of Theorem 1.3 and Propositions 1.2 and 1.4. Section 3 first gives examples of complete location families, including those generated by a symmetric stable law, by applying Theorem 1.1. Theorem 1.3 is illustrated by giving a simultaneous proof of a theorem on translated moments by P. Hall and a uniqueness result of M. Riesz in potential theory. In Sect. 4, we compare Theorem 1.1 with a classical theorem of Wiener and recent work of Isenbeck and Rüschendorf. The possbility of omitting the conditions (i) and (ii) in Theorem 1.1 is also discussed in that context.

2 Proofs

This section assumes some familiarity with the theory of Fourier transforms of tempered distributions as treated in [17]. In particular, we assume as known the spaces $\mathscr{D}$, $\mathscr{D}'$, $\mathscr{S}$, $\mathscr{S}'$, $\mathscr{O}_M$ and $\mathscr{O}'_C$.

In the following, the term convolution and the symbol $*$ will be reserved for the convolution of two tempered distributions of which at least one belongs to the space $\mathscr{O}'_C$ (see [17, p. 247]).

Returning to the heuristic equivalence of (3), (4) and (5), the first idea is to make sure that g is a tempered distribution so that its Fourier tranform $\hat{g}$ exists, again as a tempered distribution. Remembering that products and convolutions of tempered distributions are not defined in general, our second idea is to consider the function h defined by (1) as a limit of convolutions of suitable tempered distributions, in order to conclude that $\hat{h}=\hat{g}\hat{f}$ holds locally wherever the product on the right is defined. This latter idea is made precise by the following lemma, which may be "well known", but for which I do not know any reference.

Lemma 2.1 *Let f, g and h be tempered distributions. Assume the existence of a sequence (g_n) with*

(i) $g_n \in \mathscr{O}'_C$,
(ii) $g_n \to g$ *in* $\mathscr{S}'$, *and*
(iii) $g_n * f \to h$ *in* $\mathscr{S}'$.

Then, for every open set $U \subset \mathbf{R}^d$ such that $\hat{f}$ is infinitely often differentiable in U,

$$\hat{h}|_U = \hat{g}|_U \cdot \hat{f}|_U, \tag{10}$$

where $|_U$ denotes restriction to U and equality is understood in the sense of distributions in $\mathscr{D}'(U)$.

Proof. Let α be any function with $\hat{\alpha} \in \mathscr{D}(U)$. Then $\alpha \in \mathscr{S} \subset \mathscr{O}'_C$ and $\hat{\alpha} \in \mathscr{O}_M$. It follows (see the argument given below) that

$$\begin{aligned}
\hat{h}\hat{\alpha} &= \widehat{h*\alpha} \\
&= ((\lim_{n\to\infty} g_n * f) * \alpha)\,\hat{} \\
&= (\lim_{n\to\infty} ((g_n * f) * \alpha))\,\hat{} \\
&= (\lim_{n\to\infty} (g_n * (f * \alpha)))\,\hat{} \\
&= (g * (f * \alpha))\,\hat{} \\
&= \hat{g}(\widehat{f*\alpha}) \\
&= \hat{g}(\hat{f}\hat{\alpha}).
\end{aligned}$$

In fact, the first equality holds by [17, Théorème XV, p. 268], since $\alpha \in \mathscr{O}'_C$. The second equality is trivial by assumption (iii). The third holds by the hypocontinuity of the convolution [17, Théorème XI, p. 247] and assumption (iii). The fourth equality follows from the associativity of convolutions [17, Théorème

XI, pp. 247–248], since g_n, $\alpha \in \mathcal{O}'_C$. Now $\widehat{f*\alpha} = \hat{f}\hat{\alpha}$ has compact support and thus belongs to $\mathcal{O}_M$. It follows from [17, Théorème XV, p. 268] that $f*\alpha \in \mathcal{O}'_C$, so that the fifth equality above follows by hypocontinuity of the convolution and assumption (ii). The remaining two equalities follow again from [17, Théorème XV, p. 268], since $f*\alpha \in \mathcal{O}'_C$ and $\alpha \in \mathcal{O}'_C$.

