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Attractors for random dynamical systems

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Summary. A criterion for existence of global random attractors for RDS is established. Existence of invariant Markov measures supported by the random attractor is proved. For SPDE this yields invariant measures for the associated Markov semigroup. The results are applied to reaction diffusion equations with additive white noise and to Navier–Stokes equations with multiplicative and with additive white noise.

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1 Introduction

One of the most important discoveries in mathematical physics during the past 20 years is that of finite-dimensional attractors in mathematical models for fluid dynamics. However, all the analysis breaks down as soon as one wants to take random influences on the system under investigation in account. In particular, when subjecting the system to additive white noise, there is no chance that bounded subsets of the state space remain invariant. White noise pushes the system out of every bounded set with probability one. The present paper is an attempt to overcome this problem. We in fact do discover compact invariant sets – however, they are not fixed, but they depend on chance, and they move with time (in a coherent, "stationary" manner). Typically they leave every bounded deterministic set. Still they are compact – in fact, often even finite-dimensional – and they attract all bounded subsets of the state space.

The approach we use is that of random dynamical systems (RDS). By taking an abstract, ergodic theoretical point of view, RDS cover some of the most common classes of systems involving randomness and time evolution. Amongst them are stochastic flows, random flows, and products of stationary random maps. See Arnold [1]; for a survey on RDS see Arnold and Crauel [3].

Since we want to apply the results to infinite-dimensional systems, we will have to take care that the respective system does generate a stochastic flow. In finite dimensions these questions are settled in a fairly satisfactory manner by the theory of stochastic flows, see Elworthy [13] and Kunita [15] (also included in Kunita [16]). A "stochastic flow" needs jointly continuous dependence of the solutions of the SDE under consideration on time and initial state outside some nullset. This often does not hold for infinite dimensions. Some infinite dimensional systems do generate a stochastic flow, others do not. For a survey on this field see Flandoli [14]. In our applications existence of the flow has to be established in each individual case.

The paper is organized as follows. In Sect. 2 we recall the concept of random dynamical system. In Sect. 3 we introduce the notions of Ω-limit set, random invariant set, absorbing set, and global attractor for a random dynamical system. The main theorem of Sect. 3 establishes the existence of a global attractor, which is a compact random invariant set. We then prove that in a connected state space a global attractor must be connected itself. In Sect. 4 we analyze some relations with the concept of flow-invariant measure, and we prove that a global attractor always supports a Markov invariant measure. Finally, in Sects. 5–7 we discuss three infinite dimensional examples: a reaction-diffusion equation with additive noise and an abstract 2-dimensional Navier–Stokes equation in both the multiplicative and the additive noise case. Although the main conceptual line in proving the existence of a global attractor is similar, these examples display remarkable differences at the technical level. In particular, the analysis of the Navier–Stokes equation with additive noise requires some non-trivial considerations. All the applications we present here address dissipative stochastic systems with a sort of parabolic structure–roughly speaking such that their flows map bounded sets into precompact sets. The main results of this paper can be applied to systems with a hyperbolic structure along the lines of Temam [20]. This extension and related examples will be treated in a forthcoming paper.

Attractors for deterministic systems are quite well investigated. Temam [20] gives a comprehensive presentation (by which we have been inspired throughout the work). In the deterministic case different concepts of attractor have been introduced. The differences between them mainly concern speed of convergence to the attractor. This amounts essentially to the question whether certain points of the phase space are elements of the attractor. For stochastic systems a greater variety of definitions is possible. Two of them, completely different from the one of this paper, have been introduced previously. Brzezniak, Capinski and Flandoli [5], consider the Ω-limit set for $t \to +\infty$ of the trajectories. Morimoto [18] and Schmalfuß [19]f are concerned with attractors for the Markov semigroup generated by a stochastic differential equation. In this paper we think of the attractor as a subset of the phase space (as in [5]), instead of the space of probability measures. However, we define the attractor as the Ω-limit set at time $t = 0$ of the trajectories "starting in bounded sets at time $t = -\infty$" (roughly speaking). Equivalently, we detect a random subset of the phase space which moves accordingly to the dynamics in a stationary manner, attracting all trajectories starting from deterministic or random L^∞ initial conditions. While the definition of [5] is of interests for systems with real noise, the notion developed here is useful for the white noise case as well.

2 The basic set-up

Let $\{\vartheta_t: \Omega \to \Omega\}$, $t \in T$, $T = \mathbb{R}$ or $T = \mathbb{Z}$, be a family of measure preserving transformations of a probability space $(\Omega, \mathcal{F}, P)$ such that $(t, \omega) \mapsto \vartheta_t \omega$ is measurable, $\vartheta_0 = \mathrm{id}$, and $\vartheta_{t+s} = \vartheta_t \circ \vartheta_s$ for all $t, s \in T$. Thus $(\vartheta_t)_{t \in \vartheta}$ is a flow, and $((\Omega, \mathcal{F}, P), (\vartheta_t)_{t \in T})$ is a (measurable) dynamical system.

2.1 Definition *Let $T = \mathbb{R}, \mathbb{R}^+, \mathbb{Z}$ or $\mathbb{N}$. A random dynamical system (RDS) with time T on a metric, complete and separable space (X, d) with Borel σ-algebra $\mathcal{B}$ over $\{\vartheta_t\}$ on $(\Omega, \mathcal{F}, P)$ is a measurable map*

$$\varphi: T \times X \times \Omega \to X$$

$$(t, x, \omega) \mapsto \varphi(t, \omega)x$$

such that $\varphi(0, \omega) = \mathrm{id}$ (identity on X) and

$$\varphi(t + s, \omega) = \varphi(t, \vartheta_s \omega) \circ \varphi(s, \omega) \tag{1}$$

for all $t, s \in T$ and for all $\omega \in \Omega$, where $\circ$ means composition. A family of maps $\varphi(t, \omega)$ satisfying (1) is called a cocycle, and (1) is the cocycle property.

An RDS is said to be *continuous* or *differentiable* if $\varphi(t, \omega)$: $X \to X$ is continuous or differentiable, respectively, for all $t \in T$ outside a P-nullset. In the present paper we will be concerned solely with continuous or differentiable RDS.

We often omit mentioning $((\Omega, \mathcal{F}, P), (\vartheta_t)_{t \in T})$ in the following, speaking of an RDS φ.

We do not assume the maps $\varphi(t, \omega)$ to be invertible a priori. By the cocycle property, $\varphi(t, \omega)$ is automatically invertible (for all $t \in T$ and for P-almost all ω) if $T = \mathbb{R}$ or $\mathbb{Z}$. In fact, then $\varphi(t, \omega)^{-1} = \varphi(-t, \vartheta_t \omega)$ for $t \in T$. Instead of assuming (1) for all $\omega \in \Omega$ it suffices to assume it for all ω from a measurable $(\vartheta_t)_t$-invariant subset of full measure.

Note that we assume ϑ_t to be defined for all $t \in \mathbb{R}$ or $\mathbb{Z}$, resp., even if φ is not invertible.

For a comprehensive exposition on RDS see Arnold [1]. For a survey see Arnold and Crauel [3].

3 Attraction and absorption

We need some notations. A set valued map $K: \Omega \to 2^X$ taking values in the closed subsets of X is said to be *measurable* if for each $x \in X$ the map $\omega \mapsto d(x, K(\omega))$ is measurable, where $d(A, B) = \sup\{\inf\{d(x, y): y \in B\}: x \in A\}$ for $A, B \in 2^X$, $A, B \neq \emptyset$; and $d(x, B) = d(\{x\}, B)$. Note that $d(A, B) = 0$ if and only if $A \subset B$, so d is not a metric. A closed set valued measurable map

$K:\Omega \to 2^X$ will be called a *random closed* set. (This coincides with the notion of a measurable multifunction as used by Castaing and Valadier [6] Definition III.10, p. 68.)
A random set K is said to be (*strictly*) *φ-forward invariant* if

$$\varphi(t,\omega)K(\omega) \subset K(\vartheta_t\omega) \quad (\varphi(t,\omega)K(\omega) = K(\vartheta_t\omega)) \quad \text{for all } t > 0.$$

3.1 Definition *Given a random set K, the set*

$$\Omega(K,\omega) = \Omega_K(\omega) = \bigcap_{T \geqq 0} \overline{\bigcup_{t \geqq T} \varphi(t,\vartheta_{-t}\omega)K(\vartheta_{-t}\omega)}$$

is said to be the **Ω-limit set of** *K. By definition $\Omega_K(\omega)$ is closed.*
(The usual notion "ω-limit set" is likely to give rise to confusion. The notion 'Ω-limit set' should not, even if Ω keeps on denoting the probability space.)

We may identify

$$\Omega_K(\omega) = \{y \in X\colon \text{there exist } t_n \to \infty \text{ and } x_n \in K(\vartheta_{-t_n}\omega) \text{ such that } \varphi(t_n,\vartheta_{-t_n}\omega)x_n \xrightarrow[n\to\infty]{} y\}.$$

The ϑ-shift of an Ω-limit set is

$$\Omega_K(\cdot)\circ\vartheta_t = \Omega(K,\vartheta_t\omega) = \{y \in X\colon \text{there exist } t_n \to \infty \text{ and } x_n \in K(\vartheta_{-t_n+t}\omega) \text{ such that } \varphi(t_n,\vartheta_{-t_n+t}\omega)x_n \xrightarrow[n\to\infty]{} y\}. \tag{2}$$

The following lemma is proved along the same lines as the corresponding one for deterministic systems, invoking continuity of $\varphi(t,\omega)$.

3.2 Lemma *The Ω-limit set of an arbitrary random set K is invariant.*

Proof. Given $y \in \Omega_K(\omega)$, there exist $t_n \to \infty$ and $x_n \in K(\vartheta_{-t_n}\omega)$ such that $y = \lim_{n\to\infty} \varphi(t_n,\vartheta_{-t_n}\omega)x_n$. For $t > 0$ thus

$$\begin{aligned}\varphi(t,\omega)y &= \lim_{n\to\infty} \varphi(t+t_n,\vartheta_{-t_n}\omega)x_n \\ &= \lim_{n\to\infty} \varphi(t+t_n,\vartheta_{-t-t_n}\vartheta_t\omega)x_n \\ &= \lim_{n\to\infty} \varphi(\tilde{t}_n,\vartheta_{-\tilde{t}_n}\vartheta_t\omega)x_n,\end{aligned}$$

where $\tilde{t}_n = t + t_n \to \infty$ and $x_n \in K(\vartheta_{-t_n}\omega) = K(\vartheta_{-\tilde{t}_n}\vartheta_t\omega)$. Hence $\varphi(t,\omega)y \in \Omega_K(\vartheta_t\omega)$ by (2). □

Lemma 3.2 does not say anything about non-voidness of Ω-limit sets.

3.3 Definition *A random set A is said to* **attract** *another random set B if P-almost surely*

$$d(\varphi(t,\vartheta_{-t}\omega)B(\vartheta_{-t}\omega), A(\omega)) \xrightarrow[n\to\infty]{} 0.$$

The proof of the following Lemma is immediate from the ϑ_t-invariance of P.

3.4 Lemma *If A attracts B then* $d(\varphi(t,\omega)B(\omega), A(\vartheta_t\omega)) \xrightarrow[n\to\infty]{} 0$ *in probability.*

If a set A attracts another set B then any set A' such that P-a.s. $A \subset A'$ also attracts B.

