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Operator-semistable operator Lévy's measures on finite dimensional vector spaces

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Summary. In the paper various characterizations of full operator-semistable operator Lévy's measures on finite dimensional vector spaces are presented. They are given in terms of: 1. some decomposability properties; 2. the characteristic functions; 3. stochastic integrals. Also a number of decomposability properties, especially for a full operator Lévy's measure, are obtained.

0 Introduction

In the class of infinitely divisible laws on a finite dimensional real vector space V, a very interesting subclass consists of so-called full (i.e. not concentrated on any proper hyperplane of V) "operator limit laws". These are the limit distributions of operator normed and centered sums of a sequence of independent random vectors in V. Among the full operator limit laws, three classes of particular interest have so far been investigated in great detail: operator-stable, operator-semistable and operator Lévy's distributions. The operator-stable laws are obviously the smallest class, and it seems an interesting question to find a description of their "closest relations", namely, the intersection of the set of the operator-semistable measures with the set of operator Lévy's ones. In the one-dimensional case, it was done in [2]; all that has been done in the multidimensional setup consists in describing two classes: operator-semistable measures that are multivariate Lévy's and operator Lévy's measures that are multivariate semistable (cf. [7]).

The main purpose of this paper is to give a complete solution to the above mentioned problem. We characterize the class of those full operator-semistable laws that are also operator Lévy's measures. Actually, three such characterizations are given. The first one is expressed in terms of some decomposability properties, the second refers to the characteristic functions, and the third is obtained by means of some stochastic integrals.

In the course of our analysis, we obtain also some decomposability properties of a full probability measure which are of interest on their own.

For the sake of making the paper as self-contained as possible, we present in the Appendix a fundamental property of compact semigroups of operators, which is employed in our considerations.

1 Preliminaries and notation

Throughout the paper, V will stand for a finite dimensional real vector space with an inner product $(\cdot,\cdot)$ and σ-algebra $\mathscr{B}(V)$ of its Borel subsets. We let End V denote the set of all linear operators on V, whereas Aut V stands for the linear invertible operators.

Let $A: V \to W$ be a linear mapping into a finite dimensional real vector space W and let μ be a (probability) measure over $(V, \mathscr{B}(V))$. The measure $A\mu$ on $(W, \mathscr{B}(W))$ is defined by

$$A\mu(E) = \mu(A^{-1}(E)),\ E \in \mathscr{B}(W) .$$

The following equalities are easily verified

$$A(B\mu) = (AB)\mu, \quad \widehat{A\mu}(v) = \hat{\mu}(A^* v), \quad A(\mu * v) = A\mu * Av ,$$

for linear operators A, B and probability measures μ, v (here $\widehat{\ }$ denotes the characteristic function and the asterisk $*$ stands for the convolution of measures or for the adjoint of an operator, as the case may be).

By $\delta(h)$ we denote the probability measure concentrated at point h.

We recall that an infinitely divisible measure μ on V has the unique representation $[m, D, M]$, where $m \in V$, D is a non-negative linear operator on V, and M is the Lévy spectral measure of μ, i.e. a Borel measure defined on $V_0 = V - \{0\}$ such that $\int_{V_0} \|v\|^2/(1 + \|v\|^2)M(\mathrm{d}v) < \infty$. The characteristic function of μ has then the form

$$\hat{\mu}(v) = \exp\left\{ i(m, v) - \frac{1}{2}(Dv, v) + \int_{V_0} \left[\mathrm{e}^{i(v,u)} - 1 - \frac{i(v, u)}{1 + \|u\|^2} \right] M(\mathrm{d}u) \right\}$$

(cf. e.g. [9]).

The main objects of this paper are full operator semistable measures and full operator Lévy's measures on V. For their definitions, being generalizations of the classical definitions of semistable and Lévy's limit laws, as well as for more detailed accounts, the reader is referred to [3, 6] (operator semistable measures) and [14, 16] (operator Lévy's measures). Here we only recall that a full probability measure μ on V is: (i) operator semistable; (ii) operator Lévy's, respectively, if and only if it is infinitely divisible and

(i) there are $0 < a < 1$, $h \in V$, and a linear operator A on V, $\|A\| < 1$, such that

(1) $\quad \mu^a = A\mu * \delta(h);$

(ii) there is a linear operator Q on V whose all eigenvalues have negative real parts, such that, for each $t \geqq 0$,

(2) $\quad \mu = \mathrm{e}^{tQ}\mu * \mu_t$

where μ_t is a probability measure on V.

To emphasize the role of a and A, the measures satisfying (1) will be called in the sequel (a, A)-quasi-decomposable.

For a probability measure μ on V, its decomposability semigroup $\mathbb{D}(\mu)$ is defined as

$$\mathbb{D}(\mu) = \{B \in \mathrm{End}\ V : \mu = B\mu * \nu \text{ for some probability measure } \nu\} .$$

An important subset of $\mathbb{D}(\mu)$ is the so-called symmetry group $\mathbb{S}(\mu)$:

$$\mathbb{S}(\mu) = \{S \in \mathrm{Aut}\ V : \mu = S\mu * \delta(h) \text{ for some } h \text{ in } V\} .$$

Let us introduce the set of exponents of measure μ

$$\mathbb{E}(\mu) = \{Q \in \mathrm{End}\ V : \mathrm{e}^{tQ} \in \mathbb{D}(\mu) \text{ for each } t \geqq 0\} .$$

Finally, two subsets of $\mathbb{E}(\mu)$ will be of special importance for our considerations:

$$\mathbb{E}_0(\mu) = \{Q \in \mathbb{E}(\mu) : \text{all eigenvalues of } Q \text{ have zero real parts}\}$$

$$\mathbb{E}_-(\mu) = \{Q \in \mathbb{E}(\mu) : \text{all eigenvalues of } Q \text{ have negative real parts}\} .$$

To clarify the mutual relations between the sets above, let us observe that, in general, we may have $\mathbb{E}(\mu) = \mathbb{E}_0(\mu) = \{0\}$, and for μ full, $\mathbb{E}_-(\mu)$ is non-void if and only if μ is an operator Lévy's measure, which is a consequence of the characterization given by formula (2). For the sake of the preliminary description, let us notice that, for μ full, $\mathbb{D}(\mu)$ and $\mathbb{S}(\mu)$ are compact (cf. [14; Propositions 1.1 and 1.2]), and $\mathbb{E}(\mu)$ is a closed cone (cf. [8; Proposition 3]).

2 Decomposability properties of full measures

This section is devoted to a more detailed analysis of the set $\mathbb{E}(\mu)$ yielding some decomposability properties of a full measure.

Lemma 1 *Let μ be full. Then $\mathbb{E}_0(\mu) = T(\mathbb{S}(\mu))$ (where $T(\mathbb{S}(\mu))$ denotes the tangent space of $\mathbb{S}(\mu)$ at the point I—the identity operator).*

Proof. Let, for a linear operator B on V, $\det B$ denote the determinant of the matrix representation of B with respect to a fixed basis in V.

Assume that $Q \in \mathbb{E}_0(\mu)$. Then $\mathrm{e}^{tQ} \in \mathbb{D}(\mu)$ for $t \geqq 0$, and $|\det \mathrm{e}^{tQ}| = |\mathrm{e}^{t\, \mathrm{trace}\, Q}| = 1$, which, on account of [14; Proposition 1.4], means that $\mathrm{e}^{tQ} \in \mathbb{S}(\mu)$ for $t \geqq 0$, hence $Q \in T(\mathbb{S}(\mu))$.

Conversely, if $Q \in T(\mathbb{S}(\mu))$, then $\mathrm{e}^{tQ} \in \mathbb{S}(\mu)$ for all $t \in \mathbb{R}$. By virtue of [14; Proposition 1.2], $\mathbb{S}(\mu)$ is a compact subgroup of Aut V, thus there is a non-singular operator C in Aut V such that, for each $S \in \mathbb{S}(\mu)$, CSC^{-1} is orthogonal. This means, in particular, that $|\det \mathrm{e}^{tQ}| = 1$. According to [8; Lemma 2], all eigenvalues of Q have non-positive real parts, so the last equality is possible only if the real parts are equal to zero, which shows that $Q \in \mathbb{E}_0(\mu)$. □

Now, we show a facial property of $\mathbb{E}_0(\mu)$.

Lemma 2 *Let μ be full and let $Q_1, Q_2 \in \mathbb{E}(\mu)$. If $Q_1 + Q_2 \in \mathbb{E}_0(\mu)$, then $Q_1, Q_2 \in \mathbb{E}_0(\mu)$.*

Proof. Since $\mathbb{E}_0(\mu)$ is a subspace, $-(Q_1 + Q_2) \in \mathbb{E}_0(\mu)$, and $-Q_1 = Q_2 - (Q_1 + Q_2) \in \mathbb{E}(\mu)$ because $\mathbb{E}(\mu)$ is a cone. On account of [8; Lemma 2], all elements in $\mathbb{E}(\mu)$ have eigenvalues with non-positive real parts, so Q_1, $-Q_1 \in \mathbb{E}(\mu)$, implies that all eigenvalues of Q_1 must have their real parts equal to zero, that is $Q_1 \in \mathbb{E}_0(\mu)$. Analogously, we get $Q_2 \in \mathbb{E}_0(\mu)$. □

As an immediate corollary, we obtain

Corollary 3 *Let μ be full and let $Q_1, \ldots, Q_n \in \mathbb{E}(\mu)$. If $\sum_{i=1}^n Q_i \in \mathbb{E}_0(\mu)$, then $Q_i \in \mathbb{E}_0(\mu)$, $i = 1, \ldots, n$.*

This is proved by induction upon observing that $\mathbb{E}(\mu)$ being a cone implies $\sum_{i=1}^{n-1} Q_i \in \mathbb{E}(\mu)$.