In order to prove equality (10), note first that both sides of it are actually defined as elements of $\mathcal{D}'$ (see [17, pp. 26, 117]). If now $\varphi \in \mathcal{D}(U)$ is given, we may choose α with $\hat{\alpha} \in \mathcal{D}(U)$ in such a way that $\hat{\alpha} \equiv 1$ in the support of φ. Then we clearly have, omitting the symbol $|_U$ and writing $\langle u, \psi \rangle$ for the value assumed by the distribution u at the test function ψ,

$$\begin{aligned}
\langle \hat{h}, \varphi \rangle &= \langle \hat{h}, \hat{\alpha}\varphi \rangle \\
&= \langle \hat{h}\hat{\alpha}, \varphi \rangle \\
&= \langle \hat{g}(\hat{f}\hat{\alpha}), \varphi \rangle \\
&= \langle \hat{g}, (\hat{f}\hat{\alpha})\varphi \rangle \\
&= \langle \hat{g}, \hat{f}\varphi \rangle \\
&= \langle \hat{g}\hat{f}, \varphi \rangle,
\end{aligned}$$

by the definition of multiplication of distributions with infinitely often differentiable functions. □

We have given the above proof in detail, since it is well known that careless formal manipulations with convolutions of distributions may easily lead to wrong results (see [17, p. 117]).

Proof of Theorem 1.3 Let $G \in M(\mathcal{F})$ be given. We may assume that

$$\int |f(-b_i - y)| \, |\mathrm{d}G(y)| < \infty \qquad (i = 1, \ldots, m) \tag{11}$$

holds, since, by (6), this will certainly be the case if $\mathrm{d}G(y)$ is replaced by $\mathrm{d}G(y+y_0)$ for some y_0, and this replacement does not affect the validity of (9). We define complex measures G_n by

$$\mathrm{d}G_n(x) = 1(|x| \leqq n)\, \mathrm{d}G(x)$$

(where 1 (*statement*) is 1 or 0 according to whether *statement* is true or false), and almost everywhere defined functions h, h_n by

$$\begin{aligned}
h(x) &= \int f(x-y)\, \mathrm{d}G(y), \\
h_n(x) &= \int f(x-y)\, \mathrm{d}G_n(y),
\end{aligned}$$

and check the hypothesis of Lemma 2.1, with G and G_n in place of g and g_n.

First of all, f is a tempered distribution, since it is locally integrable and, by (8) with some fixed y making the right hand side finite, dominated by a polynomial outside some compact set. By (7) and (11),

$$\int \frac{|\mathrm{d}G(x)|}{(1+|x|^2)^k} < \infty, \tag{12}$$

so that G is a tempered distribution in view of [17, Théorème VII, p. 242]. We have $G_n \in \mathcal{O}'_C$ because G_n has compact support. In order to prove that $G_n \to G$

in $\mathscr{S}'$, let (see [17, pp. 238, 71]) Φ be any bounded set in $\mathscr{S}$. By [17, p. 235], there exists a function k with $k(x)=o(|x|^{-\nu})$ as $|x|\to\infty$ for every ν, such that $|\varphi(x)|\leqq k(x)$ for every $x\in\mathbf{R}^d$ and every $\varphi\in\Phi$. This implies

$$\sup_{\varphi\in\Phi}|\langle G-G_n,\varphi\rangle|\leqq\int k(x)\,1(|x|>n)\,|\mathrm{d}G(x)|\to 0$$

(by (12) and dominated convergence), i.e. $G_n\to G$ in $\mathscr{S}'$.

It remains to show that $h\in\mathscr{S}'$ and $h_n\to h$ in $\mathscr{S}'$. To this end write

$$\begin{aligned} h(x) &= \int f(x-y)\,1(|x-y|\leqq R)\,\mathrm{d}G(y) \\ &\quad+\int f(x-y)\,1(|x-y|>R)\,\mathrm{d}G(y) \\ &= h^{(1)}(x)+h^{(2)}(x). \end{aligned} \tag{13}$$

$h^{(1)}$ is the convolution of the tempered distribution G with a compactly supported distribution and hence again tempered, e.g. by [17, Théorème XI, p. 247]. Further, by (8) and (11),

$$\begin{aligned}\int\frac{|h^{(2)}(x)|}{(1+|x|^2)^l}\,\mathrm{d}x &\leqq\int\frac{1}{(1+|x|^2)^l}\,C(1+|x|^2)^k\int\sum_{i=1}^{m}|f(-y-b_i)|\,|\mathrm{d}G(y)|\,\mathrm{d}x \\ &<\infty\end{aligned} \tag{14}$$

if l is large enough, so that $h^{(2)}$ is also tempered. If we now replace G by $G-G_n$ with $n>R$ in (13) and (14), then $h^{(1)}$ vanishes and an argument similar to one given above yields the convergence $h_n\to h$ in $\mathscr{S}'$. Thus Lemma 2.1 is applicable in the present situation.