3.5 Definition *If K and B are random sets such that for P-almost all ω there exists a time* $t_B(\omega)$ *such that for all* $t \geqq t_B(\omega)$

$$\varphi(t, \vartheta_{-t}\omega)B(\vartheta_{-t}\omega) \subset K(\omega)$$

then K is said to absorb B, and t_B *is called the absorption time.*

The following result is concerned with properties of the Ω-limit set Ω_B of a set B which is absorbed by some compact set K.

3.6 Proposition *Suppose K and B are random sets with K absorbing B, and K is compact P-as. Then for P-almost all ω*

(i) $\Omega_B(\omega)$ *is nonvoid, and* $\Omega_B(\omega) \subset K(\omega)$*, hence it is compact.*
(ii) $\Omega_B(\omega)$ *is strictly invariant.*
(iii) $\Omega_B(\omega)$ *attracts B.*

Proof. First note that if $(t_n)_{n\in\mathbb{N}}$ is any sequence of times with $t_n \xrightarrow[n\to\infty]{} \infty$ and $(b_n)_{n\in\mathbb{N}}$ is any sequence with $b_n \in B(\vartheta_{-t_n}\omega)$, $n \in \mathbb{N}$, then for all n big enough such that $t_n \geqq t_B(\omega)$ absorption implies $\varphi(t_n, \vartheta_{-t_n}\omega)b_n \in K(\omega)$. Compactness of $K(\omega)$ entails existence of a convergent subsequence of $(\varphi(t_n, \vartheta_{-t_n}\omega)b_n)_{n\in\mathbb{N}}$ to some $y \in X$.

(i) Using any sequences t_n and b_n with the properties of the preceeding paragraph, the limit $y = \lim \varphi(t_n, \vartheta_{-t_n}\omega)b_n$ satisfies $y \in \Omega_B(\omega)$, so $\Omega_B(\omega) \neq \emptyset$. Furthermore,

$$\Omega_B(\omega) \subset \bigcap_{T \geqq t_B(\omega)} \overline{\bigcup_{t \geqq T} \varphi(t, \vartheta_{-t}\omega)B(\vartheta_{-t}\omega)} \subset K(\omega),$$

where $t_B(\omega)$ is the absorption time, hence $\Omega_B(\omega)$ is compact.

(ii) Suppose $y \in \Omega_B(\vartheta_s\omega)$ for some $s \geqq 0$. Then $y = \lim_{n\to\infty} \varphi(t_n, \vartheta_{-t_n+s}\omega)b_n$ for some sequence $t_n \to \infty$ and $b_n \in B(\vartheta_{-t_n+s}\omega)$, hence

$$y = \lim_{n\to\infty} \varphi(s,\omega)\varphi(t_n - s, \vartheta_{-t_n+s}\omega)b_n. \tag{3}$$

For n big enough such that $t_n - s \geqq t_B(\omega)$ absorption implies $k_n := \varphi(t_n - s, \vartheta_{-(t_n-s)})b_n \in K(\omega)$, hence there is a convergent subsequence k_{n_j} converging to some $u \in \Omega_B(\omega)$. Continuity of $\varphi(t,\omega)$ implies $y = \varphi(s,\omega)u$, hence $y \in \varphi(s,\omega)\Omega_B(\omega)$. We have proved $\Omega_B(\vartheta_s\omega) \subset \varphi(s,\omega)\Omega_B(\omega)$ for all $s > 0$, so strict invariance follows together with Lemma 3.2.

(iii) If $\Omega_B(\omega)$ would not attract B there were $\delta > 0$, a sequence $t_n \to \infty$, and $b_n \in B(\vartheta_{-t_n}\omega)$, such that for all $n \in \mathbb{N}$

$$d(\varphi(t_n, \vartheta_{-t_n}\omega)b_n, \Omega_B(\omega)) \geqq \delta. \tag{4}$$

But $(\varphi(t_n, \vartheta_{-t_n}\omega)b_n)_{n\in\mathbb{N}}$ has a convergent subsequence, converging to a limit in $\Omega_B(\omega)$, which, by continuity of $\varphi(t,\omega)$, contradicts (4). □

3.7 Remark Strict invariance of Ω_B follows even without assuming B to be compactly absorbed as soon as φ is invertible (i.e., $\varphi(t,\omega)^{-1}$ exists and is continuous for all $t>0$). In fact, if $y \in \Omega_B(\vartheta_s\omega)$ for some $s>0$, $y = \lim_{n\to\infty} \varphi(t_n, \vartheta_{-t_n+s}\omega)b_n$, then also

$$z = \lim_{n\to\infty} \varphi(s,\omega)^{-1}\varphi(t_n, \vartheta_{-t_n+s}\omega)b_n = \lim_{n\to\infty} \varphi(t_n - s, \vartheta_{-t_n+s}\omega)b_n$$

exits, and $z \in \Omega_B(\omega)$. Since $y = \varphi(s,\omega)z \in \varphi(s,\omega)\Omega_B(\omega)$ we obtain $\Omega_B(\vartheta_s\omega) \subset \varphi(s,\omega)\, \Omega_B(\omega)$, and again strict invariance follows together with Lemma 3.2.

The next result is concerned with the Ω-limit set Ω_K of a compact set K which absorbs some set B.

3.8 Proposition *Suppose K and B are random sets such that K is compact P-a.s. and absorbs B. Then P-almost surely $\Omega_B \subset \Omega_K$. In particular, Ω_K is nonvoid P-a.s. (since Ω_B is nonvoid by Proposition 3.6 (i)), and Ω_K attracts B.*

Proof. Suppose $y \in \Omega_B(\omega)$. Then $y = \lim_{n\to\infty} \varphi(t_n, \vartheta_{-t_n}\omega)b_n$ for some sequence $t_n \to \infty$, $b_n \in B(\vartheta_{-t_n}\omega)$.
Fix $T \geqq 0$ and put $N_0 = \min\{n \in \mathbb{N} : t_n \geqq T + t_B(\vartheta_{-T}\omega)\}$. Then for all $n \geqq N_0$

$$\varphi(t_n, \vartheta_{-t_n}\omega)b_n = \varphi(T, \vartheta_{-T}\omega)\varphi(t_n - T, \vartheta - t_n\omega)b_n.$$

For $n \geqq N_0$ we have $t_n - T \geqq t_B(\vartheta_{-T}\omega)$, hence $k_n := \varphi(t_n - T, \vartheta_{-t_n}\omega)b_n = \varphi(t_n - T, \vartheta_{-(t_n-T)}\vartheta_{-T}\omega)b_n \in K(\vartheta_{-T}\omega)$, since $b_n \in B(\vartheta_{-t_n}\omega) = B(\vartheta_{-(t_n-T)}\vartheta_{-T}\omega)$. Consequently, for all $n \geqq N_0$

$$\varphi(t_n, \vartheta_{-t_n}\omega)b_n \in \varphi(T, \vartheta_{-T}\omega)K(\vartheta_{-T}\omega) \subset \bigcup_{t \geqq T} \varphi(t, \vartheta_{-t}\omega)K(\vartheta_{-t}\omega),$$

which implies $y \in \Omega_K(\omega)$. □

Note that under the conditions of Proposition 3.8 Ω_K need not be compact.

3.9 Definition *Suppose φ is an RDS such that there exists a random compact set $\omega \mapsto A(\omega)$ which satisfies the following conditions:*
(i) *$\varphi(t,\omega)A(\omega) = A(\vartheta_t\omega)$ for all $t>0$*
(ii) *A attracts every bounded deterministic set $B \subset X$.*
Then A is said to be a **universally** *or* **globally attracting** *set for φ.*

3.10 Remarks (i) The notions ‘attraction’ and ‘absorption’ are very close. If a compact K absorbs some B, then Ω_K attracts B. On the other hand, if a set A attracts some B, then every K containing an open neighbourhood of A absorbs B– for instance, $K = \overline{U_\delta(A)}$ for $\delta > 0$. In particular, if A attracts B, then $\Omega_B \subset A$.

(ii) Another way to define a globally attracting random set would impose the stronger condition that every *random* bounded set is to be attracted instead of the weaker condition that only nonrandom, deterministic sets are attracted. These two notions are in general not equivalent. We chose this definition since for our applications we can establish only the weaker requirement.

(iii) A globally attracting set must contain every invariant set, so it need not really be an attractor in an intuitive sense. For a discussion see Eckmann and Ruelle [12] II.B, p. 623 (cf. also II.D-F, pp. 624–627). Nevertheless we shall speak in the following simply of attractors instead of attracting sets, and call a globally attracting set a *global attractor*.

(iv) The notion of a global attractor is not a topological but a metric concept.

The following theorem is the main result of this section.

3.11 Theorem *Suppose φ is an RDS on a Polish space X, and suppose that there exists a compact set $\omega \mapsto K(\omega)$ absorbing every bounded nonrandom set $B \subset X$. Then the set*

$$A(\omega) = \overline{\bigcup_{B \subset X} \Omega_B(\omega)} \tag{5}$$

is a global attractor for φ. Furthermore, A is measurable with respect to $\mathscr{F}$ if T is discrete, and it is measurable with respect to the completion of $\mathscr{F}$ (with respect to P) if T is continuous.

Proof. For any bounded $B \subset X$ we have $\Omega_B \subset K$(P-a.s.) by Proposition 3.6(i), hence A is compact P-a.s. Since $\omega \mapsto \bigcup_{B \subset X} \Omega_B(\omega)$ is strictly invariant by Proposition 3.6(ii), continuity of φ implies that A is invariant. Strict invariance of A follows from compactness of A.

To prove measurability, first note that for any $x \in X$ and any (nonrandom) $B \subset X$ the map $(t, \omega) \mapsto d(x, \varphi(t, \vartheta_{-t}\omega)B) = \inf\{d(x, \varphi(t, \vartheta_{-t}\omega)y) : y \in B\}$ is measurable by separability of X and continuity of φ. For each $\tau \geqq 0$

$$d\left(x, \bigcup_{t \geqq \tau} \varphi(t, \vartheta_{-t}\omega)B\right) = \inf_{t \geqq \tau} d(x, \varphi(t, \vartheta_{-t}\omega)B).$$

If time T is discrete, measurability of Ω_B is immediate. For T continuous note that for $a \in \mathbb{R}$ arbitrary

$$\left\{\omega : \inf_{t \geqq \tau} d(x, \varphi(t, \vartheta_{-t}\omega)B) < a\right\} = \pi_\Omega\{(t, \omega) : d(x, \varphi(t, \vartheta_{-t}\omega)B) < a, t \geqq \tau\},$$

where π_Ω denotes the canonical projection of $T \times \Omega$ to Ω. Measurability of $\omega \mapsto d(x, \overline{\bigcup_{t \geqq \tau} \varphi(t, \vartheta_{-t}\omega)B})$ with respect to the P-completion of $\mathscr{F}$ follows by the projection theorem (see Castaing and Valadier [6] Theorem III.23, p. 75). Taking the intersection

$$\bigcap_{\tau \geqq 0} \overline{\bigcup_{t \geqq \tau} \varphi(t, \vartheta_{-t}\omega)B}$$

over τ from a countable unbounded set (e.g., $\tau \in \mathbb{N}$), Ω_B is seen to be measurable. Since A can be obtained using only a countable number of B's in (5), the assertion is proved. □

In case $t \mapsto \varphi(t, \vartheta_{-t}\omega)$ is continuous P-a.s. (hence separable), the attractor is measurable without completion. For invertible random or stochastic flows this holds, see Arnold [1].

