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**Probability Theory and Related Fields**



# Limit distributions for minimal displacement of branching random walks

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**Summary.** We study the minimal displacement $(X_n)$ of branching random walk with non-negative steps. It is shown that $(X_n - EX_n)$ is tight under a mild moment condition on the displacements. For supercritical B.R.W. $(X_n)$ converges almost surely. For critical B.R.W. we determine the possible limit points of $(X_n - EX_n)$, and we prove a generalization of Kolmogorov's theorem on the extinction probability of a critical branching process. Finally we generalize Bramson's results on the almost sure convergence of $X_n \log 2/\log\log n$.

## 1 Introduction

We consider discrete time and discrete space branching diffusion. At time 0 there is a single particle at the origin. At time 1 this particle splits into a random number $N$ of particles, which move to positions $D_1, D_2, \ldots, D_N$, where the $D_j$ are (not necessarily independent) integer valued random variables. At time $n$ a particle $v$ with position $k$ gives rise to a random number $N(v)$ of particle at positions $k+D_1(v), k+D_2(v), \ldots, k+D_{N(v)}(v)$, where the vector $(N(v); D_1(v), \ldots, D_{N(v)}(v))$ is distributed as $(N; D_1, \ldots, D_N)$, and is independent of the evolution of all other particles at the same and previous times.

Let $\Gamma_n$ be the set of all particles at time $n$, and let $N_n = \text{Card}\,\Gamma_n$. Then $(N_n)$ is a Galton-Watson process with offspring distribution $N_1 = N$. Furthermore, let $d(v)$ be the position of a particle $v$ in $\Gamma_n$. The random variable

$$X_n = \min\{d(v) : v \in \Gamma_n\}$$

is the *minimal displacement* of the branching random walk process determined by the random vector $(N; D_1, \ldots, D_N)$. Here we put $X_n = \infty$ if $\Gamma_n$ is empty. Note that $X_0 \equiv 0$. The random variables $(X_n)$ have been studied by several authors, sometimes in unexpected contexts [4], [8]. The following basic result holds for the sequence $(X_n)$.

**H.K.B.-Theorem** ([2, 6, 9]) *Let* $H(t) = E \sum_{j=1}^{N} 1_{(-\infty,t]}(D_j)$, *and* $m(\theta) = \int_{\mathbb{R}} e^{-\theta t}\, dH(t)$. *Suppose that* $m(\theta) < \infty$ *for some* $\theta > 0$. *Then*

$$\frac{X_n}{n} \to \gamma \quad \textit{almost surely on } S, \quad \textit{and in } L_1, \tag{1.1}$$

*where* $S$ *is the set of nonextinction of the branching process* $(N_n)$, *and* $\gamma = \inf\{a : \mu(a) > 1\}$, *with* $\mu(a) = \inf\{e^{\theta a} m(\theta) : \theta \geqq 0\}$.

In this paper we consider the case where the displacements are non- negative (this is no loss of generality if the displacements are bounded)

$$D_j \geqq 0 \quad j = 1, \ldots, N .$$

This obviously implies that $(X_n)$ is increasing

$$X_{n+1} \geqq X_n \quad n = 0, 1, \ldots \tag{1.2}$$

It is convenient to introduce the random variables $N(k)$ defined for $k \geqq 0$ by

$$N(k) = \operatorname{Card}\{j : D_j = k\} .$$

With this notation we have $m(\theta) = \sum_{k=0}^{\infty} EN(k)\, e^{-k\theta}$. Note that the constant $\gamma$ in (1.1) is zero if and only if $\mu(0) \geqq 1$. But since $m'(\theta) < 0$ in any non-trivial case, we have that $\mu(0) = \lim_{\theta \to \infty} m(\theta) = EN(0)$. As $\mu$ is continuous on the set where $\mu$ is positive, we obtain the following.

**Observation.** *Let* $(X_n)$ *be the minimal displacement of a branching random walk with non-negative steps and* H.K.B.*-constant* $\gamma$. *Then* $\gamma = 0$ *iff* $EN(0) \geqq 1$.

We call a branching random walk determined by $(N(k))_{k=0}^{\infty}$ *subcritical, critical, or supercritical* if, respectively, $EN(0) < 1$, $EN(0) = 1$, or $EN(0) > 1$.

In Sect. 2 we prove that $(X_n)$ converges a.s. on the set of nonextinction (without any norming), if the process is supercritical (this answers a question by Durrett [5, p. 118]). We also determine the distribution of the limit. In Sect. 3 we show quite generally that the sequence $(X_n - EX_n)$ is tight (travelling wave phenomenon). The remainder of the paper is devoted to the critical case. In Sects. 4 to 7 we determine the possible limit distributions of $(X_n - d_n)$, where $(d_n)$ is an integer sequence such that $(d_n - EX_n)$ is bounded. In the final part of the paper we generalize Bramson's results on the almost sure behaviour of $(X_n)$ ([3]). Except for a strong form of tightness, we do not have results for the subcritical case.

To shorten our statements and arguments, we assume from now on that

$$P[N = 0] = 0, \tag{1.3}$$

i.e., that the process survives with probability 1. The usual techniques (see e.g. [1, I.12]) will extend our results to the case where one only assumes

$$EN > 1 \,. \tag{1.4}$$

As the case $EN = 1$ (with (1.3)) corresponds to ordinary random walk, we assume henceforth that (1.4) holds. Finally there is one trivial case we want to exclude in the whole paper, we assume from now on that

$$P[N(0) = 1] < 1 \,. \tag{1.5}$$

## 2 Supercritical branching random walk

First we give another characterization of the supercritical case.

**Lemma 1** *Let* $\pi = P[X_n \to \infty]$. *Then* $\pi = 0$ *or* $\pi = 1$, *and* $\pi = 0$ *iff* $EN(0) > 1$.

*Proof.* Let $X_n^{(j)}$ be the minimal displacement of the $j^{\text{th}}$ particle in the first generation $j = 1, \ldots, N_1$. Then

$$[X_n \to \infty] = \bigcap_{j=1}^{N_1} [X_n^{(j)} \to \infty] \,.$$

Since the $X_n^{(j)}$ are independent, $\pi$ satisfies $\pi = E\pi^N$. By (1.3) it follows that $\pi = 0$ or 1. Let us denote

$$N_n(0) = \text{Card}\{v \in \Gamma_n : d(v) = 0\}$$

for the number of particles at zero at time $n$. We observe (as in [7]) that $(N_n(0))$ is a Galton-Watson process with offspring distribution $N(0)$. Now if $EN(0) > 1$, then

$$1 - \pi \geqq P[X_n = 0, \;\; n = 0, 1, 2, \ldots] = P[N_n(0) > 0, \;\; n = 1, 2, \ldots] > 0,$$

hence $\pi = 0$. On the other hand, by (1.2)

$$[X_n \not\to \infty] = \bigcup_p [X_n \to p] = \bigcup_p \bigcup_m [n \geqq m : X_n = p] \,.$$

Hence if $\pi = 0$, then there are $p$ and $m$ such that $P[X_n = p : n \geqq m] \geqq 0$. This implies that with positive probability there is at least one particle in the $m^{\text{th}}$ generation at $p$ such that the minimal displacement of the branching random walk generated by this particle is 0 for all subsequent times. We then have that $(N_n(0))$ is supercritical, hence $EN(0) > 1$. (recall (1.5)).

**Theorem 1.** *Let $(X_n)$ be the minimal displacement of a supercritical branching random walk with non-negative steps. Then there exists an almost surely finite random variable $X$ such that*

$$X_n \to X \quad \textit{almost surely.}$$

*If $F(k) = P[X \geqq k]$, then for $k > 0$ $F(k)$ is the unique solution in $[0, 1)$ to*

$$F(k) = EF(k)^{N(0)} F(k-1)^{N(1)} \dots F(1)^{N(k-1)} . \tag{2.1}$$

*Moreover, if $\inf_j D_j$ is a.s. bounded, then $X_n \to X$ in $L_p$ for all $p > 0$.*

*Proof.* The almost sure convergence follows immediately from Lemma 1 and the monotonicity of the $X_n$.

Let $F_n(k) = P[X_n \geqq k]$. Conditioning on the positions of the first generation particles, we find

$$F_{n+1}(k) = EF_n(k)^{N(0)} F_n(k-1)^{N(1)} \dots F_n(1)^{N(k-1)} . \tag{2.2}$$

Almost sure convergence of $(X_n)$ implies $F_n(k) \to F(k)$ for all $k$ as $n \to \infty$, and (2.1) follows by bounded convergence. Now for $k = 1$, (2.1) reduces to $F(1) = EF(1)^{N(0)}$, which has a unique solution in $[0, 1)$, in fact $F(1) = P[X \geqq 1]$ equals the extinction probability of $(N_n(0))$. For $k = 2$, let $\varphi(x) = Ex^{N(0)} F(1)^{N(1)}$. Then $F(2)$ is a fixed point of $\varphi$. This map is a increasing convex map from [0,1] to itself, and has a unique fixed point in $[0, 1)$, since either $\varphi(1) < 1$ or $\varphi(1) = 1$ and $\varphi'(1) = EN(0) > 1$. An analogous argument applies for $k \geqq 3$.

