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Probability Theory and Related Fields

The exponent for the mean square displacement of self-avoiding random walk on the Sierpinski gasket

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Summary. The authors rigorously prove that the exponent for the mean square displacement of self-avoiding random walk on the Sierpinski gasket is $\log 2/\log\left(\frac{7-\sqrt{5}}{2}\right)=0.79862>0.5>\log 2/\log 5$.

Mathematics Subject Classifications: 60K35, 82A99

0 Introduction

Let us define the pre-Sierpinski Gasket as follows. Let $O=(0,0)$. $a_0=\left(\frac{1}{2},\frac{\sqrt{3}}{2}\right)$, $b_0=(1,0)$, and let F_0 be a graph which consists of the vertices and the edges of the equilateral triangle $\Delta O\,a_0\,b_0$. Let us define a sequence of graphs inductively by

$$F_{n+1}=F_n\cup(F_n+2^n a_0)\cup(F_n+2^n b_0), \qquad n=0,1,2,\ldots,$$

where, $A+a=\{x+a\,|\,x\in A\}$, and $kA=\{kx\,|\,x\in A\}$. Let $F=\bigcup_{n=0}^{\infty}F_n$. F is the pre-Sierpinski Gasket. Let G_0 be the set of the vertices in F, and $a_n=2^n a_0$, $b_n=2^n b_0$.

We define the set of self-avoiding paths W_0 on G_0 to be the set of mappings $w\colon \mathbf{Z}_+\to G_0$ such that $w(0)=0$ and that there exists $L(w)\in\mathbf{Z}_+\cup\{\infty\}$ for which $w(i)=w(L(w))$, $i\geqq L(w)$, $w(i_1)\neq w(i_2)$, $0\leqq i_1<i_2\leqq L(w)$, $\overline{w(i)\,w(i+1)}\subset F$, and $|w(i)-w(i+1)|=1$, $0\leqq i\leqq L(w)-1$. We call $L(w)$ the length of the path w. Also, we define $\|w\|$, $w\in W_0$, by $\|w\|=\max\{|w(k)|;\,k=1,2,\ldots,L(w)\}$.

Let $N_n=\#(\{w\in W_0;\,L(w)=n\})$. We define probability measures P_n, $n=1,2,\ldots$, on W_0 given by

$$P_n(A)=N_n^{-1}\cdot\#(\{w\in A;\,L(w)=n\}), \qquad A\subset W_0.$$

We show the following results in the present paper.

Theorem. (1) $\lim_{n\to\infty}\left(\frac{1}{n}\right)\cdot\log N_n=\beta_c$. *Here* β_c *is the critical inverse temperature given in Hattori et al.* [4] (*See Proposition* 1.1 *in this paper also*), *and* $\beta_c=0.827691$ $(e^{\beta_c}=2.2880)$.

(2) *For any* $s>0$, $\lim_{n\to\infty}(\log n)^{-1}\cdot\log E^{P_n}[|w(n)|^s]=\nu\cdot s$. *Here* $\nu=\log 2/\log\left(\frac{7-\sqrt{5}}{2}\right)=0.79862$.

(3) There is an $\alpha>0$ such that

$$P_n[(\log n)^{-\alpha_n\nu}\leqq\|w\|\leqq(\log n)^{\alpha_n\nu}]\to 1,\qquad n\to\infty.$$

This theorem says that "the exponent for mean square displacement of self-avoiding random walk in Sierpinski Gasket" is 2ν. This number has been given in Ben et al. in [2], Dhar [3], Klein and Seiz [5] and Rammal et al. [7]. However, we believe that this is the first time a mathematically rigorous proof has been given.

Unfortunately, our results are not so sharp. For example, we could not prove that

$$0<\varliminf_{n\to\infty} n^{-s\nu}E^{P_n}[|w(n)|^s]\leqq\varlimsup_{n\to\infty} n^{-s\nu}E^{P_n}[|w(n)|^s]<\infty.$$

So several problems are still open from a mathematical point of view.

1 Preliminary

In this section, we summarize some facts which were proven in Hattori et al. [4] for later use and give some additional results. The proof of the main theorem relies strongly on the results in [4], and the reader is requested to be familiar with [4] to comprehend the detail of the proof.

