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Shock fluctuations in asymmetric simple exclusion

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Summary. The one dimensional nearest neighbors asymmetric simple exclusion process in used as a microscopic approximation to the Burgers equation. We study the process with rates of jumps $p > q$ to the right and left, respectively, and with initial product measure with densities $\varrho < \lambda$ to the left and right of the origin, respectively (with shock initial conditions). We prove that a second class particle added to the system at the origin at time zero identifies microscopically the shock for all later times. If this particle is added at another site, then it describes the behavior of a characteristic of the Burgers equation. For vanishing left density ($\varrho = 0$) we prove, in the scale $t^{1/2}$, that the position of the shock at time t depends only on the initial configuration in a region depending on t. The proofs are based on laws of large numbers for the second class particle.

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Introduction

The inviscid Burgers equation

$$\frac{\partial u}{\partial t} + \theta \frac{\partial [u(1-u)]}{\partial r} = 0 \tag{1.1.a}$$

with $\theta > 0$ admits travelling wave (weak) solutions $u(r - vt)$, where, for $\varrho < \lambda$,

$$\begin{aligned} u(r) &= \varrho 1\{r < 0\} + \lambda 1\{r \geqq 0\} \\ v &= \theta(1 - \varrho - \lambda) \ . \end{aligned} \tag{1.1.b}$$

These solutions are called the entropic solutions. In this paper we continue the study of Ferrari, Kipnis and Saada [fks] of the microscopic approximation of these solutions by the asymmetric simple exclusion process. This is a Markov process on the state space $X := \{0.1\}^{\mathbb{Z}}$. For configurations $\eta \in X$ we say that there is a particle at x if $\eta(x) = 1$, otherwise x is empty, so that at each site of the one dimensional lattice $\mathbb{Z}$ there is at most one particle. Informally the process is described by saying that if there is particle at site x, then it jumps to site $x+1$ (respectively $x-1$) with rate p (resp. q) if $x+1$ (resp. $x-1$) is empty and with rate 0 if it is cccupied. We assume $p+q=1$ and $p>q$. The generator of the process is given by

$$Lf(\eta) = \sum_{x \in \mathbb{Z}} \sum_{y=x\pm 1} p(x,y)\eta(x)(1-\eta(y))[f(\eta^{x,y}) - f(\eta)]$$

where f is a cylindric function, $\eta^{x,y}$ is the configuration obtained when the values of η at x and y are interchanged, $p(x,x+1) = p, p(x,x-1) = q = 1-p$ and $p(x,y) = 0$ if $|x-y|>1$. We call η_t the resulting Markov process and $S(t)$ the semigroup generated by L. Liggett proved that all the translation invariant and (time) invariant measures for this process are convex combinations of the product measures ν_α, $0 \leqq \alpha \leqq 1$, for which the probability that a given site is occupied is given by α. Under the invariant measure ν_α the average velocity of the particles is $(p-q)(1-\alpha)$. There are also so called "blocking" invariant measures $\nu^{(n)}$, $n \in \mathbb{Z}$. These measures are also product, have marginals

$$\nu^{(n)}(\eta(x)) = \frac{(p/q)^{x-n}}{1+(p/q)^{x-n}} \tag{1.2}$$

and are even reversible for the process. They approach exponentially fast the densities 0 and 1 to the left and right of the origin respectively [L], [l]. We consider these measures as a first example of a microscopic shock: at any time the process with initial measure $\nu^{(n)}$ has a measure (which is the same $\nu^{(n)}$) which, if shifted by x, approaches ν_0 as $x \to -\infty$ and ν_1 as $x \to \infty$. This corresponds to the case $\varrho = 0$ and $\lambda = 1$ in the Burgers equation. The explicit formula for $\nu^{(n)}$ plays a crucial role in this paper as it did in [fks]: we use (1.2) to find shocks for other values of ϱ and λ.

In order to derive Eq. (1.1) in the general case one starts with the product measure $\nu_{\varrho,\lambda}$ with densities ϱ and λ to the left and right of the origin respectively, $\varrho < \lambda$. Under this measure the particles initially to the left of the origin have average velocity $(p-q)(1-\varrho)$ which exceeds the initial velocity of the particles to the right of the origin $(p-q)(1-\lambda)$. This together with the exclusion interaction is the reason of the formation of shocks in this model. In fact it has been proven by Benassi and Fouque [bf] and Andjel and Vares [av] that the hydrodynamical limit of this process is given by (1.1) with $\theta = p-q$. They proved that $\tau_{\varepsilon^{-1}r}\nu_{\varrho,\lambda}S(\varepsilon^{-1}t)$ converges, as $\varepsilon \to 0$, to ν_ϱ if $r < vt$ and to ν_λ if $r > vt$, where $v = (p-q)(1-\lambda-p)$ and τ_x is the translation by x. We call v the schock velocity. At $r = vt$ it is expected to see what Wick calls a "dynamical phase transition", i.e. a convex combination of ν_ϱ and ν_λ. This was in fact proven by Wick [w] and De Masi et al. [dkps] for the case $\varrho = 0$ and by Andjel et al. [abl] for the case $\lambda + \varrho = 1$. Notice however that this is not true when $\varrho = 0$ and $\lambda = 1$, as $\nu^{(n)}$ are invariant and not a convex combination of ν_1 and ν_0. Some of these results were reviewed by Bramson [b].

The question that arises naturally then is what does the shock look like? The hydrodynamic limit shows that it is rigid in the scale ε^{-1}. Is there another scale such that it is smooth? [fks] answered this question negatively. Indeed they proved that the shock is rigid on the microscopic level by defining a random position $X(t)$ and a measure $\mu \sim \nu_{\varrho,\lambda}$ such that at any time t the system as seen from $X(t)$ is distributed according to μ, where $\mu \sim \nu_{\varrho,\lambda}$ means that for all cylindric f on $\mathbf{X}$,

$$\lim_{x \to -\infty} \tau_x \mu f = \nu_\varrho f \quad \text{and} \quad \lim_{x \to \infty} \tau_x \mu f = \nu_\lambda f \,. \tag{1.3}$$

In this paper we show that the microscopic shock is also described by a "second class particle". One of the advantages of this approach is that the shock can be shortly described and we proceed to do that now. We use the graphic construction and the basic coupling. We refer to Liggett's book [L] for both techniques, widely used in interacting particle systems. The graphic construction of the simple exclusion process η_t is determined by the following rules. At each pair of sites $(x, x+1)$ associate two Poisson point processes (Ppp), one with rate p and the other with rate q. Each of these Ppp is a sequence of times on $\mathbb{R}^+$. We say that an arrow going from x to $x+1$ is present at each of these times for the Ppp with rate p and a left arrow going from $x+1$ to x for the Ppp with rate q. These Ppp are mutually independent. When an arrow occurs going from x to y if x is occupied and y is empty, then the particle jumps to the empty site. Otherwise nothing happens. In this way, for each realization of the arrows and each initial configuration we construct a version of η_t.

The basic coupling consists in realizing jointly two or more versions of the process with different initial configurations using the same realization of the arrows. In this way each marginal has the distribution of the simple exclusion process and we can learn other properties by comparing the same realization with different initial configurations. In particular, consider two initial configurations that differ at only one site, say the origin. The reader can check that the coupling has the property that at later times the two marginals also differ at only one site. This site is called a second class particle. The name comes from the way this "particle" interacts with the other particles that we call first class: First and second class particles at rate p jump to the right site if it is empty and at rate q do the same to the left. None of them jump to sites occupied by first class particles. But, when a first class particle tries to jump to a site occupied by the second class, the jump is realized and the two particles interchange positions.

We describe now our results. We show in Sect. 2 that starting with the second class particle at the origin and distributing the other particles according to $\nu_{\varrho,\lambda}$, the process as seen from that particle at time t has a distribution $\sim \nu_{\varrho,\lambda}$ uniformly in t, where $\sim$ is defined in (1.3). Furthermore, as $t \to \infty$, this process converges weakly to an invariant measure $\hat{\mu} \sim \nu_{\varrho,\lambda}$. The process as seen from the second class particle is a Markov process. This is an improvement over the process as seen from the random position described by [fks].

In Sect. 3 we show laws of large numbers for the second class particle: call $R(t)$ the position at time t of the second class particle ($R(0) = 0$). We prove that $R(t)/t$ converges almost surely to $v = (p-q)(1-\lambda-\varrho)$ when the initial measure is either the product measure $v_{\varrho,\lambda}$ or the invariant measure $\hat{\mu}$. To prove these laws we use and improve laws of large numbers of [fks]. Furthermore we show that if the first class particles are distributed according to the (translation invariant) mesure v_α, then $R(t)/t$ converges almost surely to $(p-q)(1-2\alpha)$.