Concerning uniqueness of the global attractor we note that if A' is another compact invariant set attracting all bounded sets, then $\Omega_B \subset A'$ by Remark 3.10(i), hence $A \subset A'$. Thus the attractor given by Theorem 3.11 is minimal with respect to set inclusion.

In Theorem 3.11 we cannot proceed in exact analogy with the deterministic case, where the Ω-limit set Ω_K of the absorbing set can be taken to be the attractor. In the present situation this does not work, since the random set K need not absorb itself, so we cannot guarantee compactness of Ω_K.

We conclude this section by showing that a global attractor in a connected space is connected. We need a lemma.

3.12 Lemma *Suppose A is a non-connected compact subset of a metric space X.*

(i) *There exists α_0 such that for $\alpha \leqq \alpha_0$ the α-neighbourhood $U_\alpha(A) = \{y \in X: d(y, A) < \alpha\}$ of A is the disjoint union of two open non-empty sets.*

(ii) *If α_0 is as in* (i), *then*

$$\inf\{d(S, A): A \subset S, \text{ and } S \text{ connected}\} \geqq \alpha_0.$$

Proof. (i) Since A is not connected there are open sets $U, V \subset X$ such that $A \subset U \cup V$, $A \cap U \neq \emptyset$, $A \cap V \neq \emptyset$, and $A \cap U \cap V = \emptyset$. Since $A \cap U$ and $A \cap V$ are closed and disjoint,

$$\alpha_0 := \frac{1}{3} \inf\{d(x, y): x \in A \cap U,\ y \in A \cap V\} > 0.$$

For $\alpha \leqq \alpha_0$ put $G_1 = U_\alpha(A \cap U)$ and $G_2 = U_\alpha(A \cap V)$. Then $G_1 \cap G_2 = \emptyset$, and $U_\alpha(A) = U_\alpha((A \cap U) \cup (A \cap V)) = U_\alpha(A \cap U) \cup U_\alpha(A \cap V) = G_1 \cup G_2$.

(ii) Denote by U and V the two open disjoint and non-empty sets such that $U_{\alpha_0}(A) = U \cup V$ according to (i). Suppose S is connected and $d(S, A) < \alpha_0$, then $S \subset U_{\alpha_0}(A)$, hence S is contained either in U or in V, and so A is not a subset of S. Thus if S is connected then $A \subset S$ enforces $d(S, A) \geqq \alpha_0$. □

3.13 Proposition *Suppose φ is an RDS on a connected space X. If φ has a global attractor A, then P-a.s. A is connected.*

Proof. Since $\varphi(t, \omega)A(\omega) = A(\vartheta_t\omega)$, either A is connected P-a.s., or A is not connected P-a.s.

Assume that A is not connected P-a.s. Then there exists a number $\omega \mapsto \alpha_0(\omega)$ such that the assertion of Lemma 3.12 is satisfied. Pick a bounded connected set B such that $P\{A(\omega) \subset B\} \geqq 1 - \zeta$ for $0 < \zeta < \frac{1}{2}$. Such B exists, since for $x \in X$ arbitrary the map $\omega \mapsto d(x, A(\omega))$ takes real values, so choosing for B a ball around x with sufficiently large radius does.
Then

$$P\{A(\omega) \subset \varphi(t, \vartheta_{-t}\omega)B\} = P\{A(\omega) \subset B\} \geqq 1 - \zeta,$$

and since $\varphi(t, \vartheta_{-t}\omega)B$ is connected this implies

$$P\{d(\varphi(t, \vartheta_{-t}\omega)B, A(\omega)) \geqq \alpha_0(\omega)\} \geqq 1 - \zeta \tag{6}$$

for all $t \geqq 0$ by Lemma 3.12 (ii).

On the other hand, $d(\varphi(t, \vartheta_{-t}\omega)B, A(\omega))$ converges to zero P-a.s., so there exists T such that

$$P\left\{d(\varphi(T, \vartheta_{-T}\omega)B, A(\omega)) \leqq \frac{1}{2}\alpha_0(\omega)\right\} \geqq 1 - \zeta. \tag{7}$$

Since (6) and (7) are contradictory, A must be connected P-a.s. □

4 Invariant measures on random sets

Let $\mu_{\cdot}(\cdot)$ be a transition probability from Ω to X, i.e., μ_ω is a Borel probability measure on X and $\omega \mapsto \mu_\omega(B)$ is measurable for every Borel set $B \subset X$. Denote by $Pr_\Omega(X)$ the set of transition probabilities with $\mu_{\cdot}$ and $\nu_{\cdot}$ identified if $P\{\omega\colon \mu_\omega \neq \nu_\omega\} = 0$.

Suppose μ is a probability measure on $X \times \Omega$ with marginal P on Ω. Then for any $\mu \in Pr_P(X \times \Omega)$ there is a disintegration $\mu_{\cdot} \in Pr_\Omega(X)$ uniquely determined by

$$\mu(B \times F) = \int_F \mu_\omega(B)\, dP(\omega)$$

for all $B \in \mathscr{B}$ and $F \in \mathscr{F}$. We will henceforth identify probability measures on $X \times \Omega$ with marginal P with their disintegration $\omega \mapsto \mu_\omega$.

4.1 Definition (i) *An invariant measure for an RDS φ is a probability measure on $X \times \Omega$ whose marginal on Ω is P, and which is invariant under the flow Θ_t: $X \times \Omega \to X \times \Omega$, $(x, \omega) \mapsto (\varphi(t, \omega)x, \vartheta_t\omega)$, for $t \geqq 0$. The flow Θ_t: $t \in T$, is called* **the skew product flow induced by** *φ.*

(ii) *A probability measure μ on $X \times \Omega$ with marginal P on Ω is said to be* **supported by a measurable random set** *$\omega \mapsto A(\omega)$, if $\mu(A) = 1$, where $A = \{(x, \omega)\colon x \in A(\omega)\}$ is the graph of the mapping $\omega \mapsto A(\omega)$. Equivalent is: P-a.s. $\mu_\omega(A(\omega)) = 1$.*

The proof of the following Lemma is completely straightforward, using invertibility of $\{\vartheta_t\}$ (i.e., the fact that ϑ_t is defined for all $t \in \mathbb{Z}$ or $t \in \mathbb{R}$, respectively). It is wrong if ϑ_t is defined for $t \geqq 0$ only.

4.2 Lemma *Suppose μ is a probability measure on $X \times \Omega$ with marginal P on Ω, and $\omega \mapsto \mu_\omega$ is its disintegration. Then the disintegration of $\Theta_t\mu$ is*

$$\omega \mapsto \varphi(t, \vartheta_{-t}\omega)\mu_{\vartheta_{-t}\omega} = (\varphi(t, \cdot)\,\mu_{\cdot}) \circ \vartheta_{-t}(\omega).$$

Denote by $C_\Omega(X)$ the set of functions $f\colon X \times \Omega \to \mathbb{R}$ such that $f(x, \cdot)$ is measurable for each $x \in X$, $f(\cdot, \omega)$ is continuous and bounded for each $\omega \in \Omega$, and $\omega \mapsto \sup\{|f(x, \omega)|\colon x \in X\}$ is integrable with respect to P, where two such functions f and g are identified if $P\{\omega\colon f(\cdot, \omega) \neq g(\cdot, \omega)\} = 0$ (measurable by continuity of f and g together with separability of X).

Define the *narrow topology* on $Pr_\Omega(X)$ to be the coarsest topology such that

$$\mu \mapsto \int_{X \times \Omega} f(x,\omega)\, d\mu(x,\omega) = \mu(f)$$

is continuous for all $f \in C_\Omega(X)$. The skew product flow $(\Theta_t)_{t \in T}$ acts as a flow of continuous transformations on $Pr_\Omega(X)$.

A subset Γ of $Pr_\Omega(X)$ is said to be *tight* if $\pi_X \Gamma \subset Pr(X)$ is tight, where π_X denotes canonical projection from $X \times \Omega$ onto X. Thus Γ is tight if for every $\varepsilon > 0$ there exists a compact $K_\varepsilon \subset X$ such that $\mu(K_\varepsilon \times \Omega) \geqq 1 - \varepsilon$ for all $\mu \in \Gamma$. The following result is due to Valadier [21] Theorem 11, p. 162.

4.3 Theorem *Suppose $\Gamma \subset Pr_\Omega(X)$ is tight. Then*
(i) *Γ is relatively compact in $Pr_\Omega(X)$.*
(ii) *Γ is relatively sequentially compact (i.e., if $(\mu^n_\bullet)_{n \in \mathbb{N}}$ is a sequence in Γ, then there exists a convergent subsequence $(\mu^{n_k}_\bullet)_{k \in \mathbb{N}}$).*

4.4 Corollary *Let φ be an RDS, and suppose $\omega \mapsto A(\omega)$ is a compact measurable forward invariant set for φ. Then there exist invariant measures for φ which are supported by A.*

Proof. Put $\Gamma = \{\mu \in Pr_\Omega(X)\colon \mu_\omega(A_\omega) = 1 \ P\text{-a.s.}\}$. Then Γ is tight and closed (Crauel [9]), hence compact by Theorem 4.3. Furthermore, Γ is convex and invariant under Θ_t, $t \geqq 0$. The assertion thus follows from the Markov-Kakutani fixed point theorem (Dunford and Schwartz [11] Thm. V. 10.6, p. 456). □

Define two σ-algebras corresponding to the future and the past, respectively, by

$$\mathscr{F}^+ = \sigma\{\omega \mapsto \varphi(\tau, \vartheta_t\omega)\colon \tau, t \geqq 0\} \text{ and } \mathscr{F}^- = \sigma\{\omega \mapsto \varphi(\tau, \vartheta_{-t}\omega)\colon 0 \leqq \tau \leqq t\}.$$

Then $\vartheta_t^{-1}\mathscr{F}^+ \subset \mathscr{F}^+$ for all $t \geqq 0$ and $\vartheta_t^{-1}\mathscr{F}^- \subset \mathscr{F}^-$ for all $t \leqq 0$.
Note that for an invertible RDS $\varphi, \mathscr{F}^+$ coincides with $\sigma(\omega \mapsto \varphi(\tau,\omega)\colon \tau \geqq 0)$ and $\mathscr{F}^-$ coincides with $\sigma(\omega \mapsto \varphi(\tau,\omega)^{-1}\colon \tau \leqq 0)$.

Provided the conditions of Theorem 3.11, it is immediate from the construction of the attractor that it is measurable with respect to the past $\mathscr{F}^-$ (since Ω_B is so for any nonrandom B).

4.5 Proposition *Suppose $\omega \mapsto A(\omega)$ is a φ-invariant compact set which is measurable with respect to the past $\mathscr{F}^-$ for an RDS φ. Then there exist invariant measures μ supported by A such that also $\omega \mapsto \mu_\omega$ is measurable with respect to $\mathscr{F}^-$.*

Proof. The set of all probability measure valued $\omega \mapsto \mu_\omega$ which are measurable with respect to an arbitrary sub-σ-algebra of $\mathscr{F}$ form a closed subset of $L^\infty(\Omega;\, Pr(X))$. This holds in particular for $\mathscr{F}^-$. Furthermore, Lemma 4.2 yields that the set of $\mathscr{F}^-$-measurable measures is invariant under the linear continuous action induced by Θ_t. The assertion follows as soon we have established existence of $\mathscr{F}^-$-measurable measures $\omega \mapsto \mu_\omega$ supported by A.