The following proposition describes the basic properties of the operators from $\mathbb{E}(\mu)$.

Proposition 4 *Let μ be full and let $Q \in \mathbb{E}(\mu)$. Then there exist projections P, T* (i.e. $P = P^2$, $T = T^2$) *in $\mathbb{D}(\mu)$ such that*

(i) $P + T = I$, $PT = TP = 0$;
(ii) $PQ = QP$, $TQ = QT$;
(iii) $P = 0$ *if and only if* $Q \in \mathbb{E}_-(\mu)$, $T = 0$ *if and only if* $Q \in \mathbb{E}_0(\mu)$;
(iv) $PQ \in \mathbb{E}_0(\mu)$, $TQ \in \mathbb{E}(\mu)$; *moreover, if* $Q \notin \mathbb{E}_0(\mu)$, *then* $TQ \notin \mathbb{E}_0(\mu)$, *in particular,* $TQ \neq 0$.

Proof. Put $\mathbb{G} = \overline{\{e^{tQ} : t \geqq 0\}}$. $\mathbb{G}$ is a compact abelian semigroup of $\mathbb{D}(\mu)$. Let $\mathbb{H}$ be the set of the limit points at infinity of e^{tQ}. According to the Appendix, $\mathbb{H}$ is a group, and let P be its unit. P is clearly a projection and $P \in \mathbb{D}(\mu)$. Put $T = I - P$. Then, by virtue of [14; Proposition 1.5], $T \in \mathbb{D}(\mu)$; moreover, $PT = TP = 0$.

Since $Pe^{tQ} = e^{tQ}P$ for $t \geqq 0$, differentiation at $t = 0$ yields $PQ = QP$, and, consequently, $TQ = QT$.

If $Q \in \mathbb{E}_-(\mu)$, then $\lim_{t \to \infty} e^{tQ} = 0$ (cf. e.g. [1, Chap. 13]), so $P = 0$. Conversely, if $P = 0$, then for some $t_n \to \infty$, $e^{t_n Q} \to 0$ which is possible only if $Q \in \mathbb{E}_-(\mu)$.

Now, if $Q \in \mathbb{E}_0(\mu)$, then $e^{tQ} \in \mathbb{S}(\mu)$, $t \geqq 0$. Let $t_n \to \infty$ be such that $e^{t_n Q} \to P$. We have

$$\mu = e^{t_n Q}\mu * \delta(h_n) .$$

Passing to the limit, we get

$$\mu = P\mu * \delta(h)$$

and the fullness of μ yields $P = I$.

Conversely, if $T = 0$, then $P = I$, and taking again $t_n \to \infty$ such that $e^{t_n Q} \to I$, we have, for any eigenvalue α of Q, $e^{t_n \alpha} \to 1$, which yields re $\alpha = 0$, i.e. $Q \in \mathbb{E}_0(\mu)$.

Next, we have the formulae

$$e^{tPQ} = Pe^{tQ} + T, \quad e^{tTQ} = Te^{tQ} + P, \quad t \geqq 0 , \tag{3}$$

and, as $Q \in \mathbb{E}(\mu)$, for μ formula (2) holds. On account of [14; Proposition 1.5],

$$\mu = P\mu * T\mu ,$$

and thus, by (3),

$$
\begin{aligned}
e^{tPQ}\mu * P\mu_t = e^{tPQ}(P\mu * T\mu) * P\mu_t &= e^{tPQ}P\mu * e^{tPQ}T\mu \\
* P\mu_t &= e^{tQ}P\mu * T\mu * P\mu_t = P(e^{tQ}\mu * \mu_t) * T\mu \\
&= P\mu * T\mu = \mu ,
\end{aligned}
$$

which means that $PQ \in \mathbb{E}(\mu)$. Taking $t_n \to \infty$ such that $e^{t_n Q} \to P$, we get

$$e^{t_n PQ} = Pe^{t_n Q} + T \to P + T = I ,$$

which, as before, shows that $PQ \in \mathbb{E}_0(\mu)$.
Using (3) again, we obtain

$$
\begin{aligned}
e^{tTQ}\mu * T\mu_t &= e^{tTQ}(P\mu * T\mu) * T\mu_t = P\mu * T(e^{tQ}\mu * \mu_t) \\
&= P\mu * T\mu = \mu ,
\end{aligned}
$$

showing that $TQ \in \mathbb{E}(\mu)$. Finally, if $TQ \in \mathbb{E}_0(\mu)$, then $Q = PQ + TQ \in \mathbb{E}_0(\mu)$ by Lemma 2 and the relation $PQ \in \mathbb{E}_0(\mu)$. □

As an important consequence of the above proposition, we get the following decomposability result.

Theorem 5 *Let μ be full and let $Q \in \mathbb{E}(\mu)$. There are decompositions*

$$V = U \oplus W, \qquad \mu = \nu' * \lambda'$$

such that the subspaces U and W are Q-invariant, ν' is concentrated on U, λ' is concentrated on W, and denoting

$$\nu = \nu' | U, \qquad \lambda = \lambda' | W ,$$

we have $Q|U \in \mathbb{E}_0(\nu)$ and $Q|W \in \mathbb{E}_-(\lambda)$ if $W \neq \{0\}$. Moreover, $U = \{0\}$ if and only if $Q \in \mathbb{E}_-(\mu)$, and $W = \{0\}$ if and only if $Q \in \mathbb{E}_0(\mu)$.

Proof. Putting, for P and T as in Proposition 4,

$$U = P(V), \qquad W = T(V), \qquad \nu' = P\mu, \qquad \lambda' = T\mu$$

we obtain the desired decompositions. It remains to prove the relations $Q|U \in \mathbb{E}_0(\nu)$ and $Q|W \in \mathbb{E}_-(\lambda)$.

For μ_t as in (2), we have, since $PQ \in \mathbb{E}_0(\mu)$,

$$\mu = e^{tPQ}\mu * \delta(h_t),$$

yielding

$$P\mu = e^{tPQ}P\mu * \delta(Ph_t) ,$$

which shows that $Q|U \in \mathbb{E}_0(\nu)$.
We also have

$$T\mu = e^{tQ}T\mu * T\mu_t = e^{tTQ}T\mu * T\mu_t ,$$

hence $Q|W \in \mathbb{E}(\lambda)$. For an arbitrary $t_n \to \infty$, there is a subsequence $\{t_{n_k}\}$ such that $e^{t_{n_k}Q} \to R$, where R is an element in $\mathbb{H}$. Since P is an identity in $\mathbb{H}$, we get, using (3),

$$e^{t_{n_k}TQ} = Te^{t_{n_k}Q} + P \to TR + P = TPR + P = P ,$$

so

$$\lim_{t\to\infty} e^{tTQ} = P\,. \tag{4}$$

If $W \neq \{0\}$, i.e. $T \neq 0$, the last equality implies $\lim_{t\to\infty}{}^{tQ|W} = 0$ (zero operator on W), showing that $Q|W \in \mathbb{E}_-(\lambda)$ (cf. e.g. [1; Chap. 13]). $\square$

Now, we wish to give a probabilistic description of the set $\mathbb{E}_-(\mu)$. To this end, we start with some simple "fullness" considerations.

Lemma 6 *Let W be a subspace of V and let μ be a probability measure on V. Then $\mu(W) = 1$ if and only if $\hat{\mu}(v) = 1$ for each $v \in W^1$.*

The proof is similar to the proof of Proposition 1 in [12]. The details are left to the reader.

Lemma 7 *Let W be a subspace of V, let μ be a probability measure on V and let A be a linear operator on V. If $\mu(W) = 1$, $A\mu(W) = 1$ and $A\mu$ is full on W, then $A(W) = W$.*

Proof. We have $A\mu(A(W)) = \mu(A^{-1}(A(W))) \geqq \mu(W) = 1$ and thus

$$A\mu(W \cap A(W)) = 1\,.$$

Since $A\mu$ is full on W, we get

$$W \cap A(W) = W\,,$$

which means that $W \subset A(W)$, and therefore $W = A(W)$. $\square$

The following result—the openness of the set of full measures — was mentioned in [12; p. 52] without proof. For the sake of completeness, we sketch its proof here.