Let $W^{(n)}$ and $\tilde{W}^{(n)}$, $n\geqq 1$, be subsets of W_0 given by

$$W^{(n)}=\{w\in W_0;\, w(L(w))=a_n,\, w(i)\neq b_n,\, i\geqq 0\},$$

and

$$\tilde{W}^{(n)}=\{w\in W_0;\, w(L(w))=a_n,\, w(i)=b_n,\text{ for some } i\geqq 0 \text{ and } w(i)\in F_n,\, i\geqq 0\}.$$

Let $Z_n(\beta)=\sum_{w\in W^{(n)}}e^{-\beta\cdot L(w)}$ and $\tilde{Z}_n(\beta)=\sum_{w\in\tilde{W}^{(n)}}e^{-\beta\cdot L(w)}$, $n\geqq 1$, $\beta\in\mathbb{R}$.

Let $G\colon\mathbb{C}^2\to\mathbb{C}^2$ be an algebraic map given by

$$G(x,y)=((x+y)^2+x^2(x+2y),\, xy(x+2y)),\qquad (x,y)\in\mathbb{C}^2.$$

Also, let $\Phi_n\colon\mathbb{C}^2\to\mathbb{C}$ and $\Theta_n\colon\mathbb{C}^2\to\mathbb{C}$, $n\geqq 1$ be given by $(\Phi_n(x,y),\Theta_n(x,y))=G^n(x,y)$, $(x,y)\in\mathbb{C}^2$, where G^n is defined by $G^1(x,y)=G(x,y)$, and $G^{n+1}(x,y)$

$=G(G^n(x,y))$ inductively. Then we have $Z_n(\beta)=\Phi_n(e^{-\beta}, e^{-2\beta})$ and $\tilde{Z}_n(\beta)=\Theta_n(e^{-\beta}, e^{-2\beta})$, $n\geqq 1$, $\beta\in\mathbb{R}$.

Also, for any $(x,y)\in(0,\infty)^2$, let $R_n(x,y)=\dfrac{\Theta_n(x,y)}{\Phi_n(x,y)}$, $n\geqq 1$, and $R_0(x,y)=\dfrac{y}{x}$.

Let $D=\left\{(x,y)\in(0,\infty)^2;\ \sup_n \Phi_n(x,y)\leqq\dfrac{\sqrt{5}-1}{2}\right\}$. Then we have the following [4, Sect. 2].

(1.1) **Proposition.** (1) *D is a closed set in $(0,\infty)^2$ and the closure of D in $\mathbb{R}^2$ contains the point $(a, 0)$. Hence $a=\dfrac{\sqrt{5}-1}{2}$.*

(2) *If $(x,y)\in\partial D\cap(0,\infty)^2$, then $\lim_{n\to\infty}\Phi_n(x,y)=a$.*

(3) *If $(x,y)\in D\backslash\partial D$, then $\lim_{n\to\infty} 2^{-n}\log\Phi_n(x,y)$ exists and is negative.*

(4) *If $(x,y)\in D$, then $\lim_{n\to\infty}\Theta_n(x,y)=0$.*

(5) *If $(x,y)\in D$, $0<x'<x$ and $0<y'<y$, then $(x',y')\in D\backslash\partial D$.*

(6) *There is a unique $\beta_c>0$ such that $(\exp(-\beta_c), \exp(-2\beta_c))\in\partial D$.*

(7) *$R_n(x,y)\leqq R_{n-1}(x,y)$, $n\geqq 1$, $x, y\in(0,\infty)^2$, and $R(x,y)=\lim_{n\to\infty}R_n(x,y)$ is continuous in $(x,y)\in(0,\infty)^2$. Moreover, $R(x,y)=0$, for any $(x,y)\in D$.*

We define probability measures $\mu_n(\beta)$, $\beta\in R$, $n\geqq 1$, on W_0 by

$$\mu_n(\beta)(A)=Z_n(\beta)^{-1}\sum_{w\in A\cap W^{(n)}} e^{-L(w)\beta},\qquad A\subset W_0.$$

Let ν_n denote the probability law of $\lambda^{-n}L(w)$ under $\mu_n(\beta_c)(dw)$, $n\geqq 1$. Here $\lambda=\dfrac{7-\sqrt{5}}{2}$. Then we have

$$\int_0^\infty \exp(-\xi x)\,\nu_n(dx)=Z_n(\beta_c)^{-1}Z_n(\beta_c+\lambda^{-n}\xi),\qquad \xi\in\mathbb{C}.$$

Moreover, we have the following [4, Theorem 0.3 and Proposition 4.17].