This follows by choosing a measurable selection $\omega \mapsto x(\omega) \in A(\omega)$ (Deimling [10] Theorem 24.3, p. 307) and putting $\mu_\omega = \delta_{x(\omega)}$. $\square$

Measures which are measurable with respect to the past are called *Markov measures*. The reason for this notion comes from the fact that in case of a stochastic flow these are precisely those flow-invariant measures which correspond to invariant measures for the Markov semigroup induced by the one-point motions of the flow, see Crauel [8]. We obtain

4.6 Corollary *Under the conditions of the Proposition suppose in addition that φ is a RDS whose one-point motions form a Markov family, and such that $\mathscr{F}^+$ and $\mathscr{F}^-$ are independent. Then there exists an invariant measure ρ for the associated Markov semigroup. Furthermore, the limit*

$$\mu_\omega = \lim_{t \to \infty} \varphi(t, \vartheta_{-t}\omega)\rho \tag{8}$$

exists P-a.s., $\rho = \int \mu_\omega dP(\omega) = E(\mu_\cdot)$, and μ is a Markov measure.

Proof. Let μ be an invariant measure for φ supported by A, such that $\omega \mapsto \mu_\omega$ is $\mathscr{F}^-$-measurable. Then $\rho = E(\mu_\cdot)$ is an invariant measure for the Markov semigroup by Crauel [8] Theorem 4.4 and 5.2.2. Assertion (8) follows from Le Jan [17] Lemme 1, p. 112, or Crauel [7] Proposition 3.1 (the argument can easily be seen to carry over to non-invertible RDS when replacing $\varphi(-t,\omega)^{-1}$ by $\varphi(t, \vartheta_{-t}\omega)$ for $t \geqq 0$.) $\square$

It should be emphasized that the Markov semigroup invariant measure ρ from Corollary 4.6 does not have compact support in general. This need not even be true if μ_ω is a random Dirac measure.

5 Reaction-diffusion equation with additive noise

5.1 Formulation

Let $D \subset \mathbb{R}^n$ be a bounded open set with regular boundary ∂D. Denote by Δ the Laplacian in $\mathbb{R}^n$, and by f a polynomial of the form

$$f(u) = \sum_{k=0}^{2p-1} a_k u^k, \quad \text{with } a_{2p-1} < 0$$

for some positive integer p and real numbers $a_0, \ldots, a_{2p-1}$. Consider the following (reaction-diffusion type) stochastic partial differential equation in D with additive noise:

$$\begin{cases} du = \Delta u dt + f(u)dt + \sum_{j=1}^{m} \phi_j dw_j(t), \\ u = 0 \quad \text{on } \partial D, \end{cases} \tag{9}$$

where $w_j(t)$, $1 \leqq j \leqq m$, are independent two-sided Wiener processes (compare Sect. 2) and $\phi_j: D \to \mathbb{R}$, $1 \leqq j \leqq m$, will be specified below. We introduce the following spaces and operators related to (9):

$$H = L^2(D); \quad V = H_0^1(D); \quad Z = L^{2p}(D), \quad Z' = L^{(2p)'}(D),$$

where $(2p)' = \frac{2p}{2p-1}$; and

$$A: D(A) \subset H \to H,$$

where $D(A) = \{u \in H^2(D): u = 0 \text{ on } \partial D\}$ and $Au = \Delta u$, and finally $F: Z \to Z'$, $F(u) = f(u)$. With these notations we assume that $\phi_j \in D(A)$ and $A\phi_j \in Z'$ (this assumption may be relaxed according to Remark 5.1 below). Moreover, we can rewrite equation (9) in the abstract form

$$du = Au\,dt + F(u)dt + \sum_{j=1}^{m} \phi_j dw_j(t). \tag{10}$$

We denote by $\langle \cdot,\cdot \rangle$ and $|\cdot|$ the inner product and norm in H, and by $\langle\langle \cdot,\cdot \rangle\rangle$ and $\|\cdot\|$ the inner product and norm in $V = H_0^1(D) = D((-A)^{1/2})$, defined as $\|x^2\| = |(-A)^{1/2}x|^2$ for all $x \in H_0^1(D)$ (hence $\|x\| = \langle -Ax, x\rangle$ for all $x \in D(A)$). By Rellich's Theorem A^{-1} is compact, and the embedding $V \subset H$ is compact. If λ_1 is the first eigenvalue of $-A$, we have

$$\|x\|^2 \geqq \lambda_1 |x|^2$$

for all $x \in V$. Let us further denote the norms in Z and Z' by $\|\cdot\|_Z$ and $\|\cdot\|_{Z'}$, respectively. Concerning the boundedness of F in these spaces, note that by the Young inequality

$$ab \leqq \frac{\varepsilon}{r} a^r + \frac{1}{r' \varepsilon^{\frac{r'}{r}}} b^{r'},$$

which holds true for all $a, b, \varepsilon > 0$, $r \in (1, \infty)$, and $r' = \frac{r}{r-1}$, there exist two constants $c_1, c_2 > 0$ such that

$$|f(u)| \leqq c_1 |u|^{2p-1} + c_2 \quad \text{for all } u \in \mathbb{R}.$$

Hence, with $c_3 = c_2 |D|^{1/(2p)'}$, we have

$$\|F(u)\|_{Z'} \leqq c_1 \|(|u|^{2p-1})\|_{Z'} + c_3 = c_1 \|u\|_Z^{2p-1} + c_3$$

for all $u \in Z$.

Let us study equation (10) by means of the classical change of variable

$$v(t) = u(t) - w(t),$$

where, for brevity, we write $w(t) = \sum_{j=1}^{m} \phi^j w^j(t)$. Formally, $v(t)$ satisfies the equation (which depends on a random parameter)

$$\frac{dv}{dt} = Av + F(u) + Aw, \tag{11}$$

or the more explicit equation

$$\frac{dv}{dt} = Av + F(v + w) + Aw. \tag{12}$$

By the same proof as that of Theorem 1.1, Chap. III, of Temam [20] one can show that for P-almost every $\omega \in \Omega$ the following holds

(i) for all $t_0 < T \in \mathbb{R}$ and all $v_0 \in H$ there exists a unique solution $v \in C([t_0, T); H) \cap L^2(t_0, T; V) \cap L^{2p}([t_0, T]; Z)$ of equation (12) with $v(t_0) = v_0$;

(ii) if $v_0 \in V$, the solution belongs to $C([t_0, \infty); V) \cap L^2_{\text{loc}}(t_0, \infty; D(A))$;

(iii) hence, for all $v_0 \in H$, $v \in C([t_0 + \varepsilon, \infty); V) \cap L^2_{\text{loc}}(t_0 + \varepsilon, \infty; D(A))$, for every $\varepsilon > 0$;

(iv) denoting such a solution by $v(t, \omega; t_0, v_0)$, the mapping $v_0 \mapsto v(t, \omega; t_0, v_0)$ is continuous for all $t \geqq t_0$.

The proof proceeds by a priori estimates on Galerkin approximations of the same form as those developed in the following subsections.
Having the mapping $v_0 \mapsto v(t, \omega; 0, v_0)$, one can define a stochastic flow $\varphi(t, \omega)$ by

$$\varphi(t, \omega) u_0 = v(t, \omega; 0, u_0) + w(t, \omega). \tag{13}$$

This will be called the stochastic flow associated with equation (9) (one can show that the process $\varphi(t, \omega) u_0$ is a solution of (9) in a suitable sense, but this fact is not needed in the following).

5.1 Remark Let $z(t)$ be the stationary solution of the Ornstein–Uhlenbeck equation

$$dz = Az\,dt + dw(t).$$

The process $z(t)$ is more regular than $w(t)$ in the space variable. With some additional technical details one can perform the following analysis using the change of variable $v(t) = u(t) - z(t)$ in place of $v(t) = u(t) - w(t)$. The advantage is that less regularity conditions on ϕ_k have to be imposed. However, for simplicity, we restrict our attention to the change of variable $v(t) = u(t) - w(t)$.

5.2 Two preliminary inequalities

By definition of A and F, for all $u \in D(A) \cap Z$ we have

$$\begin{aligned} -\langle A(u - w(t)), F(u) \rangle &= -\langle Au, F(u) \rangle + \langle Aw(t), F(u) \rangle \\ &\quad - \int_D \Delta u f(u) + \int_D \Delta w(t) f(u) \\ &= \int_D f'(u) \nabla u \cdot \nabla u + \int_D \Delta w(t) f(u). \end{aligned}$$

By the assumptions on the polynomial f its derivative is bounded from above, i.e.

$$f'(r) \leqq \beta \quad \text{for all } r \in \mathbb{R} \tag{14}$$

for some constant β. [This may be deduced either from the Young inequality or from elementary considerations on the graph of f'.] Then

$$\begin{aligned} -\langle A(u-w(t)), F(u)\rangle &\leqq \beta\|u\|^2 + \|F(u)\|_Z \|\Delta w(t)\|_{Z'} \\ &\leqq \beta\|u\|^2 + (c_1\|u\|_Z^{2p-1} + c_3)\|\Delta w(t)\|_{Z'} \\ &\leqq \beta\|u\|^2 + \|u\|_Z^{2p} + p_1(t,\omega) \end{aligned}$$

(by the Young inequality), where, for some constant $c_4 > 0$,

$$p_1(t,\omega) = c_4\|\Delta w(t)\|_{Z'}^{2p} + c_3\|\Delta w(t)\|_{Z'}$$

has at most polynomial growth as $t \to -\infty$, for P-a.s. $\omega \in \Omega$ (by the law of large numbers), and $p_1(t,\omega) \geqq 0$. This is the first inequality we will need in the following. Next, by arguments similar to those yielding (14), there are positive constants δ_0 and c_5 such that

$$f(r)r \leqq -\delta_0 r^{2p} + c_5 \quad \text{for all } r \in \mathbb{R} \tag{15}$$

Hence, for all $u \in Z$,

$$\begin{aligned} \langle u - w(t), F(u)\rangle = \langle u, F(u)\rangle - \langle w(t), F(u)\rangle &= \int_D uf(u) - \int_D w(t)f(u) \\ &\leqq -\delta_0\|u\|_Z^{2p} + c_5|D| + (c_1\|u\|_Z^{2p-1} + c_3)\|w(t)\|_{Z'} \\ &\leqq -\frac{\delta_0}{2}\|u\|_Z^{2p} + c_5|D| + c_6\|w(t)\|_{Z'}^{2p} + c_3\|w(t)\|_{Z'} \\ &\quad \text{(by the Young inequality, for some constant } c_6 > 0) \\ &= -\delta\|u\|_Z^{2p} + p_2(t,\omega) \end{aligned}$$

with $\delta = \frac{\delta_0}{2}$ and $p_2(t,\omega) = c_5|D| + c_6\|w(t)\|_{Z'}^{2p} + c_3\|w(t)\|_{Z'}$, where p_2 has the same properties as p_1 above. Summarizing these results we obtain

5.2 Lemma *There exist two functions $p_i(t,\omega) \geqq 0$, $i = 1, 2$, with at most polynomial growth as $t \to -\infty$ for P-a.e. $\omega \in \Omega$, and there exist constants $\beta, \gamma > 0$, and $\delta > 0$, such that*

(i) *for all $u \in D(A) \cap Z$*

$$-\langle A(u - w(t,\omega)), F(u)\rangle \leqq \beta\|u\|^2 + \gamma\|u\|_Z^{2p} + p_1(t,\omega); \tag{16}$$

(ii) *for all $u \in Z$*

$$\langle u - w(t), F(u)\rangle \leqq -\delta\|u\|_Z^{2p} + p_2(t,\omega). \tag{17}$$

When proving the existence of a compact absorbing set (which implies existence of a global stochastic attractor) we shall only use these two inequalities. Therefore, the proof and result of this section can be reformulated in a more abstract form and applied to other equations or systems satisfying (16) and (17). At the abstract level, this would need suitable assumptions to ensure solvability of the abstract equations (10) and (11). In order not to overload the presentation, we limit ourselves to the concrete problem (9).