For $L_p$-convergence it suffices to show that the limit variable $X$ has an exponentially decreasing tail (the result will follow by monotone convergence). If $\inf_j D_j$ is a.s. bounded, then there exists an integer $L$ such that $P[N(0) + \cdots + N(L) \geqq 1] = 1$. Let $\tau = P[N(0) + \cdots + N(L) = 1]$ if this probability is positive, or any number in (0,1) otherwise. Since $EN(0) > 1$, $P[N(0) \geqq 2] > 0$, and hence $\tau$ is less than 1. Define

$$\psi(x) = Ex^{N(0) + \cdots + N(L)} .$$

Then $\psi(0) = 0$ and $\psi'(0) \leqq \tau < 1$. Hence for all positive $x$ small enough, say $x < \delta$, we have $\psi(x) \leqq \gamma x$, for some $\gamma < 1$. Now choose $k_0 \geqq L$ such that $F(k) = P[X \geqq k] < \delta$ for all $k \geqq k_0$. then for $k \geqq k_0 + L$ we have

$$\begin{aligned} F(k) &= EF(k)^{N(0)} \dots F(1)^{N(k-1)} \\ &\leqq EF(k-L)^{N(0) + \cdots + N(L)} \\ &= \psi(F(k-L)) \leqq \gamma F(k-L) . \end{aligned}$$

Iterating shows that $X$ has an exponentially decreasing tail.

Note that the proof of (2.1) gives a method to compute the distribution of $X$ step by step.

*Example.* Let $N \equiv 2$, $1 - P[D_1 = D_2 = 0] = P[D_1 = D_2 = 1] = p$, for some $p \in (0, 1)$. Then $EN(0) = 2(1-p)$, so the process is supercritical iff $p < \frac{1}{2}$.

Application of Theorem 1 yields

$$F(k) = (1-p)F^2(k) + pF^2(k-1) .$$

Solving this equation we find $F(1) = P[X \geqq 1] = \frac{p}{1-p}$, and in general $F(k) = \varphi_p^k(1)$, the $k^{\text{th}}$ iterate of $\varphi_p(x) = (1-\sqrt{1-4p(1-p)x^2})(2(1-p))^{-1}$. We say that $X$ has an *elliptic* distribution (this generalizes the geometric distribution where $\varphi_p$ is linear).

It might seem more natural to consider the process where $N \equiv 2$, and $D_1, D_2$ are i.i.d. with $1 - P[D_i = 0] = P[D_i = 1] = p$ ([4], [7]), but this process behaves in the same way as the one above. The limiting distribution $\widetilde{X}$ with $P[\widetilde{X} \geqq k] = \widetilde{F}(k)$ satisfies $\widetilde{F}(k) = F^2(k)$ for all integers $k$.

Theorem 1 settles a special case of old questions and conjectures by Hammersley ([6]) on certain refinements of the H.K.B.-Theorem of Sect. 1. Let $\gamma = \lim_{n\to\infty} \frac{1}{n} EX_n$. Then [6, p. 677] the questions are
(i) How does $EX_n - n\gamma$ behave? (ii) Does $EX_{n+1} - EX_n \to \gamma$? (iii) Is $\text{Var}(X_n)$ bounded? Theorem 1 gives the answers in the case $\gamma = 0$ and $EN(0) > 1$. In Sect. 4 we show that $EX_{n+1} - EX_n \to 0$ in case $\gamma = 0$ and $EN(0) = 1$ and the displacements are bounded, and in Sect. 6 there is an example for this case where $\text{Var}\, X_n$ is bounded but does not converge. For $\gamma > 0$ we can only prove (for bounded displacements) that $\text{Var}\, X_n$ is bounded (this follows from Proposition 2).

## 3 Tightness

In this section it is convenient to assume that the displacements are ordered

$$D_1 \leqq D_2 \leqq \ldots \leqq D_N .$$

Our goal is to prove tightness of the sequence $(X_n - EX_n)$ under a moment condition. We define

$$\widetilde{D} = D_1 1_{[N=1]} + D_2 1_{[N \geqq 2]} .$$

**Proposition 1** *Suppose $E\widetilde{D} < \infty$. Then for all $n$*

$$E|X_n' - X_n''| \leqq \frac{2E\widetilde{D}}{P[N > 1]} , \tag{3.1}$$

*if $X_n', X_n''$ are independent and distributed as $X_n$.*

*Proof.* Note first that $EX_n$ is finite because $E\widetilde{D} < \infty$. Argueing as before we have

$$X_{n+1} = \min_{1 \leqq j \leqq N} (X_n^{(j)} + D_j) , \tag{3.2}$$

where the $X_n^{(j)}$ are the independent minimal displacements of the first generation particles. From this we obtain

$$\begin{aligned}EX_{n+1} &\leqq E(X_n^{(1)}+D_1)1_{[N=1]}+E\min(X_n^{(1)}+D_2, X_n^{(2)}+D_2)1_{[N\geqq 2]}\\ &= E(X_n^{(1)}+D_1)1_{[N=1]}+\frac{1}{2}E(X_n^{(1)}+X_n^{(2)}+2D_2-|X_n^{(1)}-X_n^{(2)}|)1_{[N\geqq 2]}\\ &= EX_n+E\widetilde{D}-\frac{1}{2}E|X_n^{(1)}-X_n^{(2)}|1_{[N\geqq 2]}\\ &= EX_n+E\widetilde{D}-\frac{1}{2}P[N\geqq 2]E|X_n^{(1)}-X_n^{(2)}|,\end{aligned}$$

where the last step holds since $N$ and $(X_n^{(1)}, X_n^{(2)})$ are independent. As $EX_{n+1}-EX_n \geqq 0$, (3.1) follows.

**Theorem 2** *Suppose $E\widetilde{D}<\infty$. Then $(X_n - EX_n)$ is a tight family.*

*Proof.* For any two i.i.d. random variables $Z_1$ and $Z_2$ with $EZ_1 = 0$, one has $E|Z_1 - Z_2| \geqq E|Z_1|$. Applying this, the Markov inequality and Proposition 1, we obtain

$$\begin{aligned}P\left[|X_n-EX_n|\geqq K\right] &\leqq \frac{E|X_n-EX_n|}{K} \leqq \frac{E|X_n'-X_n''|}{K}\\ &\leqq \frac{2E\widetilde{D}}{KP[N>1]}\end{aligned} \tag{3.3}$$

for all $K > 0$, which is tightness of $(X_n - EX_n)$.

If the displacements are bounded, we can obtain a much stronger result than (3.3), which will be used in Sect. 4.

**Proposition 2** *Suppose that $D_j \leqq L$ for $j = 1, \ldots, N$ a.s. for some integer $L$. Then there exist $\gamma < 1$ and $C > 0$ such that for all $n$*

$$P[|X_n - EX_n| \geqq k] \leqq C\gamma^k \quad k = 0, 1, 2, \ldots . \tag{3.4}$$

*Proof.* Let $\psi(x) = Ex^N$. Then $\psi(0) = P[N = 0] = 0$, $\psi'(0) = P[N = 1] < 1$, $\psi(1) = 1$ and $\psi'(1) = EN > 1$ (possibly infinite). Hence there exist $\delta > 0$ and $\widetilde{\gamma} < 1$ such that

$$\begin{cases}\psi(x)\leqq\widetilde{\gamma}x & x\in[0,\delta],\\ 1-x\leqq\widetilde{\gamma}(1-\psi(x)) & x\in[1-\delta,1].\end{cases} \tag{3.5}$$

By Theorem 2, we can choose $M > 0$ such that for all $n \geqq 1$ and $k \geqq M$

$$P[|X_n - e_n| \geqq k] = 1 - F_n(e_n - k - 1) + F_n(e_n + k) \leqq \delta, \tag{3.6}$$

where $e_n = [EX_n]$.

For all $n$ and $k$ we deduce with (2.2) that

$$F_n(k) \leqq F_{n+1}(k) \leqq EF_n(k-L)^N = \psi(F_n(k-L)) .$$

Iterating this inequality $p$ times, where $p$ for each $k \geqq M$ is determined by $0 \leqq k - M - pL \leqq L$, one obtains with (3.5) and (3.6)

$$F_n(e_n + k) \leqq F_{n+1}(e_n + k) \leqq \tilde{\gamma}^p F_n(e_n + k_k - pL) \leqq \gamma^{pL} \leqq \gamma^{k-M-L}, \quad (3.7)$$

putting $\gamma = \tilde{\gamma}^{1/L}$.

Similarly one obtains for all $k \geqq M$, choosing $p$ such that $-L \leqq -k + M + pL \leqq 0$,

$$1 - F_n(e_n - k) \leqq \tilde{\gamma}^p [1 - F_n(e_n - k + pL)] \leqq \gamma^{pL} \leqq \gamma^{k-M-L}. \quad (3.8)$$

Accounting for $EX_n - e_n$, (3.4) follows from (3.7) and (3.8) with $C = \gamma^{-M-L-2}$.

## 4 The critical case, first results

In the critical case, i.e. $EN(0) = 1$, we shall make a precise asymptotic analysis of $F_n(k) = P[X_n \geqq k]$. For this we need stronger conditions. From now on we assume that the displacements are bounded almost surely, so there exists an integer $L$ such that

$$D_j \leqq L \quad j = 1, \ldots, N. \quad (4.1)$$

The fundamental relation (2.2) then simplifies to

$$F_{n+1}(k) = \Phi(F_n(k), F_n(k-1), \ldots, F_n(k-L)) \quad n \geqq 0, \; k \in \mathbb{Z} \quad (4.2)$$

where

$$\Phi(x_0, \ldots, x_L) = Ex_0^{N(0)} x_1^{N(1)} \ldots x_L^{N(L)} \quad (4.3)$$

is the generating function of $(N(k))_{k=0}^{L}$.

Our goal is to determine all possible limit points of the sequences $(X_n - d_n)$, where the $d_n$ are integers, and $(d_n - EX_n)$ is a bounded sequence. By Theorem 2 these sequences are tight, and the only interesting normings of $X_n$. We remark that the condition of bounded displacements is not needed till Theorem 4.