In Sect. 4 we consider the situation when the second class particle starts at the position $\varepsilon^{-1}r > 0$ and study its motion in the hydrodynamic limit described above. We prove that its macroscopic motion can be described as follows: it starts at r and has velocity $(p-q)(1-2\lambda)$ up to the moment that it meets the shock that started at the origin and has velocity $(p-q)(1-\lambda-\varrho)$. At this moment the second class particle adopts the velocity of the shock. On the other hand, when the initial point is $\varepsilon^{-1}r < 0$, the velocity is $(p-q)(1-2\varrho)$ up to the moment it meets the shock. This is the behavior that the characteristics of the Burgers equation have.

The fluctuations of the shock are at least diffusive. In fact we prove in Sect. 6 that

$$\liminf_{t\to\infty} t^{-1} E_{\hat{v}_{\varrho,\lambda}}(R(t)-vt)^2 \geqq (p-q)\frac{\varrho(1-\varrho)+\lambda(1-\lambda)}{\lambda-\varrho}, \tag{1.4}$$

where E_v is the expectation of the process with initial measure v; the measure $\hat{v}_{\varrho,\lambda}$ is the measure that puts a second class particle at the origin and the first class particles are distributed according to $v_{\varrho,\lambda}$; finally, in this paper we write $E(.)^2$ for $E[(.)^2]$.

Equation (1.4) proves half of the conjecture of Spohn [lps] [S], who has heuristic arguments that justify the existence of the limit and the identity in (1.4). Also Boldrighini et al. performed computer simulations that support the conjecture [bcfg]. The proof of (1.4) is based on a result shown in Sect. 5: if a particle is added to the system at time zero at any site, then, as $t \to \infty$, the shock shifts to a random position that differs in average $-(\lambda-\varrho)^{-1}$ from the shock position of the original process.

More can be said when $\varrho = 0$. In Sect. 6 we show that the fluctuations of the shock depend only on the initial distribution. We prove that

$$\lim_{t\to\infty} t^{-1} \int d\hat{v}_{0,\lambda}(\eta) E_\eta \left(R(t) - \frac{n_0(\eta, r^+ t)}{\lambda} \right)^2 = 0 \tag{1.5}$$

where $n_0(\eta, y) := \sum_{x=1}^{y} (1-\eta(x))$ is the number of empty sites of the configuration η between the origin and y and $r^+ = (p-q)\lambda$. This extends a result by Gärtner and Presutti [gp] who proved (1.5) for $p = 1$. Equation (1.5) implies a central limit theorem for $R(t)$ because $n_0(\eta, r^+ t)$ is a sum of independent identically distributed random variables. In Remark 6.6 we show that for the tagged particle [f], [k], [df], i.e. a regular particle, (1.5) is also true substituting $\nu_{0,\lambda}$ by ν_λ. For any ϱ and λ we conjecture that the following is true

$$\lim_{t\to\infty} t^{-1}\int d\widehat{\nu}_{\varrho,\lambda}(\eta)E_\eta\left(R(t)-\frac{n_0(\eta,r^+t)}{\lambda-\varrho}-\frac{n_1(\eta,r^-t)}{\varrho-\lambda}\right)^2=0 \qquad (1.6)$$

where $n_1(\eta, x)$ is minus the number of particles between the origin and $x < 0$ and $r^+ = -r^- = (p-q)(\lambda-\varrho)$.

Finally let us mention that the second class particle is conjectured to have the "$1/f$ noise" behavior. The conjecture is that when $\lambda = \varrho = \alpha$ the fluctuations are superdiffusive:

$$\lim_{t\to\infty} t^{-4/3}E_{\widehat{\nu}_{\alpha,\alpha}}(R(t)-ER(t))^2 = \text{ constant } > 0\,.$$

See [S] and [vb] for a heuristic justification and [bcfg] for computer simulations.

2 A second class particle identifies the shock

We define the process with a second class particle described in the introduction. Define $(\eta_t, R(t))$ on the state space $X \times \mathbb{Z}$ as the process with generator

$$\begin{aligned} L^*f(\eta,r) = &\sum_{x\neq r}\sum_{y=x\pm 1\neq r} p(x,y)\eta(x)(1-\eta(y))[f(\eta^{x,y},r)-f(\eta,r)] \\ &+\sum_{x=r\pm 1}(p(x,r)\eta(x)+p(r,x)(1-\eta(x)))[f(\eta^{x,r},x)-f(\eta,r)]\,. \end{aligned} \qquad (2.1)$$

Consider also $\widehat{\eta}_t := \tau_{R(t)}\eta_t$, the process as seen from the second class particle. This is a Markov process with state space $\widehat{X} := \{0,1\}^{\mathbb{Z}\setminus\{0\}} \times \{0\}$, the space with a second class particle at the origin and with first class particles and empty sites on the other sites. Call $\widehat{L}$ and $\widehat{S}(t)$ the corresponding generator and semigroup. Let $\widehat{\nu}_{\varrho,\lambda}$ be the product measure on $\widehat{X}$ for which the density of first class particles to the right of the origin is λ and the density to the left of the origin is ϱ. The symbol $\sim$ is defined in (1.3) above.

Theorem 2.2 *The following holds uniformly in t*

$$\widehat{\nu}_{\varrho,\lambda}\widehat{S}(t) \sim \nu_{\varrho,\lambda}\,.$$

Furthermore, $\widehat{\nu}_{\varrho,\lambda}\widehat{S}(t)$ *converges weakly, as* $t \to \infty$, *to an invariant measure* $\widehat{\mu} \sim \nu_{\varrho,\lambda}$.

In the remainder of this section we prove Theorem 2.2 The proof uses the identification of the interface given by [fks]. They proved that there exists a position $X(t)$ such that if η_0 is $\nu_{\varrho,\lambda}$ distributed, then the process $\tau_{X(t)}\eta_t$ has distribution $\sim \nu_{\varrho,\lambda}$ uniformly in t. We will prove here that there exists a coupling for which $E|R(t) - X(t)| \leqq C$ for all t. The major difference between $X(t)$ and $R(t)$ is that the process $\tau_{R(t)}\eta_t$ is Markovian while $\tau_{X(t)}\eta_t$ is not.

Now we recall the results of [fks]. The position $X(t)$ is obtained from the basic coupling described in the introduction between two copies of the simple exclusion process. The first copy has initial (marginal) measure ν_ϱ and the second ν_λ. Under the initial measure the configuration of the first copy is coordinatewise less or equal than the configuration of the second one. We call second class particles those occupied sites of the second copy that do not have a corresponding particle in the first copy, and we call first class particles the common occupied sites. This nomenclature is justified in the introduction. The resulting process is denoted (σ_t, ξ_t), where the σ particles are the first class and the ξ particles are the second class. We say as in [ak] that the σ particles have priority over the ξ particles and denote the priority $\sigma_t \vdash \xi_t$. By construction σ_t is the simple exclusion process with initial measure ν_ϱ while $\sigma_t + \xi_t$ (cordinatewise) is the simple exclusion process with initial measure ν_λ. Call ν_2 the initial measure for the process (σ_t, ξ_t) and $\nu_2' := \nu_2(.|\xi(0) = 1)$, so that under ν_2' there is a second class particle at the origin. Let $X(t)$ be the position of that particle at time t. Each realization of $X(t)$ is univoquely determined by the corresponding realization of the path ξ_s, $0 \leqq s \leqq t$.