Lemma 8 *Let μ be a full probability measure on V and let $\{\mu_n\}$ be a sequence of probability measures on V weakly convergent to μ. Then μ_n are full for all sufficiently large n.*

Proof. If μ_{n_k} are not full, then

$$|\hat{\mu}_{n_k}(\alpha v_k)| = 1$$

for some $v_k \in V$, $\|v_k\| = 1$ and all $\alpha \in \mathbb{R}$. We may assume that $v_k \to v_0$ and since for any fixed $\alpha \in \mathbb{R}$ the set $\{v \in V : \|v\| \leqq \alpha\}$ is compact,we get

$$\hat{\mu}(\alpha v_0) = \lim_{k\to\infty} \hat{\mu}_{n_k}(\alpha v_k)\,,$$

showing that

$$|\hat{\mu}(\alpha v_0)| = 1\,,$$

which contradicts the fullness of μ. $\square$

Let us recall that, for a probability measure μ on V, $\bar{\mu}$ is defined as

$$\bar{\mu}(E) = \mu(-E), \qquad E \in \mathscr{B}(V)\,,$$

and the symmetrization $\mathring{\mu}$ of μ is $\mathring{\mu} = \mu * \bar{\mu}$. Obviously, $\hat{\mathring{\mu}} = |\hat{\mu}|^2$.

Now, if μ is concentrated on a hyperplane $W + h$, where W is a subspace of V, $h \in V$, and if μ is full on $W + h$ in the sense that no other hyperplane contained in $W + h$ has μ measure equal to one, then $\mu * \delta(-h)$ is concentrated and full on W, thus $\overline{\mu * \delta(-h)} = \bar{\mu} * \delta(h)$ is concentrated and full on W. Consequently,

$$[\mu * \delta(-h)] * [\bar{\mu} * \delta(h)] = \mathring{\mu}$$

is concentrated and full on W, which shows that any symmetrized measure is concentrated and full on some subspace (which may be the whole space V in the case when μ is full).

Now we are in a position to prove a characterization of a full operator Lévy's measure in terms of some decomposability properties.

Theorem 9 *Let μ be a full probability measure on V and let $Q \in \mathbb{E}(\mu)$ such that formula* (2) *holds. The following conditions are equivalent*:
(i) $Q \in \mathbb{E}_-(\mu)$ (i.e. *μ is an operator Lévy's measure*);
(ii) *μ_t is full for all $t > 0$*;
(iii) *μ_{t_0} is full for some $t_0 > 0$.*

Proof. (i) $\Rightarrow$ (ii). From formula (2), we get

$$e^{sQ}\mu = e^{sQ}e^{tQ}\mu * e^{sQ}\mu_t = e^{(t+s)Q}\mu * e^{sQ}\mu_t$$

and thus

$$\mu = e^{sQ}\mu * \mu_s = e^{(t+s)Q}\mu * e^{sQ}\mu_t * \mu_s \, .$$

On the other hand,

$$\mu = e^{(t+s)Q}\mu * \mu_{t+s} \, ,$$

and since μ is infinitely divisible, $\hat{\mu}(v) \neq 0$, which yields the equality

$$\mu_{t+s} = e^{sQ}\mu_t * \mu_s \, , \qquad t, s \geqq 0 \, ,$$

and, after symmetrization,

$$\mathring{\mu}_{t+s} = e^{sQ}\mathring{\mu}_t * \mathring{\mu}_s \, , \qquad t, s \geqq 0 \, . \tag{5}$$

Assume that, for some t', $\mu_{t'}$ is not full. Then $\mathring{\mu}_{t'}$ is not full, and let $\mathring{\mu}_{t'}$ be concentrated and full on a subspace W. For $v \in W^{\perp}$, we have, on account of Lemma 6 and (5),

$$1 = \widehat{\mathring{\mu}}_{t'}(v) = \widehat{e^{sQ}\mathring{\mu}_t}(v)\widehat{\mathring{\mu}}_s(v), \qquad t + s = t' \, .$$

Since the characteristic functions on the right hand side of the above equality are positive, we get

$$\widehat{e^{sQ}\mathring{\mu}_t}(v) = \widehat{\mathring{\mu}}_s(v) = 1 \, ,$$

which, again by Lemma 6, means that

$$e^{sQ}\mathring{\mu}_t(W) = \mathring{\mu}_s(W) = 1$$

for all $t, s \leqq t'$, $t + s = t'$. Equality (5) together with the inequality $\widehat{\mu}(v) \neq 0$ show that $\mathring{\mu}_t \Rightarrow \mathring{\mu}_{t_0}$ as $t \to t_0$. In particular,

$$e^{Q/n}\mathring{\mu}_{t' - \frac{1}{n}} \Rightarrow \mathring{\mu}_{t'} \, ,$$

so, according to Lemma 8 applied to measures considered only on W, we obtain that $e^{Q/n}\mathring{\mu}_{t'-\frac{1}{n}}$ is full on W for all sufficiently large n. As $\mathring{\mu}_{t'-\frac{1}{n}}(W) = 1$, Lemma 7 implies that $e^{Q/n}(W) = W$ for large n. The formula

$$Qv = \lim_{n\to\infty} n(e^{Q/n} - I)v$$

yields $Q(W) \subset W$, thus $Q(W) = W$. Consequently, $e^{tQ}(W) = W$ for all t, which gives

$$e^{t'Q}\mathring{\mu}_{t'}(W) = \mathring{\mu}_{t'}(e^{-t'Q}(W)) = \mathring{\mu}_{t'}(W) = 1 .$$

Thus

$$\mathring{\mu}_{2t'}(W) = (e^{t'Q}\mathring{\mu}_{t'} * \mathring{\mu}_{t'})(W) = 1 .$$

Proceeding further that way, we obtain $\mathring{\mu}_{nt'}(W) = 1$ for all positive integers n. But the equality

$$\mathring{\mu} = e^{tQ}\mathring{\mu} * \mathring{\mu}_t$$

and the relation $\lim_{t\to\infty} e^{tQ} = 0$ yield $\mathring{\mu}_t \Rightarrow \mathring{\mu}$ as $t \to \infty$; in particular, $\mathring{\mu}_{nt'} \Rightarrow \mathring{\mu}$, which gives a contradiction, since $\mathring{\mu}$ is full and $\mathring{\mu}_{nt'}$ are concentrated on W.

(ii) $\Rightarrow$ (iii). Obvious.

(iii) $\Rightarrow$ (i). Assume, on the contrary, that $Q \notin \mathbb{E}_-(\mu)$. Let P be the projection as in Proposition 4. By this proposition point (iii), $P \neq 0$, moreover, we have

$$P\mu = e^{tQ}P\mu * P\mu_t = e^{tPQ}P\mu * P\mu_t .$$

On the other hand, $PQ \in \mathbb{E}_0(\mu)$, so

$$\mu = e^{tPQ}\mu * \delta(h_t) ,$$

giving

$$P\mu = e^{tPQ}P\mu * \delta(Ph_t) ,$$

and thus

$$e^{tPQ}P\mu * P\mu_t = e^{tPQ}P\mu * \delta(Ph_t) ,$$

which yields the equality $P\mu_t = \delta(Ph_t)$, showing that, for any t, μ_t is not full. This contradiction finishes the proof. □

3 Description of operator Lévy's measures in terms of the decomposability semigroup

For the purposes of this section, we recall that a probability measure μ is called operator Lévy's if it is the limit of sequence $A_n(\nu_1 * \ldots * \nu_n) * \delta(h_n)$, where $h_n \in V$, A_n's are linear operators on V and $\{\nu_n\}$ is an arbitrary sequence of probability measures on V, such that the measures $A_n\nu_k$, $k = 1, \ldots, n$; $n = 1, 2, \ldots$, form a uniformly infinitesimal triangular array. It is shown in [14; Proposition 5.2] that if $\mathbb{D}(\mu) \supset \{e^{tQ} : t \geqq 0\}$ such that $\lim_{t\to\infty} e^{tQ} = 0$, then μ is operator Lévy's, which, as mentioned in Sect. 1, is a characterization in the case of full μ. Since the condition $\{e^{tQ} : t \geqq 0\} \subset \mathbb{D}(\mu)$ is usually not easily verifiable, we want to give in this section other conditions on $\mathbb{D}(\mu)$ which would guarantee μ being operator Lévy's. In order to keep close to our main objects which are full measures, as well as to simplify the following considerations, we assume in this section that the measures in question are full. However, it is worth noticing that the next two lemmas remain true for arbitrary μ.