We first prove some properties of the function $\Phi$. Two important quantities are

$$\theta = P[N(0) > 1], \quad \tau = \frac{1}{2} \operatorname{Var} N(0).$$

Note that $\theta$ and $\tau$ are positive by assumption (1.5). Throughout this section we assume that

$$EN < \infty, \quad \operatorname{Var} N(0) < \infty. \quad (4.4)$$

The function $\Phi(\boldsymbol{x})$ is convex and differentiable in each variable $x_i$, where $\boldsymbol{x} = (x_0, \ldots, x_L)$. We denote $\Phi_i(\boldsymbol{x}) = \frac{\partial \Phi}{\partial x_i}(\boldsymbol{x})$ for $i = 0, \ldots, L$.

**Lemma 2** (i) *For all* $\boldsymbol{x} \in [0, 1]^{L+1}$

$$1 - \Phi(\boldsymbol{x}) \geqq \sum_{i=0}^{L} (1 - x_i)\Phi_i(\boldsymbol{x}) + \theta(1 - x_0)^2 .$$

(ii) *For all* $\varepsilon > 0$ *there exist* $\delta > 0$ *such that for all* $\boldsymbol{x} \in [1 - \delta, 1] \times [0, 1]^L$

$$1 - \Phi(\boldsymbol{x}) \geqq \sum_{i=0}^{L} (1 - x_i)\Phi_i(\boldsymbol{x}) + (\tau - \varepsilon)(1 - x_0)^2 .$$

*Proof.* Let $f(x) = Ex^{N(0)}$. Then

$$1 - f(x) \geqq (1 - x)f'(x) + \theta(1 - x)^2 \tag{4.5}$$

since $f$ is convex, and $\theta = 1 - f(0) - f'(0)$. By convexity of $\Phi$, for $i = 1, \ldots, L$

$$\Phi(x_0, \ldots, x_{i-1}, 1, \ldots, 1) - \Phi(x_0, \ldots, x_i, 1, \ldots, 1)$$
$$\geqq (1 - x_i)\Phi_i(x_0, \ldots, x_i, 1, \ldots, 1) \geqq (1 - x_i)\Phi_i(\boldsymbol{x}) .$$

Adding these equations and (4.5) with $x = x_0$, we obtain Lemma 2 (i), noting that $f(x_0) = \Phi(x_0, 1, \ldots, 1)$, and $f'(x_0) = \Phi_0(x_0, 1, \ldots, 1) \geqq \Phi_0(\boldsymbol{x})$. Lemma 2 (ii) is derived in a similar way.

Because of relation (4.2) the set

$$A = \{\boldsymbol{x} \in [0, 1]^{L+1} : x_0 \leqq \ldots \leqq x_L , \Phi(\boldsymbol{x}) \geqq x_0\}$$

is of special interest to us. We denote $\mathbf{1} = (1, \ldots, 1)$.

**Lemma 3** *Let* $g = \min\{i > 0 : EN(i) > 0\}$. *Then* $\frac{1-x_g}{1-x_0} \to 0$ *as* $\boldsymbol{x} \to \mathbf{1}$ *in* $A$.

*Proof.* Note that $g$ exists by condition (1.4) since $EN(0) = 1$. By Lemma 2 (i),

$$1 \geqq \frac{1 - \Phi(\boldsymbol{x})}{1 - x_0} \geqq \Phi_0(\boldsymbol{x}) + \frac{1 - x_g}{1 - x_0}\Phi_g(\boldsymbol{x}) \geqq 0 .$$

Since $\Phi_0(\boldsymbol{x}) \to EN(0) = 1$, and $\Phi_g(\boldsymbol{x}) \to EN(g) > 0$ as $\boldsymbol{x} \to \mathbf{1}$, the conclusion of Lemma 3 follows.

For vectors $\boldsymbol{x}, \boldsymbol{y}$ we denote $\boldsymbol{x} \leqq \boldsymbol{y}$ if $x_i \leqq y_i$ for $0 \leqq i \leqq L$.

**Lemma 4** *Let* $\boldsymbol{x}, \boldsymbol{y} \in A$ *with* $\boldsymbol{x} \leqq \boldsymbol{y}$, *and suppose that* $\boldsymbol{y} - \boldsymbol{x} \leqq c(\mathbf{1} - \boldsymbol{y})$ *for some real number* $c$. *Then*

(i) $\Phi(\boldsymbol{y}) - \Phi(\boldsymbol{x}) \leqq c[1 - \Phi(\boldsymbol{y})][1 - \frac{\theta}{EN}(1 - y_0)]$,

(ii) *for all* $\varepsilon > 0$ *there exists* $\eta > 0$ *such that for* $y_0 \geqq 1 - \eta$
$\Phi(\boldsymbol{y}) - \Phi(\boldsymbol{x}) \leqq c[1 - \Phi(\boldsymbol{y})][1 - \frac{\tau-\varepsilon}{1+\varepsilon}(1 - y_0)]$.

*Proof.* By convexity of $\Phi$,

$$1 - \Phi(\boldsymbol{y}) \leqq \sum_{i=0}^{L} \Phi_i(\mathbf{1})(1 - y_i) = \sum_{i=0}^{L} EN(i)(1 - y_i) . \tag{4.6}$$

Since $y_0 \leqq \dots \leqq y_L$, it follows that $1-\Phi(y) \leqq (1-y_0)EN$, and hence, with Lemma 2 (i), that

$$\sum_{i=0}^{L}(1-y_i)\Phi_i(y) \leqq [1-\Phi(y)]\left[1-\frac{\theta}{EN}(1-y_0)\right].$$

Also, by convexity and the hypotheses,

$$\Phi(y)-\Phi(x) \leqq \sum_{i=0}^{L}(y_i-x_i)\Phi_i(y) \leqq c\sum_{i=0}^{L}(1-y_i)\Phi_i(y),$$

and (i) follows. On the other hand (4.6) implies (as $EN(0)=1$) that

$$1-\Phi(y) \leqq 1-y_0+(1-y_g)EN = (1-y_0)\left[1+\frac{1-y_g}{1-y_0}EN\right] \leqq (1-y_0)(1+\varepsilon),$$

for a given $\varepsilon > 0$, if $y_0$ is close enough to 1, by Lemma 3. Now (ii) follows as above with Lemma 2 (ii).

Our main tool in the analysis of the critical case is the following result.

**Proposition 3** *Let $F_n(k) = P[X_n \geqq k]$. For all $\beta < 1$ there exists a real number $K$ such that for all $k \geqq 1, n > 0$*

$$\frac{F_{n+1}(k)-F_n(k)}{1-F_{n+1}(k)} \leqq \frac{K}{n^{\beta}}. \tag{4.7}$$

*Proof.* By Kolmogorov's theorem on the extinction probability of a critical branching process (see e.g. [1.p.19])

$$1-F_n(1) = P[X_n = 0] = P[N_n(0) > 0] \sim \frac{2}{n \operatorname{Var} N(0)} = \frac{1}{\tau n} \quad \text{as } n \to \infty.$$

Choose $\varepsilon > 0$ such that $\tau - \varepsilon > \beta\tau(1+\varepsilon)^2$. There exists $n_0$ such that for $n \geqq n_0$

$$1-F_n(k) \geqq 1-F_n(1) \geqq \frac{1}{\tau(1+\varepsilon)n} \quad \text{for all } k \geqq 1.$$

Let $\eta \leqq 1$ be associated to $\varepsilon$ as in Lemma 4, and let $n_1 = \max(n_0, [\beta EN/\eta\theta]+1)$. Also, let

$$Q_n(k) = \frac{F_{n+1}(k)-F_n(k)}{1-F_{n+1}(k)}, \quad M_n = \sup_{k>0} Q_n(k).$$

For $n \geqq n_1$ and all $k$ with $F_{n+1}(k) \geqq 1-\eta$ it holds by Lemma 4 (ii) that

$$\begin{aligned} Q_{n+1}(k) &\leqq M_n\left[1-\left(\frac{\tau-\varepsilon}{1+\varepsilon}\right)(1-F_{n+1}(k))\right] \\ &\leqq M_n\left[1-\frac{(\tau-\varepsilon)}{\tau(1+\varepsilon)^2(n+1)}\right] \leqq M_n\left[1-\frac{\beta}{n+1}\right]. \end{aligned}$$

On the other hand, if $n \geqq n_1$, and $F_{n+1}(k) < 1-\eta$, then by Lemma 4 (i)

$$Q_{n+1}(k) \leqq M_n\left[1-\frac{\theta}{EN}(1-F_{n+1}(k))\right]$$
$$\leqq M_n\left[1-\frac{\theta_\eta}{EN}\right] \leqq M_n\left[1-\frac{\beta}{n+1}\right].$$

So for $n \geqq n_1$, $M_{n+1} \leqq M_n[1-\frac{\beta}{n+1}]$, which implies the existence of $K$ such that $M_n \leqq Kn^{-\beta}$ for all $n$.

As corollaries we obtain the following theorems.

**Theorem 3** *Let $X$ be a weak limit point of a sequence $(X_n - d_n)$, where $(d_n - EX_n)$ is bounded. Then $F(k) = P[X \geqq k]$ satisfies $\Phi F = F$.*

*Proof.* Suppose $X_{n'} - d_{n'} \to X$ weakly as $n' \to \infty$. Then

$$F_{n'+1}(k+d_{n'}) = \Phi(F_{n'}(k+d_{n'}), \ldots, F_{n'}(k+d_{n'}-L))$$

by (4.2). The right side tends to $\Phi(F(k), \ldots, F(k-L))$ by dominated convergence, the left side to $F(k)$, since $F_{n'+1}(k+d_{n'}) - F_{n'}(k+d_{n'}) \to 0$ by Proposition 3.