(1.2) **Proposition.** *ν_n converges in law to a certain probability measure ν_∞ in $(0,\infty)$. The laplace transform $g(\xi)=\int_0^\infty \exp(-\xi x)\,\nu_\infty(dx)$ is an entire function in ξ and satisfies*

$$g(\lambda\xi)=a^2\cdot g(\xi)^3+a\cdot g(\xi)^2,\qquad \xi\in\mathbb{C}, \tag{1.3}$$

and $g'(0)<0$. Here $a=\dfrac{\sqrt{5}-1}{2}$ again. Moreover,

$$\int_0^\infty \exp(-\xi x)\,\nu_n(dx)\to g(\xi),\qquad n\to\infty,$$

uniformly in ξ on any bounded set in $\mathbb{C}$.

We have the following regularity result for the probability measure ν_∞.

(1.4) **Proposition.** *There is a smooth function* $f: \mathbb{R} \to \mathbb{R}$ *such that* $f(x)=0$, $x \leqq 0$, $f(x)>0$, $x>0$, *and* $\nu_\infty(dx)=f(x)\,dx$.

Proof. By the fact that $g'(0)<0$ and g satisfies the function equality (1.3), we see that the probability measure ν_∞ has a smooth density f (see the argument [6, Sect. 3, Theorem 3.19] or [1, Lemmas 3.4 and 3.6]). Then the formula (1.3) implies that

$$\lambda^{-1} f(\lambda^{-1} x) = a \cdot (f * f)(x) + a^2 \cdot (f * f * f)(x), \qquad x>0. \tag{1.5}$$

So it is sufficient to show that $\operatorname{supp} f = [0, \infty)$.

Let A be the support of f. Then A is a closed set in $[0, \infty)$ and (1.5) implies that if $x, y, z \in A$, then $\lambda^{-1}(x+y)$, $\lambda^{-1}(x+y+z) \in A$. Note that $2<\lambda<3$. There is an $x_0 \in A$ such that $x_0>0$. Then we see that $(2\lambda^{-1})^n x_0 \in A$, $n \geqq 1$. So we see that $0 \in A$. Therefore we see that if $x, y, z \in A$, then 0, $\lambda^{-1} x$, $\lambda^{-1}(x+y)$, $\lambda^{-1}(x+y+z) \in A$. So by induction, we see that $m \cdot \lambda^{-n} x_0 \in A$, $m=0, 1, \ldots, 3^n$. This implies that $A=[0, \infty)$.

This completes the proof.

(1.6) **Proposition.** (1) $C_0 = \sup Z_n(\beta_c - \lambda^{-n}) < \infty$, *and* $\tilde{Z}_n(\beta_c - \lambda^{-n}) \to 0$.
(2) *There are* $n_0 \geqq 1$, $C_1>0$ *and* $\gamma_1>0$ *such that*

$$|Z_{n+m}(\beta_c + \lambda^{-n}\xi)| \leqq C_1 \cdot \exp(-\gamma_1 \cdot 2^m)$$

for any $n \geqq n_0$, $m \geqq 1$ *and* $\xi \in \mathbb{C}$ *with* $\operatorname{Re}\xi \geqq 0$ *and* $\lambda^{-1} \leqq |\xi| \leqq \lambda^2$.
(3) *There are* $C_2>0$ *and* $\gamma_2>0$ *such that*

$$Z_{n+m}(\beta_c + \lambda^{-n}) + \tilde{Z}_{n+m}(\beta_c + \lambda^{-n}) \leqq C_2 \cdot \exp(-\gamma_2 \cdot 2^m)$$

for any $n \geqq n_0$ *and* $m \geqq 1$. *Here* n_0 *is as in the assertion* (2).

Proof. (1) By Proposition 1.2, we see that $\lim_{n\to\infty} Z_n(\beta_c - \lambda^{-n}) = g(-1)>0$. By Proposition 1.1 (7), we see that

$$Z_n(\beta_c - \lambda^{-n})^{-1} \tilde{Z}_n(\beta_c - \lambda^{-n}) \leqq R_m(\exp(-(\beta_c - \lambda^{-n})), \exp(-2(\beta_c - \lambda^{-n})))$$

for any $n \geqq m$. So we see that

$$\overline{\lim_{n\to\infty}}\, Z_n(\beta_c - \lambda^{-n})^{-1} \tilde{Z}_n(\beta_c - \lambda^{-n}) \leqq R_m(\exp(-\beta_c), \exp(-2\beta_c)).$$