To recover the original process η_t [fks] define yet a new process $(\sigma_t, \gamma_t, \zeta_t)$ where $\gamma_t + \zeta_t = \xi_t$ with priorities $\sigma_t \vdash \gamma_t \vdash \zeta_t$. This means that the particles follow the arrows to jump either to empty sites or to sites occupied by another particle of lower priority interchanging positions. The initial distribution of $(\sigma_t, \gamma_t, \zeta_t)$ is given as follows. Pick (σ_0, ξ_0) from the distribution ν_2', and label the n-th ξ particle counted from the origin a γ particle with probability $(p/q)^n/(1+(p/q)^n)$, otherwise a ζ particle. Do this independently for each n. Call ν_3' the resulting distribution of $(\sigma_0, \gamma_0, \zeta_0)$. The notable property of this construction is that the distribution of the γ_t and ζ_t labeling is the same for all $t \geqq 0$. In Lemma 3.26 of [fks] it is proved that the shifted process $\tau_{X(t)}(\sigma_t, \gamma_t, \zeta_t)$ is Markovian and has measure $\nu_3' S_3'(t)$ with the property that the $\sigma + \gamma$ marginal has distribution $\sim \nu_{\varrho,\lambda}$ uniformly in t. Using compactness, [fks] proved that there exists an invariant measure μ_3' with the same properties. The original process is recovered by defining $\eta_t = \sigma_t + \gamma_t$ but with a different initial distribution. In fact note that at time zero the projection of ν_3 over the η coordinate is not exactly $\nu_{\varrho,\lambda}$ but a measure equivalent to it. Hence $\tau_{X(t)}\eta_t$ has distribution $\sim \nu_{\varrho,\lambda}$ uniformly in t. Projecting the invariant measure μ_3' over $\sigma + \gamma$ one obtains a measure $\mu \sim \nu_{\varrho,\lambda}$ such that, starting with μ, $\tau_{X(t)}\eta_t$ is distributed according to μ for all t. The position of $X(t)$ can be recovered from the process $\tau_{X(t)}(\sigma_t, \xi_t)$ by counting the algebraic number of translations of this process in $[0, t]$.

Now we establish the relationship between $R(t)$ and $X(t)$. For $p = 1$ we simply label the ξ-particles to the right of the origin at time 0 as γ particles and those to the left as ζ particles. In this case notice that $X(t)$ is just a second class particle with respect to the η_t process (which is the same as $\sigma_t + \gamma_t$). So in this case $R(t) \equiv X(t)$, and $\hat{\mu} = \mu$.

When $p < 1$ we have the following

Proposition 2.3 *There exists a coupling between* $(\eta_t, R(t))$ *with initial measure* $\widehat{\nu}_{\varrho,\lambda}$ *and* $(\sigma_t\ \xi_t, X(t))$ *with initial measure* ν_2' *such that for all* $t \geqq 0$

$$E|R(t) - X(t)| \leqq C < \infty\,. \tag{2.4}$$

Proof. At time $t = 0$ distribute (σ_0, ξ_0) according to ν_2' and set $(\eta_0, R(0))$ as follows

$$\begin{aligned} &R(0) = X(0) = 0 \\ &\eta_0(x) = \sigma_0(x)\,, \quad \text{for } x < 0 \\ &\eta_0(x) = \sigma_0(x) + \xi_0(x)\,, \quad \text{for } x > 0\,. \end{aligned} \tag{2.5}$$

Now call γ and ζ the ξ particles to the right and left of the origin respectively. For later times consider the following priorities:

$$\sigma_t \vdash \gamma_t \vdash R(t) \vdash \zeta_t$$

(we identify $R(t)$ with the configuration with a particle at $R(t)$ and no particles elsewhere). Hence $(\eta_t, R(t)) = (\sigma_t + \gamma_t, R(t))$ and $(\sigma_t, \xi_t, X(t)) = (\sigma_t, (\gamma_t + \zeta_t) \cup R(t), X(t))$ have the right distribution. (We identify the configuration η with the set of occupied sites $\{x : \eta(x) = 1\}$ and abuse notion by writing $R(t)$ for $\{R(t)\}$.)

We now study the process $\tau_{X(t)}(\sigma_t, \gamma_t, \zeta_t, R(t))$. For the initial distribution of this process it is convenient to consider a new measure ν_4' for which $\nu_4'(.|\mathbf{A})$ satisfies (2.5), with $\mathbf{A} := \{R(0) = 0;\ \gamma(x) = 0,\ x < 0;\ \zeta(x) = 0,\ x > 0\}$. To define ν_4', put the σ and ξ particles according to ν_2'. Define $x_i :=$ position of i-th particle ($x_0 = X(0) = 0$). Choose $R(0)$ to be equal to x_i, with probability

$$m(i) := M\left(\left(1 + (p/q)^{i-\frac{1}{2}}\right)\left(1 + (q/p)^{i+\frac{1}{2}}\right)\right)^{-1} \tag{2.6.a}$$

independently of the configuration (σ, ξ), where M is a normalizing constant making $\sum m(i) = 1$. Finally decide which ξ particles different from $R(0)$ are γ particles: label the j-th ξ particle as γ with probability

$$m(j|i) := \begin{cases} (p/q)^{j-\frac{1}{2}} / \left(1 + (p/q)^{j-\frac{1}{2}}\right) & \text{if } j < i \\ (p/q)^{j+\frac{1}{2}} / \left(1 + (p/q)^{j+\frac{1}{2}}\right) & \text{if } j > i \end{cases} \tag{2.6.b}$$

independently of everything, otherwise as ζ. A formal definition is given in (2.11) below. The values of $m(i)$ and $m(j|i)$ are chosen so that (2.16) below holds. Other choices are possible but this one makes $m(i)$ symmetric with respect to the origin.

Call $x_i(t)$ the position at time t of the i-th ξ_t particle ($x_0(t) \equiv X(t)$). We prove below that for all $t \geqq 0$,

$$P_{\nu_4'}\left(R(t) = x_i(t) \middle| \mathscr{F}_{2,t}\right) = m(i) \tag{2.7}$$

where $\mathscr{F}_{2,t}$ is the sigma algebra generated by $\{(\sigma_s, \xi_s) : 0 \leqq s \leqq t\}$. Before proving (2.7) we finish the proof of the proposition. Equation (2.7) implies that $\{R(t) = x_i(t)\}$ is independent of (σ_t, ξ_t). Hence

$$E_{\nu_4'}|R(t) - X(t)| = \sum_{i \in \mathbb{Z}} E_{\nu_4'}\left(|R(t) - X(t)| \,\Big|\, R(t) = x_i(t)\right) P\left(R(t) = x_i(t)\right)$$
$$= \sum_{i \in \mathbb{Z}} \frac{|i|}{\lambda - \varrho} m(i) = \text{ constant } < \infty \tag{2.8}$$

where in the second identity we used that for all i and all $t \geqq 0$, $E_{\nu_4'}(x_i(t) - x_{i-1}(t)) = 1/(\lambda - \varrho)$. This proves (2.4) when the initial measure is ν_4'. In fact our initial measure is $\nu_4'(.|\mathbf{A})$. Since ν_4' gives positive mass to $\mathbf{A}$, (2.8) implies the proposition. □

Proof of (2.7). The process $\tau_{X(t)}(\sigma_t, \gamma_t, \zeta_t, R(t))$ is a Markov process with generator $L_4' := L_2'' + L_3'' + L_4''$, where L_2'' is the generator of the motion of the σ and ξ particles and the translations due to the jumps of $X(t)$:

$$L_2'' f(\sigma, \gamma, \zeta, r)$$
$$= \sum_{x \neq 0} \sum_{y = x \pm 1 \neq 0} \Big\{ \sigma(x)(1 - \sigma(y)) p(x, y)$$
$$\times \left[f(\sigma^{x,y}, \gamma^{x,y}, \zeta^{x,y}, r^{x,y}) - f(\sigma, \gamma, \zeta, r) \right]$$
$$+ \Big(\gamma(x) + \zeta(x) + 1\{r = x\}\Big)\Big(1 - \sigma(y) - \gamma(y) - \zeta(y) - 1\{r = y\}\Big) p(x, y)$$
$$\times \left[f(\sigma, \gamma^{x,y}, \zeta^{x,y}, r^{x,y}) - f(\sigma, \gamma, \zeta, r) \right]\Big\}$$
$$+ \sum_{y = \pm 1} \Big\{ \sigma(y) p(y, 0) \left[f\Big(\tau_y \sigma^{0,y}, \tau_y \gamma^{0,y}, \tau_y \zeta^{0,y}, r^{0,y} - y\Big) - f(\sigma, \gamma, \zeta, r) \right]$$
$$+ \Big(1 - \sigma(y) - \gamma(y) - \zeta(y) - 1\{r = y\}\Big) p(0, y)$$
$$\times \left[f(\tau_y \sigma, \tau_y \gamma^{0,y}, \tau_y \zeta^{0,y}, r^{0,y} - y) - f(\sigma, \gamma, \zeta, r) \right]\Big\} \tag{2.9}$$

where $r^{x,y} = r$ for $r \neq x$ and $r^{x,y} = y$ if $r = x$; L_3'' is the generator of the exchanges between γ and ζ particles:

$$L_3'' f(\sigma, \gamma, \zeta, r)$$
$$= \sum_{x \neq r} \sum_{y = x \pm 1 \neq r} \gamma(x) \zeta(y) p(x, y) \left[f(\sigma, \gamma^{x,y}, \zeta^{x,y}, r) - f(\sigma, \gamma, \zeta, r) \right] \tag{2.10}$$

and L_4'' is the generator which describes the motion of $R(t)$ on the ξ particles:

$$L_4''(\sigma, \gamma, \zeta, r)$$
$$= \Big(p\gamma(r-1) + q\zeta(r-1)\Big)\Big[f(\sigma, \gamma^{r-1,r}, \zeta^{r-1,r}, r-1) - f(\sigma, \gamma, \zeta, r)\Big]$$
$$+ \Big(q\gamma(r+1) + p\zeta(r+1)\Big)\Big[f(\sigma, \gamma^{r+1,r}, \zeta^{r+1,r}, r+1) - f(\sigma, \gamma, \zeta, r)\Big].$$

The point here is that the generators L_3'' and L_4'' do not affect the position of the σ and ξ particles as they describe interchanges of ξ particles. On the other hand, L_2'' does not affect the γ, ζ and R labeling of $x_i(t)$.