**Theorem 4** *Let $(X_n)$ be the minimal displacement of a critical branching random walk satisfying (4.4). Then $EX_{n+1} - EX_n \to 0$ as $n \to \infty$.*

*Proof.* Let $e_n = [EX_n]$. Note that

$$\begin{aligned} EX_{n+1} - EX_n &= \sum_{k=-\infty}^{+\infty} [F_{n+1}(k) - F_n(k)] \\ &= \sum_{k=-\infty}^{+\infty} [F_{n+1}(e_n+k) - F_n(e_n+k)] \\ &\leqq \sum_{k=-\infty}^{-M-1} [1-F_n(e_n+k)] + \sum_{k=-M}^{M} [F_{n+1}(e_n+k) - F_n(e_n+k)] \\ &\quad + \sum_{k=M+1}^{\infty} F_{n+1}(e_n+k), \end{aligned}$$

for any positive integer $M$. The result now follows from Proposition 2 (more precisely from (3.7) and (3.8)) and Proposition 3.

We end this section with a noteworthy property of the total number of particles $U_n = \mathrm{Card}\{v \in \Gamma_n : d(v) = X_n\}$ at $X_n$: the expectation of $U_n$ tends to infinity. This follows from Theorem 4 by noticing that

$$P[N(0)>0]^{EU_n} \leqq EP[N(0)>0]^{U_n} = P[X_{n+1} > X_n] \leqq E(X_{n+1} - X_n).$$

We conjecture that $U_n \not\to \infty$ almost surely.

## 5 Shape of limit distributions

In this section we shall refine some of the results of the previous section replacing (4.4) by

$$EN^2 < \infty , \tag{5.1}$$

and prove a generalization of Kolmogorov's theorem on the extinction probability of critical branching processes (Theorem 6).

**Theorem 5** *Let* $\tau = \frac{1}{2} \operatorname{Var} N(0)$ *and suppose* $\mu = EN(1) > 0$. *Then uniformly in* $k \geqq 2$ *and* $n \geqq 1$,

$$\frac{1 - F_n(k-1)}{[1 - F_n(k)]^2} \to \frac{\tau}{\mu} \quad as\ F_n(k) \to 1 .$$

*Proof.* Since $EN^2 < \infty$, $\Phi(\boldsymbol{x})$ can be expanded as

$$\Phi(\boldsymbol{x}) = 1 - \sum_{i=0}^{L} (1 - x_i) EN(i) + \sum_{i,\, j=0}^{L} (1 - x_i)(1 - x_j)\Phi_{ij}(\mathbf{1}) + R(\boldsymbol{x}) , \tag{5.2}$$

where $\Phi_{ij}(\boldsymbol{x}) = \frac{1}{2} \frac{\partial^2 \Phi(\boldsymbol{x})}{\partial x_i \partial x_j}$ and $R(\boldsymbol{x}) = o(\|\mathbf{1} - \boldsymbol{x}\|^2)$.

On $A$, the remainder term $R(\boldsymbol{x}) = o((1 - x_0)^2)$, and all $(1 - x_i)(1 - x_j)$ are $O((1 - x_0)^2)$. Since $\Phi(\boldsymbol{x}) \geqq x_0$, $EN(0) = 1$, $EN(1) = \mu$, and all $EN(i) \geqq 0$, it follows that $\mu(1 - x_1) = O((1 - x_0)^2)$ on the compact set $A$, and hence that on $A$

$$(1 - x_1) \leqq B(1 - x_0)^2 \tag{5.3}$$

for some constant $B$.

By Proposition 3 with $\beta = 2/3$

$$F_{n+1}(k) - F_n(k) \leqq Kn^{-2/3}(1 - F_{n+1}(k)) \leqq Kn^{-2/3}(1 - F_n(k)) . \tag{5.4}$$

But by monotonicity of $F_n$ and (5.3) with $\boldsymbol{x} = (F_n(2), F_n(1), 1, \dots, 1)$ we have

$$1 - F_n(k) \geqq 1 - F_n(2) \geqq [B^{-1}(1 - F_n(1))]^{1/2} \geqq \hat{c} n^{-\frac{1}{2}} ,$$

for some $\hat{c} > 0$, again by Kolmogorov's result. With (5.4) this gives (putting $\tilde{c} = K\hat{c}^{-4/3}$)

$$F_{n+1}(k) - F_n(k) \leqq K[\hat{c}^{-1}(1 - F_n(k))]^{4/3}(1 - F_n(k)) = \tilde{c}(1 - F_n(k))^{7/3} . \tag{5.5}$$

We apply (5.2) with $\boldsymbol{x} = (F_n(k), \dots, F_n(k - L))$. We just showed that $\Phi(\boldsymbol{x}) - x_0 \leqq \tilde{c}(1 - x_0)^{7/3}$. From (5.3) it follows that $(1 - x_i)(1 - x_j) \leqq B(1 - x_0)^3$ for $(i, j) \neq (0, 0)$. Furthermore $1 - x_i \leqq 1 - x_2 \leqq B(1 - x_1)^2$

$\leqq B^3(1-x_0)^4$ for $i > 1$, applying (5.3) twice. Using these estimations in (5.2), noting that $\Phi_{00}(1) = \frac{1}{2}EN(0)[N(0)-1] = \tau$, we find that

$$\mu(1-x_1) - \tau(1-x_0)^2 = O((1-x_0)^{7/3}) .$$

This implies the statement of the theorem.

**Corollary.** *Let* $\mu = EN(1) > 0$. *Uniformly on the set of limit distributions*

$$\frac{1-F(k-1)}{[1-F(k)]^2} \to \frac{\tau}{\mu} \qquad as \quad F(k) \to 1 .$$

As another corollary to Theorem 5 we have

**Theorem 6** *Let* $\tau = \frac{1}{2}\operatorname{Var} N(0)$, *and suppose* $\mu = EN(1) > 0$. *Then for all* $k \geqq 0$

$$P[X_n \leqq k] \sim \frac{\mu}{\tau}\frac{1}{[\mu n]^{2^{-k}}} \qquad as \quad n \to \infty .$$

## 6 An example

We apply the results of the previous section to our standard example $N \equiv 2$, $P[D_1 = D_2 = 0] = P[D_1 = D_2 = 1] = \frac{1}{2}$. Let $X$ be a weak limit point of $(X_n - d_n)$, where $(d_n - EX_n)$ is bounded sequence. According to Theorem 3, $F(k) = P[X \geqq k]$ satisfies

$$F(k) = \frac{1}{2}F^2(k) + \frac{1}{2}F^2(k-1)$$

for all integers $k$. Hence if we define for $x \in [0, 1]$

$$\varphi(x) = 1 - \sqrt{1-x^2} ,$$

then $F(k) = \varphi(F(k-1))$ for all integers $k$. For any $\alpha \in (0, 1)$ let the integer valued random variable $X(\alpha)$ be defined by

$$P[X(\alpha) \geqq k] = \varphi^k(\alpha), \quad k \in \mathbb{Z} . \tag{6.1}$$

Here $\varphi^0 = Id$, $\varphi^k = \varphi \circ \ldots \circ \varphi$, $\varphi^{-k} = \varphi^{-1} \circ \ldots \circ \varphi^{-1}$ ($k$ times) for $k > 0$. We call $X(\alpha)$ a *circle* law. By the remarks above any limit point $X$ has to be a circle law. We shall show later (Theorem 7), that any circle law does occur as a limit point. We give a list of some properties of circle laws, the proofs of which are left to the reader.

**Observation.** *Let* $X(\alpha)$ *be a circle law for* $\alpha \in (0, 1)$. *Then* (i) $X(\varphi(\alpha)) = X(\alpha) - 1$. (ii) $\alpha \to EX(\alpha)$ *is a strictly monotone function from* $(0, 1)$ *onto* $\mathbb{R}$. (iii) $X(1-\alpha)$ *is distributed as* $-X(\alpha) - 1$ (iv) $EX(1/2) = -1/2, EX(\sqrt{1/2}) = 0$. (v) $\operatorname{Var} X(\varphi(\alpha)) = \operatorname{Var} X(\alpha)$, $\operatorname{Var}(X(1-\alpha)) = \operatorname{Var}(X(\alpha))$.

Hammersley conjectures in [6] that $\operatorname{Var} X_n$ converges for this example. Since any circle law $X(\alpha)$ can occur as a limit this is not true, as $\operatorname{Var} X(\alpha)$ is not constant. See the following table (whose entries were computed with (6.1))

| $\alpha$ | $\operatorname{Var} X(\alpha)$ | $\alpha$ | $\operatorname{Var} X(\alpha)$ |
|---|---|---|---|
| 0.1 | 0.85851501 | 0.4 | 0.85852356 |
| 0.2 | 0.85852356 | 0.5 | 0.85850792 |
| 0.3 | 0.85854271 | | |

## 7 Parametrisation of limit distributions

We continue the analysis of the critical case. Although condition (4.4) will suffice for some of the results in this section we will assume throughout that $EN^2 < \infty$. The goal of this section is to prove a converse to Theorem 3, i.e., that if $a = (a_k)_{k \in \mathbb{Z}}$ is a sequence of real numbers satisfying

$$a_{k+1} \geqq a_k \quad \text{for all} \quad k, \qquad \lim_{k \to -\infty} a_k = 0, \qquad \lim_{k \to \infty} a_k = 1 \tag{7.1}$$

and for all integers $k$

$$\Phi(a_k, a_{k+1}, \dots, a_{k+L}) = a_k, \tag{7.2}$$

then $F$ defined by $F(k) = a_{-k}$ is a limit distribution. (We inverted the direction of space as we are more interested in the left tails of the limit distributions). We call sequences $a$ satisfying (7.1) and (7.2) *admissible* sequences. Note that (7.1) and (7.2) imply that $0 < a_k < 1$ for all $k$.