5.3 Absorption in H at time $t = -1$

In the following computation $\omega \in \Omega$ is fixed; the results will hold P-a.s. Let $t_0 < -1$ and $u_0 \in H$ be given, and let v be the solution of Eq. (12) for $t \geqq t_0$, with $v(t_0) = u_0 - w(t_0)$ (denoted above by $v(t, \omega; t_0, u_0 - \omega(t_0, \omega))$). From (11) we get

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}|v|^2 &= \left\langle v, \frac{dv}{dt}\right\rangle - \|v\|^2 + \langle v, F(u)\rangle + \langle v, Aw\rangle \\ &\leqq -\|v\|^2 - \delta\|u\|_Z^{2p} + p_2(t,\omega) + \frac{1}{2}\|v\|^2 + \frac{1}{2}\|w\|^2 \\ &\leqq -\frac{1}{2}\|v\|^2 - \delta\|u\|_Z^{2p} + p_3(t,\omega),\end{aligned}$$

where $p_3(t,\omega) = p_2(t,\omega) + \frac{1}{2}\|w\|^2$ has the same properties as p_1 and p_2 from Lemma 5.2. Therefore

$$\frac{d}{dt}|v|^2 + \frac{1}{2}\|v\|^2 + 2\delta\|u\|_Z^{2p} \leqq -\frac{\lambda_1}{2}|v|^2 + 2p_3(t,\omega). \tag{18}$$

By the Gronwall Lemma

$$\begin{aligned}|v(-1)|^2 &\leqq e^{-\frac{\lambda_1}{2}(-1-t_0)}|v(t_0)|^2 + \int_{t_0}^{-1} e^{-\frac{\lambda_1}{2}(-1-s)}2p_3(s,\omega)\,ds \\ &\leqq 2e^{-\frac{\lambda_1}{2}(-1-t_0)}|u_0|^2 \\ &\quad + 2e^{-\frac{\lambda_1}{2}(-1-t_0)}|w(t_0)|^2 + \int_{-\infty}^{-1} e^{-\frac{\lambda_1}{2}(-1-s)}2p_3(s,\omega)\,ds.\end{aligned} \tag{19}$$

Hence we have

5.3 Lemma *There exists a random radius $r_1(\omega) > 0$, such that for all $\rho > 0$ there exists (a deterministic) $\bar{t} \leqq -1$ such that the following holds P-a.s. For all $t_0 \leqq \bar{t}$ and for all $u_0 \in H$ with $|u_0| \leqq \rho$, the solution $v(t, \omega; t_0, u_0 - w(t_0, \omega))$ of Eq. (12) over $[t_0, \infty)$, with $v(t_0) = u_0 - w(t_0)$, satisfies the inequality*

$$|v(-1, \omega; t_0, u_0 - w(t_0, \omega))|^2 \leqq r_1^2(\omega).$$

Of course one can deduce a similar absorption result for $u(-1)$ instead of $v(-1)$ from this Lemma, but we will not need it in the following.

Proof. Put

$$r_1^2(\omega) = 2 + 2\sup_{t_0 \leqq -1} e^{-\frac{\lambda_1}{2}(-1-t_0)}|w(t_0)|^2 + \int_{-\infty}^{-1} e^{-\frac{\lambda_1}{2}(-1-s)}2p_3(s,\omega)ds;$$

this is finite P-a.s. since $|w(t_0)|^2$ and $p_3(s,\omega)$ have at most polynomial growth for t_0 and s, respectively, tending to $-\infty$. Given $\rho > 0$, choose $\bar{t}$ such that

$$e^{-\frac{\lambda_1}{2}(-1-t_0)}\rho^2 \leqq 1$$

for all $t_0 \leqq \bar{t}$. The claim then follows from (19). □

5.4 Auxiliary estimates in $[-1, 0]$

Integrating (18) over $[-1, 0]$ we have

$$\frac{1}{2}\int_{-1}^{0}\|v(s)\|^2 ds + 2\delta\int_{-1}^{0}\|u(s)\|_Z^{2p} ds \leqq |v(-1)|^2 + \int_{-1}^{0} 2p_3(s,\omega)ds. \quad (20)$$

Consequently, from Lemma 5.3 we immediately get

5.4 Lemma *There exist two random variables* $c_1(\omega)$ *and* $c_2(\omega)$ *such that for all* $\rho > 0$ *there exists* $\bar{t} \leqq -1$ *such that the following holds P-a.s. For all* $t_0 \leqq \bar{t}$ *and all* $u_0 \in H$ *with* $|u_0| \leqq \rho$, *the solution* $v(t, \omega; t_0, u_0 - w(t_0, \omega))$ *of Eq.* (12) *over* $[t_0, \infty)$, *with* $v(t_0) = u_0 - w(t_0)$, *satisfies*

$$\int_{-1}^{0}\|v(s)\|^2 ds \leqq c_1(\omega), \quad \textit{and} \quad \int_{-1}^{0}\|u(s)\|_Z^{2p} ds \leqq c_2(\omega),$$

where $u(s) = v(s) + w(s)$.

5.5 Absorption in V at time $t = 0$

From (11) we also have

$$\frac{1}{2}\frac{d}{dt}\|v\|^2 = -\left\langle Av, \frac{dv}{dt}\right\rangle = -|Av|^2 - \langle Av, F(u)\rangle - \langle Av, Aw\rangle$$

$$\leqq -|Av|^2 + \beta\|u\|^2 + \gamma\|u\|_Z^{2p} + p_1(t,\omega) + \frac{1}{2}|Av|^2 + \frac{1}{2}|Aw|^2$$

$$\leqq -\frac{1}{2}|Av|^2 + \beta\|u\|^2 + \gamma\|u\|^2 + p_4(t,\omega)$$

for some function $p_4(t,\omega) \geqq 0$ with the same properties as p_1 and p_2 from Lemma 5.2. Integrating over an arbitrary interval $[s, 0]$ we get

$$\|v(0)\|^2 \leqq \|v(s)\|^2 + \int_{s}^{0}\{2\beta\|u(\sigma)\|^2 + 2\gamma\|u(\sigma)\|_Z^{2p} + 2p_4(\sigma,\omega)\}d\sigma.$$

Integrating again in s over $[-1, 0]$ we finally have

$$\|v(0)\|^2 \leqq \int_{-1}^{0}\|v(s)\|^2 ds + \int_{-1}^{0}\{4\beta\|v(\sigma)\|^2 + 4\beta\|w(\sigma)\|^2 + 2\gamma\|u(\sigma)\|_Z^{2p} + 2p_4(\sigma,\omega)\}d\sigma.$$

From Lemma 5.4 we readily have

5.5 Lemma *There exists a random radius* $r_2(\omega)$ *such that for all* $\rho > 0$ *there exists* $\bar{t} \leqq -1$ *in such a way that the following holds P-a.s. For all* $t_0 \leqq \bar{t}$ *and all* $u_0 \in H$ *with* $|u_0| \leqq \rho$ *denote by* $v(t, \omega; t_0, u_0 - w(t_0, \omega))$ *the solution of Eq.* (12)

over $[t_0, \infty)$ *with* $v(t_0) = u_0 - w(t_0)$, *and put* $u(t, \omega; t_0, u_0) = w(t, \omega) + v(t, \omega; t_0, u_0 - w(t_0, \omega))$. *Then*

$$\|u(0, \omega; t_0, u_0)\|^2 \leqq r_2^2(\omega).$$

5.6 Compact Attractor and invariant measures

5.6 Theorem *The stochastic flow associated with the reaction-diffusion equation with additive noise (9) has a compact stochastic attractor, in the sense of Theorem 3.11.*

Moreover, the Markov semigroup induced by the flow on H has an invariant measure ρ. The associated flow-invariant Markov measure μ on $H \times \Omega$ (cf. Corollary 4.6) has the property that its disintegration $\omega \mapsto \mu_\omega$ is supported by the attractor.

Proof. Recall that, in the language of the stochastic flow associated with Eq. (9),

$$u(0, \omega; t_0, u_0) = \varphi(t_n, \vartheta_{-t_0}\omega) u_0 .$$

Hence by Lemma 5.5 there exists a random ball in V which absorbs the bounded sets of H. Since V is compactly embedded in H, we have proved the existence of a compact absorbing set. Therefore Theorem 3.11 applies to the stochastic flow associated with Eq. (24).

The existence of an invariant Markov measure is a direct consequence of Corollary 4.6, provided we know that the one-point motions associated with the flow $\varphi(t, \omega)$ define a family of Markov processes. The proof of this fact is classical, so that we only give the idea. As a general remark, we note that all the properties used below for $\varphi(t, \omega)x$ can be easily proved using only the definition of $\varphi(t, \omega)x$ in terms of $v(t)$ and $w(t)$, without explicit reference to the equation (9).

Let $\varphi_{s,t}$ be defined as in Sect. 2. Let $\mathscr{F}_{s,t}$ be the σ-algebra generated by $w(r) - w(s)$ for $r \in [s, t]$, and let $\mathscr{F}_t = \mathscr{F}_{0,t}$. Define the operators P_t in the space of bounded measurable functions over H as $P_t(f)(x) = E(f(\varphi(t)x))$. If we show that

$$E(f(\varphi(t)x) | \mathscr{F}_s) = P_{t-s}(f)(\varphi(s)x)$$

for all $0 \leqq s \leqq t$ and all bounded continuous functions f over H, then clearly $\varphi(t)x$ is a Markov process with transition semigroup P_t. Since $\varphi(t, \omega)x = \varphi_{s,t}(\omega)\varphi(s, \omega)x$, it is sufficient to show that for a generic square integrable H-valued random variable η, measurable with respect to $\mathscr{F}_s$, we have

$$E(f(\varphi_{s,t}\eta) | \mathscr{F}_s) = P_{t-s}(f)(\eta).$$

If $\eta = x_0$ a.s. for some $x_0 \in H$, this is true because $\varphi_{s,t}x_0$ is $\mathscr{F}_{s,t}$-measurable, hence independent of $\mathscr{F}_s$. Here we have used the fact that $P_{t-s}(f)(x) = E(f(\varphi(t-s)x)) = E(f(\varphi_{s,t}x))$ (since the coefficients of the equation for $\varphi_{s,t}$ are time independent, one can see that the H-valued random variables $\varphi_{s,t}x$ and $\varphi(t-s)x$ have the same law). If $\eta = \sum_{i=1}^N x_i \mathbf{1}_{A_i}$, where $A_i \in \mathscr{F}_s$, one can show that $\varphi_{s,t}\eta = \sum_{i=1}^N \mathbf{1}_{A_i}\varphi_{s,t}x_i$; by this fact it is easy to prove the claim also in this case. Finally, for a general η there exists a sequence of η_n of the previous form which converges to η in $L^2(\Omega; H)$ and almost surely. By continuity of f one can pass to the limit, which completes the proof. □

6 Navier–Stokes equations with multiplicative noise

6.1 Formulation

Let H be a real separable Hilbert space (inner product $\langle \cdot,\cdot \rangle$, norm $|\cdot|$), and let $A: D(A) \subset H \to H$ be a selfadjoint strictly negative linear operator in H. Denote by V the Hilbert space $D((-A)^{1/2})$, endowed with the norm $\|x^2\| = |(-A)^{1/2}x|^2$ $(= -\langle Ax, x\rangle$ for $x \in D(A))$. Identifying H with its dual H', we can write $V \subset H \subset V'$, with dense continuous injections. We assume also that A^{-1} is compact, whence the embedding $V \subset H$ is compact. If λ_1 is the first eigenvalue of $-A$, we have

$$\|x\|^2 \geqq \lambda_1 |x|^2$$

for all $x \in V$.