Lemma 10 *Assume that* $\mathbb{D}(\mu)$ *contains a sequence of linear operators* B_n *with properties*
(i) $\lim_{n\to\infty} B_n = I$;
(ii) $\lim_{n\to\infty} (B_n \dots B_1) = 0$.
Then μ *is an operator Lévy's measure.*

Proof. In view of (i), we may assume, taking sufficiently large n, that B_n's are non-singular. We have

$$\mu = B_n\mu * \mu_n, \qquad n = 1, 2, , \dots ,$$

which yields the relation

$$\mu_n \Rightarrow \delta(0) . \tag{6}$$

The following equality is easily verified

$$\mu = B_n \dots B_1\mu * B_n \dots B_2\mu_1 * \dots * B_n\mu_{n-1} * \mu_n , \quad n = 1, 2, \dots .$$

Put

$$A_n = B_n \dots B_1, \quad \nu_0 = \mu, \quad \nu_n = A_n^{-1}\mu_n, \quad n = 1, 2, \dots .$$

We then have

$$\mu = A_n(\nu_0 * \nu_1 * \dots * \nu_n) . \tag{7}$$

Moreover, by (ii), for each k

$$\lim_{n\to\infty} A_nA_k^{-1} = \lim_{n\to\infty} (B_n \dots B_{k+1}) = 0 . \tag{8}$$

Since the operators $A_nA_k^{-1}$ for $k = 1, \dots, n$, $n = 1, 2, \dots$ are in $\mathbb{D}(\mu)$ which is compact, we have

$$\| A_nA_k^{-1} \| \leqq c, \quad k = 1, \dots , n, \quad n = 1, 2, \dots . \tag{9}$$

Consequently, by (6), (8) and (9), the measures $\{A_n\nu_k : k = 0, 1, \dots , n;\ n = 1, 2, \dots\} = \{A_n\nu_0, \dots , A_nA_k^{-1}\mu_k : k = 1, \dots , n; n = 1, 2, \dots\}$ form a uniformly infinitesimal triangular array, and (7) implies that μ is an operator Lévy's measure. □

Lemma 11 *Assume that* $\mathbb{D}(\mu)$ *contains a sequence of linear operators* C_n *with properties*
(i) $\lim_{n\to\infty} C_n = I$;
(ii) $\lim_{k\to\infty} C_n^k = 0, \quad n = 1, 2, \dots$
Then μ *is an operator Lévy's measure.*

Proof. Choose positive integers k_i such that

$$\| C_i^{k_i} \| \leqq \frac{1}{i}, \quad i = 1, 2, \ldots,$$

and put (with $k_0 = 0$)

$$B_n = C_i \text{ for } k_0 + \cdots + k_{i-1} + 1 \leqq n \leqq k_0 + \cdots + k_i, \quad i = 1, 2, \ldots .$$

Clearly, $\lim_{n \to \infty} B_n = I$. Moreover, for each n we can find an indice i_n and a positive integer $1 \leqq l \leqq k_{i_n}$, such that $n = k_0 + \cdots + k_{i_{n-1}} + l$. We then have, since $C_i^k \in \mathbb{D}(\mu)$,

$$\begin{aligned} \| B_n \ldots B_1 \| &= \| C_{i_n}^l C_{i_n-1}^{k_{i_n-1}} \ldots C_1^{k_1} \| \\ &\leqq c \| C_{i_n-1}^{k_{i_n-1}} \| \ldots \| C_1^{k_1} \| \leqq c \frac{1}{i_{n-1}}, \end{aligned}$$

where $c = \sup\{ \| A \| : A \in \mathbb{D}(\mu)\} < \infty$. The last inequality means that $\lim_{n \to \infty} (B_n \ldots B_1) = 0$, and by Lemma 10, the result follows. □

As an interesting corollary, we get the following property of projections of full operator Lévy's measures.

Proposition 12 *Let μ be a full operator Lévy's measure and let P be a non-zero projection in $\mathbb{D}(\mu)$. Then $P\mu | P(V)$ is a full operator Lévy's measure (consequently, $P\mu$ as a measure on V is also operator Lévy's).*

Proof. According to [14; Lemma 4.2], for a given non-zero projection P from $\mathbb{D}(\mu)$ there exists a sequence $\{D_n\}$ of operators from $\mathbb{D}(\mu)$ satisfying the conditions

(i) $D_n P = P D_n = D_n$;

(ii) $\lim_{n \to \infty} D_n = P$;

(iii) $\lim_{k \to \infty} D_n^k = 0$, $n = 1, 2, \ldots$.

Put

$$U = P(V), \quad \nu = P\mu | U, \quad C_n = D_n | U .$$

We have

$$\mu = D_n \mu * \mu_n ,$$

so

$$P\mu = D_n P\mu * P\mu_n ,$$

which means that $C_n \in \mathbb{D}(\nu)$. Clearly, ν is full on U; moreover,

$$\lim_{n \to \infty} C_n = I_U \quad \text{and} \quad \lim_{k \to \infty} C_n^k = 0, \quad n = 1, 2, \ldots .$$

Now, Lemma 11 applied to the measure ν on U yields the claim. □

4 A basic property of $\mathbb{E}(\mu)$ for a full operator-semistable operator Lévy's measure

Throughout this section, μ is assumed to be a full (a, A)-quasi-decomposable measure on V, i.e. for μ formula (1) holds.

Lemma 13 *Let $Q \in \mathbb{E}(\mu)$ be such that $AQ = QA$ and let U, W, v, λ be the subspaces and the measures, respectively, given by Theorem 5. Then U and W are A-invariant, v is $(a, A|U)$-quasi-decomposable, and λ is $(a, A|W)$-quasi-decomposable.*

Proof. Since $AQ = QA$, we get $Ae^{tQ} = e^{tQ}A$ for all t. Let P and T be the projections as in the proof of Theorem 5; recall that $U = P(V)$, $W = T(V)$ and P is a limit point at infinity of the semigroup $\{e^{tQ}: t \geqq 0\}$. Consequently, we have $AP = PA$, and thus $AT = TA$, which yields the A-invariance of U and W.

The measure μ is (a, A)-quasi-decomposable and $Q \in \mathbb{E}(\mu)$, so assume that for μ formulae (1) and (2) hold. Then we have

$$\widehat{A\mu}(v) = \widehat{e^{tQ}A\mu}(v)\widehat{A\mu_t}(v)$$

and

$$[\hat{\mu}(v)]^a = [\widehat{e^{tQ}\mu}(v)]^a[\hat{\mu}_t(v)]^a$$
$$= e^{i(e^{tQ}h, v)}\widehat{e^{tQ}A\mu}(v)[\hat{\mu}_t(v)]^a .$$

These two equalities give

$$e^{i(e^{tQ}h, v)}\widehat{e^{tQ}A\mu}(v)[\hat{\mu}_t(v)]^a = [\hat{\mu}(v)]^a = e^{i(h, v)}\widehat{A\mu}(v)$$
$$= e^{i(h, v)}\widehat{e^{tQ}A\mu}(v)\widehat{A\mu_t}(v)$$

and since μ is infinitely divisible, $\hat{\mu}(v) \neq 0$ for all $v \in V$, which gives

$$[\hat{\mu}_t(v)]^a = e^{i(h - e^{tQ}h, v)}\widehat{A\mu_t}(v) ,$$

meaning that μ_t is (a, A)-quasi-decomposable. In particular, $T\mu_t$ is (a, A)-quasi-decomposable for $t \geqq 0$.
Furthermore, we have

$$T\mu = e^{tQ}T\mu * T\mu_t = e^{tTQ}T(T\mu) * T\mu_t .$$

According to (4),

$$\lim_{t \to \infty} e^{tTQ}P = P ,$$

thus

$$\lim_{t \to \infty} e^{tTQ}T = \lim_{t \to \infty} (e^{tTQ} - e^{tTQ}P) = 0 .$$

Take t_0 such that $\|e^{t_0TQ}T\| < 1$ and put $B = e^{t_0TQ}T$, $\rho = T\mu_{t_0}$. Then

$$T\mu = B(T\mu) * \rho \tag{10}$$

and iterating the last equality, we get

$$T\mu = B^{n+1}(T\mu) * B^n\rho * \cdots * B\rho * \rho, \qquad n = 1, 2, \ldots .$$

Since $B^n \to 0$, by putting

$$\rho_n = B^n\rho * \cdots * B\rho * \rho ,$$

we obtain

$$\lim_{n \to \infty} \rho_n = T\mu .$$

Since

$$\rho^a = A\rho * \delta(h') \quad \text{for some} \quad h' \in W ,$$

Lemma 13 *Let $Q \in \mathbb{E}(\mu)$ be such that $AQ = QA$ and let U, W, v, λ be the subspaces and the measures, respectively, given by Theorem 5. Then U and W are A-invariant, v is $(a, A|U)$-quasi-decomposable, and λ is $(a, A|W)$-quasi-decomposable.*

Proof. Since $AQ = QA$, we get $Ae^{tQ} = e^{tQ}A$ for all t. Let P and T be the projections as in the proof of Theorem 5; recall that $U = P(V)$, $W = T(V)$ and P is a limit point at infinity of the semigroup $\{e^{tQ} : t \geqq 0\}$. Consequently, we have $AP = PA$, and thus $AT = TA$, which yields the A-invariance of U and W.