Assuming now that h vanishes almost everywhere, its conclusion reads

$$0=\hat{G}|_{\mathbf{R}^d\setminus A}\,\hat{f}|_{\mathbf{R}^d\setminus A},$$

which, since $\hat{f}$ is zerofree outside of A, implies

$$0=\hat{G}|_{\mathbf{R}^d\setminus A}.$$

This means that the support of $\hat{G}$ is contained in the finite set A and implies (see [17, Théorème XXXV, p. 100]) that $\hat{G}$ is a finite linear combination of derivatives of Dirac measures, located at pairwise different points $a_1,\dots,a_n\in\mathbf{R}^d$ for some nonnegative integer n (where $n=0$ is admitted and means that $\hat{G}$ is zero). Hence G has a λ^d-density g given by

$$g(x)=\sum_{\nu=1}^{n}\sum_{j}c_{j,\nu}\,x^j e^{ia_\nu x}, \tag{15}$$

where the inner sum ranges over some finite set of multiindices $j=(j_1,\dots,j_d)$, the $c_{j,\nu}$ are complex numbers, and $a_\nu x$ denotes the euclidean inner product.

In order to prove that g vanishes identically, we assume the contrary, i.e. $n\geqq 1$ in (15), and for every ν there is some j with $c_{j,\nu}>0$. We will show in a moment that this implies

$$\int e^{ia_1(x-y)}f(y)\,\mathrm{d}y=0\quad[\lambda^d], \tag{16}$$

which gives the desired contradiction: In case that f is integrable, the left hand side of (16) is $\exp(ia_1 x)\,\hat{f}(-a_1)$, and $\hat{f}(-a_1)\neq 0$ by assumption, whereas, in case that f is not integrable, the left hand side of (16) simply does not exist for any x.

To derive (16) from

$$\int g(x-y)\,f(y)\,\mathrm{d}y=0 \quad [\lambda^d] \tag{17}$$

and (15), consider the operator $T=T_{\alpha,\eta}$ which transforms a function γ in $M(\mathscr{F})$ (i.e., the λ^d-density of an absolutely continous element of $M(\mathscr{F})$) into the function $T\gamma$ defined by

$$(T\gamma)(x)=\mathrm{e}^{-ia\eta}\gamma(x+\eta)-\gamma(x)$$

for $a\in\mathbf{R}^d$ and $\eta\in\mathbf{R}^d\setminus\{0\}$. Clearly, if γ satisfies (17) when substituted for g, so does $T\gamma$. In particular, $T\gamma$ is again a function in $M(\mathscr{F})$. For $\eta=(0,\ldots,0,\eta_d)$, we have

$$T_{a,\eta}\,x_1^{j_1}\cdot\ldots\cdot x_d^{j_d}\,e^{ia_\nu x}=x_1^{j_1}\cdot\ldots\cdot x_{d-1}^{j_{d-1}}\,p(x_d)\,e^{ia_\nu x}$$

where

$$p(t)=(e^{i(a_\nu-a)\eta_d}-1)\,t^{j_d}+e^{-ia\eta}t^{j_d-1}+\ldots$$

is a polynomial. If η_d is sufficiently small, p has degree j_d or j_d-1, according to whether a_ν equals a or not (we attribute degree -1 to the polynomial identically 0).

Hence it is clear that first we may repeatedly apply $T_{a_\nu,\eta}$ for $\nu=2,\ldots,n$ and

$$\eta=(\eta_1,0,\ldots,0),(0,\eta_2,0,\ldots,0),\ldots,(0,\ldots,0,\eta_d),$$

with η_ν sufficiently small, to g in such a way as to obtain a function

$$\tilde{g}(x)=\sum_j \tilde{c}_{j,1}\,x^j e^{ia_1 x}$$

satisfying (17) when substituted for g and with coefficients $\tilde{c}_{j,1}$, nonzero for the same j as $c_{j,1}$. In a second step we can apply $T_{a_1,\eta}$ with the same vectors η as above so often to $\tilde{g}$ as to obtain a function

$$\tilde{\tilde{g}}(x)=\tilde{\tilde{c}}_{0,1}\,e^{ia_1 x}$$

with $\tilde{\tilde{c}}_{0,1}\neq 0$ satisfying (17) when substituted for g, i.e. (16) is true. □