Let $B(u, v)$ be a bilinear continuous operator from $V \times V$ to V'. Suppose that there exists a constant $c_B > 0$ such that

$$\langle B(u,v), z\rangle \leqq c_B |u|^{\frac{1}{2}} \|u\|^{\frac{1}{2}} |v|^{\frac{1}{2}} \|v\|^{\frac{1}{2}} \|z\| \tag{21}$$

for all $u, v, z \in V$,

$$\langle B(u,v), v\rangle = 0 \tag{22}$$

for all $u, v \in V$, and

$$\langle B(u,v), z\rangle \leqq c_B |u|^{\frac{1}{2}} |Au|^{\frac{1}{2}} \|v\| \, |z| \tag{23}$$

for all $u \in D(A)$, $v \in V$, and $z \in H$.

Finally, let $f \in H$ and $b_1, \dots, b_m \in \mathbb{R}$ be given. Under these hypotheses, consider the stochastic evolution equation with Stratonovich multiplicative noise

$$du = \{Au + B(u,u) + f\}dt + \sum_{j=1}^m b_j u \circ dw_j(t). \tag{24}$$

This equation represents an abstract form of the stochastic Navier–Stokes equation in two space dimensions, describing the motion of an incompressible fluid in a bounded domain, with Dirichlet or periodic boundary conditions (see Temam [20] for details in the deterministic case).

Consider the process

$$\alpha(t) = e^{-\sum_{j=1}^m b_j w_j(t)}.$$

It satisfies the Stratonovich equation

$$d\alpha(t) = -\sum_{j=1}^m b_j \alpha(t) \circ dw_j(t).$$

Hence, formally, the process $v(t)$, defined by the time change

$$v(t) = \alpha(t)u(t),$$

satisfies the equation (which depends on a random parameter)

$$\frac{dv}{dt} = Av + \alpha B(u,u) + \alpha f, \tag{25}$$

or, in a more explicit form, the equation

$$\frac{dv}{dt} = Av + \alpha^{-1}B(v,v) + \alpha f. \tag{26}$$

By the same proof as that of Theorem 2.1, Chapter III, of Temam [20] one can show that for P-almost every $\omega \in \Omega$ the following holds

(i) for all $t_0 \in \mathbb{R}$ and for all $v_0 \in H$ there exists a unique solution $v \in C([t_0, \infty); H) \cap L^2_{\mathrm{loc}}(t_0, \infty; V)$ of equation (26) with $v(t_0) = v_0$;

(ii) if $v_0 \in V$, the solution belongs to $C([t_0, \infty); V) \cap L^2_{\mathrm{loc}}(t_0, \infty; D(A))$;

(iii) hence, for every $\varepsilon > 0$, $v \in C([t_0 + \varepsilon, \infty); V) \cap L^2_{\mathrm{loc}}(t_0 + \varepsilon, \infty; D(A))$, for all $v_0 \in H$;

(iv) denoting such solution by $v(t, \omega; t_0, v_0)$, the mapping $v_0 \mapsto v(t, \omega; t_0, v_0)$ is continuous for all $t \geqq t_0$.

The proof proceeds by a priori estimates on Galerkin approximations of the same form as those developed in the following subsections.
Having the mapping $v_0 \mapsto v(t, \omega; 0, v_0)$, one can define a stochastic flow $\varphi(t, \omega)$ by

$$\varphi(t,\omega)u_0 = \alpha(t,\omega)^{-1} v(t,\omega; 0, u_0). \tag{27}$$

This will be called the stochastic flow associated with equation (24).

6.2 *Absorption in H at time $t = -1$*

In the following computations $\omega \in \Omega$ is fixed; the results will hold P-almost surely. Let $t_0 < -1$ and $u_0 \in H$ be given, and let v be the solution of Eq. (26) for $t \geqq t_0$, with $v(t_0) = \alpha(t_0)u_0$ (denoted above by $v(t, \omega; t_0, \alpha(t_0, \omega)u_0)$). From (22) and (25) we get

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}|v|^2 = \left\langle v, \frac{dv}{dt} \right\rangle &= -\|v\|^2 + \langle v, \alpha B(u,u) + \alpha f \rangle \\ &= -\|v\|^2 + \langle v, \alpha f \rangle \leqq -\|v\|^2 + |v|\,|\alpha f|. \end{aligned}$$

Hence there exists a constant c_0, depending only on λ_1, such that

$$\frac{d}{dt}|v|^2 + \|v\|^2 \leqq -\frac{\lambda_1}{2}|v|^2 + c_0|\alpha f|^2. \tag{28}$$

By the Gronwall Lemma

$$|v(-1)|^2 \leqq e^{-\frac{\lambda_1}{2}(-1-t_0)}|\alpha(t_0)u_0|^2 + \int_{t_0}^{-1} e^{-\frac{\lambda_1}{2}(-1-\sigma)} c_0|\alpha(\sigma)f|^2\, d\sigma$$

$$\leqq e^{\frac{\lambda_1}{2}}\{e^{\frac{\lambda_1}{2}t_0}\alpha(t_0)^2|u_0|^2 + c_0|f|^2 \int_{-\infty}^{-1} e^{\frac{\lambda_1}{2}\sigma}\alpha(\sigma)^2\, d\sigma\}. \tag{29}$$

Now by standard arguments (using, e.g., the law of the iterated logarithm)

$$\lim_{t\to-\infty} \frac{1}{t}\sum_{j=1}^{m} b_j w_j(t) = 0 \quad P\text{-a.s.}$$

It easily follows that $\sigma \mapsto e^{\frac{\lambda_1}{2}\sigma}\alpha(\sigma)^2$ is pathwise integrable over $(-\infty, 0]$. Similarly,

$$e^{\frac{\lambda_1}{2}\sigma}\alpha(\sigma)^2 \to 0 \quad \text{as } \sigma \to -\infty \quad P\text{-a.s.} \tag{30}$$

Hence from (29) we have

6.1 Lemma *There exists a random radius $r_1(\omega) > 0$, depending only on λ_1, $b_1, \ldots, b_m$, and $|f|$, such that for all $\rho > 0$ there exists $t(\omega) \leqq -1$ such that the following holds P-a.s. For all $t_0 \leqq t(\omega)$ and all $u_0 \in H$ with $|u_0| \leqq \rho$, the solution $v(t,\omega; t_0, \alpha(t_0,\omega)u_0)$ of Eq. (26) over $[t_0, \infty)$, with $v(t_0) = \alpha(t_0)u_0$, satisfies the inequality*

$$|v(-1,\omega; t_0, \alpha(t_0,\omega)u_0)|^2 \leqq r_1^2(\omega).$$

Proof. Put

$$r_1^2(\omega) = e^{\frac{\lambda_1}{2}}\{1 + c_0|f|^2 \int_{-\infty}^{-1} e^{\frac{\lambda_1}{2}\sigma}\alpha(\sigma)^2\, d\sigma\},$$

which is finite P-a.s. by the above considerations. Given $\rho > 0$, by (30) there exists $t(\omega)$ such that

$$e^{\frac{\lambda_1}{2}t_0}\alpha(t_0)^2\rho^2 \leqq 1$$

for all $t_0 \leqq t(\omega)$. The claim of the Lemma follows from (29). □

6.3 Auxiliary estimates in $[-1, 0]$

From (28) we also have, by the Gronwall Lemma with $t \in [-1, 0]$ and by integrating over $[-1, 0]$, respectively:

$$|v(t)|^2 \leqq e^{-\frac{\lambda_1}{2}(t+1)}|v(-1)|^2 + c_0|f|^2 \int_{-1}^{t} e^{-\frac{\lambda_1}{2}(t-\sigma)}\alpha(\sigma)^2 d\sigma,$$

$$\int_{-1}^{0} \|v(s)\|^2 ds \leqq |v(-1)|^2 + c_0|f|^2 \int_{-1}^{0} \alpha(s)^2 ds.$$

Therefore, from Lemma 6.1 we deduce

6.2 Lemma *There exist two random variables $c_1(\omega)$ and $c_2(\omega)$, depending only on $\lambda_1, b_1, \ldots, b_m$, and $|f|$, such that for all $\rho > 0$ there exists $t(\omega) \leqq -1$ such that the following holds P-a.s. For all $t_0 \leqq t(\omega)$ and all $u_0 \in H$ with $|u_0| \leqq \rho$, the solution $v(t,\omega; t_0, \alpha(t_0,\omega)u_0)$ of Eq. (26) over $[t_0, \infty)$, with $v(t_0) = \alpha(t_0)u_0$, satisfies*

$$|v(t,\omega; t_0, \alpha(t_0,\omega)u_0)|^2 \leqq c_1(\omega) \quad \textit{for all } t \in [-1, 0]$$

$$\int_{-1}^{0} \| v(s,\omega; t_0, \alpha(t_0,\omega)u_0)\|^2 \, ds \leqq c_2(\omega).$$

Proof. Put

$$c_1(\omega) = e^{-\frac{\lambda_1}{2}(t+1)} r_1^2(\omega) + c_0 |f|^2 \int_{-1}^{t} e^{-\frac{\lambda_1}{2}(t-\sigma)} \alpha(\sigma)^2 \, d\sigma,$$

$$c_2(\omega) = r_1^2(\omega) + c_0 |f|^2 \int_{-1}^{0} \alpha(s)^2 \, ds,$$

where $r_1(\omega)$ is from Lemma 6.1. Then, given $\rho > 0$, it is sufficient to choose the same $t(\omega)$ as in the proof of Lemma 6.1. □

6.4 Absorption in V at time $t = 0$

From (23) and (25) we further obtain

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\| v \|^2 = -\left\langle Av, \frac{dv}{dt} \right\rangle &\leqq -|Av|^2 + |Av|\,|\alpha B(u,u)| + |Av|\,|\alpha f| \\
&\leqq -\frac{1}{2}|Av|^2 + |\alpha B(u,u)|^2 + |\alpha f|^2 \\
&\leqq -\frac{1}{2}|Av|^2 + c_B \alpha^2 |u|\,|Au|\,\| u \|^2 + |\alpha f|^2 \\
&= -\frac{1}{2}|Av|^2 + c_B |u|\,|Av|\,\| u \|\,\| v \| + |\alpha f|^2 \\
&\leqq -\frac{1}{4}|Av|^2 + (c_B^2 |u|^2 \| u \|^2)\| v \|^2 + |\alpha f|^2.
\end{aligned}$$

