Probab. Th. Rel. Fields 88, 47–75 (1991)

Probability Theory and Related Fields

Rates of clustering for some Gaussian self-similar processes

Victor Goodman[1,*] **and James Kuelbs**[2,**]

[1] Department of Mathematics, Indiana University, Bloomington, IN 47405, USA
[2] Department of Mathematics, University of Wisconsin, Madison, WI 53706, USA

Received December 12, 1989; in revised form June 26, 1990

Summary. The analogue of Strassen's functional law of the iterated logarithm is known for many Gaussian processes which have suitable scaling properties, and here we establish rates at which this convergence takes place. We provide a new proof of the best upper bound for the convergence to $\mathscr{K}$ by suitably normalized Brownian motion, and then continue with this method to get similar bounds for the Brownian sheet and other self-similar Gaussian processes. The previous method, which produced these results for Brownian motion in $\mathbb{R}^1$, was highly dependent on many special properties unavailable when dealing with other Gaussian processes.

1. Introduction

Let $\{W(t)\colon 0 \leqq t < \infty\}$ denote standard Brownian motion in $\mathbb{R}^1$. If $C[0, 1]$ denotes the continuous functions on $[0, 1]$, and

$$(1.1) \qquad \mathscr{K} = \left\{ \int_0^t g(s)\,ds,\ 0 \leqq t \leqq 1 \colon \int_0^1 g^2(s)\,ds \leqq 1 \right\},$$

then $\mathscr{K}$ is a compact, convex, symmetric subset of $C[0, 1]$ such that with probability one the random sequence given by

$$(1.2) \qquad \eta_n(t) = W(nt)/(2nL_2 n)^{1/2} \qquad (0 \leqq t \leqq 1),$$

converges to and clusters throughout $\mathscr{K}$ in the uniform norm. This is Strassen's functional LIL for Brownian motion, and in [1, 2, 5, 7] the rate at which this convergence and clustering takes place is examined.

* Supported in part by NSF Grant NSF-88-07121
** Supported in part by NSF Grant DMS-85-21586

Prior to [5], the best convergence rates to $\mathscr{K}$ were due to K. Grill in [7], and assert that with probability one

$$\limsup_n \inf_{f\in\mathscr{K}} \|f-\eta_n\| (L_2 n)^\alpha = \begin{cases} 0, & \alpha<2/3 \\ \infty, & \alpha>2/3. \end{cases} \tag{1.3}$$

Here $\|f\| = \sup_{0\leq t\leq 1} |f(t)|$, $L_2 x = L(Lx)$ where $Lx = \max(1, \log_e x)$, and $L_k x = L(L_{k-1} x)$ for $k \geqq 3$.

If $A \subseteq C[0, 1]$, $A^\varepsilon = \{g\in C[0, 1]: \inf_{f\in A} \|g-f\| < \varepsilon\}$, and $\varepsilon_n = (L_2 n)^{-\alpha}$, then the two statements in (1.3) are equivalent to

$$P(\eta_n \in \mathscr{K}^{\varepsilon_n} \text{ eventually}) = 1 \qquad (\alpha < 2/3), \tag{1.4}$$

and

$$P(\eta_n \notin \mathscr{K}^{\varepsilon_n} \text{ i.o.}) = 1 \qquad (\alpha > 2/3). \tag{1.5}$$

More precisely, (1.4) indicates an upper bound for the convergence of $\{\eta_n\}$ to $\mathscr{K}$, and (1.5) provides a lower bound. Here we will only consider analogues of the upper bound result as in (1.4). The paper [5] presented results related to (1.5), but at present (1.5) is the best uniform lower bound available.

Using a construction of Grill appearing in [7], the main result of [5] demonstrated that for Brownian motion in $\mathbb{R}^1$ and $\varepsilon_n = \gamma(L_3 n/L_2 n)^{2/3}$, there is a $\gamma > 0$ sufficiently large such that

$$P(\eta_n \in \mathscr{K}^{\varepsilon_n} \text{ eventually}) = 1. \tag{1.6}$$

The construction of Grill depended on the fact that Brownian motion in $\mathbb{R}^1$ has the strong Markov property, stationary independent increments, and also several other crucial properties (see (2.2) to (2.5) of [5]). Hence the method to establish (1.6) does not apply to other common Gaussian processes.

The main theoretical tool of this paper is contained in Theorem 2.1. This result makes use of Borell's inequality for Gaussian measures [3], and can be viewed as a quantification of the elegant argument employed by Talagrand in [14]. To state and prove Theorem 2.1 we need some notation which is given at the beginning of Sect. 2. The various applications to specific Gaussian processes are contained in Sects. 3 and 4.

We write $a_n \sim b_n$ if $\lim_n a_n/b_n = 1$, and $a_n \approx b_n$ if $0 < \underline{\lim}_n a_n/b_n \leqq \overline{\lim}_n a_n/b_n < \infty$.

2. The basic limit theorem

Let B denote a real separable Banach space with norm $\|\cdot\|$ and topological dual B^*. If μ is a mean zero Gaussian measure on B, then it is well known that there is a unique Hilbert space $H_\mu \subseteq B$ such that μ is determined by considering the pair (B, H_μ) as an abstract Wiener space (see [8]).

The limit set K in our results will always be the unit ball of H_μ for an appropriately chosen mean-zero Gaussian measure μ. Lemma 2.1 in [11] or [6] presents a construction of H_μ along with various properties of the relationship between H_μ and B. In particular, we will use the continuous linear operators

$$\Pi_d(x)=\sum_{k=1}^{d}\alpha_k(x)\,S\alpha_k \quad\text{and}\quad Q_d(x)=x-\Pi_d(x) \quad (d\geqq 1) \tag{2.1}$$

taking B into B. Here $\{\alpha_k\colon k\geqq 1\}$ is a sequence in B^* orthonormal in $L^2(\mu)$, $\{S\alpha_k: k\geqq 1)$ is a CONS in $H_\mu\subseteq B$ defined by the Bochner integral

$$S\alpha_k=\int_B x\,\alpha_k(x)\,d\mu(x),$$

and when restricted to H_μ, Π_d and Q_d are orthogonal projections onto their ranges. Furthermore, if X is a B-valued random vector with $\mu=\mathscr{L}(X)$ a mean zero Gaussian measure, then it is well known that $\lim_d \|Q_d(X)\|=0$ with probability one,

$$E\|Q_d(X)\|\downarrow 0 \quad \text{as } d\uparrow\infty, \tag{2.2}$$

and $\Pi_d(x)$ and $Q_d(X)$ are independent centered Gaussian random vectors.

Theorem 2.1. *Let $X, X_1, X_2, \ldots$ be a sequence of identically distributed centered Gaussian random vectors with values in B. Let Q_d $(d\geqq 1)$ be the linear operators of* (2.1), *K the unit ball of H_μ where $\mu=\mathscr{L}(X)$, and*

$$\Gamma=\sup_{x\in K}\|x\|. \tag{2.3}$$

Let $\{d_n\}$ be a sequence of integers such that

$$d_n\geqq \inf\{m\geqq 1: E\|Q_m X\|/m\leqq (\Gamma L_2 n)/(2Ln)^{1/2}\}, \tag{2.4}$$

and for $\gamma>0$ let

$$\varepsilon_n=(\gamma d_n L_2 n)/Ln. \tag{2.5}$$

Then for $\gamma>3\Gamma$

$$P(X_n/(2Ln)^{1/2}\in K^{\varepsilon_n}\ \textit{eventually})=1. \tag{2.6}$$

Remark. The proof shows $\varepsilon_n=o((Ln)^{-1/2})$ is always possible in (2.6).

Remark. The constant 3Γ is clearly not best possible, and was chosen for notational convenience.

Proof. It is clear from the definition of ε_n in (2.5) that it suffices to prove (2.6) when equality holds in (2.4). Let $U=\{x\colon \|x\|<1\}$ and set

$$p_n=P(X/(2Ln)^{1/2}\notin K+3\varepsilon_n\Gamma U) \tag{2.7}$$

where K, Γ, and ε_n are as above. Since the sequence $X, X_1, X_2, \ldots$ is identically distributed we have by the Borel-Cantelli lemma that

(2.8) $$P(X_n/(2Ln)^{1/2} \in K^{3\Gamma\varepsilon_n} \text{ eventually}) = 1$$

provided

(2.9) $$\sum_{n=1}^{\infty} p_n < \infty.$$

Thus if we can verify (2.9) for all $\gamma > 1$ in the definition of ε_n, then (2.8) holds, and this will yield (2.6) whenever $\gamma > 3\Gamma$. Hence it suffices to prove (2.9) for $\gamma > 1$.

Since $K \subseteq 2\Gamma U$ we first observe that

(2.10) $$\begin{aligned} p_n &\leqq P(X/(2Ln)^{1/2} \notin (1+\varepsilon_n)K + \varepsilon_n \Gamma U) \leqq P(\Pi_{d_n}(X) \notin b_n K) \\ &\quad + P(\Pi_{d_n}(X) \in b_n K, X/(2Ln)^{1/2} \notin (1+\varepsilon_n)K + \varepsilon_n \Gamma U) \end{aligned}$$

where $b_n = (2Ln)^{1/2}(1+\varepsilon_n)$, and then that

(2.11) $$\begin{aligned} &P(\Pi_{d_n}(X) \in b_n K, X \notin b_n K + (2Ln)^{1/2} \varepsilon_n \Gamma U) \\ &\quad \leqq P(\|\Pi_{d_n}(X)\|_{H_\mu} \leqq b_n, Q_{d_n}(X) \notin (2Ln)^{1/2} \varepsilon_n \Gamma U \\ &\qquad + (b_n^2 - \|\Pi_{d_n}(X)\|_{H_\mu}^2)^{1/2} Q_{d_n}(K)). \end{aligned}$$

To verify (2.11) assume $\Pi_{d_n}(x) \in b_n K$, write $x = \Pi_{d_n}(x) + Q_{d_n}(x)$, and assume that $Q_{d_n}(x) = (2Ln)^{1/2} \varepsilon_n \Gamma u + (b_n^2 - \|\Pi_{d_n}(x)\|_{H_\mu}^2)^{1/2} Q_{d_n}(k)$ for some $u \in U$ and $k \in K$. Then

(2.12) $$x = \tilde{k} + (2Ln)^{1/2} \varepsilon_n \Gamma u$$

where

(2.13) $$\tilde{k} = \Pi_{d_n}(x) + (b_n^2 - \|\Pi_{d_n}(x)\|_{H_\mu}^2)^{1/2} Q_{d_n}(k) \in H_\mu$$

because $\Pi_{d_n}\colon B \to H_\mu$ and $Q_{d_n}\colon H_\mu \to H_\mu$. Furthermore, the orthogonality of the projections Π_{d_n} and Q_{d_n} on H_μ with $\|\Pi_{d_n}(x)\|_{H_\mu} \leqq b_n$ and $\|Q_{d_n}(k)\|_{H_\mu} \leqq 1$ (since $k \in K$) implies

(2.14) $$\|\tilde{k}\|_{H_\mu}^2 = \|\Pi_{d_n}(x)\|_{H_\mu}^2 + (b_n^2 - \|\Pi_{d_n}(x)\|_{H_\mu}^2) \|Q_{d_n}(k)\|_{H_\mu}^2 \leqq b_n^2,$$

and thus (2.12) and (2.14) together imply

(2.15) $$x = \tilde{k} + (2Ln)^{1/2} \varepsilon_n \Gamma u \in b_n K + (2Ln)^{1/2} \varepsilon_n \Gamma U.$$

Hence $\Pi_{d_n}(x) \in b_n K$ and $x \notin b_n K + (2Ln)^{1/2} \varepsilon_n \Gamma U$ together imply $\Pi_{d_n}(x) \in b_n K$ and $Q_{d_n}(x) \notin (2Ln)^{1/2} \varepsilon_n \Gamma U + (b_n^2 - \|\Pi_{d_n}(x)\|_{H_\mu}^2)^{1/2} Q_{d_n}(K)$, so (2.11) holds.