The measure μ is (a, A)-quasi-decomposable and $Q \in \mathbb{E}(\mu)$, so assume that for μ formulae (1) and (2) hold. Then we have

$$\widehat{A\mu}(v) = \widehat{e^{tQ}A\mu}(v)\widehat{A\mu_t}(v)$$

and

$$\begin{aligned}[\hat{\mu}(v)]^a &= [\widehat{e^{tQ}\mu}(v)]^a[\hat{\mu}_t(v)]^a \\ &= e^{i(e^{tQ}h, v)}\widehat{e^{tQ}A\mu}(v)[\hat{\mu}_t(v)]^a .\end{aligned}$$

These two equalities give

$$\begin{aligned}e^{i(e^{tQ}h, v)}\widehat{e^{tQ}A\mu}(v)[\hat{\mu}_t(v)]^a = [\hat{\mu}(v)]^a &= e^{i(h, v)}\widehat{A\mu}(v) \\ &= e^{i(h, v)}\widehat{e^{tQ}A\mu}(v)\widehat{A\mu_t}(v)\end{aligned}$$

and since μ is infinitely divisible, $\hat{\mu}(v) \neq 0$ for all $v \in V$, which gives

$$[\hat{\mu}_t(v)]^a = e^{i(h - e^{tQ}h, v)}\widehat{A\mu_t}(v) ,$$

meaning that μ_t is (a, A)-quasi-decomposable. In particular, $T\mu_t$ is (a, A)-quasi-decomposable for $t \geqq 0$.
Furthermore, we have

$$T\mu = e^{tQ}T\mu * T\mu_t = e^{tTQ}T(T\mu) * T\mu_t .$$

According to (4),

$$\lim_{t \to \infty} e^{tTQ}P = P ,$$

thus

$$\lim_{t \to \infty} e^{tTQ}T = \lim_{t \to \infty} (e^{tTQ} - e^{tTQ}P) = 0 .$$

Take t_0 such that $\|e^{t_0TQ}T\| < 1$ and put $B = e^{t_0TQ}T$, $\rho = T\mu_{t_0}$. Then

$$T\mu = B(T\mu) * \rho \tag{10}$$

and iterating the last equality, we get

$$T\mu = B^{n+1}(T\mu) * B^n\rho * \cdots * B\rho * \rho, \quad n = 1, 2, \ldots .$$

Since $B^n \to 0$, by putting

$$\rho_n = B^n\rho * \cdots * B\rho * \rho ,$$

we obtain

$$\lim_{n \to \infty} \rho_n = T\mu .$$

Since

$$\rho^a = A\rho * \delta(h') \quad \text{for some} \quad h' \in W ,$$

and since $AB = BA$, it follows that

$$\begin{aligned}\rho_n^a &= B^n\rho^a * \cdots * \rho^a = B^n A\rho * \delta(B^n h') * \cdots * A\rho * \delta(h') \\ &= A(B^n\rho * \cdots * \rho) * \delta(B^n h' + \cdots + h') \\ &= A\rho_n * \delta((B^n + \cdots + I)h') .\end{aligned}$$

Passing to the limit yields

$$(T\mu)^a = A(T\mu) * \delta(h'') \quad \text{for} \quad h'' = (I - B)^{-1} h' , \tag{11}$$

that is, $T\mu$ is (a, A)-quasi-decomposable.
Moreover,

$$\mu^a = (P\mu)^a * (T\mu)^a \quad \text{and} \quad A\mu = A(P\mu) * A(T\mu) ,$$

hence

$$\begin{aligned}(P\mu)^a * A(T\mu) * \delta(h'') &= (P\mu)^a * (T\mu)^a = \mu^a \\ &= A(P\mu) * A(T\mu) * \delta(h) ,\end{aligned}$$

which implies

$$(P\mu)^a = A(P\mu) * \delta(h - h'') ,$$

that is, $P\mu$ is (a, A)-quasi-decomposable. Taking into account the equalities $\nu = P\mu | U$, $\lambda = T\mu | W$ and the A-invariance of U and W, we conclude the proof. □

In the remaining part of this section, we assume also that μ is an operator Lévy's measure. Then we have

Lemma 14 *There exists $Q' \in \mathbb{E}(\mu) - \mathbb{E}_0(\mu)$ such that $AQ' = Q'A$.*

Proof. Take $Q'' \in \mathbb{E}_-(\mu)$ and put $Q_n = A^{-n}Q''A^n$. Since μ satisfies (1) and the equality

$$\mu = e^{tQ''}\mu * \mu_t, \qquad t \geqq 0 ,$$

with μ_t infinitely divisible (cf. [14; Proposition 5.2]), we have

$$\mu^a = e^{tQ''}\mu^a * \mu_t^a ,$$

so

$$\begin{aligned}A\mu * \delta(h) &= e^{tQ''}[A\mu * \delta(h)] * \mu_t^a \\ &= e^{tQ''}A\mu * \mu_t^a * \delta(e^{tQ''}h)\end{aligned}$$

and thus

$$\mu = A^{-1}e^{tQ''}A\mu * A^{-1}\mu_t^a * \delta(A^{-1}(e^{tQ''}h - h)) .$$

As

$$A^{-1}e^{tQ''}A = e^{tA^{-1}Q''A} ,$$

we infer that $A^{-1}Q''A \in \mathbb{E}_-(\mu)$. Repeating this procedure, we obtain that $Q_n \in \mathbb{E}_-(\mu)$.

First, we shall show that the set $\{Q_n : n = 0, 1, \ldots\}$ is bounded in norm.

If this was not the case, we would have $\|Q_{n_k}\| \to \infty$ for some subsequence $\{n_k\}$ of positive integers. Since $\mathbb{E}(\mu)$ is a closed cone, $Q_{n_k}/\|Q_{n_k}\| \in \mathbb{E}(\mu)$ and passing to a subsequence of $\{n_k\}$, if necessary, we may assume that $Q_{n_k}/\|Q_{n_k}\| \to \tilde{Q}$, where

$\tilde{Q} \in \mathbb{E}(\mu)$, $\|\tilde{Q}\| = 1$. Upon observing that the eigenvalues of Q_n are the same for all n, we get that all the eigenvalues of $\tilde{Q}$ must be zero; in particular, $\tilde{Q} \in \mathbb{E}_0(\mu)$. Consequently, $e^{t\tilde{Q}} \in \mathbb{S}(\mu)$ for all $t \in \mathbb{R}$. Since $\mathbb{S}(\mu)$ is compact, there is a non-singular linear operator C such that $C\mathbb{S}(\mu)C^{-1} \subset \mathbb{O}$, where $\mathbb{O}$ is the group of orthogonal operators on V. This means that $Ce^{t\tilde{Q}}C^{-1} = e^{tC\tilde{Q}C^{-1}}$ is orthogonal for all $t \in \mathbb{R}$, hence $D = C\tilde{Q}C^{-1}$ is skew-symmetric. Since the eigenvalues of D are all equal to zero, the same is true for D^2. But

$$D^*D = -D^2 ,$$

which means that D^*D is a hermitean operator having all the eigenvalues equal to zero, so $D^*D = 0$. Consequently, $D = 0$, and $\tilde{Q} = 0$, which contradicts the equality $\|\tilde{Q}\| = 1$. Thus

$$\|Q_n\| \leqq c, \quad n = 0, 1, \ldots .$$

Now, put

$$R_n = \frac{1}{n} \sum_{i=0}^{n-1} Q_i .$$

We have $\|R_n\| \leqq c$ and $R_n \in \mathbb{E}(\mu)$ because $\mathbb{E}(\mu)$ is a cone, so the closedness of $\mathbb{E}(\mu)$ implies that there is a $Q' \in \mathbb{E}(\mu)$ such that

$$Q' = \lim_{k \to \infty} R_{n_k}$$

for some subsequence $\{n_k\}$ of positive integers. Furthermore,

$$\|A^{-1}R_nA - R_n\| = \frac{1}{n}\left\|\sum_{i=0}^{n-1} (AQ_iA^{-1} - Q_i)\right\|$$

$$= \frac{1}{n}\left\|\sum_{i=0}^{n-1} (Q_{i+1} - Q_i)\right\| = \frac{1}{n}\left\|Q_n - Q_0\right\| \leqq \frac{2c}{n} ,$$

and passing to the limit for the sequence $\{n_k\}$ gives

$$A^{-1}Q'A - Q' = 0 ,$$

that is

$$AQ' = Q'A .$$

It remains to prove that $Q' \notin \mathbb{E}_0(\mu)$.

First, let us observe that all the limit points of the sequence $\{Q_n : n = 0, 1, \ldots\}$ have the same eigenvalues as Q'', so $\mathbb{F} = \overline{\{Q_n : n = 0, 1, \ldots\}}$ is a compact subset of $\mathbb{E}_-(\mu)$. Now, if $\sum_{i=1}^m \alpha_i F_i$ is a convex combination of elements from $\mathbb{F}$, $0 < \alpha_i < 1$, $\sum_{i=1}^m \alpha_i = 1$, $F_i \in \mathbb{F}$, then $\alpha_i F_i \in \mathbb{E}_-(\mu)$, $i = 1, \ldots, m$, and thus, by virtue of Corollary 3, $\sum_{i=1}^m \alpha_i F_i \notin \mathbb{E}_0(\mu)$. This means that for the convex hull of $\mathbb{F}$, we have the relation

$$\operatorname{conv} \mathbb{F} \cap \mathbb{E}_0(\mu) = \varnothing .$$

Since $\mathbb{F}$ is compact, [13; 3.2.8, p. 92] yields the equality

$$\overline{\operatorname{conv} \mathbb{F}} = \operatorname{conv} \mathbb{F},$$

and so

$$\overline{\operatorname{conv}\{Q_n : n = 0, 1, \ldots\}} \subset \overline{\operatorname{conv} \mathbb{F}} = \operatorname{conv} \mathbb{F} ,$$

which means that

$$\overline{\operatorname{conv}\{Q_n : n = 0, 1, \ldots\}} \cap \mathbb{E}_0(\mu) = \varnothing .$$

Since $Q' \in \overline{\operatorname{conv}\{Q_n : n = 0, 1, \ldots\}}$, the conclusion follows. □

Now, we shall prove a basic property of full quasi-decomposable operator Lévy's measures.