However, by Proposition 1.1 (6) and (7) we have

$$\lim_{m\to\infty} R_m(\exp(-\beta_c), \exp(-2\beta_c)) = R(\exp(-\beta_c), \exp(-2\beta_c)) = 0.$$

These imply the assertion (1).
(2) Let $A = \{\xi \in \mathbb{C};\ \operatorname{Re}\xi \geqq 0,\ \lambda^{-1} \leqq |\xi| \leqq \lambda^2\}$. Then by Proposition 1.2, we see that $Z_n(\beta_c)^{-1} \cdot Z_n(\beta_c + \lambda^{-n}\xi)$ converges to $\int_0^\infty \exp(-\xi x) f(x)\,dx$ uniformly in $\xi \in A$ as $n \to \infty$. Also we see that $|\tilde{Z}_n(\beta_c + \lambda^{-n}\xi)| \leqq \tilde{Z}_n(\beta_c)$, $\xi \in A$. Note that by Proposi-

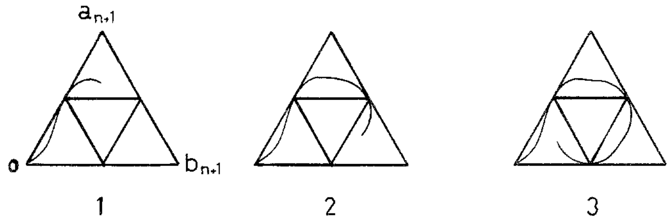

Fig. 1

tion 1.4 $\sup\left\{\left|\int_0^\infty \exp(-\xi x) f(x)\, dx\right|;\ \xi\in A\right\}<1$. Therefore there is an $\varepsilon>0$ and $n_1\geqq 1$ such that $|Z_n(\beta_c+\lambda^{-n}\xi)|\leqq a-\varepsilon$ for all $n\geqq n_1$ and $\xi\in A$. By Proposition 1.1, there is a $\delta>0$ such that $(a-\varepsilon,\delta)\in D\setminus\partial D$. There is an $n_0\geqq n_1$ such that $|Z_n(\beta_c+\lambda^{-n}\xi)|\leqq a-\varepsilon$ and $|\tilde{Z}_n(\beta_c+\lambda^{-n}\xi)|\leqq\delta$ for any $n\geqq n_0$ and $\xi\in A$. Then we see that

$$\sup\{|Z_{n+m}(\beta_c+\lambda^{-n}\xi)|;\ \xi\in A\}\leqq\Phi_m(a-\varepsilon,\delta),\qquad n\geqq n_0,\qquad m\geqq 1.$$

By Proposition 1.1(3), we have $\overline{\lim}_{m\to\infty} 2^{-m}\log\Phi_m(a-\varepsilon,\delta)<0$. This implies our assertion (2).

(3) By Proposition 1.1(7), we see that

$$Z_n(\beta)^{-1}\tilde{Z}_n(\beta)\leqq R_0(e^{-\beta},e^{-2\beta})=e^{-\beta}\leqq 1,\qquad n\geqq 1,\qquad \beta>0.$$

So the assertion (3) follows from the assertion (2).

This completes the proof.

2 Estimates for the number of self-avoiding paths

Let $N_n=\#(\{w\in W_0;\ L(w)=n\})$, $n\geqq 1$. We will estimate N_n from above and below. Let $D\colon W_0\to\{0,1,\dots\}$ be a map given by

$$D(w)=\min\{n\geqq 0;\ w(i)\in F_n \text{ for all } i\geqq 0\},\qquad w\in W_0. \tag{2.1}$$

Now for each $n\geqq 0$, let

$$M_n=\sum_{w\in W_0,\ D(w)\leqq n}\exp(-\beta_c L(w)).$$

Obviously $M_{n+1}-M_n$ is the summation of $\exp(-\beta_c L(w))$ for $w\in W_0$ such that $D(w)=n+1$. The path $w\in W_0$ with $D(w)=n+1$ can be classified into three types (see Fig. 1). The summation of $\exp(-\beta_c L(w))$ for $w\in W_0$ of type 1 (resp. type 2, type 3) is dominated by $2(Z_n(\beta_c)+\tilde{Z}_n(\beta_c))\,M_n$ (resp. $2(Z_n(\beta_c)+\tilde{Z}_n(\beta_c))^2 M_n$, $2Z_n(\beta_c)(Z_n(\beta_c)+\tilde{Z}_n(\beta_c))^2 M_n$). So we have

$$\begin{aligned}M_{n+1}\leqq M_n&+2(Z_n(\beta_c)+\tilde{Z}_n(\beta_c))\,M_n+2(Z_n(\beta_c)+\tilde{Z}_n(\beta_c))^2 M_n\\&+2Z_n(\beta_c)(Z_n(\beta_c)+\tilde{Z}_n(\beta_c))^2 M_n\end{aligned}$$

for any $n\geqq 1$.