Thus (2.10) and (2.11) imply

(2.16) $$\begin{aligned} p_n &\leqq P(\|\Pi_{d_n}(X)\|_{H_\mu} > b_n) \\ &\quad + P(\|\Pi_{d_n}(X)\|_{H_\mu} \leqq b_n, Q_{d_n}(X) \notin (2Ln)^{1/2} \varepsilon_n \Gamma U \\ &\quad + (b_n^2 - \|\Pi_{d_n}(X)\|_{H_\mu}^2)^{1/2} Q_{d_n}(K)), \end{aligned}$$

with X Gaussian making $\Pi_{d_n}(X)$ and $Q_{d_n}(X)$ independent. Now for $\gamma \geqq 1$, Chebyshev's inequality, (2.5), and (2.4) imply that

$$(2.17)\qquad \begin{aligned} P(Q_{d_n}(X) \notin (2Ln)^{1/2}\varepsilon_n \Gamma U) &= P(\|Q_{d_n}(X)\| \geqq (2Ln)^{1/2}\varepsilon_n \Gamma) \\ &\leqq E\|Q_{d_n}(X)\|/(\Gamma \varepsilon_n (2Ln)^{1/2}) \\ &= \frac{E\|Q_{d_n}(X)\|}{d_n} \cdot \frac{(Ln)^{1/2}}{2^{1/2}\gamma \Gamma (L_2 n)} \leqq 1/2. \end{aligned}$$

Hence by Borell's inequality [3] we have for any number a, $0 \leqq a \leqq b_n^2$, that

$$(2.18)\qquad \begin{aligned} &P(Q_{d_n}(X) \notin (2Ln)^{1/2}\varepsilon_n \Gamma U + (b_n^2 - a)^{1/2} Q_{d_n}(K)) \\ &\leqq 1 - \Phi((b_n^2 - a)^{1/2}), \end{aligned}$$

and by the independence of $\Pi_{d_n}(X)$ and $Q_{d_n}(X)$ we thus have by (2.11) that

$$(2.19)\qquad \begin{aligned} &P(\Pi_{d_n}(X) \in b_n K, X \notin b_n K + (2Ln)^{1/2}\varepsilon_n \Gamma U) \\ &\leqq E(I(\|\Pi_{d_n}(X)\|_{H_\mu} \leqq b_n)(1 - \Phi((b_n^2 - \|\Pi_{d_n}(X)\|_{H_\mu}^2)^{1/2})). \end{aligned}$$

In (2.18) and (2.19) Φ denotes the distribution function of a $N(0, 1)$ random variable, and by combining (2.16) and (2.19) we thus have

$$(2.20)\qquad p_n \leqq P(\|\Pi_{d_n+1}(X)\|_{H_\mu} > b_n)$$

provided $\gamma \geqq 1$.

Now $P(\|\Pi_{d_n+1}(X)\|_{H_\mu} > b_n) = P(g_1^2 + \dots + g_{d_n+1}^2 > b_n^2)$ where $g_1, g_2, \dots$ are i.i.d. $N(0, 1)$. Furthermore, (2.4) and $E\|Q_m(X)\| \downarrow 0$ as $m \nearrow \infty$ (see (2.2)) implies that

$$d_n = o((Ln)^{1/2}/L_2 n),$$

and hence $\varepsilon_n = o((Ln)^{-1/2})$ with $b_n = (2Ln)^{1/2}(1+\varepsilon_n)$ implies $d_n = o(b_n)$. Now standard calculus estimates imply that if $g_1, g_2, \dots$ are i.i.d. $N(0, 1)$ and $d = o(x^2)$ as $x \to \infty$, then as $x \to \infty$, $d \to \infty$

$$(2.21)\qquad P(g_1^2 + \dots + g_d^2 > x^2) \sim (2\pi d)^{-1/2} \exp\{-x^2/2 + (d/2 - 1)\log(x^2 e/(d-2)\}.$$

Thus applying (2.21) to (2.20) with $d_n = o(b_n)$, we get for $\gamma \geqq 1$ and n sufficiently large that

$$(2.22)\qquad \begin{aligned} p_n &\leqq (d_n)^{-1/2} \exp\{-b_n^2/2 + d_n L_2 n\} \\ &\leqq (d_n)^{-1/2} \exp\{-Ln - 2Ln\varepsilon_n + d_n L_2 n\} \\ &= (d_n)^{-1/2} \exp\{-Ln - 2\gamma d_n L_2 n + d_n L_2 n\} \\ &\leqq n^{-1}(d_n)^{-1/2} \exp\{-\gamma d_n L_2 n\} \\ &\leqq n^{-1} \exp\{-\gamma L_2 n\}. \end{aligned}$$

Thus for $\gamma > 1$ we get $\sum_{n=1}^{\infty} p_n < \infty$ and the theorem is proved.

3. Some upper bounds for rates in the LIL for Brownian motion and the Brownian sheet

To apply Theorem 2.1 we need only choose $\{d_n\}$ such that (2.4) holds. Hence we need some knowledge of the sequences $\{\alpha_k: k \geqq 1\}$ and $\{S\alpha_k: k \geqq 1\}$ used for the operators Π_d of (2.1).

Part I: Brownian motion in $\mathbb{R}^p$

Let $W = \{W(t): 0 \leqq t \leqq 1\}$ be standard Brownian motion in $\mathbb{R}^p$ having continuous sample paths on [0, 1]. The linear mapping of a vector in $\mathbb{R}^p$ to its j^{th} coordinate (with respect to the canonical basis) is denoted by λ_j for $j = 1, \ldots, p$. Since W has continuous paths, we set $B = C_p[0, 1]$, the space of $\mathbb{R}^p$-valued continuous functions on [0, 1], and consider two norms on B, namely

$$\|f\| = \sup_{0 \leqq t \leqq 1} \left(\sum_{j=1}^{p} |\lambda_j \circ f(t)|^2 \right)^{1/2} \tag{3.1}$$

and

$$\|f\|_2 = \left(\sum_{j=1}^{p} \int_0^1 |\lambda_j \circ f(t)|^2 \, dt \right)^{1/2} \tag{3.2}$$

Of course, B is not complete in $\|\cdot\|_2$, but this does not cause difficulties.

The limit set in Strassen's LIL for $\mathbb{R}^p$-valued Brownian motion is

$$\mathcal{K}_p = \left\{ \int_0^t g(s)\, ds, 0 \leqq t \leqq 1 : g: [0,1] \to \mathbb{R}^p, \|g\|_2 \leqq 1 \right\}, \tag{3.3}$$

and we consider the open unit balls

$$U = \{f \in C_p[0,1]: \|f\| < 1\} \tag{3.4}$$

$$V = \{f \in C_p[0,1]: \|f\|_2 < 1\}. \tag{3.5}$$

Then, when using the sup-norm $\|\cdot\|$,

$$\mathcal{K}_p^\varepsilon = \mathcal{K}_p + \varepsilon U,$$

and for the inner product norm $\|\cdot\|_2$,

$$\mathcal{K}_p^\varepsilon = \mathcal{K}_p + \varepsilon V.$$

The upper bounds for the rate of convergence in Strassen's LIL for $\mathbb{R}^p$-valued Brownian motion are given in our next theorem. When $p = 1$ and the sup-norm is used, this result was obtained in [5] by entirely different methods.

Theorem 3.1. *Let* $\{W(t): 0 \leqq t < \infty\}$ *be standard* $\mathbb{R}^p$*-valued Brownian motion with continuous sample paths, and define*

$$\eta_n(t) = W(nt)/(2nL_2 n)^{1/2} \qquad (0 \leqq t \leqq 1).$$

Then, the following hold:

(*A*) *If U is as in* (3.4) *and* $\varepsilon_n = \gamma(L_3 n/L_2 n)^{2/3}$, *then for* $\gamma > 0$ *sufficiently large*

$$P(\eta_n \in \mathscr{K}_p + \varepsilon_n U \text{ eventually}) = 1. \tag{3.6}$$

(*B*) *If V is as in* (3.5) *and* $\varepsilon_n = \gamma(L_3 n)^{1/3}/(L_2 n)^{2/3}$, *then for* $\gamma > 0$ *sufficiently large*

$$P(\eta_n \in \mathscr{K}_p + \varepsilon_n V \text{ eventually}) = 1. \tag{3.7}$$

The first step of the proof is to establish the following proposition.

Proposition 3.1. *If* $W, W_1, W_2, \ldots$ *are standard* $\mathbb{R}^p$*-valued Brownian motions with sample paths in* $C_p[0, 1]$, *then the following hold:*

(*A*) *If* $\varepsilon_n = \gamma(L_2 n/Ln)^{2/3}$, *then for* $\gamma > 0$ *sufficiently large*

$$P(W_n/(2Ln)^{1/2} \in \mathscr{K}_p + \varepsilon_n U \text{ eventually}) = 1. \tag{3.8}$$

(*B*) *If* $\varepsilon_n = \gamma(L_2 n)^{1/3}/(Ln)^{2/3}$, *then for* $\gamma > 0$ *sufficiently large*

$$P(W_n/(2Ln)^{1/2} \in \mathscr{K}_p + \varepsilon_n V \text{ eventually}) = 1. \tag{3.9}$$

Proof of Proposition 3.1-A. The process $\{W(t): 0 \leqq t \leqq 1\}$ can be expressed as a uniformly convergent random series on [0, 1] as follows:

$$W(t) = (b_1(t), \ldots, b_p(t)) \tag{3.10}$$

where, for $j = 1, \ldots, p$,

$$b_j(t) = \sum_{n=0}^{\infty} \sum_{k \in \Gamma(n)} g_{j,kn} \phi_{kn}(t). \tag{3.11}$$

In (3.11) $\{g_{j,kn}: j = 1, \ldots, p, k \in \Gamma(n), n \geqq 0\}$ consists of independent $N(0, 1)$ random variables, and the following hold:

$$\begin{aligned} &\Gamma(0) = \{0\}, \\ &\Gamma(n) = \{1, 2, \ldots, 2^{n-1}\} \qquad (n \geqq 1), \\ &\phi_{kn}(t) = \int_0^t f_{kn}(s)\, ds \qquad (k \in \Gamma(n), n \geqq 0), \end{aligned} \tag{3.12}$$

where

$$f_{00}(s) = 1 \qquad (0 \leqq s \leqq 1),$$

and for $k \in \Gamma(n)$ and $n \geqq 1$

$$f_{kn}(s) = \begin{cases} 2^{(n-1)/2} & (2k-2)\,2^{-n} \leqq t < (2k-1)\,2^{-n} \\ -2^{(n-1)/2} & (2k-1)\,2^{-n} \leqq t < (2k)\,2^{-n} \\ 0 & \text{otherwise.} \end{cases}$$