Theorem 15 *There exists* $Q \in \mathbb{E}_-(\mu)$ *such that* $AQ = QA$.

Proof. Let Q' be the exponent constructed in the preceding lemma. By virtue of Theorem 5, there are decompositions

$$V = U \oplus W; \qquad \mu = \nu' * \lambda'$$

where U and W are Q'-invariant, $\nu'(U) = \lambda'(W) = 1$, and by Lemma 13, U and W are also A-invariant.
For notational convenience, put

$$U_1 = U, \quad V_1 = W, \quad A'_1 = A|U_1, \quad A_1 = A|V_1, \quad Q_1 = Q'|V_1 ,$$

$$\nu_1 = \nu'|U_1, \qquad \lambda_1 = \lambda'|V_1 .$$

Since $Q' \notin \mathbb{E}_0(\mu)$, Theorem 5 gives

$$V_1 \neq \{0\} \quad \text{and} \quad Q_1 \in \mathbb{E}_-(\lambda_1) ;$$

moreover, the relation $AQ' = Q'A$ yields

$$A_1 Q_1 = Q_1 A_1 .$$

Consider now the measure ν_1. According to Proposition 12, ν_1 is a full operator Lévy's measure on U_1, and on account of Lemma 13, ν_1 is (a, A'_1)-quasi-decomposable. Applying, for measure ν_1, the procedure employed above, we find, by Lemma 14, an operator $Q'_1 \in \mathbb{E}(\nu_1) - \mathbb{E}_0(\nu_1)$ such that $A'_1 Q'_1 = Q'_1 A'_1$, and, accordingly, we shall get the decompositions

$$U_1 = U_2 \oplus V_2, \qquad \nu_1 = \nu'_2 * \lambda'_2$$

and the operators

$$A'_2 = A'_1|U_2 = A|U_2, \quad A_2 = A'_1|V_2 = A|V_2, \quad Q_2 = Q'_1|V_2 ,$$

such that, putting $\lambda_2 = \lambda'_2|V_2$, we have

$$V_2 \neq \{0\}, \quad Q_2 \in \mathbb{E}_-(\lambda_2), \quad A_2 Q_2 = Q_2 A_2 .$$

Moreover, if ρ is a measure on a subspace X of V and we define the canonical extension $\tilde{\rho}$ of ρ by

$$\tilde{\rho}(E) = \rho(E \cap X), \qquad E \in \mathscr{B}(V) ,$$

then the formula

$$\mu = \tilde{\lambda}_1 * \tilde{\lambda}_2 * \tilde{\nu}_2$$

holds. Proceeding further that way, we obtain, by virtue of the condition $V_i \neq \{0\}$ and the finiteness of $\dim V$, non-trivial subspaces $V_1, \ldots, V_m$, operators $A_1, \ldots, A_m$ and $Q_1, \ldots, Q_m$, and measures $\lambda_1, \ldots, \lambda_m$ with the properties

(i) $V = V_1 \oplus \ldots \oplus V_m$;
(ii) λ_i is a probability measure on V_i, $i = 1, \ldots, m$;
(iii) A_i, Q_i are linear operators on V_i such that $A_i Q_i = Q_i A_i$, $i = 1, \ldots, m$;
(iv) $A_i = A | V_i$, $i = 1, \ldots, m$;
(v) $Q_i \in \mathbb{E}_-(\lambda_i)$, $i = 1, \ldots, m$;

moreover,

$$\mu = \tilde{\lambda}_1 * \cdots * \tilde{\lambda}_m .$$

Put

$$Q = Q_1 \oplus \cdots \oplus Q_m .$$

Since, by (iv),

$$A = A_1 \oplus \cdots \oplus A_m ,$$

we have, taking (iii) into account,

$$AQ = QA .$$

From (v), we get

$$\lambda_i = e^{tQ_i} \lambda_i * \lambda_i^{(t)}, \quad t \geqq 0, \quad i = 1, \ldots, m .$$

As $Q | V_i = Q_i$, for the canonical extensions we have

$$\widetilde{e^{tQ_i} \lambda_i} = e^{tQ} \tilde{\lambda}_i, \quad i = 1, \ldots, m ,$$

which implies the equality

$$\tilde{\lambda}_i = e^{tQ} \tilde{\lambda}_i * \tilde{\lambda}_i^{(t)}, \quad t \geqq 0, \quad i = 1, \ldots, m .$$

Consequently,

$$\begin{aligned} \mu &= \tilde{\lambda}_1 * \cdots * \tilde{\lambda}_m = e^{tQ} \tilde{\lambda}_1 * \cdots * e^{tQ} \tilde{\lambda}_m * \tilde{\lambda}_i^{(t)} * \cdots * \tilde{\lambda}_m^{(t)} \\ &= e^{tQ} (\tilde{\lambda}_1 * \cdots * \tilde{\lambda}_m) * \tilde{\lambda}_1^{(t)} * \cdots * \tilde{\lambda}_m^{(t)} \\ &= e^{tQ} \mu * \lambda_1^{(t)} * \cdots * \tilde{\lambda}_m^{(t)}, \quad t \geqq 0 , \end{aligned}$$

showing that $Q \in \mathbb{E}(\mu)$.

For $v_i \in V_i$, $i = 1, \ldots, m$, we have, by (v),

$$\lim_{t \to \infty} e^{tQ_i} v_i = 0 ,$$

which implies

$$\lim_{t \to \infty} e^{tQ} v = 0, \quad v \in V ,$$

that is $Q \in \mathbb{E}_-(\mu)$ and the proof is finished. □

5 Operator-semistable operator Lévy's measures

This section is devoted to characterizations of full operator-semistable operator Lévy's measures. The first one is given in terms of some decomposability properties.

Theorem 16 *Let μ be a full probability measure on V. μ is an (a, A)-quasi-decomposable operator Lévy's measure if and only if, for each $t > 0$, decomposition* (2) *holds with some $Q \in \mathbb{E}_-(\mu)$ and μ_t being a full (a, A)-quasi-decomposable measure on V.*

Proof. Assume that μ is (a, A)-quasi-decomposable and operator Lévy's. According to Theorem 15, there exists $Q \in \mathbb{E}_-(\mu)$ such that $AQ = QA$. We have

$$\begin{aligned}\mu^a &= e^{tQ}\mu^a * \mu_t^a = e^{tQ}(A\mu * \delta(h)) * \mu_t^a \\ &= e^{tQ}A\mu * \mu_t^a * \delta(e^{tQ}h)\end{aligned}$$

and

$$\begin{aligned}A\mu * \delta(h) &= A(e^{tQ}\mu * \mu_t) * \delta(h) = Ae^{tQ}\mu * A\mu_t * \delta(h) \\ &= e^{tQ}A\mu * A\mu_t * \delta(h) .\end{aligned}$$

Taking into account relation (1) and the inequality $\hat{\mu}(v) \neq 0$, we get

$$\mu_t^a = A\mu_t * \delta(h - e^{tQ}h) ,$$

which shows that μ_t, is (a, A)-quasi-decomposable. The fullness of μ_t is a consequence of Theorem 9.

Conversely, let in (2) $Q \in \mathbb{E}_-(\mu)$ and μ_t be (a, A)-quasi-decomposable. Obviously, μ is an operator Lévy's measure. Since $\lim_{t\to\infty} e^{tQ} = 0$, we can find t_0 such that $\| e^{t_0 Q} \| < 1$. Put $B = e^{t_0 Q}$, $\rho = \mu_{t_0}$. Then

$$\mu = B\mu * \rho$$

and ρ is (a, A)-quasi-decomposable. The (a, A)-quasi-decomposability of μ follows exactly as in the proof of Lemma 13, lines from formula (10) to (11). □

Now, we are going to characterize full operator-semistable operator Lévy's measures in terms of the characteristic functions.

For a non-singular linear operator A with norm less than one, denote

$$\begin{aligned}Z_A &= \{v \in V : \|v\| \leqq 1\} \cap \{v \in V : \|A^{-1}v\| > 1\} \\ &= \{v \in V : \|v\| \leqq 1\} - A(\{v \in V : \|v\| \leqq 1\}) .\end{aligned}$$

Theorem 17 *Let μ be a full probability measure on V. μ is an operator-semistable operator Lévy's measure if and only if its characteristic function has the form*

$$\begin{aligned}(12) \qquad \hat{\mu}(v) = \exp\Bigg\{ & i(m, v) - \tfrac{1}{2}(Dv, v) \\ & + \sum_{n=-\infty}^{\infty} a^{-n} \int_{Z_A \cap W} \int_0^{\infty} \left[e^{i(v, e^{tQ}A^n u)} - 1 - \frac{i(v, e^{tQ}A^n u)}{1 + \|e^{tQ}A^n u\|^2} \right] dt\gamma(du) \Bigg\} ,\end{aligned}$$

where $m \in V$, $0 < a < 1$, V is decomposable as

$$V = U \oplus W ,$$

such that U and W are linear subspaces of V invariant with respect to A and Q, γ is a finite Borel measure on $Z_A \cap W$, and the linear operators D, A, Q satisfy:

(i) *D is non-negative and such that*

$$D(U) = U, \quad D(W) = \{0\}, \quad aD = ADA^*, \quad QD + DQ^* \leqq 0;$$

(ii) *A is non-singular and such that*

$$\|A\| < 1, \quad \sigma(A|U) \subset \{\alpha \in \mathbb{C} : |\alpha|^2 = a\},$$
$$\sigma(A|W) \subset \{\alpha \in \mathbb{C} : |\alpha|^2 < a\},$$

where σ stands for the spectrum of the operator in question;
(iii) $\sigma(Q) \subset \{\alpha \in \mathbb{C} : \operatorname{Re} \alpha < 0\}$;
(iv) $AQ = QA$.