Let π_2 be a measure on X^2 with the good marginals, i.e. $\int d\pi_2(\sigma, \xi) f(\sigma) = \nu_\varrho f$ and $\int d\pi_2(\sigma, \xi) f(\sigma + \xi) = \nu_\lambda f$. Let $\pi_2' := \pi_2(.|\xi(0) = 1)$. Define the measure π_4' on $X^3 \times \mathbb{Z}$ as follows. Let A, B, C, $\{r\}$ be pairwise disjoint subsets of $\mathbb{Z}$ and $f_{A,B,C,r}(\sigma, \gamma, \zeta, R) := \prod_{x\in A} \sigma(x) \prod_{x\in B} \gamma(x) \prod_{x\in C} \zeta(x) 1\{R = r\}$. The fourth coordinate stands for the position of $R(t) - X(t)$.

$$\pi_4' f_{A,B,C,r} := \int d\pi_2'(\sigma, \xi) \prod_{x\in A} \sigma(x) \prod_{x\in B} \xi(x) m(n(\xi, x)|n(\xi, r))$$
$$\times \prod_{x\in C} \xi(x)(1 - m(n(\xi, x)|n(\xi, r))) \tag{2.11}$$
$$\times\, \xi(r) m(n(\xi, r))$$

where $m(i)$ and $m(j|i)$ are defined in (2.6) and $n(\xi, x)$ is the signed number of ξ particles between the origin and x:

$$n(\xi, x) := \begin{cases} \sum_{y=1}^{x} \xi(y) & \text{if } x > 0 \\ 0 & \text{if } x = 0 \\ -\sum_{y=x}^{-1} \xi(y) & \text{if } x < 0 \end{cases}$$

i.e. the label of the ξ particle at x. We use the notation π_2 for any measure with the good marginals and ν_2 for the product measure with the good marginals. The measures π_4' and ν_4' are constructed in the same manner from π_2 and ν_2 respectively. Call $S_i'(t)$ the semigroup corresponding to the generator L_i'. We shall prove that $\nu_4' S_4'(t)$ can be constructed from $\nu_2' S_2'(t)$ as π_4' is constructed from π_2' in (2.11). To do that we prove that if π_4' is defined as in (2.11) then it is reversible for L_3'' and L_4'':

$$\pi_4'(g L_3'' f) = \pi_4'(f L_3'' g) \tag{2.12}$$
$$\pi_4'(g L_4'' f) = \pi_4'(f L_4'' g) \tag{2.13}$$

for all f and g cylindric. We prove first (2.13) that is equivalent to prove that for all finite pairwise disjoint sets A, B, C, $\{r, r+1\}$,

$$\pi_4'\left(q1\{R=r\}\gamma(r+1)\prod_{x\in A}\sigma(x)\prod_{x\in B}\gamma(x)\prod_{x\in C}\zeta(x)\right)$$
$$=\pi_4'\left(p1\{R=r+1\}\gamma^{r,r+1}(r)\prod_{x\in A}\sigma(x)\prod_{x\in B}\gamma^{r,r+1}(x)\prod_{x\in C}\zeta^{r,r+1}(x)\right) \tag{2.14.a}$$

$$\pi_4'\left(p1\{R=r\}\zeta(r+1)\prod_{x\in A}\sigma(x)\prod_{x\in B}\gamma(x)\prod_{x\in C}\zeta(x)\right)$$
$$=\pi_4'\left(q1\{R=r+1\}\zeta^{r,r+1}(r)\prod_{x\in A}\sigma(x)\prod_{x\in B}\gamma^{r,r+1}(x)\prod_{x\in C}\zeta^{r,r+1}(x)\right). \tag{2.14.b}$$

Notice however that π_4' is not even invariant for L_4', the generator of the whole process. By the definition of π_4', (2.14a) is equivalent to the following identity when integrated with respect to π_2':

$$\begin{aligned}&\xi(r)\xi(r+1)q\,m(n(\xi,r))\,m(n(\xi,r+1)|n(\xi,r))\\ &\quad\times\prod_{x\in B}m(n(\xi,x)|n(\xi,r))\prod_{x\in C}(1-m(n(\xi,x)|n(\xi,r)))\\ &=\xi(r)\xi(r+1)p\,m(n(\xi,r+1))\,m(n(\xi,r)|n(\xi,r+1))\\ &\quad\times\prod_{x\in B}m(n(\xi,x)|n(\xi,r+1))\prod_{x\in C}(1-m(n(\xi,x)|n(\xi,r+1)))\,.\end{aligned} \tag{2.15}$$

Equation (2.14.b) is equivalent to an analogous identity. Now (2.15) holds if $\xi(r)=0$ or $\xi(r+1)=0$. Otherwise $n(\xi,r+1)=n(\xi,r)+1$. Hence it suffices to check that, for all i, the first of the following identities hold:

$$\begin{aligned}q\,m(i)m(i+1|i)&=p\,m(i+1)m(i|i+1)\\ p\,m(i)(1-m(i+1|i))&=q\,m(i+1)(1-m(i|i+1))\,.\end{aligned} \tag{2.16}$$

The second identity in (2.16) implies (2.14b). The proof of (2.16) is left to the reader. This proves (2.13). The proof of (2.12) is similar, the last step being to show that for $j, j+1\neq i$

$$p\,m(j|i)(1-m(j+1|i))=q\,m(j+1|i)(1-m(j|i))\,.$$

The motion related to the generator L_2'' does not change the labels of the ξ particles and by (2.12) and (2.13) the distribution of these labels remains invariant under the motion related to the generators L_3'' and L_4''. Equation (2.7) is then consequence of the Trotter-Kato formula:

$$S_4'(t)=\lim_{n\to\infty}(S_2''(t/n)S_3''(t/n)S_4''(t/n))^{n/3}\,.$$

where $S_i''(t)$ is the semigroup corresponding to the generator L_i'', $i=2, 3, 4$. □

Remark. We have proven that $\nu_4' S_4'(t)$ is reversible for $L_3'' + L_4''$. This implies that, letting $J, K \subset \mathbb{Z}$, $i \in \mathbb{Z}$, $\{i\}, J, K$ pairwise disjoint,

$$P_{\nu_4'}\left(R(t) = x_i(t)\,;\, \gamma_t(x_j(t)) = 1, \quad j \in J\,;\ \ \zeta_t(x_k(t)) = 1,\ \ k \in K \,\middle|\, \mathscr{F}_{2,t}\right)$$
$$= m(i) \prod_{j \in J} m(j|i) \prod_{k \in K} (1 - m(k|i))$$

or, in other words, using the notation of (2.11), for all $t \geqq 0$,

$$\nu_4' S_4'(t) f_{A,B,C,r} = \int \mathrm{d}\nu_2' S_2'(t)(\sigma, \xi) \prod_{x \in A} \sigma(x) \prod_{x \in B} \xi(x) m(n(\xi, x)|n(\xi, r))$$
$$\times \prod_{x \in C} \xi(x)(1 - m(n(\xi, x)|n(\xi, r)))$$
$$\times\, \xi(r) m(n(\xi, r))\ . \tag{2.17}$$

The next proposition is used to prove the convergence to an invariant measure in Theorem 2.2. Its proof uses again the basic coupling, but now between two copies of the process (η_t, ξ_t) which was constructed itself using that coupling. What we mean for basic couplig is again the fact that the two copies use the same realization of arrows. This guarantees that each marginal has the correct distribution. The same remark is valid for the future, where we will copy up to three processes each with different priorities.