The proof that any admissible sequence yields a limit distribution relies on finding good parametrizations of these sets. Note for example that (by Theorem 4) for each real number $e$ there exists a limit point $X$ with $EX = e$.

**Lemma 5** *For all $s \in (0, 1)$ there is a limit distribution $F$ with $F(0) = s$.*

*Proof.* For all $m \geqq 1$ the set $\{F_n(k) : n \geqq m, k \geqq 0\}$ is dense in [0,1]. To see this, let $\varepsilon > 0$. By Proposition 3 we can chose $p \geqq m$ such that $F_{n+1}(k) - F_n(k) < \varepsilon$ for all $n \geqq p$. Now choose a $k$ such that $F_p(k) < \varepsilon$. Since $F_n(k) \to 1$ as $n \to \infty$, the sequence $\{F_n(k) : n \geqq p\}$ is $\varepsilon$-dense in [0,1]. Let $s \in (0, 1)$. By our first remark there exists a sequence of the form $(F_{n_j}(d_j))$ such that $F_{n_j}(d_j) \to s$ as $j \to \infty$. Since $0 < s < 1$, it follows by Theorem 2 that $(d_j - EX_{n_j})$ is bounded, and that the sequence $(X_{n_j} - d_j)$ has a weakly converging subsequence. The limit distribution $F$ satisfies $F(0) = s$.

In the sequel we want to show that a limit distribution $F$ is uniquely determined by $F(0)$. This is not the case in general. For example take $N \equiv 2, P[D_0 = D_1 = 0] = P[D_0 = D_1 = 2] = \frac{1}{2}$, and $k_n = [EX_n]$ if this is even, or $k_n = [EX_n] + 1$ otherwise. Then any limit point $X$ of $(X_n - k_n)$ is concentrated on $2\mathbb{Z}$, but there are limit points $Y$ of $(X_n - k_n - 1)$ concentrated on $2\mathbb{Z} + 1$ with $P[X \geqq 0] = P[Y \geqq 0]$, while obviously $X \neq Y$.

Formulated in another way, we want to avoid (near) sign changes in the sequence $(a_k - b_k)$ if $a$ and $b$ are admissible. Here the case $L > 1$ is essentially more complicated than the case $L = 1$. For $L = 1$ it is easy to see that sign changes in these differences are not possible. If $\Psi(x, y) = \Phi(x, y) - x$, then $dy/dx = -\partial\Psi/\partial x/\partial\Psi/\partial y = (1 - \partial\Phi/\partial x)/\partial\Phi/\partial y \geqq 0$, since $EN(0) = 1$. To study the sign changes for general $L$ we define

$$\alpha_k = \max\{0, a_{k+1} - b_{k+1}, \ldots, a_{k+L} - b_{k+L}\},$$
$$\beta_k = \max\{0, b_{k+1} - a_{k+1}, \ldots, b_{k+L} - a_{k+L}\}.$$

**Lemma 6** *Suppose $EN(i) > 0$ for $1 \leqq i \leqq L$, and let $a$, $b$ be admissible sequences. Then there exists a constant $c > 0$ and all integer $k_0$ such that for all $k \geqq k_0$*

$$\text{if } a_k \geqq b_k \quad \text{then} \quad \alpha_k \geqq c\beta_k,$$
$$\text{if } b_k \geqq a_k \quad \text{then} \quad \beta_k \geqq c\alpha_k.$$

*Proof.* The second assertion follows from the first by symmetry. Let $\varepsilon = \frac{1}{2}\min\{EN(i) : 0 \leqq i \leqq L\}$, and define $c = \varepsilon/EN$. Since $\Phi_i(\mathbf{1}) = EN(i) \geqq 2\varepsilon$, there exists $\delta > 0$ such that $\Phi_i(\boldsymbol{x}) \geqq \varepsilon$ for $\boldsymbol{x} \in [1-\delta, 1]^{L+1}$ and $0 \leqq i \leqq L$.

Choose $k_0$ such that for $k \geqq k_0$ $a_k \geqq 1 - \delta$ and $b_k \geqq 1 - \delta$. Let $c_k = \max(a_k, b_k)$. Then $\max\{c_{k+i} - a_{k+i} : 1 \leqq i \leqq L\} = \beta_k$. Therefore if $k \geqq k_0$ and $a_k \geqq b_k$, then we obtain from the mean value theorem

$$\Phi(b_k, c_{k+1}, \ldots, c_{k+L}) \geqq \Phi(b_k, a_{k+1}, \ldots, a_{k+L}) + \beta_k\varepsilon \geqq b_k + \beta_k\varepsilon,$$

since $\Phi_0(\boldsymbol{x}) \leqq 1$ and $\Phi(a_k, \ldots, a_{k+L}) = a_k$.

On the other hand, since $EN(0) + \cdots + EN(L) = EN$,

$$\Phi(b_k, c_{k+1}, \ldots, c_{k+L}) \leqq \Phi(b_k, b_{k+1}, \ldots, b_{k+L}) + \alpha_k EN.$$

Combination of these inequalities yields $\alpha_k EN \geqq \beta_k\varepsilon$ or $\alpha_k \geqq c\beta_k$.

We need yet another simple lemma for the proof of Proposition 4.

**Lemma 7** *Let $a$, $b$ be admissible, $k \in \mathbb{Z}$. If $a_{k+i} \geqq b_{k+i}$ for $i = 1, \ldots L$ then $a_k \geqq b_k$; if moreover $a_{k+i} > b_{k+i}$ for at least one i, and $EN(i) > 0$, then $a_k > b_k$.*

*Proof.* Let $h(x) = \Phi(x, a_{k+1}, \ldots, a_{k+L})$. Then $h(a_k) = a_k$, and $h'(x) \leqq 1$ for all $x \in [0, 1]$, since $EN(0) = 1$. But $h(b_k) \geqq \Phi(b_k, b_{k+1}, \ldots, b_{k+L}) = b_k$, hence $b_k \leqq a_k$. If $a_{k+i} > b_{k+i}$ and $EN(i) > 0$, for some $i$ then the inequalities are strict.

**Proposition 4** *Suppose $EN(i) > 0$ for $1 \leqq i \leqq L$. Let $a$, $b$ be two admissible sequences with $a \neq b$. Then either $a_k > b_k$ for all k, or $a_k < b_k$ for all k.*

*Proof.* Suppose on the contrary, that there are integers $p$ and $q$ such that $a_p > b_p$ and $a_q \leqq b_q$. Let $k \geqq \max(p, q)$. If $\alpha_k = 0$, then by Lemma 7, $a_k \leqq b_k$. Applying the same lemma repeatedly, we arrive at the contradiction $a_p \leqq b_p$. So for all $k \geqq \max(p, q)$ there exists $i$ with $1 \leqq i \leqq L$ and $a_{k+i} > b_{k+i}$. But then $\beta_k = 0$ implies $a_k > b_k$ by Lemma 7, and again repeated application of this lemma yields the contradiction, $a_q > b_q$. We conclude that $\alpha_k > 0$ and $\beta_k > 0$ for *all* $k \geqq \max(p, q)$. Let $k_1 = \max(k_0, p, q)$, where $k_0$ is given by Lemma 6, and let

$$I_+ = \{k \geqq k_1 : a_k \geqq b_k, a_{k+1} < b_{k+1}\} .$$

We showed above that at least one of every $L$ consecutive integers belongs to $I_+$. Let $k \in I_+$. By Lemma 6 there exists $i$ with $1 \leqq i \leqq L$ such that $a_{k+i} - b_{k+i} \geqq c\beta_k$. Let $j = \max\{\ell : a_{k+\ell} \leqq b_{k+\ell}\}$. Since $a_{k+1} < b_{k+1}$ and $a_{k+i} > b_{k+i}$ we have $1 \leqq j < i \leqq L$. Therefore $\alpha_{k+j} \geqq a_{k+i} - b_{k+i}$. Let

$$I_- = \{k \geqq k_1 : a_k \leqq b_k, a_{k+1} > b_{k+1}\} .$$

We just showed that for each $k \in I_+$ there exists $\widetilde{k} \in I_-$ with $k < \widetilde{k} \leqq k + L$ and $\alpha_{\widetilde{k}} \geqq c\beta_k$. By symmetry there exists for each $\ell \in I_-$ and $\widetilde{\ell} \in I_+$ with $\ell < \widetilde{\ell} \leqq \ell + L$ such that $\beta_{\widetilde{\ell}} \geqq c\alpha_\ell$. Thus there exists for each $k \in I_+$ an $m \in I_+$ with $k < m \leqq 2k + L$ such that $\beta_m \geqq c^2\beta_k$. In this way we obtain a sequence of elements $(m_j)$ of $I_+$ such that $0 < m_{j+1} - m_j \leqq 2L$ for all $j$ and

$$\beta_{m_{j+1}} \geqq c^2\beta_{m_j} .$$

Choose $k_j \in \{m_{j+1}, m_j + L\}$ such that $b_{k_j} - a_{k_j} = \beta_{m_j}$. Then $k_j \to \infty$, $k_j \leqq m_0 + 2Lj$, and

$$\liminf_{j\to\infty} \frac{b_{k_j} - a_{k_j}}{c^{2j}} > 0 . \tag{7.3}$$

But by Lemma 8 below, $a_k$ and $b_k$ tend to 1 faster than geometrically, in particular $1 - a_k$ and $1 - b_k$ are bounded by $c^{2k/L}$ for $k$ large enough, and so $b_k - a_k \leqq c^{2k/L}$ for $k$ large enough, implying that $b_{k_j} - a_{k_j} \leqq c^{2m_0/L}c^{4j}$, which contradicts (7.3).