That the series arising from (3.10) and (3.11) converges uniformly to $\mathbb{R}^p$-valued Brownian motion on [0, 1] is obvious once one knows that (3.11) converges uniformly to standard Brownian motion in $\mathbb{R}^1$ for each $j = 1, \ldots, p$. This, however, is a well known fact, and [1] and [13] contain a nice discussion of this as well as material applying to the Brownian sheet. Furthermore, once the process is defined, standard properties of stochastic integrals allow us to represent the independent $N(0, 1)$ random variables $\{g_{j,kn}\}$ as

$$g_{j,kn} = \int_0^1 f_{kn}(s)\, db_j(s). \tag{3.13}$$

Henceforth we make this identification and define, for $h \in C[0, 1]$, the linear functionals

$$\alpha_{kn}(h) = \int_0^1 f_{kn}(s)\, dh(s) \qquad (k \in \Gamma(n), n \geqq 0). \tag{3.14}$$

Then, for $j = 1, \ldots, p$, $k \in \Gamma(n)$, and $n \geqq 0$,

$$\alpha_{kn} \circ \lambda_j(W) = g_{j,kn}. \tag{3.15}$$

Since $Sf = E(Wf(W))$ for each $f \in B^*$ (see Lemma 2.1 of [11] for details), standard properties of stochastic integrals easily imply that

$$S(\alpha_{kn} \circ \lambda_j)(t) = \phi_{kn}(t) \qquad (j = 1, \ldots, p, k \in \Gamma(n), n \geqq 0). \tag{3.16}$$

The reader should also notice that the so called Schauder functions $\phi_{kn}(t)$ are piecewise linear, continuous, non-negative, and have non-overlapping support for different $k \in \Gamma(n)$. Furthermore,

$$\sup_{0 \leqq t \leqq 1} |\phi_{kn}(t)| = \begin{cases} 1 & k = 0, n = 0 \\ 2^{-(n+1)/2} & k \in \Gamma(n), n \geqq 1. \end{cases} \tag{3.17}$$

Thus we can write (3.11) in the form

$$b_j(\cdot) = \sum_{n=0}^{\infty} \sum_{k \in \Gamma(n)} \alpha_{kn} \circ \lambda_j(W)\, S(\alpha_{kn} \circ \lambda_j(\cdot)) \tag{3.18}$$

for $j = 1, \ldots, p$, and letting e_j $(1 \leqq j \leqq p)$ denote the standard basis for $\mathbb{R}^p$ we have

$$W(t) = \sum_{j=1}^{p} \left(\sum_{n=0}^{\infty} \sum_{k \in \Gamma(n)} \alpha_{kn} \circ \lambda_j(W)\, S(\alpha_{kn} \circ \lambda_j)(t) \right) e_j. \tag{3.19}$$

To express (3.19) in terms of a single index d as in (2.1) we order the sequence $\{\alpha_{kn} \circ \lambda_j : j = 1, \ldots, p, k \in \Gamma(n), n \geqq 0\}$ lexicographically by the triples (j, n, k). That is, $(j_1, n_1, k_1) < (j_2, n_2, k_2)$ if $j_1 < j_2$, or $j_1 = j_2$ and $n_1 < n_2$, or $j_1 = j_2$, $n_1 = n_2$, and $k_1 < k_2$. Then $\Pi_d(W)$ is the sum of the first d terms of (3.19) in this ordering.

Combining (3.15), (3.16), and (3.19) we see that for $p2^m \leqq d < p2^{m+1}$, Jensen's inequality implies

$$(3.20)\qquad \begin{aligned} E\|W-\Pi_d(W)\| &\leqq E\|W-\Pi_{p2^m}(W)\| \\ &= E\left\|\sum_{j=1}^{p}\left(\sum_{n=m}^{\infty}\sum_{k\in\Gamma(n)}\alpha_{kn}\circ\lambda_j(W)\,S(\alpha_{kn}\circ\lambda_j)\right)e_j\right\| \\ &\leqq \sum_{j=1}^{p}\sum_{n=m}^{\infty}E\|\sum_{k\in\Gamma(n)}\alpha_{kn}\circ\lambda_j(W)\,S(\alpha_{kn}\circ\lambda_j)e_j\| \\ &\leqq \sum_{j=1}^{p}\sum_{n=m}^{\infty}2^{-(n+1)/2}E(\sup_{k\in\Gamma(n)}|\alpha_{kn}\circ\lambda_j(W)|) \end{aligned}$$

by (3.16), (3.17), and the disjoint supports for ϕ_{kn}, $k\in\Gamma(n)$.

$$\begin{aligned} &\leqq Cp\sum_{n=m}^{\infty}2^{-(n+1)/2}(\log 2^n)^{1/2} \\ &\leqq C^2p((\log 2^m)\,2^{-m})^{1/2} \end{aligned}$$

where C is an absolute constant such that

$$E(\sup_{1\leqq j\leqq n}|g_j|)\leqq C(Ln)^{1/2}$$

for all $n\geqq 1$ and $(g_j\colon j\geqq 1)$ i.i.d. $N(0,1)$, and

$$\sum_{n=m}^{\infty}2^{-(n+1)/2}(\log 2^n)^{1/2}\leqq C((\log 2^m)\,2^{-m})^{1/2}.$$

Thus from (3.20) we have for all d sufficiently large that

$$(3.21)\qquad E\|W-\Pi_d(W)\|\leqq 2C^2p^{3/2}\left(\frac{Ld}{d}\right)^{1/2}.$$

Hence for (2.4) we may choose d_n as a large constant multiple of the solution of

$$d^{-1}\left(\frac{Ld}{d}\right)^{1/2}=L_2n/(Ln)^{1/2}.$$

Thus

$$d_n\approx((Ln)/(L_2n))^{1/3},$$

and solving for ε_n in (2.5) we see it is possible to take

$$\varepsilon_n=\gamma(L_2n/Ln)^{2/3}$$

for $\gamma>0$ sufficiently large. Applying Theorem 2.1 now yields Proposition 3.1-A.

Proof of Proposition 3.1-B. Let λ_j and e_j be as in the proof of part A, and assume

$$\gamma_n(t)=\frac{2\sqrt{2}}{\pi(2n+1)}\,\mathrm{Sin}((2n+1)\pi t/2) \qquad (0\leqq t\leqq 1, n\geqq 0). \tag{3.22}$$

For h in $C[0,1]$ we define the linear functionals

$$\alpha_n(h)=\int_0^1 \gamma_n'(s)\,dh(s) \qquad (n\geqq 0).$$

Then elementary properties of stochastic integrals imply $\{\alpha_n\circ\lambda_j(W)\colon n\geqq 0, j=1,\ldots,p\}$ are i.i.d. $N(0,1)$ and

$$S(\alpha_n\circ\lambda_j)(t)=\gamma_n(t) \qquad (0\leqq t\leqq 1, n\geqq 0).$$

As a result of [10, p. 126] the series

$$\sum_{j=1}^{p}\left(\sum_{n=0}^{\infty}\alpha_n\circ\lambda_j(W)\,S(\alpha_n\circ\lambda_j)(\cdot)\right)e_j$$

represents $\mathbb{R}^p$-valued Brownian motion and converges uniformly on $[0,1]$. Hence we can write

$$W(t)=\sum_{j=1}^{p}\left(\sum_{n=0}^{\infty}\alpha_n\circ\lambda_j(W)\,S(\alpha_n\circ\lambda_j)(t)\right)e_j \qquad (0\leqq t\leqq 1), \tag{3.23}$$

and to express the series (3.23) in terms of a single index d as in (2.1), we order the pairs $\{(j,n)\colon j=1,\ldots,p,\ n\geqq 0\}$ lexicographically. That is, $(j_1,n_1)<(j_2,n_2)$ if $j_1<j_2$ or $j_2=j_2$ and $n_1<n_2$. Then $\Pi_d(W)$ is the sum of the first d terms of (3.23) with respect to this ordering.

Thus for

$$pm\leqq d<p(m+1),$$

Jensen's inequality and the orthogonality of the sequence $\{S\alpha_n\circ\lambda_j\colon n\geqq 0\}$ with respect to Lebesgue measure implies that for all $m\geqq 2$

$$\begin{aligned} E\|W-\Pi_d W\|_2^2 &\leqq E\|W-\Pi_{pm}(W)\|_2^2 \\ &=E\left\|\sum_{j=1}^{p}\left(\sum_{n=m}^{\infty}\alpha_n\circ\lambda_j(W)\,\gamma_n(\cdot)\right)e_j\right\|_2^2 \\ &=E\left(\sum_{j=1}^{p}\int_0^1\sum_{n=m}^{\infty}(\alpha_n\circ\lambda_j(W)\,\gamma_n(s))^2\,ds\right) \\ &=p\sum_{n=m}^{\infty}\int_0^1\gamma_n^2(s)\,ds \\ &=p\sum_{n=m}^{\infty}4/(\pi^2(2n+1)^2) \\ &\leqq (4p^2/\pi^2)/d. \end{aligned} \tag{3.24}$$

Hence for (2.4) we may choose d_n as a large constant multiple of the solution of

$$d^{-1} d^{-1/2} = L_2 n/(Ln)^{1/2}.$$

Thus

$$d_n \approx (Ln)^{1/3}/(L_2 n)^{2/3},$$

and solving for ε_n in (2.5) we see it is possible to take

$$\varepsilon_n = \gamma (L_2 n)^{1/3}/(Ln)^{2/3}$$

for $\gamma > 0$ sufficiently large. Applying Theorem 2.1 now yields Proposition 3.1-B.

Completion of the proof of Theorem 3.1. Let $n_r = \exp\{r/(Lr)^2\}$ for $r \geqq 1$ and set

$$W_r = W(n_r(\cdot))/n_r^{1/2} \qquad (r \geqq 1). \tag{3.25}$$

Then Proposition 3.1-A implies

$$P(W_r/(2Lr)^{1/2} \in \mathscr{K}_p + \beta(L_2 r/Lr)^{2/3}\, U \text{ eventually}) = 1 \tag{3.26}$$

for $\beta > 0$ sufficiently large. Furthermore, $\sup_{f \in \mathscr{K}_p} \|f\| = 1$ and

$$|1 - (Lr)^{1/2}(L_2 n_r)^{-1/2}| \sim L_2 r/Lr, \tag{3.27}$$

so (3.25), (3.26), and (3.27) together imply

$$P(\eta_{n_r} \in \mathscr{K}_p + 2\beta(L_2 r/Lr)^{2/3}\, U \text{ eventually}) = 1. \tag{3.28}$$

The verification of (3.6) is completed with the following lemma by taking $\gamma \geqq 3\beta$.

Lemma 3.1. *Let* $n_r = \exp\{r/(Lr)^2\}$, $\delta_r = 2\beta(L_2 r/Lr)^{2/3}$ $(\beta > 0)$, *and* $d(n) = (2nL_2 n)^{1/2}$. *If* (3.28) *holds, then*

$$P(\eta_n \in \mathscr{K}_p + 3\beta(L_3 n/L_2 n)^{2/3}\, U \text{ eventually}) = 1.$$

The proof of Lemma 3.1 is exactly that of Lemma 3.1 in [5]. We indicate a brief outline for the sake of completeness. Further details can be found in [5]. The reader should note that the proof involves a simple rescaling, and henceforth we refer to Lemma 3.1, and related results, as rescaling lemmas.