Proposition 2.18 *Let π_2 be a measure on X^2 with the good marginals,* i.e. $\int \mathrm{d}\pi_2(\sigma, \xi) f(\sigma) = \nu_\varrho f$ *and* $\int \mathrm{d}\pi_2(\sigma, \xi) f(\sigma + \xi) = \nu_\lambda f$, *and let* $\pi_2' := \pi_2(.|\xi(0) = 1)$. *Let μ_2 be an invariant measure for $S_2(t)$ with the good marginals (its existence is proven in [fks]). Then $\pi_2 S_2(t)$ converges weakly to μ_2 and $\pi_2' S_2'(t)$ converges weakly to $\mu_2' := \mu_2(.|\xi(0) = 1)$.*

Proof. Consider the basic coupling between two copies of (σ_t, ξ_t):

$$(\sigma_t^0, \xi_t^0) \quad \text{with initial measure } \pi_2$$
$$(\sigma_t^1, \xi_t^1) \quad \text{with initial measure } \mu_2\ .$$

Since the σ and the $\sigma + \xi$ marginals are the same for π_2 and μ_2, we can assume that under our coupling, at time 0 (and hence for all later times), $\sigma_0^0 + \xi_0^0 = \sigma_0^1 + \xi_0^1$. In that way the marginal (σ_0^0, σ_0^1) has a translation invariant measure and we can apply Lemma 3.2 of Chap. VIII of Liggett [L] to the marginal coupling (σ_t^0, σ_t^1) to obtain that any weak limit $\tilde{\mu}$ of $(\sigma_t^0, \sigma_t^1, \xi_t^0, \xi_t^1)$ satisfies $\tilde{\mu}(\sigma^0(x) = \sigma^1(y) = 1, \sigma^0(y) = \sigma^1(x) = 0) = 0$ for all x, y. Hence $\tilde{\mu}(\sigma^0 \geqq \sigma^1$ or $\sigma^0 \leqq \sigma^1) = 1$. This implies that either $\tilde{\mu}\{\sigma^0 \geqq \sigma^1\} > 0$ or $\tilde{\mu}\{\sigma^0 \leqq \sigma^1\} > 0$. Assume $\tilde{\mu}(\mathbf{A}) > 0$, where $\mathbf{A} := \{\sigma^0 \geqq \sigma^1\}$, the other case being similar. We want to prove that $\tilde{\mu}(\sigma^0 = \sigma^1) = 1$. We proceed by contradiction. Assume that $\tilde{\mu}(\sigma^0 > \sigma^1, \mathbf{A}) > 0$. Then, by translation invariance,

$\widetilde{\mu}(\sigma^0(x) > \sigma^1(x), \mathbf{A}) = c > 0$ for all x, and for all $n \geqq 0$,

$$\int_{\mathbf{A}} d\widetilde{\mu}(\sigma^0, \sigma^1)\left(\frac{1}{n}\sum_{x=1}^{n}(\sigma^0(x) - \sigma^1(x))\right) = c > 0 . \qquad (2.19)$$

But his leads to a contradiction because the first two marginals of $\widetilde{\mu}$ are ν_{ϱ}, and the law of large numbers plus dominate convergence imply that the limit as $n \to \infty$ of the left hand side of (2.19) is zero. Then $\widetilde{\mu}(\sigma^0 = \sigma^1) = 1$. Since $\sigma_t^1 + \xi_t^1 = \sigma_t^0 + \xi_t^0$ for all t, this implies that $\pi_2 S_2(t)$ converges to μ_2. This and the fact that $\pi_2' S_2'(t) = \pi_2 S_2(t)(.|\xi(0) = 1)$ (Lemma 3.6 of [fks]) imply that $\pi_2' S_2'(t)$ converges to μ_2'. □

Proof of Theorem 2.2. Proposition 2.3 guarantees that since ν_2' has the good marginals, then, under initial measure ν_4', $X(t) - R(t)$ is tight. This and the fact that the projection over $\sigma + \gamma$ of $\nu_4' S_4'(t)$ is $\sim \nu_{\varrho,\lambda}$ uniformly in t (Lemma 3.26 of [fks]) imply that $\tau_{R(t)}\eta_t$ has distribution $\sim \nu_{\varrho,\lambda}$ uniformly in t. This proves the first part of the theorem.

For the second part, we observe that Proposition 2.18 and Eq. (2.17) imply that $\nu_4' S_4'(t)$ converges to a unique measure for the process as seen from $X(t)$. This implies that $X(t) - R(t)$ converges in distribution. Since $E|X(t) - R(t)|$ is uniformly bounded, $\tau_{R(t)}\eta_t = \tau_{R(t)-X(t)}\tau_{X(t)}(\sigma_t + \gamma_t)$ converges in law to the invariant measure $\widehat{\mu}$. □

3 Laws of large numbers

In this section we prove laws of large numbers for $R(t)$ and others microscopic shocks. We start with a strong law of large numbers for $X(t)$ when the initial measure in any measure with the good marginals. This extends the results of [fks], who proved a weak law when the initial measure is product and a strong law when the initial measure is the invariant measure μ_2'. We give here an unified proof. Let $v = (p - q)(1 - \lambda - \varrho)$.

Theorem 3.1 *Let π_2 be a measure with the good marginals as in Proposition 2.18, and $\pi_2' = \pi_2(.|\xi(0) = 1)$. Then*

$$\lim_{t\to\infty} \frac{X(t)}{t} = v, \qquad P_{\pi_2'} \text{ almost surely.}$$

Proof. Let $U(t) \in \mathbb{Z}$ and define $F(\xi_t, U(t)) :=$ number of ξ particles that at time zero were to the left of the origin and at time t are to the right of $U(t)$ minus the number of ξ particles that at time zero were to the right of the origin and at time t are to the left of $U(t)$. In other words, $F(\xi_t, U(t))$ is the net flux of ξ particles throught the space time line $[(0,0), (U(t), t)]$. Notice that $F(\xi_t, X(t)) \equiv 0$. Define analogously $F(\sigma_t, U(t))$ for the process σ_t and $F(\sigma_t + \xi_t, U(t))$ for the process $\sigma_t + \xi_t$. Now assume that $U(t)$ is a random walk on $\mathbb{Z}$ that jumps to the right neighbor at rate w independent of (σ_t, ξ_t) and study the process $\tau_{U(t)}\sigma_t$. Using the techniques of Liggett [l], it can be proven that all invariant measures for this process are translation invariant and that $\nu_\alpha, 0 \leqq \alpha \leqq 1$, are extremal invariant for this process. Notice that $F(\tau_{U(t)}\sigma_t, 0) = F(\sigma_t, U(t))$. Now we can use the martingale decomposition of $F(\tau_{U(t)}\sigma_t, 0)$ and the fact what ν_α is extremal for $\tau_{U(t)}\sigma_t$ to prove – as in the proof of Theorem 6 of [k] or Theorem 1 of [s] – that

$$\lim_{t\to\infty} \frac{F(\sigma_t, U(t))}{t} = (p-q)\varrho(1-\varrho) - w\varrho, \quad P_{\pi'_2} \text{ a.s.}$$

where we used that the σ marginal of π'_2 is absolutely continuous with respect to ν_ϱ. Analogously using the fact that $\sigma_t + \xi_t$ is also the simple exclusion process with measure (absolutely continuous with respect to) ν_λ,

$$\lim_{t\to\infty} \frac{F(\sigma_t + \xi_t, U(t))}{t} = (p-q)\lambda(1-\lambda) - w\lambda, \quad P_{\pi'_2} \text{ a.s.}$$

from where, using the fact that $F(\xi_t, U(t)) = F(\sigma_t + \xi_t, U(t)) - F(\sigma_t, U(t))$,

$$\lim_{t\to\infty} \frac{F(\xi_t, U(t))}{t} = (p-q)(\lambda(1-\lambda) - \varrho(1-\varrho)) - w(\lambda - \varrho), \quad P_{\pi'_2} \text{ a.s.}$$

Now check that this limit is negative for $w > v = (p-q)(1-\lambda-\varrho)$ and positive for $w < v$. On the other hand, $\lim_{t\to\infty} U(t)/t = w$ a.s. because $U(t)$ is a Poisson random variable with mean wt. Also $F(\xi_t, x)$ is a non increasing function of x. Hence, since $F(\xi_t, X(t)) \equiv 0$,

$$\limsup_{t\to\infty} \frac{X(t)}{t} \leqq \lim_{t\to\infty} \frac{U(t)}{t} = w, \quad P_{\pi'_2} \text{ a.s.}$$

for all $w > v$, and analogously,

$$\liminf_{t\to\infty} \frac{X(t)}{t} \geqq \lim_{t\to\infty} \frac{U(t)}{t} = w, \quad P_{\pi'_2} \text{ a.s.}$$

for all $w < v$. This proves the Theorem. □

The next lemma is a corollary to Proposition 2.3.