Hence

$$\frac{d}{dt}\| v \|^2 \leqq (2c_B^2 |u|^2 \| u \|^2)\| v \|^2 + 2|\alpha f|^2,$$

and for $s < t$ we have

$$\| v(t) \|^2 \leqq \| v(s) \|^2 + 2|f|^2 \int_s^t \alpha(\sigma)^2 \, d\sigma + \int_s^t (2c_B^2 |u(\sigma)|^2 \| u(\sigma) \|^2) \| (v(\sigma) \|^2 \, d\sigma.$$

Applying the Gronwall Lemma over an arbitrary interval $[s,0] \subset [-1,0]$ we obtain

$$\|v(0)\|^2 \leqq e^{\int_s^0 c_B^2 |u(\sigma)|^2 \|u(\sigma)\|^2 d\sigma} \left\{ \|v(s)\|^2 + 2|f|^2 \int_s^0 \alpha(\sigma)^2 d\sigma \right\}.$$

Integrating in s over the interval $[-1,0]$ yields

$$\|u(0)\|^2 = \|v(0)\|^2 \leqq e^{\int_{-1}^0 c_B^2 |u(\sigma)|^2 \|u(\sigma)\|^2 d\sigma} \left\{ \int_{-1}^0 \|v(s)\|^2 ds + 2|f|^2 \int_{-1}^0 \alpha(\sigma)^2 d\sigma \right\}. \tag{31}$$

Here,

$$\int_{-1}^0 |u(\sigma)|^2 \|u(\sigma)\|^2 d\sigma \leqq \sup_{-1 \leqq t \leqq 0} \alpha(t)^{-4} \sup_{-1 \leqq t \leqq 0} |v(t)|^2 \int_{-1}^0 \|v(\sigma)\|^2 d\sigma. \tag{32}$$

Hence we finally have

6.3 Lemma *There exists a random radius $r_2(\omega)$, depending only on $\lambda_1, b_1, \ldots, b_m$, and $|f|$, such that for all $\rho > 0$ there exists $t(\omega) \leqq -1$ such that the following holds P-a.s. For all $t_0 \leqq t(\omega)$ and all $u_0 \in H$ with $|u_0| \leqq \rho$, let $v(t,\omega; t_0, \alpha(t_0,\omega)u_0)$ be the solution of Eq. (26) over $[t_0, \infty)$, with $v(t_0) = \alpha(t_0)u_0$, and put $u(t,\omega; t_0, u_0) := \alpha(t,\omega)^{-1} v(t,\omega; t_0, \alpha(t_0,\omega)u_0)$. Then*

$$\|u(0,\omega; t_0, u_0)\|^2 \leqq r_2^2(\omega).$$

Proof. Put

$$r_2^2(\omega) = \left(c(\omega) + 2|f|^2 \int_{-1}^0 \alpha(\sigma)^2 d\sigma \right) e^{c_B^2 \sup_{-1 \leqq t \leqq 0} \alpha(t)^{-4} c(\omega)^2}$$

with $c(\omega)$ from the previous Lemma. Given $\rho > 0$, choose $t(\omega)$ as in the proof of Lemma 6.1. Then, by (32),

$$\int_{-1}^0 |u(\sigma)|^2 \|u(\sigma)\|^2 d\sigma \leqq \sup_{-1 \leqq t \leqq 0} \alpha(t)^{-4} c(\omega)^2.$$

The assertion follows from (31). □

6.5 Final result

6.4 Theorem *The stochastic flow associated with the Navier-Stokes equation with multiplicative noise (24) has a compact stochastic attractor, in the sense of Theorem 3.11.*

Moreover, the Markov semigroup induced by the flow on H has an invariant measure ρ. The associated flow-invariant Markov measure μ on $H \times \Omega$ (cf. Corollary 4.6) has the property that its disintegration $\omega \mapsto \mu_\omega$ is supported by the attractor.

The proof proceeds exactly as that of Theorem 5.6.

7 Navier–Stokes equation with additive noise

7.1 Formulation

Let $H, A: D(A) \subset H \to H$, $V = D((-A)^{1/2})$, and $B(u, v): V \times V \to V'$ be the spaces and operators introduced in the previous Section. Moreover, let $f \in H$ and $\phi_1, \ldots, \phi_m \in H$ be given. Consider the equation

$$du = \{Au + B(u, u) + f\}dt + \sum_{j=1}^{m} \phi_j dw_j(t). \tag{33}$$

We assume that $\phi_k \in D(A)$, $1 \leqq k \leqq m$, and that there exists a constant $\beta > 0$ such that

$$|\langle B(u, \phi_k), u\rangle| \leqq \beta |u|^2 \quad \text{for all } u \in V, k = 1, \ldots, m. \tag{34}$$

Remark. For a concrete 2-dimensional Navier–Stokes equation in a bounded domain $D \subset \mathbb{R}^2$ we have

$$\langle B(u, \phi_k), u\rangle = \sum_{i,j=1}^{2} \int_D u_i \frac{\partial(\phi_k)_j}{\partial x_i} u_j \, dx.$$

In this case assumption (34) is satisfied provided the functions ϕ_k are Lipschitz continuous in D.

Put $w(t) = \sum_{j=1}^{m} \phi_j w_j(t)$. If we follow the method employed for the reaction-diffusion Eq. (9), based on the change of variable $v = u - w$, we end up with the problem of finding estimates for $\langle B(v(t), w(t)), v(t)\rangle$ when analyzing $\frac{d}{dt}|v(t)|^2$. This yields a term which, roughly speaking, behaves like $|v(t)|^2 w(t)$, so that we cannot deduce any bound in H for $|v(t)|^2$. To overcome this difficulty we introduce a different change of variable.

7.2 Auxiliary Ornstein–Uhlenbeck process

Let $\alpha > 0$ be given; we shall impose condition (36) below on α. For each $k = 1, \ldots, m$, let z_k be the stationary (ergodic) solution of the one-dimensional equation

$$dz_k = -\alpha z_k \, dt + dw_k(t);$$

so

$$z_k(t) = \int_{-\infty}^{t} e^{-\alpha(t-s)} dw_k(s).$$

Putting $z(t) = \sum_{k=1}^{m} \phi_k z_k(t)$ we have

$$dz = -\alpha z \, dt + dw(t).$$

Since the trajectories of z_k are P-a.s. continuous, and $\phi_k \in D(A)$, we have

$$\sup_{-1 \leqq t \leqq 0} \{|z(t)|^2 + \|z(t)\|^2 + |Az(t)|^2\} < \infty \quad P\text{-a.s.} \tag{35}$$

We now choose $\alpha > 0$ such that

$$4\beta m E|z_1(0)| \leqq \frac{\lambda_1}{4}, \tag{36}$$

where λ_1 is the first eigenvalue of $-A$. This is possible since $(E|z_1(0)|)^2 \leqq E|z_1(0)|^2 = \mathrm{Var}(z_1(0))$, and $\mathrm{Var}(z_1(0)) \to 0$ as $\alpha \to \infty$. From (36) and the Ergodic Theorem we obtain

$$\lim_{t_0 \to -\infty} \frac{1}{-1-t_0} \int_{t_0}^{-1} 4\beta \sum_{k=1}^{m} |z_k(s)| ds = 4\beta m E|z_1(0)| \leqq \frac{\lambda_1}{4} \quad P\text{-a.s.}$$

Putting $\gamma(t) = -\frac{\lambda_1}{2} + 4\beta \sum_{k=1}^{m} |z_k(t)|$, we get

$$\lim_{t_0 \to -\infty} \frac{1}{-1-t_0} \int_{t_0}^{-1} \gamma(s) ds \leqq -\frac{\lambda_1}{4} \quad P\text{-a.s.}$$

From this fact and by stationarity of z_k we finally obtain

$$\lim_{t_0 \to -\infty} e^{\int_{t_0}^{-1} \gamma(s) ds} = 0 \quad P\text{-a.s.} \tag{37}$$

$$\sup_{t_0 < -1} e^{\int_{t_0}^{-1} \gamma(s) ds} |z(t_0)|^2 < \infty \quad P\text{-a.s.} \tag{38}$$

$$\int_{-\infty}^{-1} e^{\int_{\sigma}^{-1} \gamma(s) ds} (1 + z_k(\sigma)^2 + z_k(\sigma)^2 |z_j(\sigma)|) d\sigma < \infty \quad P\text{-a.s.} \tag{39}$$

for all $1 \leqq j, k \leqq m$. Indeed, note for instance that for $t < 0$

$$\frac{z_k(t)}{t} = \frac{z_k(0)}{t} - \frac{1}{t} \alpha \int_t^0 z_k(s) ds + \frac{w_k(t)}{t},$$

whence $\lim_{t \to -\infty} \frac{z_k(t)}{t} = 0$ (P-a.s.), which implies (38) and (39).

7.3 Stochastic flow

We now use the change of variable $v(t) = u(t) - z(t)$. Then, formally, v satisfies the equation (which depends on a random parameter)

$$\frac{dv}{dt} = Av + B(u, u) + f + \alpha z + Az, \tag{40}$$

or, more explicitly,

$$\frac{dv}{dt} = Av + B(v, v) + B(v, z) + B(z, v) + B(z, z) f + \alpha z + Az. \tag{41}$$

By the same proof as that of Theorem 2.1, Chap. III, of Temam [20] one can show that (i)–(iv) as stated in Section 6.1 for the multiplicative case also hold under the present conditions. Thus there is a continuous mapping from H into itself, $v_0 \mapsto v(t, \omega; t_0, v_0)$, where $v(t, \omega; t_0, v_0)$ is the solution of equation (41) with $v(t_0) = v_0$. We can now define a stochastic flow $\varphi(t, \omega)$ in H by putting

$$\varphi(t, \omega)u_0 = v(t, \omega; 0, u_0 - z(0, \omega)) + z(t, \omega). \tag{42}$$

This will be called the stochastic flow associated with the Navier–Stokes Eq. (33) with additive noise.