Take an $A_0 \in \mathbb{R}$ with $A_0 > 1 + 2a + 2a^2 + 2a^3 (= 1 + 4a)$ and fix it. Since $Z_n(\beta_c) \to a$ and $\tilde{Z}_n(\beta_c) \to 0$ as $n \to \infty$, we see that there is a constant C_3 such that

$$(2.2) \qquad M_n \leqq C_3 \cdot A_0^n, \qquad n \geqq 1.$$

It is easy to see that

$$\log L(w)/\log 3 \leqq D(w) \leqq \log L(w)/\log 2, \qquad w \in W_0.$$

Therefore we have

$$\exp(-\beta_c n) N_n [\log n/\log 2] \leqq \sum_{k=[\log n/\log 3]} M_k \leqq C_2 A_0 (A_0 - 1)^{-1} A_0 \log n/\log 2.$$

Therefore we have the following.

(2.3) **Proposition.** *There are constants C_4 and γ_4 such that*

$$N_n \leqq C_4 \cdot n^{\gamma_4} \cdot \exp(\beta_c n), \qquad n \geqq 1.$$

To obtain lower estimate, we make some preparations.

(2.4) **Proposition.** *Let $g(x) = (2\pi)^{-1/2} \exp(-x^2/2)$, $x \in \mathbb{R}$, and $g(x; h) = h^{-1} g(h^{-1} x)$, $x \in \mathbb{R}$, $h > 0$. Also, let $h_n = b \cdot \lambda^{-n} \cdot n^{1/2}$, $b > 0$, $n \geqq 1$. If b is sufficiently large, then*

$$(v_n * g(\cdot\,; h_n))(x) \to f(x) \quad \textit{uniformly in } x \in \mathbb{R} \quad \textit{as } n \to \infty.$$

Proof. Let $\varphi_n(\eta) = \int_{\mathbb{R}} e^{i\eta x} (v_n * g(\cdot\,; h_n))(x)\,dx$, and $\varphi(\eta) = \int_{\mathbb{R}} e^{i\eta x} f(x)\,dx$, $\eta \in \mathbb{R}$. Then we see that

$$\varphi_n(\eta) = Z_n(\beta_c)^{-1} Z_n(\beta_c - i\lambda^{-n}\eta) \exp(-h_n^2 \eta^2/2).$$

Let n_0 be an integer as in Proposition 1.6. Assume that $|\eta| \in [1, \lambda^{n-n_0}]$. Then there is an $m \in \{1, \ldots, n-n_0\}$ such that $\lambda^{-1} \leqq \lambda^{-m}|\eta| \leqq 1$. Then we see that

$$\begin{aligned} |Z_n(\beta_c - i\lambda^{-n}\eta)| &= |Z_{n-m+m}(\beta_c - i\lambda^{-(n-m)}(\lambda^{-m}\eta))| \\ &\leqq C_1 \cdot \exp(-\gamma_1 \cdot 2^m) \\ &\leqq C_1 \cdot \exp(-\gamma_1 \cdot |\eta|^{\log 2/\log \lambda}) \end{aligned}$$

for any $\eta \in [-\lambda^{n-n_0}, -1] \cup [1, \lambda^{n-n_0}]$. Since $\varphi_n(\eta) \to \varphi(\eta)$, $n \to \infty$, for each $\eta \in \mathbb{R}$, we have by the dominated convergence theorem that

$$\int_{\mathbb{R}} |\chi_{[0, \lambda^{n-n_0}]}(|\eta|) \cdot \varphi_n(\eta) - \varphi(\eta)|\,d\eta \to 0 \quad \text{as} \quad n \to \infty.$$