Proof of Lemma 3.1. Let $I(r) = [n_r, n_{r+1}]$. If (3.28) holds, then there exists $h_r \in \mathscr{K}_p$ such that

$$\|\eta_{n_{r+1}} - h_r\| < \delta_{r+1}. \tag{3.29}$$

For $n \in I(r)$, set

$$g(t) = h_r(nt/n_{r+1}) \qquad (0 \leqq t \leqq 1). \tag{3.30}$$

Then $g \in \mathscr{K}_p$, and g depends on r, but we supress that. Furthermore, for $n \in I(r)$

$$\begin{aligned} \|\eta_n - g\| = \|g - W(n(\cdot)/d(n_{r+1})\| \\ + \|(W(n(\cdot))/d(n))(1 - d(n)/d(n_{r+1}))\|. \end{aligned} \tag{3.31}$$

Now by rescaling and (3.29)

$$\|g - W(n(\cdot))/d(n_{r+1})\| \leqq \delta_{r+1}. \tag{3.32}$$

Furthermore, the classical LIL for i.i.d. $N(0, 1)$ random variables and rescaling implies that for all r sufficiently large

$$\|\eta_n(1 - d(n)/d(n_{r+1}))\| \leqq 2|1 - d(n)/d(n_{r+1})|. \tag{3.33}$$

If $n \in I(r)$, then for large r

$$|1 - d(n)/d(n_{r+1})| \leqq 2/(Lr)^2, \tag{3.34}$$

and by combining (3.31), (3.32), (3.33), and (3.34) we obtain for large r and all $n \in I(r)$ that

$$\|\eta_n - g\| \leqq \delta_{r+1} + 4/(Lr)^2 \leqq \varepsilon_n.$$

Since $g \in \mathscr{K}_p$ this completes the proof of Lemma 3.1.

Thus Theorem 3.1-A is proved and the proof of part B is exactly the same if $\|\cdot\|$ is replaced by $\|\cdot\|_2$. Thus Theorem 3.1 is established.

Part II: The Brownian sheet

Let $B = C([0, 1]^p)$ denote the Banach space of real continuous functions on the p-dimensional unit cube $Q = [0, 1]^p$, endowed with the sup-norm

$$\|f\| = \sup_{\mathbf{t} \in Q} |f(\mathbf{t})|. \tag{3.35}$$

Then the p-parameter Brownian sheet $\{W(\mathbf{t}): \mathbf{t} \in Q\}$ is a mean zero Gaussian process with sample paths in B determined by the covariance function

$$E(W(\mathbf{s})\, W(\mathbf{t})) = \min(s_1, t_1) \cdots \min(s_p, t_p) \tag{3.36}$$

for $\mathbf{s} = (s_1, \ldots, s_p)$ and $\mathbf{t} = (t_1, \ldots, t_p)$. Furthermore, since the paths are continuous and the variance function is zero at any point $\mathbf{t} = (t_1, \ldots, t_p)$ where some $t_j = 0$, if follows that $W(\mathbf{t}) = 0$ for such $\mathbf{t}$. The papers [1] and [13] contain further details regarding the Brownian sheet, and are excellent references for our purposes.

Our upper bounds for rates of convergence in the functional LIL for the Brownian sheet are contained in the next two theorems.

Theorem 3.2. *Let $W, W_1, W_2, \ldots$ be Brownian sheets on $Q = [0, 1]^p$ with continuous sample paths, and let*

$$\mathscr{K} = \{f(\mathbf{t}) = \int_{[0,\mathbf{t}]} g(\mathbf{s})\, d\mathbf{s}, \mathbf{t} \in Q: \int_Q g^2(\mathbf{s})\, d\mathbf{s} \leqq 1\}$$

where $[0, \mathbf{t}] = [0, t_1] \times \ldots \times [0, t_p]$ for $\mathbf{t} = (t_1, \ldots, t_p)$. If $U = \{f \in C(Q): \|f\| < 1\}$ and $\varepsilon_n = \gamma(L_2 n)^{p-1/3}/(Ln)^{2/3}$, then the following hold:

(*A*) *For* $\gamma > 0$ *sufficiently large*

(3.37) $$P(W_n/(2Ln)^{1/2} \in \mathscr{K} + \varepsilon_n U \text{ eventually}) = 1.$$

(*B*) *If* $\{W(\mathbf{t}): \mathbf{t} \geqq 0\}$ *is a continuous Brownian sheet on* $[0, \infty)^p$,

(3.38) $$\eta_n(\mathbf{t}) = W(n\mathbf{t})/(2n^p L_2 n)^{1/2} \qquad \mathbf{t} \in [0, 1]^p,$$

and $\varepsilon_n = \gamma (L_3 n)^{p - 1/3}/(L_2 n)^{2/3}$ *then for* $\gamma > 0$ *sufficiently large*

(3.39) $$P(\eta_n \in \mathscr{K} + \varepsilon_n U \text{ eventually}) = 1.$$

If we use the unit ball in $L^2(Q)$ rather than the sup-norm ball U, the following improvement of (3.39) is available when $p = 2$. The cases $p \geqq 3$ can also be studied, but we do not do that here.

Theorem 3.3. *Let* $\{W(\mathbf{t}): \mathbf{t} \geqq 0\}$ *be a continuous Brownian sheet on* $[0, \infty)^2$,

(3.40) $$\eta_n(\mathbf{t}) = W(n\mathbf{t})/(2n^2 L_2 n)^{1/2} \qquad (\mathbf{t} \in [0, \infty)^2,$$

and let

(3.41) $$\varepsilon_n = \gamma L_3 n/(L_2 n)^{2/3}.$$

Then, for $\gamma > 0$ *sufficiently large,*

(3.42) $$P(\eta_n \in \mathscr{K} + \varepsilon_n V \text{ eventually}) = 1$$

where

(3.43) $$V = \{f \in C([0, 1]^2): \|f\|_2 = (\int_{[0,1]^2} f^2(\mathbf{t})\, d\mathbf{t})^{1/2} < 1\},$$

and $\mathscr{K}$ *is as in Theorem* 3.2 *with* $p = 2$.

Proof of Theorem 3.2-A. Let $\{f_{kn}: k \in \Gamma(n), n \geqq 0\}$ and $\{\phi_{kn}: k \in \Gamma(n), n \geqq 0\}$ denote the Haar and Schauder functions, respectively, defined in (3.12). The set D denotes all p-tuples $\mathbf{n} = (n_1, \ldots, n_p)$ where $n_1, \ldots, n_p$ are non-negative integers, and for $\mathbf{n} \in D$ we define

$$|\mathbf{n}| = n_1 + \ldots + n_p.$$

For each $\mathbf{n} \in D$, define

$$\Gamma(\mathbf{n}) = \{(k_1, \ldots, k_p): k_1 \in \Gamma(n_1), \ldots, k_p \in \Gamma(n_p)\}$$

where $\Gamma(\cdot)$ is as in (3.12) for a single integer. If $\mathbf{k} = (k_1, \ldots, k_p) \in \Gamma(\mathbf{n})$ and $\mathbf{t} \in Q$, define, using the notation of (3.12)

(3.44) $$f_{\mathbf{kn}}(\mathbf{t}) = \prod_{j=1}^{p} f_{k_j n_j}(t_j)$$

and

$$\Phi_{\mathbf{kn}}(\mathbf{t})=\prod_{j=1}^{p}\phi_{k_j n_j}(t_j). \tag{3.45}$$

Then the support of $\Phi_{\mathbf{kn}}$ is the same as that of $f_{\mathbf{kn}}$, and for fixed $\mathbf{n}\in D$ these supports are disjoint as $\mathbf{k}$ varies over $\Gamma(\mathbf{n})$. Furthermore, for $\mathbf{k}\in\Gamma(\mathbf{n})$

$$\|f_{\mathbf{kn}}\|=\exp\left\{\log 2\cdot\sum_{j=1}^{p}(n_j-1)\,I(n_j>0)/2\right\}\leqq 2^{|\mathbf{n}|/2} \tag{3.46}$$

and

$$\|\Phi_{\mathbf{kn}}\|\leqq\prod_{j=1}^{p}\{2^{-(n_j+1)/2}I(n_j>0)+I(n_j=0)\}\leqq 2^{-|\mathbf{n}|/2}. \tag{3.47}$$

Letting $\{g_{\mathbf{kn}}\colon \mathbf{k}\in\Gamma(\mathbf{n}),\,\mathbf{n}\in D\}$ denote an i.i.d. sequence of $N(0,1)$ random variables, the series

$$\sum_{r=0}^{\infty}\sum_{|\mathbf{n}|=r}\sum_{\mathbf{k}\in\Gamma(\mathbf{n})}g_{\mathbf{kn}}\,\Phi_{\mathbf{kn}}(\mathbf{t}) \tag{3.48}$$

converges uniformly on Q, and is a Brownian sheet (see [1] or [13] for details).

For $\mathbf{k}\in\Gamma(\mathbf{n})$ and $\mathbf{n}\in D$ we define a continuous linear functional on B by

$$\alpha_{\mathbf{kn}}(h)=\int_Q f_{\mathbf{kn}}(\mathbf{s})\,dh(\mathbf{s})\qquad(h\in B). \tag{3.49}$$

Since $f_{\mathbf{kn}}$ is a simple function of bounded variation on Q, the stochastic integral in (3.49) exists, and elementary properties of these stochastic integrals as given in [1] or [13] imply that $\{\alpha_{\mathbf{kn}}(W)\colon \mathbf{k}\in\Gamma(\mathbf{n}),\,\mathbf{n}\in D\}$ is an i.i.d. sequence of $N(0,1)$ random variables with

$$\begin{aligned}S\alpha_{\mathbf{kn}}(\mathbf{t})&=E(W\alpha_{\mathbf{kn}}(W))(\mathbf{t})\\&=E(W(\mathbf{t})\int_Q f_{\mathbf{kn}}(\mathbf{s})\,dW(\mathbf{s}))\\&=E(\int_Q I(\mathbf{s}\leqq\mathbf{t})\,dW(\mathbf{s})\int_Q f_{\mathbf{kn}}(\mathbf{s})\,dW(\mathbf{s}))\\&=\int_Q I(\mathbf{s}\leqq\mathbf{t})f_{\mathbf{kn}}(\mathbf{s})\,d\mathbf{s}\\&=\Phi_{\mathbf{kn}}(\mathbf{t})\end{aligned} \tag{3.50}$$

where $\mathbf{s}\leqq\mathbf{t}$ if $s_i\leqq t_i$ for $i=1,\dots,p$.

Thus we can represent the Brownian sheet $\{W(\mathbf{t})\colon \mathbf{t}\in Q\}$ as

$$W(\mathbf{t})=\sum_{r=0}^{\infty}\sum_{|\mathbf{n}|=r}\sum_{\mathbf{k}\in\Gamma(\mathbf{n})}\alpha_{\mathbf{kn}}(W)\,S\alpha_{\mathbf{kn}}(\mathbf{t}), \tag{3.51}$$

and to express (3.51) in terms of a single index d as in (2.1) we order the $(2p+1)$-tuples $(r, \mathbf{n}, \mathbf{k})$ with $|\mathbf{n}|=r$ and $\mathbf{k}\in\Gamma(\mathbf{n})$ lexicographically. Hence $\Pi_d(W)$ is the sum of the first d terms of (3.51) in this ordering.