Proof of Proposition 1.2 We use the elementary inequalities

$$1+t^2\leqq(1+t)^2\leqq 2(1+t^2) \qquad (t\geqq 0), \tag{18}$$

and

$$1+|x+y|\leqq(1+|x|)(1+|y|) \qquad (x,y\in\mathbf{R}^d). \tag{19}$$

In (19) replace x by $x+y$ and y by $-x$ in order to obtain

$$1+|x+y|\geqq\frac{1+|y|}{1+|x|} \qquad (x,\,y\in\mathbf{R}^d). \tag{20}$$

(i′) implies (i): If (i′) is true, then

$$\begin{aligned} f(x+y) &\leqq C(1+|x+y|)^{-\alpha} \\ &\leqq C\left(\frac{1+|y|}{1+|x|}\right)^{-\alpha} \\ &\leqq 2\frac{C}{c}(1+|x|^2)^{\frac{\alpha}{2}} f(y) \end{aligned}$$

by (20), positivity of α, and (18).

(i) implies (i″): The first inequality in (i″) follows by taking $y=-x$ in (i) and subsequently replacing x by $-x$. The second follows from (i) by taking $y=0$. □

Proof of Proposition 1.4 We have, under the hypothesis of Proposition 1.4, for $m=2$, $b_1=0$, *and* $b_2=(2R, 0, \ldots, 0)$

$$\sum_{i=1}^{m} |f(x-b_i)| \geqq c \min\left(\frac{1}{(1+|x|)^{\alpha}}, \frac{1}{(1+|x-b_2|)^{\alpha}}\right),$$

which, by applying (19) or (20) to $1+|-b_2+x|$ according to whether α is positive or not, is seen to be

$$\geqq \frac{\tilde{c}}{(1+|x|)^{\alpha}} \tag{21}$$

for some positive $\tilde{c}$. This implies the inequality in condition (i). Further, if $|x-y| \geqq R$, we get

$$\begin{aligned} |f(x-y)| &\leqq \frac{C}{1+|x-y|)^{\alpha}} \\ &\leqq C(1+|x|)^{|\alpha|} \frac{1}{(1+|y|)^{\alpha}}, \\ &\leqq \tilde{C}(1+|x|^2)^{\left|\frac{\alpha}{2}\right|} \sum_{i=1}^{m} |f(x-b_i)|, \end{aligned}$$

by first using (19) or (20) according to whether α is negative or not, and then (18) and (21). □

3 Examples

Considering the six examples of the introduction, we see that only for the Cauchy distribution completeness of the generated location family may be deduced with the present method. The other densities do not fulfill condition (i″) of Proposition 1.2 and hence not condition (i) of the Theorem 1.1. Concerning the Cauchy density we have in fact the following more general result, including in particular Student's t-densities.

Theorem 3.1 *The location family in* $\mathbf{R}^d$ *generated by the density*

$$f(x)=\frac{\Gamma\left(\frac{d+\beta}{2}\right)}{\pi^{\frac{d}{2}}\Gamma\left(\frac{\beta}{2}\right)}\frac{1}{(1+|x|^2)^{\frac{d+\beta}{2}}},$$

with fixed positive β, *is complete.*

Proof. Condition (i′) of Proposition 1.2 is fulfilled and the characteristic function of f may be expressed as (compare [2, p. 294])

$$\hat{f}(t)=\frac{1}{\Gamma\left(\frac{\beta}{2}\right)}\int_0^\infty e^{-(|t|^2/4)\lambda}e^{-1/\lambda}\lambda^{-\beta/2}\frac{d\lambda}{\lambda},$$

which is everywhere positive and real analytic away from the origin. Hence Theorem 1.1 applies. □

It is easy to construct numerous other densities for which completeness may be deduced using Theorem 1.1. For example, we may take any nondegenerate mixture of the Cauchy density and a density g with $g(x)\leqq C(1+x^2)^{-1}$, $\hat{g}(x)\geqq 0$, and $\hat{g}$ infinitely often differentiable. Then condition (i) in Theorem 1.1 is easily checked via Proposition 1.2 and the location family generated by f is seen to be complete. For the special case $g(x)=(1-\cos x)/\pi x^2$, $\hat{g}$ is triangle shaped (compare e.g. [3, p. 503]) and the set A in Theorem 1.1 consists of three points (note that it will always contain the origin).

More interesting examples are provided by the symmetric stable laws.