Proof. Assume that μ is operator-semistable and operator Lévy's. According to [3; Theorem], μ is (a, A)-quasi-decomposable with some $0 < a < 1$ and a non-singular linear operator A which, by [6; Remark 1.1], may be chosen such that $\|A\| < 1$. Furthermore, there are decompositions

$$V = U \oplus W, \qquad \mu = \nu * \lambda,$$

such that U and W are A-invariant, $\sigma(A|U) \subset \{\alpha \in \mathbb{C} : |\alpha|^2 = a\}$, $\sigma(A|W) \subset \{\alpha \in \mathbb{C} : |\alpha|^2 < a\}$, and ν is an (a, A)-quasi-decomposable Gaussian measure concentrated on U and full there, λ is an (a, A)-quasi-decomposable purely Poissonian measure concentrated on W and full there. Denoting by P the projection on U along W, and by T the projection on W along U, we see that

$$\nu = P\mu, \qquad \lambda = T\mu,$$

so $P, T \in \mathbb{D}(\mu)$; in particular, we infer, by Proposition 12, that $\nu|U$ as well as $\lambda|W$ are operator Lévy's measures.

For simplicity of notation, regard ν and λ as measures on U and W, respectively. Then $\nu = [m_1, D_1, 0]$, $\lambda = [m_2, 0, M_2]$, where $m_1 \in V$, D_1 is a non-singular covariance operator on U, $m_2 \in W$, and M_2 is a Lévy spectral measure on $W_0 = W - \{0\}$. Put

$$A_1 = A|U, \qquad A_2 = A|W.$$

By [7; Lemma 1.1], we have

$$aD_1 = A_1 D_1 A_1^*, \qquad aM_2 = A_2 M_2.$$

On account of Theorem 15, there are operators $Q_1 \in \mathbb{E}_-(\nu)$, $Q_2 \in \mathbb{E}_-(\lambda)$ such that

$$A_1 Q_1 = Q_1 A_1, \qquad A_2 Q_2 = Q_2 A_2.$$

Moreover, by virtue of [14; Theorem 7.1], Q_1 satisfies

$$Q_1 D_1 + D_1 Q_1^* \leqq 0.$$

Consider now the measure λ. According to [14; Theorem 7.1, formula (7.7)], the formula

$$\int_{W_0} f(u) M(\mathrm{d}u) = \int_{W_0} \int_0^\infty f(e^{tQ_2} u) \mathrm{d}t N(\mathrm{d}u) \tag{13}$$

for every continuous function f such that $|f(u)| \leq c \frac{\|u\|^2}{1+\|u\|^2}$, establishes a one-to-one correspondence between the Lévy spectral measure M of an operator Lévy's measure and a Borel measure N with the property $\int_{W_0} \log(1+\|u\|^2) N(\mathrm{d}u) < \infty$. Thus we have

$$\int_{W_0} f(u)(aM_2)(\mathrm{d}u) = \int_{W_0} \int_0^\infty f(\mathrm{e}^{tQ_2}u)\mathrm{d}t(aN)(\mathrm{d}u)\,,$$

and, since $A_2 Q_2 = Q_2 A_2$, also

$$\int_{W_0} f(u)(A_2 M_2)(\mathrm{d}u) = \int_{W_0} f(A_2 u) M(\mathrm{d}u) = \int_{W_0} \int_0^\infty f(A_2 \mathrm{e}^{tQ_2} u)\mathrm{d}t N(\mathrm{d}u)$$

$$= \int_{W_0} \int_0^\infty f(\mathrm{e}^{tQ_2} A_2 u)\mathrm{d}t N(\mathrm{d}u) = \int_{W_0} \int_0^\infty f(\mathrm{e}^{tQ_2} u)\mathrm{d}t(A_2 N)(\mathrm{d}u).$$

The relation $aM_2 = A_2 M_2$ yields

$$aN = A_2 N\,. \tag{14}$$

By virtue of [6; Theorem 1.2], N has the form

$$N(E) = \sum_{n=-\infty}^{\infty} a^n \gamma(A_2^n E \cap Z_{A_2}), \qquad E \in \mathscr{B}(W_0)\,, \tag{15}$$

where γ is a finite Borel measure on $Z_{A_2} = Z_A \cap W$. Consequently, formula (13) becomes

$$\int_{W_0} f(u) M(\mathrm{d}u) = \sum_{n=-\infty}^{\infty} a^{-n} \int_{Z_A \cap W} \int_0^\infty f(\mathrm{e}^{tQ_2} A_2^n u)\mathrm{d}t\gamma(\mathrm{d}u)\,. \tag{16}$$

Now, putting

$$D = D_1 \oplus 0, \qquad Q = Q_1 \oplus Q_2, \quad M(E) = M_2(E \cap W), \qquad E \in \mathscr{B}(v_0)\,,$$

we easily verify that D, A and Q satisfy all the conditions (i)–(iv). Moreover, returning to v and λ as measures on V, we have

$$\hat{v}(v) = \exp\{i(m_1, v) - \tfrac{1}{2}(Dv, v)\}$$

and

$$\hat{\lambda}(v) = \exp\left\{i(m_2, v) + \int_{V_0} \left[\mathrm{e}^{i(v,u)} - 1 - \frac{i(v,u)}{1+\|u\|^2}\right] M(\mathrm{d}u)\right\}$$

$$= \exp\left\{i(m_2, v) + \sum_{n=-\infty}^{\infty} a^{-n} \int_{Z_A \cap W} \int_0^\infty [\mathrm{e}^{i(v,\mathrm{e}^{tQ}A^n u)} - 1 \right.$$

$$\left. - \frac{i(v, \mathrm{e}^{tQ} A^n u)}{1 + \|\mathrm{e}^{tQ} A^n u\|^2}\right] \mathrm{d}t\,\gamma(\mathrm{d}u)\Bigg\}$$

since M is concentrated on W_0 and $e^{tQ}A|W = e^{tQ_2}A_2$. The equality

$$\hat{\mu}(v) = \hat{v}(v)\hat{\lambda}(v)$$

yields formula (12) with $m = m_1 + m_2$.

To prove the converse implication, assume that for $\hat{\mu}$ formula (12) holds with operators D, A and Q satisfying (i)–(iv).

Define measure N by (15). According to [6; Theorem 1.1], N is a Lévy spectral measure on W_0 satisfying (14); moreover, by [6; Lemma 3.2 and Theorem 3.1], there exists an $\alpha > 0$ such that

$$\int_{\|v\|>1} \|v\|^{\alpha} N(\mathrm{d}v) < \infty .$$

It follows that

$$\int_{\|v\|>1} \log(1 + \|v\|^2) N(\mathrm{d}v) < \infty ,$$

and since N is a Lévy spectral measure, we get

$$\int_{W_0} \log(1 + \|v\|^2) N(\mathrm{d}v) < \infty .$$

By virtue of [14; Theorem 7.1], M defined by (13) is a Lévy spectral measure of some operator Lèvy's measure on W, moreover, the commutation property $A_2Q_2 = Q_2A_2$ with A_2 and Q_2 as in the first part of the proof, yields, as before,

$$aM = A_2M.$$

If we write

$$m = m_1 + m_2, \qquad m_1 \in U, \qquad m_2 \in W ,$$

and put $\lambda = [m_2, 0, M]$ with M regarded as a measure on V_0, then we shall get that λ is an (a, A)-quasi-decomposable measure concentrated on W and such that

$$\lambda = e^{tQ}\lambda * \lambda_t, \qquad t \geqq 0 ,$$

where λ_t is concentrated on W. Accordingly, putting $v = [m_1, D, 0]$, we get, by condition (i), that v is an (a, A)-quasi-decomposable Gaussian measure concentrated on U such that

$$v = e^{tQ}v * v_t, \qquad t \geqq 0 ,$$

where v_t is concentrated on U. Since $\mu = v * \lambda$, we infer that μ is (a, A)-quasi-decomposable and satisfies (2) with $\mu_t = \lambda_t * v_t$, which finishes the proof. □

Our final aim is to describe the operator-semistable operator Lévy's measures as distributions of some stochastic integrals. To this end, let us recall that for μ formula (2) holds with $Q \in \mathbb{E}_-(\mu)$ if and only if μ is the distribution of the stochastic integral $\int_0^\infty e^{tQ} \mathrm{d}Z(t)$ where $(Z(t): t \geqq 0)$ is a homogeneous process with independent increments such that $E \log(1 + \|Z(1)\|) < \infty$.