By considering the number of ways to place r objects into p cells it is easy to see that for $r\geqq 0$

$$\text{(3.52)} \qquad \operatorname{Card}\{\mathbf{n}\in D: |\mathbf{n}|=r\}=\binom{p+r-1}{p-1}.$$

Furthermore,

$$\operatorname{Card}\Gamma(\mathbf{n})=\exp\left\{\log 2\sum_{j=1}^{p}(n_j-1)\,I(n_j>0)\right\},$$

and hence with $|\mathbf{n}|=r$,

$$\text{(3.53)} \qquad \max(1, 2^{r-p})\leqq \operatorname{Card}\Gamma(\mathbf{n})\leqq 2^r.$$

Thus, if

$$\text{(3.54)} \qquad \sum_{r=0}^{m}\sum_{|\mathbf{n}|=r}\sum_{\mathbf{k}\in\Gamma(\mathbf{n})}1\leqq d<\sum_{r=0}^{m+1}\sum_{|\mathbf{n}|=r}\sum_{\mathbf{k}\in\Gamma(\mathbf{n})}1,$$

Jensen's inequality implies

$$\text{(3.55)} \qquad E\,\|W-\Pi_d(W)\|\leqq E\left\|\sum_{r=m}^{\infty}\sum_{|\mathbf{n}|=r}\sum_{\mathbf{k}\in\Gamma(\mathbf{n})}\alpha_{\mathbf{kn}}(W)\,S\alpha_{\mathbf{kn}}\right\|.$$

Hence for some positive finite constant C, possibly differing from line to line, we obtain

$$\text{(3.56)} \quad E\,\|W-\Pi_d(W)\|\leqq\sum_{r=m}^{\infty}\binom{p+r-1}{p-1}\sup_{|\mathbf{n}|=r}E\,\Big\|\sum_{\mathbf{k}\in\Gamma(\mathbf{n})}\alpha_{\mathbf{kn}}(W)\,S\alpha_{\mathbf{kn}}\Big\|$$

$$\leqq\sum_{r=m}^{\infty}(p+r)^{p-1}2^{-r/2}\sup_{|\mathbf{n}|=r}E\left(\sup_{\substack{\mathbf{k}\in\Gamma(\mathbf{n})\\|\mathbf{n}|=r}}|\alpha_{\mathbf{kn}}(W)|\right)$$

since $S\alpha_{\mathbf{kn}}=\Phi_{\mathbf{kn}}$ have disjoint support when $\mathbf{k}\in\Gamma(\mathbf{n})$, and by applying (3.47) with $|\mathbf{n}|=r$.

$$\leqq C\sum_{r=m}^{\infty}r^{p-1}2^{-r/2}(\log 2^r)^{1/2}$$

since $|\mathbf{n}|=r$ implies $\log(\operatorname{card}\Gamma(\mathbf{n}))\sim r\log 2$ by (3.53).

$$\leqq C\sum_{r=m}^{\infty}r^{p-1/2}2^{-r/2}$$

$$\leqq C2^{-m/2}m^{p-1/2}.$$

Combining (3.52), (3.53) and (3.54) we see that d, as a function of m satisfies

$$d \approx m^{p-1} 2^m \quad \text{as } m \to \infty. \tag{3.57}$$

Hence,

$$\log d \approx m \quad \text{and} \quad d(\log d)^{-(p-1)} \approx 2^m,$$

and (3.56) implies there is a finite constant C such that

$$d^{-1} E\,\|Q_d(W)\| \leqq C d^{-3/2} (\log d)^{(3p-2)/2}. \tag{3.58}$$

Hence for (2.4) we may choose d_n as a large constant multiple of the solution of

$$d^{-3/2}(\log d)^{(3p-2)/2} = L_2 n/(Ln)^{1/2}.$$

Thus

$$d_n \approx (Ln)^{1/3}(L_2 n)^{p-4/3},$$

and

$$\varepsilon_n \approx (L_2 n)^{p-1/3}/(Ln)^{2/3}. \tag{3.59}$$

Thus by applying Theorem 2.1 we have established Theorem 3.2-A.

Proof of Theorem 3.2-B. Let $n_r = \exp\{r/(Lr)^2\}$ for $r \geqq 1$, and set

$$W_r = W(n_r(\cdot))/(n_r)^{p/2} \qquad (r \geqq 1). \tag{3.60}$$

Then (3.37) implies

$$P(W_r/(2Lr)^{1/2} \in \mathscr{K} + \beta (L_2 r)^{p-1/3}/(Lr)^{2/3}\, U \text{ eventually}) = 1 \tag{3.61}$$

for $\beta > 0$ sufficiently large. Since $\sup_{f \in \mathscr{K}} \|f\| = 1$ and (3.27) holds, (3.60) and (3.61) together imply

$$P(\eta_{n_r} \in \mathscr{K} + 2\beta (L_2 r)^{p-1/3}/(Lr)^{2/3}\, U \text{ eventually}) = 1. \tag{3.62}$$

The proof of part B is now completed by proving a rescaling result as in Lemma 3.1. The details are straightforward and hence omitted.

Proof of Theorem 3.3. To prove (3.42) one first proves the following analogue of (3.37) for identically distributed copies of $\{W(\mathbf{t})\colon \mathbf{t} \in [0,1]^2\}$:

$$P(W_n/(2Ln)^{1/2} \in \mathscr{K} + \varepsilon_n V) \text{ eventually}) = 1 \tag{3.63}$$

where V is as in (3.43) and ε_n is as in (3.41). Once this is done, one proceeds exactly as before via a rescaling lemma. We do not include the details of the rescaling, but we prove (3.63).

To establish (3.63) we apply Theorem 2.1 as before. To do this we let

$$\gamma_n(t) = \frac{2\sqrt{2}}{\pi(2n+1)} \operatorname{Sin}((2n+1)\pi t/2) \qquad (0 \leqq t \leqq 1,\, n \geqq 0), \tag{3.64}$$

and define the linear functionals

$$\alpha_{ij}(h) = \int_Q \gamma_1'(s)\,\gamma_j'(t)\,dh(s,t) \tag{3.65}$$

for $h \in C(Q)$ and $Q = [0, 1]^2$. Then elementary properties of 2-parameter stochastic integrals, as developed in [1] or [13], imply

$$\alpha_{ij}(W): i \geqq 0, j \geqq 0\} \tag{3.66}$$

is an i.i.d. $N(0, 1)$ sequence, and

$$S\alpha_{ij}(s, t) = \gamma_i(s)\,\gamma_j(t). \tag{3.67}$$

Hence $\{W(s, t): (s, t) \in Q\}$ can be written as

$$W(s, t) = \sum_{n=1}^{\infty} \sum_{(i,j) \in M_n} \alpha_{ij}(W)\, S\alpha_{ij}(s, t) \tag{3.68}$$

where

$$m_1 = L_1$$
$$M_n = L_n \cap L_{n-1}^c \qquad (n \geqq 2),$$

and for $n \geqq 1$

$$L_n = \{(i, j): i, j \geqq 0 \text{ are integers satisfying (a), (b), or (c)}\}: \tag{3.69}$$

(a) $i \geqq 1, j \geqq 1$, and $ij \leqq n$.
(b) $i = 0$ and $j = 0, 1, \ldots, n$.
(c) $j = 0$ and $i = 1, \ldots, n$.

In summing the series in (3.68), we order the triples $(n, (i, j))$, where $(i, j) \in L_n$, as follows:

$$(n_1, (i_1, j_1)) < (n_2, (i_2, j_2)) \tag{3.70}$$

if $n_1 < n_2$, or if $n_1 = n_2$ and $i_1 < i_2$, or $n_1 = n_2$, $i_1 = i_2$, and $j_1 < j_2$. Then, in this ordering, the partial sums of (3.68) easily converge in $L^2(Q)$ to W, but they also converge uniformly on Q by using the Ito-Nisio theorem (see the comment prior to (2.2)).

Let $\Pi_d(W)$ denote the sum of the first d terms of the series (3.68) in this ordering. Then Π_d is as in (2.1), and for

$$\operatorname{Card}(L_m) < d \leqq \operatorname{Card}(L_{m+1}) \tag{3.71}$$

we have by Jensen's inequality that

$$E\|W-\Pi_d(W)\|_2^2 \leqq E\left\|\sum_{n=m+1}^{\infty}\sum_{(i,j)\in M_n}\alpha_{ij}(W)\,S\alpha_{ij}\right\|_2^2 \tag{3.72}$$

$$=\sum_{n=m+1}^{\infty}\sum_{(i,j)\in M_n}\frac{4}{\pi^2(2i+1)^2}\cdot\frac{4}{\pi^2(2j+1)^2}$$

$$=\frac{16}{\pi^4}\sum_{(i,j)\notin L_m}(2i+1)^{-2}(2j+1)^{-2}$$

where L_m is the truncated hyperbolic shaped region determined by (3.69). Hence (3.72) implies

$$E\|W-\Pi_d(W)\|_2^2 \leqq 16\pi^{-4}(I_1+I_2+I_3)$$

with

$$I_1=\sum_{j=0}^{1}\sum_{i=m+1}^{\infty}(2i+1)^{-2}(2j+1)^{-2}\approx m^{-1}$$

$$I_2=\sum_{j=2}^{m}\sum_{i=\left[\frac{m}{j}\right]+1}^{\infty}(2i+1)^{-2}(2j+1)^{-2}\approx\sum_{j=2}^{m}j^{-2}(j/m),$$

$$I_3=\sum_{j=m+1}^{\infty}\sum_{i=0}^{\infty}(2i+1)^{-2}(2j+1)^{-2}\approx m^{-1}.$$

Thus, for d as in (3.71), $d\approx m\log m$, and (3.72) implies there is a finite constant C such that

$$E\|W-\Pi_d(W)\|_2 \leqq C(\log m/m)^{1/2}\approx(\log d)/d^{1/2}. \tag{3.73}$$

Hence for (2.4) we may choose d_n as a large constant multiple of the solution of

$$d^{-1}(\log d)/d^{1/2}=L_2 n/(Ln)^{1/2}.$$

Thus, we have $d_n\approx(Ln)^{1/3}$, and $\varepsilon_n=\gamma L_2 n/(Ln)^{2/3}$ for $\gamma>0$ sufficiently large will yield (3.63) by applying Theorem 2.1.

As indicated previously, once (3.63) holds, rescaling will yield (3.42) of Theorem 3.3 and the theorem is proved.

4. Fractional Brownian motion and other self-similar processes

Let $\{X(t)\colon 0\leqq t<\infty\}$ be a Gaussian process of the form

$$X(t)=\int_0^t (t-s)^{H-1/2}\,dW(s)\qquad(0\leqq t<\infty) \tag{4.1}$$

where $0<H<\infty$ and $\{W(t): -\infty<t<\infty\}$ is standard Brownian motion in $\mathbb{R}^1$. These processes all have continuous versions, and are self-similar of index H, i.e. for each $a>0$ the process $\{X(at): t\geqq 0\}$ has the same distribution as the process $\{a^H X(t): t\geqq 0\}$. This was pointed out in [12], where the mean-zero, self-similar Gaussian processes with stationary increments were also introduced. They are the so-called fractional Brownian motions, i.e. the processes

$$Y(t)=Z(t)+X(t) \qquad (t\geqq 0), \tag{4.2}$$

where $\{X(t): t\geqq 0\}$ is as in (4.1),

$$Z(t)=\int_{-\infty}^{0} \{(t-s)^{H-1/2}-(-s)^{H-1/2}\}\, dW(s) \qquad (0\leqq t<\infty), \tag{4.3}$$

and $0<H<1$. When $H=1/2$, $Z=0$ and hence both $\{X(t): t\geqq 0\}$ and $\{Y(t): t\geqq 0\}$ are Brownian motion.