**Theorem 7** *Suppose $EN(i) > 0$ for $1 \leqq i \leqq L$. For each admissible sequence $a$, $F(k) = a_{-k}$ is a limit distribution.*

*Proof.* Let $a$ be admissible. By Lemma 5 there is a limit distribution $F$ such that $F(0) = a_0$. By Proposition 4, $F(k) = a_{-k}$ for all integers $k$.

Note that Proposition 4 also implies that $F \mapsto F(0)$ is one to one, and hence a true parametrization. In the next sections we shall need a parametrization with a more global character.

**Lemma 8** *Let $\mu = EN(1) > 0$, and let $a$ be an admissible sequence. Then there exists a positive real number $\chi(a)$, such that, uniformly on the set of admissible sequences*

$$\log\left(\frac{\tau}{\mu}(1 - a_k)\right) + 2^k\chi(a) \to 0 \qquad \text{as} \quad a_k \to 1 .$$

*Proof.* Following the proof of Theorem 5 with $x = (a_k, a_{k+1}, \ldots, a_{k+L})$, except for the argument leading to (5.5), which is not needed here as $\Phi(x) = x_0$

by (7.2), we find that

$$\mu(1-a_{k+1}) - \tau(1-a_k)^2 = O((1-a_k)^3)\ . \tag{7.4}$$

Putting $u_k = \log \frac{\tau}{\mu}(1-a_k)$, (7.4) implies that $u_{k+1} - 2u_k = O(1-a_k)$. It is straightforward to show that this implies the existence of a real number $t$ such that $u_k + t2^k = O(1-a_k)$. Since $u_k \to -\infty$ as $k \to \infty, t = \chi(a)$ is strictly positive.

The remainder of this section is devoted to proving that $a \mapsto \chi(a)$ is a bijection from the set of admissible sequences to $\mathbb{R}$.

**Lemma 9** *Let $a$, $b$ be admissible sequences with $a_k \geqq b_k$ for all $k$. Then*

$$\limsup_{k\to\infty} \frac{a_k - b_k}{1-a_k} \geqq \frac{a_0 - b_0}{1-a_0}\ .$$

*Proof.* We apply Lemma 4 (i) with $\boldsymbol{x} = (b_0, \ldots, b_L)$ and $\boldsymbol{y} = (a_0, \ldots a_L)$. Since $\Phi a = a$ and $\Phi b = b$ we obtain with $c_0 = \sup_{k\geqq 0} \frac{a_k - b_k}{1-a_k}$

$$\frac{a_0 - b_0}{1-a_0} \leqq c_0\left(1 - \frac{\theta}{EN}(1-a_0)\right)\ .$$

As the factor on the right is strictly smaller than 1, this implies that

$$\frac{a_0 - b_0}{1-a_0} \leqq c_1 := \sup_{k\geqq 1} \frac{a_k - b_k}{1-a_k}\ .$$

Repeatedly applying Lemma 4 (i) we obtain that

$$\frac{a_0 - b_0}{1-a_0} \leqq c_n := \sup_{k\geqq n} \frac{a_k - b_k}{1-a_k}\ .$$

Letting $n \to \infty$ yields the conclusion of Lemma 9.

**Lemma 10** *Let $a$, $b$ admissible sequences. The following statements are equivalent.*
*1) $a_k > b_k$ for all integers $k$, 2) $(1-a_k)/(1-b_k) \to 0$ as $k \to \infty$,*
*3) $\chi(a) > \chi(b)$.*

*Proof.* The equivalence of 2) and 3) is immediate from Lemma 8. If 2) holds, then $a_k > b_k$ for all large $k$, and hence for all $k$ by Lemma 7. It remains to prove that 1) implies 3). But this follows from Lemma 8 and Lemma 9: $\limsup_{k\to\infty} \frac{1-a_k}{1-b_k} - 1 = \limsup_{k\to\infty} \frac{a_k - b_k}{1-a_k} > 0$ implies $\chi(a) > \chi(b)$.

It is for the proof of the following that we needed the uniform estimation in Lemma 8.

**Proposition 5** *Suppose $EN(i) > 0$ for $1 \leqq i \leqq L$. For each positive real number $t$ there is a unique sequence $a$ such that $\chi(a) = t$, and $\chi$ is continuous in $a_0$.*

*Proof.* By Proposition 4, the sequence $a$ $(a_k = F(-k))$ in Lemma 5 is unique for each $s \in (0, 1)$. Hence there exists a function $g : (0, 1) \to (0, 1)$ with $g(a_0) = a_1$

for each admissible sequence $a$. The function $g$ is strictly increasing (apply Proposition 4 with $(b_k) = (a_{k+1})$), and surjective, hence continuous. More generally, $a_k$ is a continuous function of $a_0$. Now let $a$ be $a$ fixed admissible sequence. Then for any admissible sequence $b$

$$|\chi(a) - \chi(b)| \leqq 2^{-k}\left[r_k(a) + r_k(b) + \left|\log\frac{1-b_k}{1-a_k}\right|\right],$$

where $r_k(a) = |\chi(a)2^k + \log(\frac{\tau}{\mu}(1-a_k))|$. Let $\varepsilon > 0$. By Lemma 8 there exists $\delta > 0$ such that $r_k(c) < \varepsilon/3$ for any admissible sequence $c$, as soon as $1 - c_k < \delta$. Fix $k$ such that $1 - a_k < \delta/2$. Then for any admissible $b$ such that $b_0$ is close enough to $a_0$, we have $|a_k - b_k| < \delta/2$ by the continuity proved above. But then also $r_k(b) \leqq \varepsilon/3$. By taking $b_0$ closer to $a_0$ (if necessary) we can also ensure that $|\log(1-b_k)/(1-a_k)| \leqq \varepsilon/3$. We conclude that $\chi$ is continuous in $a_0$. By Proposition 4 and Lemma 10, $\chi$ is strictly increasing. Moreover $\chi(b) = 2\chi(a)$ if $b = (b_k) = (a_{k+1})$. Hence there exists for each $t > 0$ a unique admissible sequence $a$ such that $\chi(a) = t$.

## 8 A martingale

In this section it is convenient to interpret the branching random walk as a labelled random tree. The particle at time zero is identified with the root of the tree. Particles at time $n$ are identified with vertices $v$ in the $n^{\text{th}}$ level $\Gamma_n$ of the tree. Each gives rise to a random number $N(v)$ of vertices in $\Gamma_{n+1}$, which obtain labels $D_1(v), D_2(v), \ldots, D_{N(v)}(v)$. There is a unique shortest path $(v^{(1)}, v^{(2)}, \ldots, v^{(n)})$, called the *family line* of $v$, connecting $v = v^{(n)}$ to be root. The position $d(v)$ of $v$ is obviously equal to the sum of the labels of $v^{(1)}, \ldots, v^{(n)}$. We write $\alpha(v) = v^{(n-1)}$ for the next to last member of the family line of $v \in \Gamma_n$. A *cutset* $\Pi$ of the tree is a set of vertices such that every infinite path from the root intersects II at exactly one vertex. Inspired by ([3]) we define the following cutsets $\Pi_0 = \{\text{root}\}$ and for $k \geqq 1$

$$\Pi_k = \{v : d(v) \geqq k, d(\alpha(v)) < k\}.$$

We partition $\Pi_k$ in the sets $\Pi_k(j) = \{v \in \Pi_k : d(v) = k + j\}$ for $0 \leqq j < L$, and write $A_k(j) = \text{Card}(\Pi_k(j))$. Finally let $\mathscr{A}_k = \sigma\{A_\ell(j) : 0 \leqq \ell \leqq k, 0 \leqq j < L\}$ be the $\sigma$-algebra generated by the $A_\ell(j)$ for $\ell \leqq k$.

**Lemma 11** *Let $(a_k)$ be an admissible sequence, and let*

$$Q_k = a_k^{A_k(0)} a_{k+1}^{A_k(1)} \cdots a_{k+L-1}^{A_k(L-1)}.$$

*Then $(Q_k)_{k=0}^{\infty}$ is a martingale* w.r.t. $(\mathscr{A}_k)_{k=0}^{\infty}$.

*Proof.* Note that $A_0(0) = 1$, $A_0(j) = 0$ for $j \geqq 1$. So $Q_0 = a_0$. We first consider $k = 1$. We then have to show

$$Ea_1^{A_1(0)} a_2^{A_1(1)} \cdots a_L^{A_1(L-1)} = a_0. \tag{8.1}$$

Now $v \in \Pi_1(j)$ iff $d(v) = j+1$ and $d(\alpha(v)) = 0$. Hence if $N(0), \ldots, N(L)$ are the number of first generation particles at $0, \ldots, L$, then we can write for $j = 0, \ldots, L-1$

$$A_1(j) = N(j+1) + \sum_{\ell=1}^{N(0)} A_{1,\ell}(j), \tag{8.2}$$

where the $A_{1,\ell}(j)$ are distributed as $A_1(j)$, and independent of $(N(0), \ldots, N(L))$. Therefore, for $x_i \in [0, 1]$

$$\begin{aligned} &E x_1^{A_1(0)} x_2^{A_1(1)} \cdots x_L^{A_1(L-1)} \\ &\qquad = E\left[x_1^{N(1)} \cdots x_L^{N(L)} \left\{E x_1^{A_1(0)} x_2^{A_1(1)} \cdots x_L^{A_1(L-1)}\right\}^{N(0)}\right]. \end{aligned} \tag{8.3}$$