7.4 Absorption in H at time $t = -1$

In the following computations $\omega \in \Omega$ is fixed; the results will hold P-almost surely. Let $t_0 \leqq -1$ and $u_0 \in H$ be given, and let v be the solution of Eq. (41) for $t \geqq t_0$, with $v(t_0) = u_0 - z(t_0, \omega)$ (which was denoted above by $v(t, \omega; t_0, u_0 - z(t_0, \omega))$). From assumption (22) and from (40) we have

$$\frac{1}{2}\frac{d}{dt}|v|^2 = -\|v\|^2 - \langle B(u,u), z\rangle + \langle f, v\rangle + \langle \alpha z, v\rangle + \langle Az, v\rangle$$

$$\leqq -\|v\|^2 + \langle B(u,z), u\rangle + |f|\,|v| + \alpha|z|\,|v| + \|z\|\,\|v\|.$$

By definition of z and assumption (34),

$$\langle B(u,z), u\rangle = \sum_{k=1}^{m} \langle B(u, \phi_k), u\rangle z_k \leqq \beta|u|^2 \sum_{k=1}^{m} |z_k|$$

$$\leqq 2\beta|v|^2 \sum_{k=1}^{m} |z_k| + 2\beta|z|^2 \sum_{k=1}^{m} |z_k|.$$

Hence there exists a constant $c > 0$ depending only on λ_1 such that

$$\frac{1}{2}\frac{d}{dt}|v|^2 + \frac{1}{2}\|v\|^2 \leqq \left\{ -\frac{\lambda_1}{4} + 2\beta \sum_{k=1}^{m} |z_k| \right\}|v|^2 + c|f|^2 + c\alpha|z|^2$$

$$+ c\|z\|^2 + 2\beta|z|^2 \sum_{k=1}^{m} |z_k|. \tag{43}$$

Put

$$p(t) = c|f|^2 + c\alpha|z(t)|^2 + c\|z(t)\|^2 + 2\beta|z(t)|^2 \sum_{k=1}^{m} |z_k(t)|,$$

and let $\gamma(t)$ be defined as in subsection 7.2. We have

$$\frac{d}{dt}|v(t)|^2 \leqq \gamma(t)|v(t)|^2 + 2p(t),$$

which implies

$$|v(t)|^2 \leqq e^{\int_\tau^t \gamma(s)ds}|v(\tau)|^2 + \int_{t_0}^{t} e^{\int_\sigma^t \gamma(s)ds} 2p(\sigma)d\sigma \tag{44}$$

for all $t_0 \leqq \tau \leqq t$. For $t = -1$ and $\tau = t_0$ we get

$$|v(-1)|^2 \leqq e^{\int_{t_0}^{-1}\gamma(s)ds}|v(t_0)|^2 + \int_{t_0}^{-1} e^{\int_\sigma^t \gamma(s)ds} 2p(\sigma)d\sigma$$

$$\leqq 2e^{\int_{t_0}^{-1}\gamma(s)ds}|u_0|^2 + 2e^{\int_{t_0}^{-1}\gamma(s)ds}|z(t_0)|^2$$

$$+ \int_{-\infty}^{-1} e^{\int_\sigma^t \gamma(s)ds} 2p(\sigma)d\sigma. \tag{45}$$

7.1 Lemma *There exists a random radius $r_1(\omega) > 0$, depending only on $\lambda_1, \phi_1, \dots, \phi_m$, and $|f|$, such that for all $\rho > 0$ there exists $t(\omega) \leqq -1$ such that the following holds P-a.s. For all $t_0 \leqq t(\omega)$ and all $u_0 \in H$ with $|u_0| \leqq \rho$, the solution $v(t, \omega; t_0, u_0 - z(t_0, \omega))$ of Eq. (41) over $[t_0, \infty)$, with $v(t_0) = u_0 - z(t_0, \omega)$, satisfies the inequality*

$$|v(-1, \omega; t_0, u_0 - z(t_0, \omega))|^2 \leqq r_1^2(\omega).$$

Proof. Put

$$r_1^2(\omega) = 2 + 2 \sup_{t_0 \leqq -1} e^{\int_{t_0}^{-1}\gamma(s)ds}|z(t_0)|^2 + \int_{-\infty}^{-1} e^{\int_\sigma^t \gamma(s)ds} 2p(\sigma)d\sigma,$$

which is finite P-a.s. due to (38) and (39). Given $\rho > 0$, choose $t(\omega)$ such that

$$e^{\int_{t_0}^{-1}\gamma(s)ds}\rho^2 \leqq 1$$

for all $t_0 \leqq t(\omega)$ (recall (37)). Then the assertion follows from (45). □

7.5 Auxiliary estimates in $[-1, 0]$

Taking $t \in [-1, 0]$ and $\tau = -1$ in (44) we have

$$|v(t)|^2 \leqq e^{\int_{-1}^{t}\gamma(s)ds}|v(-1)|^2 + \int_{-1}^{t} e^{\int_\sigma^t \gamma(s)ds} 2p(\sigma)d\sigma. \tag{46}$$

Moreover, integrating (43) over $[-1, 0]$ we have

$$\int_{-1}^{0} \|v(s)\|^2 ds \leqq |v(-1)|^2 + \left(\int_{-1}^{0}\gamma(s)ds\right)\left(\sup_{-1 \leqq t \leqq 0}|v(t)|^2\right) + \int_{-1}^{0} 2p(\sigma)d\sigma. \tag{47}$$

Therefore, from Lemma 7.1 we deduce

7.2 Lemma *There exist two random variables $c_1(\omega)$ and $c_2(\omega)$, depending only on $\lambda_1, \phi_1, \dots, \phi_m$, and $|f|$, such that for all $\rho > 0$ there exists $t(\omega) \leqq -1$ such that the following holds P-a.s. For all $t_0 \leqq t(\omega)$ and for all $u_0 \in H$ with*

$|u_0| \leqq \rho$, *the solution* $v(t, \omega; t_0, u_0 - z(t_0, \omega))$ *of Eq. (41) over* $[t_0, \infty)$, *with* $v(t_0) = u_0 - z(t_0, \omega)$, *satisfies*

$$|v(t, \omega; t_0, u_0 - z(t_0, \omega))|^2 \leqq c_1(\omega) \quad \textit{for all } t \in [-1, 0],$$

$$\int_{-1}^{0} \| v(s, \omega; t_0, u_0 - z(t_0, \omega)) \|^2 ds \leqq c_2(\omega).$$

Proof. Put

$$c_1(\omega) = e^{\int_{-1}^{t} \gamma(s) ds} r_1^2(\omega) + \int_{-1}^{t} e^{\int_{\sigma}^{t} \gamma(s) ds} 2p(\sigma) d\sigma,$$

$$c_2(\omega) = r_1^2(\omega) \left\{ 1 + \int_{-1}^{0} \gamma(s) ds \right\} + \int_{-1}^{0} 2p(\sigma) d\sigma,$$

with $r_1(\omega)$ as in Lemma 7.1. Then, given $\rho > 0$, it suffices to choose $t(\omega)$ as in the proof of that Lemma 7.1. □

7.6 Absorption in V at time $t = 0$

From (40) we further get

$$\frac{1}{2}\frac{d}{dt}\| v \|^2 = -\left\langle Av, \frac{dv}{dt} \right\rangle$$

$$= -|Av|^2 + \langle B(u, u), Av \rangle + \langle f, Av \rangle + \langle \alpha z, Av \rangle + \langle Az, Av \rangle$$

$$\leqq -|Av|^2 + c_B |u|^{\frac{1}{2}} |Au|^{\frac{1}{2}} \| u \| \, |Av| + |f| \, |Av| + |\alpha z| \, |Av| + |Az| \, |Av|$$

$$\leqq -\frac{1}{2}|Av|^2 + 2c_B^2 |u| \, |Au| \, \| u \|^2 + 2\{ |f|^2 + |\alpha z|^2 + |Az|^2 \}.$$

With $q(= q(t)) = 2\{ |f|^2 + |\alpha z|^2 + |Az|^2 \}$ we obtain

$$\frac{1}{2}\frac{d}{dt}\| v \|^2 \leqq -\frac{1}{4}|Av|^2 4c_B^4 |u|^2 \| u \|^2 \| u \|^2 + 2c_B^2 |u| \, |Az| \, \| u \|^2 + q$$

$$\leqq (8c_B^4 |u|^2 \| u \|^2) \| v \|^2 + 8c_B^4 |u|^2 \| u \|^2 \| z \|^2 + 2c_B^2 |u| \, |Az| \, \| u \|^2 + q$$

.

By the Gronwall Lemma we get for $s \in [0, 1]$

$$\| v(0) \|^2 \leqq e^{\int_s^0 16 c_B^4 |u(s)|^2 \| u(s) \|^2 ds}$$

$$\times \left[\| v(s) \|^2 + \int_s^0 \{ 16 c_B^4 |u|^2 \| u \|^2 \| z \|^2 + 4c_B^2 |u| \, |Az| \, \| u \|^2 + 2q \} \, d\sigma \right]$$

Integrating in s over $[0, 1]$ we obtain

$$\| v(0) \|^2 \leqq \left[\int_{-1}^{0} \| v(s) \|^2 ds + \int_{-1}^{0} \{16c_B^4 |u|^2 \| u \|^2 \| z \|^2 + 4c_B^2 |u| \, |Az| \, \| u \|^2 + 2q\} \, d\sigma \right] e^{\int_{-1}^{0} 16c_B^4 |u(s)|^2 \| u(s) \|^2 ds} . \tag{48}$$

Now we can prove

7.3 Lemma *There exists a random radius* $r_2(\omega)$, *depending only on* $\lambda_1, \phi_1, \ldots, \phi_m$, *and* $|f|$, *such that for all* $\rho > 0$ *there exists* $t(\omega) \leqq -1$ *such that the following holds P-a.s. For all* $t_0 \leqq t(\omega)$ *and all* $u_0 \in H$ *with* $|u_0| \leqq \rho$, *let* $v(t, \omega; t_0, u_0 - z(t_0, \omega))$ *be the solution of Eq. (41) over* $[t_0, \infty)$, *with* $v(t_0) = u_0 - z(t_0, \omega)$, *and let* $u(t, \omega; t_0, u_0) = z(t, \omega) + v(t, \omega; t_0, u_0 - z(t_0, \omega))$. *Then we have*

$$\| u(0, \omega; t_0, u_0) \|^2 \leqq r_2^2(\omega).$$

Proof. Put

$$c_3(\omega) = c_1(\omega) + \sup_{-1 \leqq t \leqq 0} |z(t)|^2,$$

$$c_4(\omega) = c_2(\omega) + \int_{-1}^{0} \| z(s) \|^2 ds,$$

$$c_5(\omega) = c_1(\omega)^{\frac{1}{2}} + \sup_{-1 \leqq t \leqq 0} |z(t)|.$$

Given $\rho > 0$, choose $t(\omega)$ as in the proof of Lemma 7.1. Then

$$\int_{-1}^{0} |u(s)|^2 \| u(s) \|^2 ds \leqq 2c_3(\omega) 2c_4(\omega), \quad \text{and} \quad \int_{-1}^{0} |u(s)| \, \| u(s) \|^2 ds \leqq c_5(\omega) 2c_4(\omega).$$

From (48) we conclude

$$\begin{aligned} \| u(0) \|^2 &\leqq 2 \| z(0) \|^2 + 2 \| v(0) \|^2 \\ &\leqq 2 \| z(0) \|^2 + \left[c_2(\omega) + 64 c_B^4 c_3(\omega) c_4(\omega) \sup_{-1 \leqq t \leqq 0} \| z(t) \|^2 \right. \\ &\quad + 8 c_B^2 c_5(\omega) c_4(\omega) \sup_{-1 \leqq t \leqq 0} |Az(t)| \\ &\quad \left. + \int_{-1}^{0} 2q(s) ds \right] 2e^{64 c_B^4 c_3(\omega) c_4(\omega)} \end{aligned}$$

This gives an expression for $r_2^2(\omega)$, and completes the proof. □

7.7 Final result

7.4 Theorem *The stochastic flow associated with the Navier–Stokes equation with additive noise (33) has a compact stochastic attractor, in the sense of Theorem 3.11.*

Moreover, the Markov semigroup induced by the flow on H has an invariant measure ρ. The associated flow-invariant Markov measure μ on $H \times \Omega$ (cf. Corollary 4.6) has the property that its disintegration $\omega \mapsto \mu_\omega$ is supported by the attractor.

The proof proceeds as that of Theorem 5.6.

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