If $0<H<1$, the process $\{Z(t): t\geqq 0\}$ also has a continuous version on $[0, \infty)$, and we assume throughout that this is the case for both X and Z, and hence also Y.

Theorem 4.1. *Let $\{X(t): t\geqq 0\}$ and $\{Y(t): t\geqq 0\}$ be path continuous mean zero Gaussian processes as in* (4.1) *and* (4.2), *and set*

$$\mathscr{K}=\left\{f(t)=\int_0^t (t-u)^{H-1/2} g(u)\, du, 0\leqq t\leqq 1: \int_0^1 g^2(u)\, du\leqq 1\right\} \tag{4.4}$$

where $0<H<\infty$. Let $\|f\|=\sup_{0\leqq t\leqq 1} |f(t)|$ and define

$$U=\{f\in C[0,1]: \|f\|<1\}. \tag{4.5}$$

Then, the following hold:

(*A*) *If $\gamma>0$ is sufficiently large, then*

$$P(X(n(\cdot))/(2n^{2H} L_2 n)^{1/2}\in\mathscr{K}+\varepsilon_n U \textit{ eventually})=1 \tag{4.6}$$

where

$$\varepsilon_n=\begin{cases}\gamma(L_3 n/L_2 n)^{2/3} & \textit{for } H\geqq 1/2\\ \gamma(L_3 n/L_2 n)^{(2H+1)/(2H+2)} & \textit{for } 0<H<1/2.\end{cases} \tag{4.7}$$

(*B*) *Let $r(\beta, M, t)=\max(M, Mt^\beta)$ for $t\geqq 0$, $M\geqq 1$, $\beta>0$, and define the sample path continuous processes*

$$\tilde{Z}(t)=\int_{-r(\beta,M,t)}^{0} ((t-s)^{H-1/2}-(-s)^{H-1/2})\, dW(s),$$

$$\tilde{Y}(t)=X(t)+\tilde{Z}(t)$$

for $t \geqq 0$. *If* $\gamma > 0$ *is sufficiently large,* $\beta < \min(1, 2H)$, $0 < H < 1$, *and* ε_n *is as in* (4.7), *then*

$$P(\tilde{Y}(n(\cdot))/(2n^{2H} L_2 n)^{1/2} \in \mathscr{K} + \varepsilon_n U \text{ eventually}) = 1. \tag{4.8}$$

(*C*) *If* $0 < H < 1$, $\varepsilon_n = \gamma (L_2 n)^{-1/2}$, *and*

$$\mathscr{K} = \left\{ f(t) = T_H g(t) : 0 \leqq t \leqq 1, \int_{-\infty}^{1} g^2(u)\, du \leqq 1 \right\}$$

where

$$T_H g(t) = \int_0^t (t-u)^{H-1/2} g(u)\, du + \int_{-\infty}^{0} ((t-u)^{H1/2} - (-u)^{H-1/2})\, g(u)\, du,$$

then for all $\gamma > 0$

$$P(Y(n(\cdot))/(2n^{2H} L_2 n)^{1/2} \in (\mathscr{K} + \varepsilon_n U) \text{ eventually}) = 1. \tag{4.9}$$

Remark. In would be of interest to know whether the ε_n defined in (4.7) suffice in the setting of (C). We have nothing to report in this direction, and point out that although $\tilde{Z}(t)$ approximates $Z(t)$ for M large, the limit sets in (B) and (C) differ because of the long range interactions in these processes.

The proof of Theorem 4.1 will be established via several lemmas. The first shows that $\{\tilde{Z}(t) \colon t > 0\}$ is always neglegible if $\beta < \min(1, 2H)$, and hence Part B follows once Part A is proved. If $H \geqq 1/2$ and $r(\beta, M, t) = \max(M, Mt/(L_2 t)^{\beta})$, then a similar result hold if $\beta > H^{-1}$.

Lemma 4.1. *Let* $\{\tilde{Z}(t) \colon t > 0\}$ *be as in part* (*B*) *of Theorem* 4.1 *with* $0 < H < 1$, *and take* β *such that* $0 < \beta < \min(1, 2H)$. *Then,*

$$\overline{\lim_n}\, n^{\rho} \sup_{0 \leqq t \leqq 1} |\tilde{Z}(nt)|/(2n^{2H} L_2 n)^{1/2} = 0 \tag{4.10}$$

where

$$\rho = \begin{cases} H(1-\beta)/2 & \text{if } H \geqq 1/2 \\ (H - \beta/2)/2 & \text{if } H < 1/2. \end{cases}$$

Proof. First observe that

$$\begin{aligned} \tilde{Z}(nt) &= n^{H-1/2} \int_{-r(\beta, M, nt)}^{0} \left(\left(t - \frac{s}{n}\right)^{H-1/2} - \left(\frac{-s}{n}\right)^{H-1/2} \right) dW(s) \\ &= n^{H-1/2}(I_1 + I_2) \end{aligned} \tag{4.11}$$

where

$$I_1 = \int_{-M}^{0} \left(\left(t - \frac{s}{n}\right)^{H-1/2} - \left(\frac{-s}{n}\right)^{H-1/2} \right) dW(s)$$

and

$$I_2 = \int_{-M(nt)^\beta}^{-M} \left(\left(t-\frac{s}{n}\right)^{H-1/2} - \left(\frac{-s}{n}\right)^{H-1/2}\right) dW(s)\, I(nt>1).$$

Since $W(0)=0$, integration by parts implies

$$|I_1| \leqq 2|W(-M)| \left(\frac{n}{M}\right)^{1/2-H} + \int_{-M}^{0} |W(s)|\,|H-1/2|\, n^{-1} \left| \left(t-\frac{s}{n}\right)^{H-3/2} - \left(\frac{-s}{n}\right)^{H-3/2} \right| ds.$$

Hence

$$|I_1| \leqq O(1)\, n^{1/2-H} + 2n^{1/2-H} \int_{-M}^{0} \frac{|W(s)|}{h(-s)} \frac{h(-s)}{(-s)^{3/2-H}}\, ds,$$

where $h(u) = (u(L_2 u^{-1}))^{1/2}$ and throughout the lemma $O(1)$ is a possibly random quantity depending on M, but independent of n. By the LIL for Brownian motion at zero

$$\int_{-M}^{0} \frac{|W(s)|}{h(-s)} \frac{h(-s)}{(-s)^{3/2-H}}\, ds = O(1) \int_{-M}^{0} \frac{(L_2(1/-s))^{1/2}}{(-s)^{1-H}}\, ds = O(1)$$

since $0<H<1$. Combining both terms we therefore have

$$|I_1| \leqq O(1)\, n^{1/2-H}.$$

Integrating by parts, we have for $nt>1$ that

$$\begin{aligned} |I_2| &\leqq 2|W(-M)| \left(\frac{n}{M}\right)^{1/2-H} + 2|W(-M(nt)^\beta)| \left(\frac{n}{M(nt)^\beta}\right)^{1/2-H} \\ &\quad + |H-1/2|\, n^{-1} \int_{-M(nt)^\beta}^{-M} |W(s)| \left(\left(t-\frac{s}{n}\right)^{H-3/2} - \left(-\frac{s}{n}\right)^{H-3/2}\right) ds \\ &\leqq O(1)\, n^{1/2-H} + I_3 + I_4 \end{aligned}$$

where

$$I_3 = \frac{2|W(-M(nt)^\beta)|}{(n^\beta L_2 n)^{1/2}} (n^\beta L_2 n)^{1/2} \left(\frac{n}{M(nt)^\beta}\right)^{1/2-H}$$

and

$$I_4 = 2n^{-1} \int_{-M(nt)^\beta}^{-M} |W(s)| \left(\frac{n}{-s}\right)^{3/2-H} ds.$$

Hence for $nt > 1$ we get by the LIL for Brownian motion at infinity that

$$I_4 \leqq O(1)\, n^{1/2-H} \int_{-M(nt)^\beta}^{-M} \frac{(L_2(-s))^{1/2}}{(-s)^{1-H}}\, ds$$
$$\leqq O(1)\, n^{1/2-H} (L_2 n)^{1/2} (M(nt)^\beta)^H$$

where $O(1)$ is independent of t and n. Thus if $t \leqq 1$ we have

$$I_4 \leqq O(1)\, n^{1/2-H} (L_2 n)^{1/2} n^{\beta H}.$$

Similarly, by the functional LIL for Brownian motion, if $nt > 1$, then $t \geqq 1/n$ and

$$I_3 \leqq O(1)(n^\beta L_2 n)^{1/2} n^{1/2-H} \sup_{1/n \leqq t \leqq 1} |M(nt)^\beta|^{H-1/2}.$$

If $H \geqq 1/2$ we get

$$I_3 \leqq O(1)(n^\beta L_2 n)^{1/2} n^{1/2-H} n^{\beta(H-1/2)}$$

and for $0 < H < 1/2$ we have

$$I_3 \leqq O(1)\, n^\beta (L_2 n)^{1/2} n^{1/2-H}.$$

Combining terms we see

$$|I_2| \leqq O(1)\, n^{1/2-H} + O(1)\, n^{1/2-H} (L_2 n)^{1/2} n^{\beta H} + O(1)(n^\beta L_2 n)^{1/2} n^{1/2-H} n^{\beta(H-1/2) I(H>1/2)}. \tag{4.12}$$

Thus we have from (4.11) and (4.12) that

$$\sup_{0 \leqq t \leqq 1} \frac{|\tilde{Z}(nt)|}{(n^{2H} L_2 n)^{1/2}} \leqq O(1)(n^{-H} + n^{\beta H - H} + n^{\beta/2 + \beta(H-1/2) I(H>1/2) - H})$$

and the lemma is proved.

Lemma 4.2. *Let* $\{X(t)\colon t \geqq 0\}$ *be as in* (4.1) *with* $H \geqq \frac{1}{2}$, *then for* $\gamma > 0$ *sufficiently large* (4.6) *holds with* $\varepsilon_n = \gamma (L_3 n / L_2 n)^{2/3}$.