Hence if we put $x_0 = E x_1^{A_1(0)} x_2^{A_1(1)} \cdots x_L^{A_1(L-1)}$, then $x_0 \in [0, 1]$, and the right side of (8.3) equals $\Phi(x_0, x_1, \ldots, x_L)$. So (8.1) follows by admissibility of $a$ ($\Phi(a_0, \ldots, a_L) = a_0$), and uniqueness of this solution ($h(x) = \Phi(x, a_1, \ldots, a_L)$ is a convex function with $h'(x) \leqq 1$). Now for $k \geqq 2$. Since $\Pi_k$ is a cutset, each vertex in $\Pi_{k+1}$ has a unique member of its family line in $\Pi_k$. The vertices in $\Pi_k(j)$ for $j = 1, \ldots, L-1$ are also vertices of $\Pi_{k+1}(j-1)$, and each vertex in $\Pi_k(0)$ gives for each $j$ rise to a random number (distributed as $A_1(j)$) of vertices in $\Pi_{k+1}(j)$. Therefore, putting $A_k(L) = 0$, we have for $j = 0, \ldots, L-1$

$$A_{k+1}(j) = A_k(j+1) + \sum_{\ell=1}^{A_k(0)} A_{1,\ell}(j), \tag{8.4}$$

where the $A_{1,\ell}(j)$ are random variables distributed as $A_1(j)$ and independent of $A_k(0), \ldots, A_k(L-1)$. Hence

$$\begin{aligned} &E[x_1^{A_{k+1}(0)} \cdots x_{L-1}^{A_{k+1}(L-2)} x_L^{A_{k+1}(L-1)} | \mathscr{A}_k] \\ &= x_1^{A_k(1)} \cdots x_{L-1}^{A_k(L-1)} \left[E x_1^{A_1(0)} \cdots x_{L-1}^{A_1(L-2)} x_L^{A_1(L-1)}\right]^{A_k(0)}. \end{aligned}$$

Now if we take $x_i = a_{k+i}$, then by (8.1) (since a shift of an admissible sequence is admissible), $E(Q_{k+1} | \mathscr{A}_k) = Q_k$.

We now define for $k \geqq 1$ (interpreting $\log 0 = -\infty$)

$$Z_k = \max_{0 \leqq i < L} \frac{\log A_k(i)}{2^i}. \tag{8.5}$$

**Proposition 6** *There exists a random variable $W$, with a continuous distribution such that $\frac{Z_k}{2^k} \to W$ almost surely as $k \to \infty$.*

*Proof.* Let $(a_k)$ be the admissible sequence with $\chi(a) = t$, for $t > 0$ (cf. Proposition 4). Let

$$Q_k = Q_k(t) = a_k^{A_k(0)} \cdots a_{k+L-1}^{A_k(L-1)}$$

be the martingale of Lemma 11. Note that $0 \leqq Q_k(t) \leqq 1$ for all $k$. Let $Q(t)$ be the almost sure limit of $(Q_k(t))$. We claim that for $t > s$

$$Q(t) = 1 \qquad \text{a.s. on} \quad [Q(s) > 0] \; . \tag{8.6}$$

Indeed, let $b$ be the admissible sequence with $\chi(b) = s$. By Lemma 8, there exist $\gamma > 0$ and $\delta > 0$ such that for $k$ large enough

$$a_k \geqq 1 - \gamma e^{-t2^k} \qquad \text{and} \quad b_k \leqq 1 - \delta e^{-s2^k} \; . \tag{8.7}$$

For $x$ close enough to 0, $-\log(1-x) \leqq 2x$, hence for $k$ large enough

$$\begin{aligned} 0 &\leqq -\log(a_{k+j}) \leqq -\log(1-\gamma e^{-t2^{k+j}}) \leqq 2\gamma e^{-t2^{k+j}} \\ &= 2\gamma e^{-s2^{k+j}} e^{-(t-s)2^{k+j}} \leqq -2\gamma\delta^{-1} \log(b_{k+f}) e^{-(t-s)2^{k+j}} \; . \end{aligned} \tag{8.8}$$

Multiplying by $A_k(j)$, and summing over $j$, we see that $\log Q_k(t) \to 0$ if $Q_s(s) \to Q(s) > 0$. By (8.6), $Q(t)$ is non-decreasing in $t$ and a.s. assumes the values 0 and 1 for all but at most one $t$. We define $W$ by $W = \inf\{t : Q(t) = 1\}$. Since $EQ(t) = EQ_0(t) = a_0$, we have $0 < W < \infty$ almost surely. Moreover,

$$P[s < W < t] = P[Q(t) - Q(s) = 1] = EQ(t) - EQ(s) = a_0 - b_0$$

if $\chi(a) = t$ and $\chi(b) = s$. Since $\chi$ is continuous in $a_0$ and conversely by Proposition 5, $W$ has a continuous distribution. Finally we have

$$-\frac{1}{L} \log Q_k(t) \leqq \max_{0 \leqq j < L} -A_k(j) \log(a_{k+j}) \leqq -\log Q_k(t) \; .$$

Since $1 - a_k \sim \frac{\mu}{\tau} e^{-t2^k}$ we deduce from this (by estimations as in (8.8)) that

$$\lim_{k \to \infty} \max_{0 \leqq j < L} A_k(j) e^{-t2^{k+j}} = \begin{cases} \infty & \text{a.s. on } [W > t], \\ 0 & \text{a.s. on } [W < t] \; . \end{cases}$$

This implies that

$$\frac{Z_k}{2^k} = \max_{0 \leqq j < L} \frac{\log A_k(j)}{2^{k+j}} \to W \;\; \text{a.s.} \, .$$

Without proof we mention the following facts. Let $N_n(k) = \mathrm{Card}\{v \in \Gamma_n : d(v) = k\}$ be the number of particles with position $k$ at time $n$. Then $M_n = \prod_{k \geqq 0} a_k^{N_n(k)}$ is a martingale (w.r.t. $\sigma\{N_m(k) : m \leqq n, k \geqq 0\}$), and $M_n$ converges to the same limit as $(Q_k)$.

## 9 Almost sure convergence

Here we generalize Bramson's result [3] on the almost sure convergence of $X_n$. Lemmas 12 and 13 are closely related to Bramson's work. Instead of $X_n$ we study the first passage times

$$T_k = \inf\{n : X_n \geqq k\}$$

for positive integers $k$. This will be done with aid of the cutsets $\Pi_k$ defined in the previous section. Recall that $\Gamma_n$ denotes the cutset of all level $n$ vertices. We define

$$\Gamma_n(\Pi_k) = \bigcup_{m \geqq n} \{v \in \Gamma_m : v^{(m-n)} \in \Pi_k\} .$$

So $\Gamma_n(\Pi_k)$ is the cutset of all $n^{\text{th}}$ generation descendents of particles of $\Pi_k$. Let

$$E_{k,n} = [d(v) \geqq k + L \quad \text{for all } v \in \Gamma_n(\Pi_k)] .$$

the clearly

$$P[E_{k,n}|A_k] = F_n(L)^{A_k(0)} F_n(L-1)^{A_k(1)} \cdots F_n(1)^{A_k(L-1)} . \tag{9.1}$$

**Lemma 12** *For all $n > 0, k \geqq 0$*

$$[T_{k+L} \leqq n] \subset E_{k,n} . \tag{9.2}$$

*On the other hand for all $k$ and $n_0 > 0, \ldots, n_k > 0$ we have that*

$$E_{0,n_0} \cap E_{1,n_1} \cap \ldots \cap E_{k,n_k} \subset [T_{k+L} \leqq n_0 + \cdots + n_k] . \tag{9.3}$$

*Proof.* Since $[T_{k+L} \leqq n] = [X_n \geqq k+L] = [d(v) \geqq k+L$ for all $v \in \Gamma_n]$, (9.2) follows immediately. It is convenient to phrase the proof of (9.3) in terms of the partial ordering of cutsets. We write $\Gamma < \triangle$ if any infinite path starting at the root intersects $\Gamma$ not later than it intersects $\triangle$. For $k = 0$ (9.3) holds. Suppose (9.3) has been proved for $\ell = 1, \ldots, k$. Then $E_{0,n_0} \cap \ldots \cap E_{k,n_k} \cap E_{k+1,n_{k+1}} \subset [X_{n_0+\cdots+n_k} \geqq k+L]$ hence $d(v) \geqq k+L$ for $v \in \Gamma_{n_0+\cdots+n_k}$, which implies that $\Pi_{k+1} < \Gamma_{n_0+\cdots+n_k}$. But then $\Gamma_{n_{k+1}}(\Pi_{k+1}) < \Gamma_{n_0+\cdots+n_k+n_{k+1}}$, and for $v \in E_{k+1,n_{k+1}}$ this implies that $d(v) \geqq k+L$ if $v \in \Gamma_{n_0+\cdots+n_k+n_{k+1}}$. So $E_{0,n_0} \cap \ldots \cap E_{k+1,n_{k+1}} \subset [X_{n_0+\cdots+n_{k+1}} \geqq k+1+L]$. This finishes the induction proof.