Theorem 3.2 *The location families generated by a density* f *in* $\mathbf{R}$ *with characteristic function* $\hat{f}(t)=\exp(-|t|^\alpha)$, $0<\alpha\leqq 2$, *are complete.*

Proof. We consider the case $\alpha=2$ of the normal distribution as known and assume $0<\alpha<2$. Then it is known that f is continuous, everywhere strictly positive (this follows from the unimodality of f in conjunction with symmetry and noncompactness of the support) and satisfies (compare [12, p. 140, formula (5.8.6) and p. 149, formula (5.9.3)])

$$f(x)=\frac{\Gamma(\alpha+1)}{\pi}\sin\left(\frac{\pi}{2}\alpha\right)\frac{1}{|x|^{1+\alpha}}+O\left(\frac{1}{|x|^{1+2\alpha}}\right)$$

for $|x|\to\infty$. Hence (i′) of Proposition 1.2 is true and Theorem 1.1 is applicable. □

Now we consider applications of Theorem 1.3.

Theorem 3.3 *Let* $p>-d$ *be fixed and* $\neq 2k$ *for* $k=0, 1, 2, \ldots$. *If* G *is a complex measure such that*

$$\int_{\mathbf{R}^d}|x-y|^p\,dG(y)$$

exists and vanishes for λ^d*-almost every* x, *then* $G=0$.

Proof. Put $f(x)=|x|^p$. Then, since $p>-d$, f is locally integrable, but not integrable, and fulfills the assumption of Proposition 1.4 and hence assumption (i) of Theorem 1.3. The Fourier transform of f is given by

$$\hat{f}(t)=2^{d+p}\pi^{\frac{d}{2}}\frac{\Gamma\left(\frac{d+p}{2}\right)}{\Gamma\left(-\frac{p}{2}\right)}\frac{1}{|t|^{d+p}}$$

(see [17, p. 257]), if the singularities of the right hand side are interpreted correctly. In any case, for p as above, the gamma quotient remains finite and nonzero, so that $\hat{f}$ fulfills the remaining conditions of Theorem 1.3 with $A=\{0\}$. □

A similar result holds for the logarithmic kernel. It may be thought of as a limiting case of Theorem 3.3 if $p\to 0$.

Theorem 3.4 *If G is a complex measure such that*

$$\int_{\mathbf{R}^d} \log|x-y|\,\mathrm{d}G(y)$$

exists and vanishes for λ^d-almost every x, then $G=0$.

Proof. Condition (i) of Theorem 1.3 is easily checked for, e.g., $m=2$, $b_1=0$, $b_2=(3,0,\ldots,0)$, $k=\frac{1}{2}$, and $R=1$.

The Fourier transform of $f(x)=\log|x|$ is away from the origin given by

$$\hat{f}(t)=-2^{d-1}\pi^{\frac{d}{2}}\Gamma\left(\frac{d}{2}\right)\frac{1}{|t|^d}$$

(see [17, p. 258]), which is strictly negative and real analytic in t. Since f is not integrable, Theorem 1.3 applies again. □

A partial history of Theorem 3.3 is as follows. It was stated without complete proof for $p<0$ by M. Riesz in [16, pp. 10, 11], and proved (in a stronger form) under the additional hypothesis that G has compact support and $p>-d$ is $\neq 2k+2-d$ for $k=0,1,2,\ldots$. P. Hall obtained Theorem 3.3 for $d=1$ and $p>0$ in case that G is assumed to be the difference of two probability measures (see [6, Theorem 1]). Note that our proof explains why the cases $p=2k$, $k=0,1,2,\ldots$ have to be exluded: In these cases $\hat{f}=(-\Delta)^k\delta$, a distribution supported at the origin, so that formal multiplication with $\hat{f}$ does not look like an injective operation on $\hat{G}$, and so neither does formal convolution of f with G.

It is interesting to note that the notoriously slightly more difficult to handle logarithmic kernel causes no pain in the present proof of Theorem 3.4.

Further applications of Theorem 1.3, including a new proof of [6, Theorem 2], can easily be given by using tables of Fourier transforms of distributions, e.g. [4, pp. 346–353] or [8].

4 Discussion

We concentrate on the simpler situation of Theorem 1.1. Returning to the heuristics of the introduction, observe that our restriction on f via condition (i) of that theorem was chosen in order to induce a desired restriction on $g \in L^1(\mathscr{F})$. A different idea is to impose restrictions on g directly, i.e. replacing $L^1(\mathscr{F})$ in the definition (2) of completeness by some subset E. The resulting concept is termed *E-completeness*. If E is the space of bounded measurable functions, E-completeness is customarily called *bounded completeness*. As observed by Ghosh and Singh [5], boundedly complete location families are characterized by a classical theorem of Wiener, making the equivalence (3)$\Leftrightarrow$(5) precise in a way.