Proof. If $H = \frac{1}{2}$, $\{X(t)\colon t \geqq 0\}$ is Brownian motion so by Theorem 3.1-A there is nothing to prove. If $H > \frac{1}{2}$, then integration by parts implies

$$X(nt) = \int_0^{nt} W(s)(H - \tfrac{1}{2})(nt - s)^{H-3/2}\, ds. \tag{4.13}$$

Hence, setting $nu = s$, we get

$$X(nt) = \int_0^t W(nu)(H - \tfrac{1}{2})(nt - nu)^{H-3/2}\, n\, du \tag{4.14}$$
$$= n^{H-1/2}(H - \tfrac{1}{2}) \int_0^t W(nu)(t-u)^{H-3/2}\, du.$$

Hence, for $f \in \mathscr{K}$ as in (4.4), we get from (4.14), by integrating by parts, that

$$(4.15) \qquad f(t) = (H - \tfrac{1}{2}) \int_0^t \int_0^u g(s)\, ds (t-u)^{H-3/2} du \qquad (0 \leqq t \leqq 1).$$

Note $h(u) = \int_0^u g(s)\, ds$ satisfies $h(0)=0$, and $H > \frac{1}{2}$ implies $H - \frac{3}{2} > -1$, so the end-points involved in the integration by parts drop out when $H > \frac{1}{2}$. Combining (4.14) and (4.15)

$$(4.16) \quad \overline{\lim_n} \inf_{f \in \mathscr{K}} \sup_{0 \leqq t \leqq 1} |X(nt)/(2n^{2H} L_2 n)^{1/2} - f(t)|$$

$$\leqq (H - \tfrac{1}{2}) \overline{\lim_n} \inf_{h \in \tilde{\mathscr{K}}} \sup_{0 \leqq t \leqq 1} \left| \int_0^t \left| \frac{W(nu)}{(2nL_2 n)^{1/2}} - h(u) \right| (t-u)^{H-3/2} du \right|$$

where

$$\tilde{\mathscr{K}} = \left\{ h(t) = \int_0^t g(u)\, du, 0 \leqq t \leqq 1 : \int_0^1 g^2(u)\, du \leqq 1 \right\}.$$

Now, as $n \to \infty$,

$$\inf_{h \in \tilde{\mathscr{K}}} \sup_{0 \leqq t \leqq 1} \left| \frac{W(nu)}{(2nL_2 n)^{1/2}} - h(u) \right| = O((L_3 n / L_2 n)^{2/3})$$

by Theorem 3.1-A, and hence (4.6) implies

$$\overline{\lim_n} \inf_{f \in \mathscr{K}} \sup_{0 \leqq t \leqq 1} |X(nt)/(2n^{2H} L_2 n)^{1/2} - f(t)|$$

$$\leqq O\left((L_3 n / L_2 n)^{2/3} \sup_{0 \leqq t \leqq 1} \int_0^t (t-u)^{H-3/2} du \right)$$

$$= O((L_3 n / L_2 n)^{2/3}$$

since $H > \frac{1}{2}$. Thus the lemma is proved.

Lemma 4.3. *Let* $\{X(t): t \geqq 0\}$ *be as in* (4.1) *with* $0 < H < \frac{1}{2}$. *Then for* $\gamma > 0$ *sufficiently large,* (4.6) *holds with* $\varepsilon_n = \gamma (L_3 n / L_2 n)^{(2H+1)/(2H+2)}$.

Proof. The first step of the proof is to show that if $\{X_n(t): 0 \leqq t \leqq 1\}$ is a sequence of identically distributed copies of $\{X(t): 0 \leqq t \leqq 1\}$ then

$$(4.17) \qquad P(X_n/(2Ln)^{1/2} \in \mathscr{K} + \varepsilon_n U \text{ eventually}) = 1$$

where $\varepsilon_n = \gamma (L_2 n / Ln)^{(2H+1)/(2H+2)}$.

The proof of (4.17) will follow by an application of Theorem 2.1. To apply Theorem 2.1 we first prove the following:

For each H, $0<H<1/2$, there exists a finite constant C_H independent of n such that for all t, $0\leqq t\leqq 1$, and constants c_k, $k\in\Gamma(n)$,

$$|\sum_{k\in\Gamma(n)} c_k\,\psi_{kn}(t)|\leqq C_H\,2^{-nH}\max_{k\in\Gamma(n)}|c_k|. \tag{4.18}$$

In (4.18), $\Gamma(n)$ is as in (3.12) and

$$\psi_{kn}(t)=\int_0^t (t-s)^{H-1/2} f_{kn}(s)\,ds \tag{4.19}$$

where $f_{kn}(s)$ is as in (3.12) for $k\in\Gamma(n)$ and $0\leqq s\leqq 1$, i.e.

$$f_{kn}(s)=2^{(n-1)/2}\left(I_{\left[\frac{2k-2}{2^n},\frac{2k-1}{2^n}\right)}(s)-I_{\left[\frac{2k-1}{2^n},\frac{2k}{2^n}\right)}(s)\right). \tag{4.20}$$

Proof of (4.18). From (4.19) and (4.20), if $t\in[0,1]$ is fixed and k satisfies $(2k)\,2^{-n}<t$, then

$$\psi_{kn}(t)=2^{(n-1)/2}\int_{(2k-2)/2^n}^{(2k-1)/2^n}\{(t-s)^{H-1/2}-(t-s-2^{-n})^{H-1/2}\}\,ds.$$

By the mean value theorem for integrals there exists $\xi\in((2k-2)\,2^{-n},(2k-1)\,2^{-n})$ such that

$$\psi_{kn}(t)=2^{(n-1)/2}\,2^{-n}\{(t-\xi)^{H-1/2}-(t-\xi-2^{-n})^{H-1/2}\}. \tag{4.21}$$

By applying the mean value theorem to the function $h(u)=(t-\xi-u)^{H-1/2}$ we obtain $\eta\in((2k-2)\,2^{-n},(2k)\,2^{-n})$ such that

$$(t-\xi)^{H-1/2}-(t-\xi-2^{-n})^{H-1/2}=2^{-n}(H-\tfrac{1}{2})(t-\eta)^{H-3/2}.$$

Combining this with (4.21), we obtain

$$\psi_{kn}(t)=2^{(n-1)/2}\,2^{-2n}(H-\tfrac{1}{2})(t-\eta)^{H-3/2}.$$

By assumption $t>(2k)\,2^{-n}>\eta$, so that

$$|\psi_{kn}(t)|\leqq|\tfrac{1}{2}-H|\,2^{(n-1)/2}\,2^{-2n}\left(t-\frac{2k}{2^n}\right)^{H-3/2}. \tag{4.22}$$

On the other hand, if $t\leqq(2k-2)\,2^{-n}$, it is immediate from (4.19) and (4.20) that

$$\psi_{kn}(t)=0. \tag{4.23}$$

Suppose that $\{c_k\colon k\in\Gamma(n)\}$ are given and choose $j\in\Gamma(n)$ such that

$$(2j-2)\,2^{-n}<t\leqq(2j)\,2^{-n}.$$

Then, from (4.23) we obtain

$$(4.24)\qquad \Big|\sum_{k\in\Gamma(n)} c_k\,\psi_{kn}(t)\Big| \leqq |c_{j-1}|\,|\psi_{j-1,n}(t)| + |c_j|\,|\psi_{jn}(t)| + \sum_{k=1}^{j-2} |c_k|\,|\psi_{kn}(t)|.$$

From (4.22),

$$\sum_{k=1}^{j-2} |c_k|\,|\psi_{kn}(t)| \leqq \max_{k\leqq j-2} |c_k|\,|\tfrac{1}{2}-H|\,2^{-(3n+1)/2} \sum_{k=1}^{j-2} ((2j-2)\,2^{-n}-(2k)\,2^{-n})^{H-3/2}$$

$$(4.25)\qquad = |\tfrac{1}{2}-H| \max_{k\leqq j-2} |c_k|\,2^{-(3n+1)/2}\,2^{(n-1)(3/2-H)} \sum_{k=1}^{j-2} (j-1-k)^{H-3/2}$$

$$\leqq |\tfrac{1}{2}-H| \max_{k\leqq j-2} |c_k|\,2^{-H(n-1)-2} \sum_{i=1}^{j-2} i^{H-3/2}.$$

Since $H<1/2$, $\sum_{i=1}^{\infty} i^{H-3/2}<\infty$, and letting $C'_H=(\frac{1}{2}-H)\,2^{H-2}\sum_{i=1}^{\infty} i^{H-3/2}$ we have from (4.24) and (4.25) that

$$(4.26)\qquad \Big|\sum_{k\in\Gamma(n)} c_k\,\psi_{kn}(t)\Big| \leqq C'_H \max_{k\leqq j-2} |c_k|\,2^{-nH} + |c_{j-1}|\,|\psi_{j-1,n}(t)| + |c_j|\,|\psi_{jn}(t)|.$$

Next we consider $\psi_{jn}(t)$ for $t\in((2j-2)\,2^{-n}, (2j)\,2^{-n}]$. Using (4.19) and (4.20)

$$(4.27)\qquad \psi_{jn}(t) = 2^{(n-1)/2} \int_0^t (t-s)^{H-1/2} \{I_{\left[\frac{2j-2}{2^n},\frac{2j-1}{2^n}\right)}(s) - I_{\left[\frac{2j-2}{2^n},\frac{2j-1}{2^n}\right)}(s+2^{-n})\}\,ds$$

$$= 2^{(n-1)/2} \int_0^t x^{H-1/2} \{I_{\left[\frac{2j-2}{2^n},\frac{2j-1}{2^n}\right)}(t-x) - I_{\left[\frac{2j-2}{2^n},\frac{2j-1}{2^n}\right)}(t-x+2^{-n})\}\,dx.$$

How $H<\frac{1}{2}$ implies that $x^{H-1/2}$ is decreasing and hence $|\psi_{jn}(t)|$ is maximum for $t=(2j-1)\,2^{-n}$. Thus

$$(4.28)\qquad |\psi_{jn}(t)| \leqq 2^{(n-1)/2}(H+\tfrac{1}{2})^{-1}\,2^{-n(H+1/2)} = (H+\tfrac{1}{2})^{-1}\,2^{-nH-1/2}.$$

Again, by the monotone nature of $x^{H-1/2}$, one sees from (4.27) that $|\psi_{j-1,n}(t)| \leqq |\psi_{jn}(t)|$, so (4.28) implies

$$(4.29)\qquad |\psi_{j-1,n}(t)| \leqq (H+\tfrac{1}{2})^{-1}\,2^{-nH-1/2}.$$

Combining (4.26), (4.28), and (4.29) we obtain a constant C_H, depending only on H such that (4.18) holds.