Moreover, if $\mu = [m, D, M]$ and $\mathscr{L}(Z(1)) = [n, B, N]$ ($\mathscr{L}$ stands for the distribution), then

$$n = -Qm + \int_{V_0}\int_0^{\infty}\left[\frac{Qe^{tQ}v}{1+\|e^{tQ}v\|^2} - \frac{Qe^{tQ}v}{1+\|v\|^2}\right]dtN(dv)\,, \tag{17}$$

$$B = -(QD + DQ^*), \qquad D = \int_0^{\infty} e^{tQ}Be^{tQ^*}dt\,, \tag{18}$$

$$\int_{V_0} f(u)M(du) = \int_{V_0}\int_0^{\infty} f(e^{tQ}u)dtN(du) \tag{19}$$

for each continuous function f such that $|f(u)| \leqq C\dfrac{\|u\|^2}{1+\|u\|^2}$. Also putting

$$\psi(u) = i(n, u) - \tfrac{1}{2}(Bu, u) + \int_{V_0}\left[e^{i(u,v)} - 1 - \frac{i(u, v)}{1+\|v\|^2}\right]N(dv)\,,$$

we have

$$\hat{\mu}(u) = \exp\int_0^{\infty}\psi(e^{tQ^*}u)dt \tag{20}$$

(cf. [4, 10, 11, 15, 16] for these facts as well as for the definition of the stochastic integral).

Theorem 18 *Let μ be full. μ is an operator-semistable operator Lévy's measure if and only if there exist $0 < a < 1$, linear operators A, Q on V with the properties $\|A\| < 1$, $\sigma(Q) \subset \{\alpha\in\mathbb{C} : \mathrm{re}\,\alpha < 0\}$, $AQ = QA$, such that*

$$\mu = \mathscr{L}\left(\int_0^{\infty} e^{tQ}dZ(t)\right), \tag{21}$$

where $(Z(t) : t \geqq 0)$ is a homogeneous process with independent increments and the distribution of $Z(1)$ is (a, A)-quasi-decomposable.

Proof. First, let us observe that if $\mathscr{L}(Z(1))$ is (a, A)-quasi-decomposable, then on account of [6; Theorem 3.1], $E\|Z(1)\|^{\alpha} < \infty$ for some $\alpha > 0$, thus $E\log(1 + \|Z(1)\|) < \infty$ and the integral $\int_0^{\infty} e^{tQ}dZ(t)$ exists.

Assume now that for μ (21) holds with a, A, Q and $Z(t)$ as in the assumption of the theorem. Let $\mathscr{L}(Z(1)) = [n, D, N]$, $\phi = \widehat{\mathscr{L}(Z(1))}$. Then $\phi = \exp\psi$; moreover, since $\mathscr{L}(Z(1))$ is (a, A)-quasi-decomposable, we have

$$[\phi(u)]^a = e^{i(h,u)}\phi(A^*, u)\,,$$

which yields

$$a\psi(u) = i(h, u) + \psi(A^*u)\,.$$

Taking into account formula (20), we get

$$[\hat{\mu}(u)]^a = \exp \int_0^\infty a\psi(e^{tQ^*}u)dt = \exp \int_0^\infty [i(h, e^{tQ^*}u)$$

$$+ \psi(A^* e^{tQ^*}u)]dt = \exp\left\{i(h', u) + \int_0^\infty \psi(e^{tQ^*}A^*u)du\right\}$$

$$= e^{i(h', u)}\hat{\mu}(A^*u) \quad \left(h' = \int_0^\infty e^{tQ}h dt\right),$$

which means that μ is (a, A)-quasi-decomposable, thus operator-semistable. As was remarked earlier, μ is also an operator Lévy's measure.

Conversely, let μ be an operator-semistable operator Lévy's measure. Then μ is (a, A)-quasi-decomposable with $0 < a < 1$ and $\|A\| < 1$, and, according to Theorem 15, there is $Q \in \mathbb{E}_-(\mu)$ such that $AQ = QA$. Let $\mu = [m, D, M]$. In the proof of Theorem 17 it was shown that for M formula (19) holds with a Lévy spectral measure N satisfying $aN = AN$.

Define n and B by formulae (17), (18), respectively, and let $(Z(t): t \geqq 0)$ be a homogeneous process with independent increments such that $\mathscr{L}(Z(t)) = [tn, tB, tN]$. Then $E \log(1 + \|Z(1)\|) < \infty$ because $\int_{\|u\| \geqq 1} \log(1 + \|u\|)N(du) < \infty$ (cf. [4]), so there exists the stochastic integral $\int_0^\infty e^{tQ}dZ(t)$. Let ν be the distribution of this integral. We have, by virtue of formulae (17), (18), (19), (20),

$$\hat{\nu}(u) = \exp \int_0^\infty \psi(e^{tQ^*}u)dt = \exp\left\{i \int_0^\infty (n, e^{tQ^*}u)dt\right.$$

$$- \frac{1}{2}\int_0^\infty (Be^{tQ^*}u, e^{tQ^*}u)dt + \int_0^\infty \int_{V_0} \left[e^{i(e^{tQ^*}u, v)} - 1\right.$$

$$\left.\left. - \frac{i(e^{tQ^*}u, v)}{1 + \|v\|^2}\right]N(du)dt\right\} = \exp\left\{i(m, u) - \frac{1}{2}(Du, u)\right.$$

$$\left. + \int_0^\infty \int_{V_0} \left[e^{i(u, e^{tQ}v)} - 1 - \frac{i(u, e^{tQ}v)}{1 + \|e^{tQ}v\|^2}\right]N(du)dt\right\}$$

$$= \exp\left\{i(m, u) - \frac{1}{2}(Du, u) + \int_{V_0} \left[e^{i(u, v)} - 1 - \frac{i(u, v)}{1 + \|v\|^2}\right]M(dv)\right\}$$

$$= \hat{\mu}(u),$$

which shows that $\mu = \mathscr{L}(\int_0^\infty e^{tQ}dZ(t))$. Also, since μ is (a, A)-quasi-decomposable and $AQ = QA$, we get

$$aB = -[Q(aD) + (aD)Q^*] = -(QADA^* + ADA^*Q^*)$$

$$= -A(QD + DQ^*)A^* = ABA^*$$

which, together with the equality $aN = AN$, implies that $\mathscr{L}(Z(1)) = [n, B, N]$ is (a, A)-quasi-decomposable, proving the theorem. □

Now, identify V as $\mathbb{R}^d$, and put

$$Y(s) = \int_0^s e^{tQ} dZ(t), \qquad s \geqq 0 .$$

If we let G_0 denote the generator of the process $(Z(t): t \geqq 0)$, i.e.

$$(G_0 f)(x) = \sum_{i=1}^d a_i D_i f(x) - \frac{1}{2} \sum_{i,k=1}^d B_{ik} D_i D_k f(x)$$
$$+ \int_{\mathbb{R}^d} \left[f(x+y) - f(x) - \sum_{i=1}^d \frac{y_i}{1+\|y\|^2} D_i f(x) \right] N(dy) ,$$

where D_i stands for the partial derivative with respect to the i-th coordinate, $[B_{ik}]$ is a positive definite matrix and $a_i \in \mathbb{R}$, then $(Y(s): s \geqq 0)$ is a Markov process with generator G of the form

$$G = G_0 + \sum_{i,k=1}^d Q_{ik} x_k D_i , \tag{22}$$

where $[Q_{ik}]$ is the matrix of the operator Q (cf. [10, 11, 16]).

A Markov process with generator as above is called a process of Ornstein-Uhlenbeck type. Taking into account the relation

$$\int_0^\infty e^{tQ} dZ(t) = \lim_{s \to \infty} \int_0^s e^{tQ} dZ(t) ,$$

we get

Corollary 19 *Let μ be full. μ is an operator-semistable operator Lévy's measure if and only if there exist $0 < a < 1$, linear operators A, Q with the properties $\|A\| < 1$, $\sigma(Q) \subset \{\alpha \in \mathbb{C}: \mathrm{re}\, \alpha < 0\}$, $AQ = QA$, such that μ is the limit distribution of a process of Ornstein-Uhlenbeck type generated by generator G of form (22), where G_0 is the generator of a homogeneous process with independent increments associated with an (a, A)-quasi-decomposable probability measure.*

6 Appendix

In this section, we characterize the set of limit points of a compact semigroup of linear operators acting in V. Let Q be a linear operator on V and let $\mathbb{G}_0 = \{e^{tQ}: t \geqq 0\}$ be a semigroup such that $\sup_{t \geqq 0} \|e^{tQ}\| < \infty$. Put $\mathbb{G} = \bar{\mathbb{G}}_0$. Then $\mathbb{G}$ is a compact abelian semigroup of linear operators on V. Let $\mathbb{H}$ denote the set of the limit points at infinity of $\mathbb{G}_0$, i.e. $T \in \mathbb{H}$ if and only if there is a sequence $t_n \to \infty$ such that $T = \lim_{n\to\infty} e^{t_n Q}$. We have

$$\mathbb{G} = \mathbb{G}_0 \cup \mathbb{H} .$$

Take an arbitrary $R \in \mathbb{G}_0$, $R = e^{sQ}$, and $T \in \mathbb{H}$, $T = \lim_{n\to\infty} e^{t_n Q}$. Obviously, $RT \in \mathbb{H}$, which means that

$$R\mathbb{H} \subset \mathbb{H} .$$

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