Lemma 3.2 *Let $G(t)$ be the position of the leftmost particle of γ_t and $Z(t)$ the position of the rightmost particle of ζ_t. Then for all $t \geqq 0$*

$$E_{\nu'_4}|R(t) - G(t)| = E_{\nu'_4}|R(t) - Z(t)| < C < \infty \tag{3.3}$$

and

$$E_{\nu'_4}(R(t) - X(t)) = 0. \tag{3.4}$$

Proof. When the initial measure is ν'_4, the average distance between succesive ξ particles is $(\lambda-\varrho)^{-1}$ for all times. Furthermore the way of choosing which ξ particles are γ particles and the R particle is independent of the position of the ξ particles. This implies that

$$E_{\nu'_3}|X(t) - G(t)| = (\lambda-\varrho)^{-1} \sum_{k\in\mathbb{Z}} |k| \frac{(p/q)^k}{1+(p/q)^k} \prod_{l>k} \frac{1}{1+(p/q)^l} < \infty. \tag{3.5}$$

This and $E|X(t)-R(t)|<C<\infty$ (Proposition 2.3) imply (3.3). Equation (3.4) is a consequence of the symmetry of $m(t)$ with respect to the origin. □

In the next theorem we show that the different positions of the shock satisfy laws of large numbers.

Theorem 3.6 *The following holds*

$$\lim_{t\to\infty}\frac{R(t)}{t}=v,\quad P_{\widehat{\nu}_{\varrho,\lambda}} \text{ and } P_{\widehat{\mu}} \text{ a.s.};\tag{3.7}$$

$$\lim_{t\to\infty}\frac{G(t)}{t}=\lim_{t\to\infty}\frac{Z(t)}{t}=v,\quad P_{\nu'_3} \text{ and } P_{\mu'_3} \text{ a.s.}.\tag{3.8}$$

Proof. By Theorem 3.1, $X(t)/t$ converges $P_{\mu'_2}$ a.s. and $P_{\nu'_2}$ a.s. to v. Proposition 2.3 says that $X(t)-R(t)$ is tight. Hence $R(t)/t$ converges $P_{\widehat{\mu}}$ a.s. and $P_{\widehat{\nu}_{\varrho,\lambda}}$ a.s. to v. This and Lemma 3.2 imply that both $G(t)/t$ and $Z(t)/t$ converge $P_{\mu'_3}$ a.s. and $P_{\nu'_3}$ a.s. to v. □

Now we prove a law of large numbers for a single second class particle when the initial density of first class particles out of the origin is α (no shock). We consider the process $(\eta_t, D(t))$ on $X\times\mathbb{Z}$, with priority $\eta_t \vdash D(t)$ and inital measure $\widehat{\nu}_\alpha$, the product measure of density α with the second class particle at the origin.

Theorem 3.9 *Let $\eta_t \vdash D(t)$. Then*

$$\lim_{t\to\infty}\frac{D(t)}{t}=(p-q)(1-2\alpha),\quad P_{\widehat{\nu}_\alpha} \text{ a.s.}$$

Proof. Couple the process $(\eta_t, D(t))$ and the process $(\sigma_t,\gamma_t,\zeta_t)$ with initial distribution ν'_3, with $\lambda>\varrho=\alpha$, in such a way that at $t=0$, $D(0)=0$, $\eta_0(0)=0$ and $\eta_0(x)=\sigma_0(x)$ for $x\neq 0$. This gives distribution $\widehat{\nu}_\alpha$ to $(\eta_0, D(0))$ and the correct distribution for later times. Letting $G(t)$ to be the position of the leftmost γ_t particle,

$$\gamma_0(0)=1 \quad\text{implies}\quad D(t)\geqq G(t)\,.\tag{3.10}$$

To prove (3.10) observe that it holds trivially for $t=0$. Then observe that if $G(t)=D(t)$, a right jump of $G(t)$ implies a right jump of $D(t)$ while a left jump of $D(t)$ implies a left jump of $G(t)$. From (3.10) we get that, for any $\lambda>\varrho$, in $\{\gamma_0(0)=1\}$,

$$\liminf_{t\to\infty}\frac{D(t)}{t}\geqq\lim_{t\to\infty}\frac{G(t)}{t}=(p-q)(1-\varrho-\lambda),\quad \text{a.s.}\tag{3.11}$$

where the identity is (3.8). Since this holds for all $\lambda>\varrho$,

$$\liminf_{t\to\infty}\frac{D(t)}{t}\geqq(p-q)(1-2\alpha),\quad P_{\widehat{\nu}_\alpha} \text{ a.s.}\tag{3.12}$$

where we have used that $\nu'_3(\gamma(0)=1)>0$.

On the other hand, couple again $(\eta_t, D(t))$ with $(\sigma_t, \gamma_t, \zeta_t)$ but now with initial distribution ν_3' with $\varrho < \lambda = \alpha$. At time $t = 0$:

$$D(0) = 0, \quad \eta_0(0) = 0$$
$$\eta_0(x) = \sigma_t(x) + \gamma_t(x) + \zeta_t(x) \quad \text{for } x \neq 0 .$$

Leting $Z(t)$ to be the rightmost ζ_t particle, analogously to (3.10),

$$\zeta_0(0) = 1 \quad \text{implies} \quad D(t) \leqq Z(t), \quad t \geqq 0,$$

and

$$\limsup_{t \to \infty} \frac{D(t)}{t} \leqq (p-q)(1-2\alpha), \quad P_{\hat{\nu}_\alpha} \text{ a.s.}$$

This and (3.12) finish the proof. □

4 Second class particle and characteristics

In this section we prove that the macroscopic motion of a second class particle coincides with a characteristic of the Burgers equation. The characteristic corresponding to $r \in \mathbb{R}$ is the curve $w(r, t)$ in space time satisfying $w(r, 0) = r$ and $u(w(r, t), t) =$ constant, where u is a solution of (1.1). In our case $\theta = p - q$ and

$$w(r, t) = \begin{cases} (p-q)(1-2\lambda)t + r & \text{for } r > 0 \\ (p-q)(1-2\varrho)t + r & \text{for } r < 0 . \end{cases} \tag{4.1}$$

Since $\lambda > \varrho$, the characteristics to the right are slower than the ones to the left. Hence they meet, developing a shock. The shock is travelling at velocity $v = (p-q)(1-\lambda-\varrho)$. The characteristics starting at r and $-r$ respectively meet the shock at time

$$t(r) := \frac{|r|}{(p-q)(\lambda-\varrho)} . \tag{4.2}$$

In the next theorem we abuse notation. The measure $\nu_{\varrho,\lambda}$ stands for a measure on $X \times \mathbb{Z}$, being a product measure on X for all but one site: at (the integer part of) $\varepsilon^{-1}r$ there is a second class particle.

Theorem 4.3 *Let $Y(x, t)$ be the position at time t of a second class particle that at time zero is at site x. Then as $\varepsilon \to 0$, $\varepsilon Y(\varepsilon^{-1}r, \varepsilon^{-1}t)$ converges $P_{\nu_{\varrho,\lambda}}$ a.s. to $w(r, t)$ for $t < t(r)$ and to vt for $t \geqq t(r)$.*

Proof. For each pair (ε, r), $r > 0$ couple the processes $(\eta_t, Y(\varepsilon^{-1}r, t))$, with priority $\eta_t \vdash Y(., t)$; $(\sigma_t, \gamma_t, \zeta_t, R(t))$ with priorities $\sigma_t \vdash \gamma_t \vdash R(t) \vdash \zeta_t$ and $(\bar{\eta}_t, D(\varepsilon^{-1}r, t))$ with priority $\bar{\eta}_t \vdash D(\varepsilon^{-1}r, t)$. At time $t = 0$ set $R(0) = 0$ and distribute $(\sigma_0, \gamma_0, \zeta_0)$ on the other sites according to ν_3'. Set $\eta_0 = \sigma_0 + \gamma_0$,