The next step of the proof is to show that if $\{\alpha_{kn}\}$ is the sequence of linear functionals on $C[0, 1]$ defined in (3.14), then the series

$$\tilde{X}(t) = \sum_{n=0}^{\infty} \sum_{k \in \Gamma(n)} \alpha_{kn}(W)\, \psi_{kn}(t) \qquad (0 \leqq t \leqq 1) \tag{4.30}$$

converges uniformly on [0, 1] with probability one. This follows immediately from (4.18) and the fact that $\{\alpha_{kn}(W)\}$ an i.i.d. $N(0, 1)$ sequence implies

$$\sup_{k \in \Gamma(n)} |\alpha_{kn}(W)| = O((L2^n)^{1/2}). \tag{4.31}$$

Thus the process $\{\tilde{X}(t)\colon 0 \leqq t \leqq 1\}$ is the uniform limit of the series (4.30) and thus it is sample continuous. Furthermore, the process $\{\tilde{X}(t)\colon 0 \leqq t \leqq 1\}$ is equivalent to $\{X(t)\colon 0 \leqq t \leqq 1\}$. To check this is easy since both are mean zero Gaussian processes with identical covariance functions:

$$E(X(s)\, X(t)) = \int_0^{s \wedge t} (s-u)^{H-1/2}(t-u)^{H-1/2}\, du \tag{4.32}$$

and

$$\begin{aligned} E(\tilde{X}(s)\, \tilde{X}(t)) &= \sum_{n=0}^{\infty} \sum_{k \in \Gamma(n)} \psi_{kn}(s)\, \psi_{kn}(t) \\ &= \sum_{n=0}^{\infty} \sum_{k \in \Gamma(n)} \int_0^s (s-u)^{H-1/2} f_{kn}(u)\, du \cdot \int_0^t (t-u)^{H-1/2} f_{kn}(u)\, du \\ &= \int_0^1 I(u \leqq s)(s-u)^{H-1/2} I(u \leqq t)(t-u)^{H-1/2}\, du \\ &\quad \text{since } \{f_{kn}\colon k \in \Gamma(n), n \geqq 0\} \text{ is a CONS in } L^2[0, 1] \\ &= \int_0^{s \wedge t} (s-u)^{H-1/2}(t-u)^{H-1/2}\, du. \end{aligned} \tag{4.33}$$

Hence $\{X(t)\colon 0 \leqq t \leqq 1\}$ and $\{\tilde{X}(t)\colon 0 \leqq t \leqq 1\}$ are equivalent processes and we now turn to the task of representing $\{\tilde{X}(t)\colon 0 \leqq t \leqq 1\}$ by a series of the form (2.1). To do this define the operator

$$\Lambda f(t) = \frac{\operatorname{Sin}(\pi(1/2 - H))}{\pi} \int_0^t (t-u)^{-(H+1/2)} f(u)\, du \qquad (0 \leqq t \leqq 1). \tag{4.34}$$

Then, $0 < H < \frac{1}{2}$, and $f \in L^\infty[0, 1]$ easily implies

$$\sup_{0 \leqq t \leqq 1} |\Lambda f(t)| \leqq \|f\| \frac{\operatorname{Sin}(\pi(1/2 - H))}{\pi} \frac{1}{1/2 - H},$$

so $\Lambda: L^\infty[0,1] \to L^\infty[0,1]$ is a bounded linear operator. If $f(t)=\sum_{j=1}^{k} b_j I_{E_j}(x)$, where $E_1, \ldots, E_k$ are disjoint intervals and $\bigcup_{j=1}^{k} E_j=[0,1]$, then it is easy to check that for $0<H<\frac{1}{2}$, $\Lambda f(\cdot)\in C[0,1]$. Of course, functions in $C[0,1]$ are uniform limits of such step functions, and hence Λ bounded implies $\Lambda: C[0,1]\to C[0,1]$.

Since Λ is bounded, and the series (4.30) converges uniformly, we have

$$\Lambda\tilde{X}(t)=\sum_{n=0}^{\infty}\sum_{k\in\Gamma(n)}\alpha_{kn}(W)(\Lambda\psi_{kn})(t) \tag{4.35}$$

converging uniformly on $[0,1]$. Now

$$\psi_{kn}(x)=\int_0^x (x-u)^{H-1/2} f_{kn}(u)\,du, \tag{4.36}$$

and hence from [9, pp. 41–42] we see that (4.36) is Abel's integral equation with "inverse operator" Λ satisfying

$$\Lambda(\psi_{kn})(t)=\int_0^t f_{kn}(x)\,dx=\phi_{kn}(t). \tag{4.37}$$

Substituting (4.37) into (4.35) we get with probability one that

$$\Lambda\tilde{X}(t)=\sum_{n=0}^{\infty}\sum_{k\in\Gamma(n)}\alpha_{kn}(W)\,\phi_{kn}(t)=W(t). \tag{4.38}$$

Using (4.38) in (4.30) we get

$$\tilde{X}(t)=\sum_{n=0}^{\infty}\sum_{k\in\Gamma(n)}\alpha_{kn}(\Lambda\tilde{X})\,\psi_{kn}(t) \qquad (0\leqq t\leqq 1). \tag{4.39}$$

Now $\{\alpha_{kn}\circ\Lambda: k\in\Gamma(n), n\geqq 0\}$ consists of linear functionals on $C[0,1]$, and $\{\alpha_{kn}\circ\Lambda(\tilde{X}): k\in\Gamma(n), n\geqq 0\}$ is i.i.d. $N(0,1)$, so $\tilde{X}$ Gaussian implies (4.39) converges in L^p for any $p\geqq 1$. Hence

$$\begin{aligned} S\alpha_{kn}\circ\Lambda(t) &= E(\alpha_{kn}\circ\Lambda(\tilde{X})\,\tilde{X}(t)) \\ &= \psi_{kn}(t) \\ &= \int_0^t (t-s)^{H-1/2} f_{kn}(s)\,ds. \end{aligned} \tag{4.40}$$

Thus (4.39) represents $\{\tilde{X}(t): 0\leqq t\leqq 1\}$ as in (2.1), so Theorem 2.1 can be applied to $\tilde{X}$.

To express (4.39) in terms of a single index d as in (2.1) we order the sequence $\{\alpha_{kn}\circ\Lambda: k\in\Gamma(n), n\geqq 0\}$ lexicographically. That is, $(n_1, k_1)<(n_2, k_2)$ if $n_1<n_2$

or $n_1 = n_2$ and $k_1 < k_2$. Hence $\Pi_d(\tilde{X})$ is the sum of the first d terms of (4.39) in this ordering.

Since Card $\Gamma(n) = 2^{n-1}$ for $n \geqq 1$ with $\Gamma(0) = \{0\}$, we see that for

$$2^m \leqq d < 2^{m+1}, \tag{4.41}$$

Jensen's inequality implies

$$\begin{aligned} E\|\tilde{X} - \Pi_d(\tilde{X}\| &\leqq E\|\tilde{X} - \Pi_{2^m}(\tilde{X})\| \\ &= E\left\| \sum_{n=m}^{\infty} \sum_{k \in \Gamma(n)} \alpha_{kn} \circ \Lambda(\tilde{X})\, \psi_{kn}(\cdot) \right\| \\ &\leqq \sum_{n=m}^{\infty} E\| \sum_{k \in \Gamma(n)} \alpha_{kn} \circ \Lambda(\tilde{X})\, \psi_{kn}(\cdot)\| \\ &\leqq \sum_{n=m}^{\infty} C_H 2^{-nH} E(\max_{k \in \Gamma(n)} |\alpha_{kn} \circ \Lambda(\tilde{X})|) \\ &\quad \text{by (4.18)} \\ &\leqq C \sum_{n=m}^{\infty} 2^{-nH} (L2^n)^{1/2} \end{aligned} \tag{4.42}$$

by (4.31) since $\Lambda(\tilde{X})$ has the same distribution as W by (4.38).

Combining (4.41) and (4.42) we see there is a constant $C < \infty$ such that

$$E\|\tilde{X} - \Pi_d(\tilde{X})\| \leqq C d^{-H} (Ld)^{1/2}. \tag{4.43}$$

Hence for (2.4) we may choose d_n as a large constant multiple of the solution of

$$d^{-(H+1)} (Ld)^{1/2} = (L_2 n)(Ln)^{-1/2}.$$

Thus

$$d_n \approx \left(\frac{Ln}{L_2 n} \right)^{\frac{1}{2H+2}},$$

and

$$\varepsilon_n \approx (L_2 n / Ln)^{(2H+1)/(2H+2)}. \tag{4.44}$$

Hence by Theorem 2.1, and that $\tilde{X}$ and X have the same distribution we have (4.17) for $\varepsilon_n = \gamma (L_2 n / Ln)^{(2H+1)/(2H+2)}$ provided $\gamma > 0$ is sufficiently large.

The proof of Lemma 4.3 is now completed by rescaling as in Lemma 3.1 with $\varepsilon_n = \gamma (L_3 n / L_2 n)^{(2H+1)/(2H+2)}$. Thus Part A is proved and Part B follows immediately from Lemma 4.1. Hence it remains to verify Part C, but this follows immediately from Theorem 2.3, the remark following its statement, and by rescaling. The rescaling in this situation is slightly more complicated as we need to know that

$$\overline{\lim_n} \|\eta_n\| < \infty \tag{4.45}$$

and

$$\overline{\lim_n} \|\tilde{\eta}_n\| < \infty \tag{4.46}$$

where

$$\eta_n(t) = X(nt)/(2n^{2H} L_2 n)^{1/2}$$
$$\tilde{\eta}_n = Y(nt)/(2n^{2H} L_2 n)^{1/2}.$$

To prove (4.46) let $n_r = \exp\{r/(\mathrm{L}\, r)^2\}$ and define

$$Y_r(t) = Y(n_r t)/n_r^H.$$

Then by self-similarity and (2.6) applied to $\{Y_r: r \geqq 1\}$, it follows that

$$\overline{\lim_r} \|\tilde{\eta}_{n_r}\| < \infty \tag{4.47}$$

as $L_2 n_r \sim Lr$. Since the increments of $\{Y(t): t \geqq 0\}$ are stationary and $n_r/n_{r-1} \sim 1$, self-similarity, (4.47), and (2.6) combine to imply (4.46).

Applying the methods of Lemma 4.1 we can also show

$$\overline{\lim_n} \frac{\|Z(n(\cdot))\|}{(2n^{2H} L_2 n)^{1/2}} < \infty,$$

and combining this with (4.46) easily yields (4.45). Hence Theorem 4.1 is proved.

References

1. Bass, R.F.: Probability estimates for multiparameter Brownian processes. Ann. Probab. **16**, 251–264 (1988)
2. Bolthausen, E.: On the speed of convergence in Strassen's law of the iterated logarithm. Ann. Probab. **6**, 668–672 (1978)
3. Borell, C.: The Brunn-Minkowski inequality in Gauss space. Invent. Math. **30**, 207–216 (1975)
4. Goodman, V., Kuelbs, J.: Rates of convergence for increments of Brownian motion. J. Theor. Probab. **1**, 27–63 (1988)
5. Goodman, V., Kuelbs, J.: Rates of clustering in Strassen's LIL for Brownian motion (to appear in the J. Theor. Probab.)
6. Goodman, V., Kuelbs, J., Zinn, J.: Somes results on the LIL in Banach spaces with applications to weighted empirical processes. Ann. Probab. **9**, 713–742 (1981)
7. Grill, K.: On the rate of convergence in Strassen's law of the iterated logarithm. Probab. Th. Rel. Fields **74**, 585–589 (1987)
8. Gross, L.: Lectures in modern analysis and applications II. (Lect. Notes Math., vol. 140) Berlin Heidelberg New York: Springer 1970
9. Hochstadt, H.: Integral equations. New York: Wiley 1973
10. Jain, N.C., Marcus, M.B.: Continuity of subgaussian processes. (Advances in Probability and Related Topics 4.) New York: Dekker 1978
11. Kuelbs, J.: A strong convergence theorem for Banach space valued random variables. Ann. Probab. **4**, 744–771 (1976)
12. Mandelbrot, B.R., Van Ness, J.W.: Fractional Brownian motions, fractional noises and applications. SIAM Rev. **10**, 422–437 (1968)
13. Park, W.J.: A multi-parameter Gaussian process. Ann. Math. Statist. **41**, 1582–1595 (1970)
14. Talagrand, M.: Sur l'integrabilite' des vecteurs gaussiens. Z. Wahrscheinlichkeitstheor. Verw. Geb. **68**, 1–8 (1984)