Theorem 4.1 (Wiener) *$\mathscr{F}$ is boundedly complete iff the characteristic function $\hat{f}$ of the generating density has no zeros.*

In order to point out the importance of the space E which g in (2) is restricted to, we state another theorem of Wiener, making (3)$\Leftrightarrow$(5) precise in a different way.

Theorem 4.2 (Wiener) *Assume that the generating density f is in $L^2(\lambda^d)$. Then $\mathscr{F}$ is $L^2(\lambda^d)$-complete iff $\lambda^d(\{\hat{f}=0\})=0$.*

For example, if f is the uniform density on $[-1,1]$, $\hat{f}(t)=\dfrac{\sin t}{t}$, and $\mathscr{F}$ is $L^2(\lambda^d)$-complete but not boundedly complete.

Wiener's theorems where generalized by Beurling to weighted L^p-spaces on the line, i.e. $E=L^p(w\lambda^1)$ where the so-called weight function w is assumed to satisfy the conditions $w(x)\geqq 1$, $w(x+y)\leqq w(x)\,w(y)$ and $w\left(\dfrac{x}{\rho}\right)\leqq w(x)$ for x, $y\in\mathbf{R}$ and $\rho>1$. A typical example is $w(x)=(1+|x|)^\alpha$ for some $\alpha>0$. Recently, the theory of Wiener and Beurling was extended by Isenbeck and Rüschendorf with the object of deciding $L^p(\mathscr{F})$-completeness of location families (where $L^p(\mathscr{F})$ is similarly defined as $L^1(\mathscr{F})$ above). Note that $r\leqq s$ implies $L^r(\mathscr{F})\supset L^s(\mathscr{F})$, so that $L^r(\mathscr{F})$-completeness of $\mathscr{F}$ implies $L^s(\mathscr{F})$-completeness of $\mathscr{F}$. In particular, for $1<p<\infty$, completeness implies $L^p(\mathscr{F})$-completeness, which in turn implies bounded completeness. Isenbeck and Rüschendorf proved the following result ([7, Corollary 1]).

Theorem 4.3 (Isenbeck and Rüschendorf) *Let w be any weight function and $p>1$. Assume that the following conditions hold for the generating density f.*

(a) $fw\in L^1(\lambda^1)$,

(b) $f^{-\frac{1}{p}}\,w^{-1}\in L^\infty(\lambda^1)\cup L^q(\lambda^1)$, $\dfrac{1}{p}+\dfrac{1}{q}=1$.

Then $\mathscr{F}$ is $L^p(\mathscr{F})$-complete iff $\hat{f}$ has no zeros.

Note that (a) and (b) would be mutually exclusive for $p=1$. [7] gives an example of a density f for which $L^p(\mathscr{F})$-completeness of $\mathscr{F}$ can be shown for every $p>1$ using Theorem 4.3.

While the Theorems 1.1, 4.1, 4.2 and 4.3 are analogues of each other, neither of them implies any other of them in an obvious way. The technical character of the conditions (i) and (ii) in Theorem 1.1 however suggests the possbility of omitting them without affecting the truth of the resulting statement, which would then include Theorem 4.1 as well as Theorem 4.3. According to Wiener's Theorem 4.1, proving such a statement would be equivalent to proving the assertion "Every boundedly complete location family on $\mathbf{R}^d$ is complete". It is known that this latter assertion becomes false if the word "location" is omitted. An early counterexample was given by Lehmann and Scheffé [11, p. 312], others where given only recently by Bar-Lev and Plachky [1], but none of them is given by a location family. Hence, it is an open and perhaps not entirely trivial problem to decide whether the conditions (i) and (ii) in Theorem 1.1 may be omitted.

Our final remark concerns the missing "only if" statement in Theorem 1.3. If f is assumed to be integrable, this may of course be added, with the same proof as given in case of Theorem 1.1 immediately after its statement. Whether or not this should be possible in general, i.e. also if f is not integrable, is not clear to the author.

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Note added in proof. (December 1991) It can be shown by an example that condition (i) in Theorem 1.1 must not be omitted without substitute. Details will appear elsewhere.