$\bar{\eta}_0 = \sigma_0 + \gamma_0 + \zeta_0$ and $Y(\varepsilon^{-1}r, 0) = D(\varepsilon^{-1}r, 0) = [\varepsilon^{-1}r]$. Define

$$T_1(\varepsilon, r) := \inf\{t : D(\varepsilon^{-1}r, t) \neq Y(\varepsilon^{-1}r, t) \text{ or } Y(\varepsilon^{-1}r, t) = R(t)\}$$
$$T_2(\varepsilon, r) := \inf\{t : Y(\varepsilon^{-1}r, t) = R(t)\}$$

if those first times do not exist, we set $T_i = \infty$. Under this coupling, $D(.,t) = Y(.,t)$ up to the first moment that they meet a ζ particle. By the laws of large numbers for $D(t)$ and $Z(t)$ (the position of the rightmost ζ particle), (Theorems 3.9 and 3.6)

$$\lim_{\varepsilon \to 0} \varepsilon T_1(\varepsilon, r) = t(r), \quad \text{a.s.} \tag{4.4}$$

After T_2, $Y(.,t) \equiv R(t)$. Hence it suffices to prove that $\lim_{\varepsilon \to 0} \varepsilon T_2(\varepsilon, r) = t(r)$. Let $\Omega' := \{\lim_{\varepsilon \to 0} \varepsilon T_1(\varepsilon, r) = t(r)\}$. Since $T_2 \geqq T_1$, it suffices to show that $P(\limsup_{\varepsilon \to 0} \varepsilon(T_2 - T_1) > 0, \Omega') = 0$. But, after T_1, $Y(\varepsilon^{-1}r, t) \leqq \max\{Z(t), R(t)\}$ by the same argument to prove (3.10). Hence

$$P\left(\limsup_{\varepsilon \to 0} \varepsilon(T_2 - T_1) > 0, \Omega'\right) \leqq P\left(Z(t) - R(t) > 0, \quad \forall t > t(r)\right). \tag{4.5}$$

By Lemma 3.2, $E(Z(t) - R(t)) < C < \infty$. Then Chevychev inequality implies that, for all $\delta > 0$, there exists $M > 0$ such that $P(Z(t) - R(t) \geqq M) \leqq \delta$, for all $t \geqq 0$, which implies, for $n \in \mathbb{N}$, $P(Z(n) - R(n) < M$, infinitely often$) \geqq 1 - \delta$. But each time that $Z(n) - R(n) < M$, they have a uniformly bounded above zero probability of meeting in a time interval of length 1. Hence

$$P\left(Z(n) = R(n), \quad \text{for some } n\right) \geqq 1 - \delta. \tag{4.6}$$

Since (4.6) holds for all δ, this implies that the right hand side of (4.5) vanishes. A similar argument wokrs for $r < 0$, by defining $\bar{\eta}_0 = \sigma_0$. □

5 Initial perturbations produce shock translations

In this section we show that, as $t \to \infty$, a perturbation at one site of the inital measure $\nu_{\varrho,\lambda}$ produces a translation of the shock of the order of $(\lambda - \varrho)^{-1}$. This behavior can also be observed in the Burgers equation with shock initial conditions. Denote $R(\eta, t)$ the (random) position at time t of a second class particle that at time 0 is at the origin when the initial configuration is η. For any configuration η, a site $y \in \mathbb{Z}$, let $\eta^{y|i}$ be defined by ($i \in \{0, 1\}$)

$$\eta^{y|i}(x) = \begin{cases} \eta(x) & \text{for } x \neq y \\ i & \text{for } x = y\,. \end{cases}$$

Define $r^+ := (\lambda - \varrho)(p - q)$, $r^- := -(\varrho - \lambda)(p - q)$.

Theorem 5.1 *For all $\varepsilon > 0$ it holds*

$$\lim_{t\to\infty} \sup_{(r^- + \varepsilon)t < y < (r^+ - \varepsilon)t} \left| E_{\nu_{\varrho,\lambda}} \left(R(\eta^{y|0}, t) - R(\eta^{y|1}, t) \right) - (\lambda - \varrho)^{-1} \right| = 0 \quad (5.2)$$

$$\lim_{t\to\infty} \frac{1}{t} \sum_{y=0}^{r^+ t} E_{\nu_{\varrho,\lambda}} \left(R(\eta^{y|0}, t) - R(\eta^{y|1}, t) \right) = p - q \quad (5.3)$$

$$\lim_{t\to\infty} \frac{1}{t} \sum_{y=r^- t} E_{\nu_{\varrho,\lambda}} \left(R(\eta^{y|0}, t) - R(\eta^{y|1}, t) \right) = p - q\,. \quad (5.4)$$

In order to prove this Theorem we need the following Lemma

Lemma 5.5 *Let $R_{-1}(t)$ be a particle such that $\eta_t \vdash R(t) \vdash R_{-1}$. Then the initial distribution of $(\eta_t, R(t), R_{-1}(t))$ can be chosen such that, for all $t \geqq 0$, both $\tau_{R_{-1}(t)}(\eta_t \cup R(t))$ and $\tau_{R(t)}\eta_t$ have distribution $\hat{\nu}_{\varrho,\lambda}\hat{S}(t)$. Furthermore, with respect to the chosen initial distribution, $E(R(t) - R_{-1}(t)) = (\lambda - \varrho)^{-1}$, $t \geqq 0$.*

Proof. First observe that $R_{-1}(t)$ is a second class particle with respect to $\eta_t \cup R(t)$. Consider the coupling $(\sigma_t, \gamma_t, \zeta_t, R(t), R_{-1}(t))$ with priorities $\gamma_t \vdash \zeta_t \vdash R(t) \vdash R_{-1}(t)$. To define the initial distribution pick (σ, ξ) from the product measure $\nu_2' = \nu_2(.|\xi(0) = 0)$ and call x_i the position of the i-th ξ particle, $x_0 = 0$. Set $\zeta(x_i) = 1$ for $i \leqq -2$, $\gamma(x_i) = 1$ for $i \geqq 1$, $R(0) = x_0 = 0$, $R_{-1}(0) = x_{-1}$. Under the resulting distribution, $\tau_{R_{-1}(t)}(\sigma_t, \gamma_t \cup R(t), \zeta_t)$ and $\tau_{R(t)}(\sigma_t, \gamma_t, \zeta_t \cup R_{-1}(t))$ are identically distributed for all $t \geqq 0$. The projections $\tau_{R_{-1}(t)}(\sigma_t \cup \gamma_t \cup R(t))$ and $\tau_{R(t)}(\sigma_t \cup \gamma_t)$ are both distributed according to $\hat{\nu}_{\varrho,\lambda}\hat{S}(t)$. Hence, denoting $\eta_t = \sigma_t + \gamma_t$, we have proven the first part of the Lemma.

By (3.4) $E(R(t) - X(t)) = 0$ and $E(R_{-1}(t) - x_{-1}(t)) = 0$, where $x_i(t)$ is the position of the i-th particle of ξ $(x_0(t) \equiv X(t))$. This implies that $E(R(t) - R_{-1}(t)) = E(X(t) - x_{-1}(t)) = (\lambda - \varrho)^{-1}$. □

Proof of Theorem 5.1 Couple $(\eta^{y|1})_t$ and $(\eta^{y|0})_t$ – the processes with initial configurations $\eta^{y|1}$ and $\eta^{y|0}$, respectively – according to the basic coupling. Let $Y(y, t)$ be the site where $(\eta^{y|1})_t$ and $(\eta^{y|0})_t$ are different. Then $Y(y, t)$ behaves like a second class particle with respect to η, i.e. $(\eta^{y|0})_t \vdash Y(y, t)$. Now, $R(\eta^{y|0}, t) = R(\eta^{y|1}, t)$ until $T_1(y) :=$ first time that $Y(y, t) = R(\eta^{y|0}, t)$. After T_1, $R(\eta^{y|1}, t) = Y(y, t)$. Define $T_2(y) :=$ first time that $Y(y, t) = R_{-1}(t)$. After T_2, $Y(y, t) = R_{-1}(t)$. Then

$$\begin{aligned} E_{\nu_{\varrho,\lambda}}(R(\eta^{y|0}, t) - R(\eta^{y|1}, t)) &= E_{\nu_{\varrho,\lambda}}\left(R(t) - R_{-1}(t),\quad t \geqq T_2(y) \right) \\ &+ E_{\nu_{\varrho,\lambda}}\left(R(t) - Y(y, t),\quad T_1 \leqq t \leqq T_2(y) \right). \end{aligned} \quad (5.6)$$

A coupling argument shows that the processes can be constructed in such a way that $\{t \geqq T_i(y)\}$ is non decreasing for positive y and non increasing for negative y. Hence, as in the proof of Theorem 4.3,

$$P\left(\lim_{t\to\infty}\left(\sup_{(r^- +\varepsilon)t<y<(r^+ -\varepsilon)t}(t-T_2(y))\right)\geqq 0\right)=1\,. \tag{5.7}$$

The first term in the right hand side of (5.6) is bounded by $E|R(t)-R_{-1}(t)| < C < \infty$. Hence, by dominated convergence and (5.7) it converges to $(\lambda-\varrho)^{-1}$. The second term in the same equation converges to zero by an argument analogous to the one we used to prove Theorem 4.3.