**Lemma 13** *Suppose $EN(i) > 0$ for $1 \leqq i \leqq L$, and let $Z_k = \max_{0 \leqq i < L} 2^{-i} \log A_k(i)$. Almost surely for $k$ large enough*

$$T_{k+L} \geqq \frac{1}{k} \exp(2^{L-1} Z_k) \tag{9.4}$$

*and*

$$T_{k+L} \leqq \sum_{0 \leqq j \leqq k} \exp(k + 2^{L-1} Z_j) . \tag{9.5}$$

*Proof.* By Theorem 6, and since $F_1(L) > 0$, there exist two positive real numbers $\alpha$ and $\beta$ such that for all $n$ and $1 \leqq k \leqq L$

$$\beta n^{-2^{1-k}} \leqq -\log F_n(k) \leqq \alpha n^{-2^{1-k}} . \tag{9.6}$$

Let us take $n = n_k = [\frac{1}{k}\exp(2^{L-1}Z_k)]$ in (9.1). Since $n_k$ is $\mathscr{A}_k$-measurable, we have

$$\begin{aligned}-\log P[E_{k,n_k}|\mathscr{A}_k] &= -A_k(0)F_{n_k}(L) - \cdots - A_k(L-1)\log F_{n_k}(1)\\ &\geqq \beta A_k(0)n_k^{-2^{1-L}} + \cdots + \beta A_k(L-1)n_k^{-1}\\ &\geqq \beta k^{2^{1-L}}\{A_k(0)\,e^{-Z_k} + \cdots + A_k(L-1)\,e^{-2^{L-1}Z_k}\}\\ &\geqq \beta k^{2^{1-L}},\end{aligned}$$

since by (8.5)

$$\max_{0\leqq i<L}[A_k(i)]^{2^{-i}}\,e^{-Z_k} = 1\,. \tag{9.7}$$

Therefore

$$\sum_{k=1}^{\infty} P(E_{k,n_k}) = \sum_{k=1}^{\infty} EP(E_{k,n_k}|\mathscr{A}_k) \leqq \sum_{k=1}^{\infty}\exp(-\beta k^{2^{1-L}}) < \infty\,.$$

By Lemma 12 ((9.2)) and Borel-Cantelli, $T_{k+L} < n_k$ only finitely many times a.s., so (9.4) follows. For the proof of (9.5) we take $n_j = [\exp(k + 2^{L-1}Z_j)]$ in (9.1), for $0 \leqq j \leqq k$. Then, since $\mathscr{A}_j \subset \mathscr{A}_k$ for $0 \leqq j \leqq k$,

$$\begin{aligned}&-\log P[E_{j,n_j}|\mathscr{A}_k]\\ &= -A_j(0)F_{n_j}(L) - \cdots - A_j(L-1)\log F_{n_j}(1)\\ &\leqq \alpha A_j(0)n_j^{-2^{1-L}} + \cdots + \alpha A_j(L-1)n_j^{-1}\\ &\leqq \alpha\, e^{-k2^{1-L}}\left\{A_j(0)\,e^{-Z_j} + \cdots + A_j(L-1)\,e^{-2^{L-1}Z_j}\right\}\\ &\leqq \alpha L e^{-k2^{1-L}},\end{aligned}$$

(the last inequality follows by (9.7)). It thus follows that

$$P[E_{j,n_j}|\mathscr{A}_k] \geqq \exp(-\alpha L e^{-k2^{1-L}}) \geqq 1 - \alpha L e^{-k2^{1-L}},$$

and hence that for all $k \geqq 1$, $P[E_{0,n_0} \cap \cdots \cap E_{k,n_k}] \geqq 1 - (k+1)\alpha L\exp(-k2^{1-L})$. Now (9.5) follows by the second half of Lemma 12 and Borel-Cantelli.

**Proposition 7** *Let* $T_k = \inf\{n : X_n \geqq k\}$. *Then*

$$2^{-k}\log(T_k) \to V \qquad \text{almost surely,}$$

*for a random variable* $V$, *with* $0 < V < \infty$ a.s. *and with a continuous distribution.*

*Proof.* From Lemma 13 we have a.s. for $k$ large enough

$$\begin{aligned}\log\left(\frac{1}{k}\right) + 2^{L-1}Z_k \leqq \log(T_{k+L}) &\leqq k + \log\left(\sum_{0\leqq j\leqq k} e^{2^{L-1}Z_j}\right)\\ &\leqq k + \log(k+1) + 2^{L-1}\max_{0\leqq j\leqq k} Z_j\,.\end{aligned}$$

After multiplying by $2^{-k-L}$, the left side and the right side converge to $V = W/2$, by Proposition 6.

**Theorem 8** *Let* $(X_n)$ *be the minimal displacement of a critical branching random walk with bounded steps determined by* $(N(0), \ldots, N(L))$. *Suppose* $EN^2 < \infty$ *and* $EN(i) > 0$ *for* $1 \leqq i \leqq L$. *Then*

$$X_n = \left[\frac{\log\log n - \log V}{\log 2} + R_n\right] \quad \text{almost surely} \tag{9.8}$$

*for all* $n$ *large enough, where* $V$ *is a random variable with a continuous distribution on* $(0, \infty)$, *and* $R_n$ *a sequence of random variables with* $R_n \to 0$ *a.s.*

*Proof.* By Proposition 7 there exists a sequence of random variables $(\varepsilon_k)$ such that $\log(T_k) = 2^{k-\varepsilon_k} V$, with $\varepsilon_k \to 0$ a.s. Since $X_n \geqq k$ iff $T_k \leqq n$ we obtain

$$\frac{\log\log n - \log V}{\log 2} + \varepsilon_{X_n+1} - 1 \leqq X_n \leqq \frac{\log\log n - \log V}{\log 2} + \varepsilon_{X_n} .$$

Since $X_n \to \infty$ by Lemma 1, there exist $R_n$ such that (9.8) holds.

With a small loss of precision in the statement (9.8) we can get rid of the condition on the $EN(i)$.

**Theorem 9** *Let* $(X_n)$ *be the minimal displacement of a critical branching random walk with bounded steps and* $EN^2 < \infty$. *Let* $g = \min\{i \geqq 1 : EN(i) > 0\}$. *Then*

$$X_n - \frac{g\log\log n}{\log 2} \quad \text{is bounded almost surely for all} \quad n .$$

*In particular* $\frac{X_n \log 2}{\log\log n} \to g$ a.s. *as* $n \to \infty$.

*Proof.* Let the steps be bounded by $L$. Let $(N^{(1)}(i))$ and $(N^{(2)}(i))$ be defined by $N^{(1)}(0) = N^{(2)}(0) = N(0)$; $N^{(1)}(g) = N(g) + \cdots + N(L)$, $N^{(1)}(i) = 0$ iff $i \neq 0$ $i \neq g$; $N^{(2)}(g) = N(g)$, $N^{(2)}(mg) = N((m-1)g+1) + \cdots + N(mg-1)$ $m = 2, \ldots, M$, where $M = [(L+1)/g]$, $N^{(2)}(i) = 0$ if $i \notin \{0, g, \ldots, Mg\}$. Here moreover, if $N^{(2)}(mg) = 0$ for some $m$, then some mass of $N^{(2)}(g) = N(g)$ is moved to $N^{(2)}(mg)$ in order to have $EN^{(2)}(mg) > 0$ for $m = 1, \ldots, M$. Let $(X_n^{(1)})$ and $(X_n^{(2)})$ be the minimal displacements of the branching random walks generated by $(N^{(1)}(\cdot))$ and $(N^{(2)}(\cdot))$. By construction, for all $n$

$$X_n^{(1)} \leqq X_n \leqq X_n^{(2)} . \tag{9.9}$$

Both new processes are concentrated on $g\mathbb{Z}$. Rescaling by $g^{-1}$ we obtain processes on $\mathbb{Z}$ which satisfy the conditions of Theorem 8, and so there exist finite random variables $U^{(1)}$ and $U^{(2)}$ such that

$$\left|\frac{X_n^{(j)}}{g} - \frac{\log\log n}{\log 2}\right| \leqq U^{(j)}$$

for $j = 1, 2$ and all $n$. The result now follows from (9.9).

The following result is a refinement of Theorem 2 in [3] for the discrete case.

**Theorem 10** *Let $(X_n)$ be the minimal displacement of a critical branching random walk with $EN^2 < \infty$, and let $g = \inf\{i \geqq 1 : EN(i) > 0\}$. Then as $n \to \infty$*

$$\frac{X_n \log 2}{\log\log n} \to g \quad \text{a.s.}$$

*Proof.* As always we assume $EN > 1$. Then for any $L \geqq g$ it holds that $\sum_{i=0}^{L} EN(i) > 1$. Let $N^{(1)}(\cdot)$ and $N^{(2)}(\cdot)$ be defined by

$$N^{(1)}(i) = N^{(2)}(i) = N(i) \quad \text{for } 0 \leqq i \leqq L-1, \quad N^{(1)}(i) = N^{(2)}(i) = 0 \quad \text{for } i > L,$$

$$N^{(1)}(L) = \sum_{i=L}^{\infty} N(i),\ N^{(2)}(L) = N(L)\ .$$

For $j = 1, 2$ let $N^{(j)} = \sum_{i=0}^{\infty} N^{(j)}(i)$, then $E(N^{(j)})^2 < \infty$, $EN^{(j)} > 1$, and the gap parameters $g^{(j)} = \inf\{i \geqq 1 : EN^{(j)}(i) > 0\}$ are both equal to $g$. Let $X_n^{(j)}$ for $j = 1, 2$ be the associated minimal displacements, then $X_n^{(1)} \log 2/\log\log n \to g$, a.s., and $X_n^{(2)} \log 2/\log\log n \to g$ a.s. on the set $S_L$ of non-extinction of $N^{(2)} = N_L^{(2)}$, by Theorem 9. Since (9.9) holds for $X_n^{(1)}, X_n^{(2)}$, this implies that $X_n \log 2/\log\log n \to g$ a.s. on $S_L$. To finish the proof it suffices to see that $P(S_L) \to 1$ as $L \to \infty$. Now $1 - P(S_L)$ is the unique solution in (0,1) of $\sum_{k \geqq 0} P[N_L^{(2)} = k]s^k = s$. Using the fact that $EN_L^{(2)} \geqq EN_g^{(2)} > 1$ and (recall (1.3))

$$P[N_L^{(2)} = 0] = P\left[\sum_{i=0}^{L} N(i) = 0\right] \leqq P\left[\sum_{i=L+1}^{\infty} N(i) \geqq 1\right] \leqq EN - EN_L^{(2)} \to 0$$

as $L \to \infty$, it follows that $P(S_L) \to 1$.

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