Since $E(|R(t)-Y(y,t)|, T_1 \leqq t \leqq T_2) < C < \infty$, Eq. (5.6) also implies that $E_{\nu_{\varrho,\lambda}}(|R(\eta^{y|0},t)-R(\eta^{y|1}t)|) < C < \infty$. Hence, by dominated convergence (5.2) implies (5.3) and (5.4). □

6 Dependence on the initial configuration

We prove here a formula relating the diffusion coefficient of the shock $R(t)$ to the conjectured diffusion coefficient. We call, as before, $v=(p-q)(1-\varrho-\lambda)$. Define

$$\bar{D} := (p-q)\frac{\varrho(1-\varrho)+\lambda(1-\lambda)}{\lambda-\varrho}\,,$$

$$F(t) := E_{\hat{\nu}_{\varrho,\lambda}}(R(t)-vt)^2\,,$$

$$I(t) := \int d\hat{\nu}_{\varrho,\lambda}(\eta)E\left(R(\eta,t)-\frac{n_0(\eta,r^+t)}{\lambda-\varrho}-\frac{n_1(\eta,r^-t)}{\varrho-\lambda}\right)^2,$$

where $R(\eta,t)$ and $r^{\pm}$ are defined at the begining of Sect. 5 and $n_0(\eta,x) := \sum_{y=0}^{x}(1-\eta(y))$ is the number of empty sites of η between 0 and x and $n_1(\eta,x) := -\sum_{y=x}^{0}\eta(y)$ is minus the number of η particles between the origin and $x<0$.

Theorem 6.1 *The following holds*

$$\lim_{t\to\infty}\frac{F(t)}{t}=\bar{D}+\lim_{t\to\infty}\frac{I(t)}{t} \tag{6.2}$$

if the limits exist. If not (6.2) holds with lim *substituted by either* lim sup *or* lim inf.

Proof. Summing and substracting vt, $I(t)$ equals

$$\int d\hat{\nu}_{\varrho,\lambda}(\eta)E(R(\eta,t)-vt)^2+\int d\hat{\nu}_{\varrho,\lambda}(\eta)\left(\frac{n_0(\eta,r^+t)}{\lambda-\varrho}-(p-q)(1-\lambda)t\right)^2$$
$$+\int d\hat{\nu}_{\varrho,\lambda}(\eta)\left(\frac{n_1(\eta,r^-t)}{\varrho-\lambda}-(p-q)\varrho t\right)^2-2\int d\hat{\nu}_{\varrho,\lambda}(\eta)$$
$$\times E\left(R(\eta,t)\left(\frac{n_0(\eta,r^+t)}{\lambda-\varrho}-(p-q)(1-\lambda)t+\frac{n_1(\eta,r^-t)}{\varrho-\lambda}-(p-q)\varrho t\right)\right) \tag{6.3}$$

where we have used that $n_0(\eta,r^+t)$ and $n_1(\eta,r^-t)$ are independent under $\hat{\nu}_{\varrho,\lambda}$. Dividing by t and taking $t\to\infty$, the first term gives $\lim(F(t)/t)$, and the second and third terms give $\overline{D}$. Then it suffices to show that dividing by t and taking $t\to\infty$ the last term equals $-2\overline{D}$. Using the definition of $n_i(.,.)$, the expectation in the last term in (6.3) equals

$$\frac{1}{\lambda-\varrho}E\left(\sum_{x=1}^{r^+t}R(\eta,t)(1-\eta(x)-(1-\lambda))+\sum_{x=r^-t}^{-1}R(\eta,t)(\eta(x)-\varrho)\right). \tag{6.4}$$

Integrating the first term of (6.4),

$$-\frac{1}{\lambda-\varrho}\int d\hat{\nu}_{\varrho,\lambda}(\eta)\sum_{x=1}^{r^+t}E\Big(R(\eta,t)(\eta(x)-\lambda)\Big)$$
$$=-\frac{1}{\lambda-\varrho}\int d\hat{\nu}_{\varrho,\lambda}(\eta)\sum_{x=1}^{r^+t}\Big[E\Big(R(\eta,t)\Big|\eta(x)=1\Big)\lambda$$
$$-\lambda\Big(E\Big(R(\eta,t)\Big|\eta(x)=1\Big)\lambda+E\Big(R(\eta,t)\Big|\eta(x)=0\Big)(1-\lambda)\Big)\Big]$$
$$=-\frac{1}{\lambda-\varrho}\lambda(1-\lambda)\int d\hat{\nu}_{\varrho,\lambda}(\eta)$$
$$\times\sum_{x=1}^{r^+t}\Big[E\Big(R(\eta,t)\Big|\eta(x)=1\Big)-E\Big(R(\eta,t)\Big|\eta(x)=0\Big)\Big]$$
$$=-\frac{1}{\lambda-\varrho}\lambda(1-\lambda)\int d\hat{\nu}_{\varrho,\lambda}(\eta)\sum_{x=1}^{r^+t}E\Big(R(\eta^{x|1},t)-R(\eta^{x|0},t)\Big). \tag{6.5}$$

Dividing by t and taking the limit as $t\to\infty$ of the first term of (6.4), we get using (5.3) on (6.5) that

$$\lim_{t\to\infty}\frac{1}{\lambda-\varrho}\frac{1}{t}\int d\hat{\nu}_{\varrho,\lambda}(\eta)\sum_{x=-1}^{r^+t}E\Big(R(\eta,t)(-\eta(x)+\lambda)\Big)=(p-q)\frac{\lambda(1-\lambda)}{\lambda-\varrho}$$

and analogously using (5.4),

$$\lim_{t\to\infty}\frac{1}{\lambda-\varrho}\int d\hat{\nu}_{\varrho,\lambda}(\eta)\sum_{x=r^-t}^{-1}E\Big(R(\eta,t)(\eta(x)-\varrho)\Big)=(p-q)\frac{\varrho(1-\varrho)}{\lambda-\varrho}\,.$$

This implies the Theorem. □

I thank Errico Presutti for telling me the above proof.

Remarks 6.6 From Theorem 6.1 we conclude:

1. The diffusion coefficient of the shock is the same as the conjectured diffusion coefficient if and only if the position of the shock at time t is given – in the scale $\sqrt{t}$ – by $(p-q)(\lambda-\varrho)^{-1}$ times the number of holes between 0 and r^+t minus the number of particles between 0 and r^-t. In any case, $I(t)$ is non negative, then $\bar{D}$ is always a lower bound for the lim inf of $F(t)/t$ as announced in (1.4).

2. Tightness of $R(t)-X(t)$ and $R(t)-G(t)$ imply that Theorem 6.1 also holds for $X(t)$ and $G(t)$. When $\varrho=0$, $X(t)$ has the distribution of a plain tagged particle in the simple exclusion process with density λ and $G(t)$ has the distribution of the leftmost particle in simple exclusion with initial distribution $\nu_{0,\lambda}$. In the case $\varrho=0$, it is known that $D:=\lim_{t\to\infty}t^{-1}E(X(t)-EX(t))^2=\bar{D}$ $=(p-q)(1-\lambda)$ [df]. This implies that $\lim_{t\to\infty}I(t)/t=0$, hence in the scale $\sqrt{t}$ the position of $R(t)$ is determined by the initial configuration in the sense discussed above. This was proved for $G(t)$ when $p=1$ by Gärtner and Presutti [gp]. This and the previous remark imply that we get for free the central limit theorem of [k] for the tagged particle and of [dkps] for the leftmost particle. Unfortunately one needs to use the precise computation of [df] for the diffusion coefficient. An independent proof that $\lim I(t)/t=0$ would give a direct proof of the central limit theorems for all these objects.

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Note added in proof. Items (*b*) and (*c*) of Proposition 1 of [bf] are false. Indeed, in our notation, the product measure ν_ϱ is not stationary for the process (σ_t, ξ_t). Hence the subadditive ergodic theorem can not be applied.

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