On the other hand,

$$\begin{aligned}
&\int_{\mathbb{R}} |1-\chi_{[0,\lambda^{n-n_0}]}(\eta)| \cdot |\varphi_n(\eta)|\, d\eta \\
&\leqq 2 \int_{\lambda^{n-n_0}}^{\infty} \exp(-h_n^2 \eta^2/2)\, d\eta \\
&\leqq 2 \cdot h_n^{-1} (h_n \lambda^{n-n_0})^{-1} \cdot \exp(-(h_n \lambda^{n-n_0})^2/2) \\
&= 2\lambda^{n_0} \cdot b^{-2} \lambda^n \cdot n^{-1} \cdot \exp(-((\lambda^{-n_0} b)^2 n)/2) \\
&\to 0,\ n \to \infty, \qquad \text{if } b \text{ is sufficiently large.}
\end{aligned}$$

So we see that if b is sufficiently large, then

$$\int_{\mathbb{R}} |\varphi_n(\eta) - \varphi(\eta)|\, d\eta \to 0, \qquad \text{as } n \to \infty.$$

This implies our assertion.

(2.5) **Proposition.** *There are constants* $C_5 > 0$ *and* $\gamma_5 > 0$ *such that*

$$N_n \geqq C_5 \cdot n^{-\gamma_5} \exp(\beta_c n), \qquad n \geqq 1.$$

Proof. Let b be a sufficiently large number satisfying Proposition 2.4. Recall that

$$(v_n * g(\cdot\,; h_n))(x) = \int_{\mathbb{R}} g(x-y; h_n)\, v_n(dy), \qquad x \in \mathbb{R}.$$

Let $k_n = b \cdot (2 \cdot \log \lambda)^{1/2} n \cdot \lambda^{-n}$. Then $g(k_n; h_n) = (2\pi b^2 n)^{-1/2} \to 0$ as $n \to \infty$. So we see that

$$\int_{\mathbb{R} \setminus [x-k_n,\, x+k_n]} g(x-y; h_n)\, v_n(dy) \leqq g(k_n; h_n) \to 0.$$

This proves that

$$\sup\{|f(x) - \int_{[x-k_n,\, x+k_n]} g(x-y); h_n)\, v_n(dy)|;\, x \in \mathbb{R}\} \to 0, \qquad \text{as } n \to \infty.$$

Since $f(x) > 0$, $x > 0$, this implies that there are $n_1 \geqq 1$ and $\varepsilon > 0$ such that

$$h_n^{-1}\, v_n([x-k_n, x+k_n]) \geqq \varepsilon$$

for any $n \geqq n_1$ and $x \in [\lambda^{-1}, \lambda^2]$.

Now let $m \in \mathbb{N}$. Then there is an $n \geqq 1$ such that $\lambda^{-n} m \in [1, \lambda]$. If m is large enough, then $n \geqq n_1$ and $k_n \leqq 1 - \lambda^{-1}$, and so we have $v_n([\lambda^{-n} m - 2k_n, \lambda^{-n} m]) \geqq h_n \varepsilon$. This implies that

$$\begin{aligned}
Z_n(\beta_c)\, h_n \varepsilon &\leqq \sum_{w \in W^{(N)},\, L(w) \in [m - 2k_n \lambda^n,\, m]} \exp(-\beta_c L(w)) \\
&\leqq \exp(\beta_c \cdot 2k_n \lambda^n) \cdot \exp(-\beta_c m)\, N_m,
\end{aligned}$$

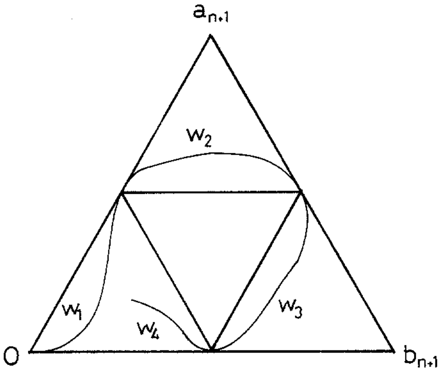

Fig. 2

because $w \in W^{(n)}$ with $L(w) \leqq m$ can be extended to a path $w' \in W_0$ with $L(w') = m$. Therefore we have

$$N_m \geqq Z_n(\beta_c)\, \varepsilon b \cdot \lambda^{-n} n^{1/2} \exp(-2\beta_c b \cdot (2 \cdot \log \lambda)^{1/2} n) \exp(\beta_c m).$$

Since $n \leqq \log m / \log \lambda$, this implies our assertion.

3 Estimates for numbers of short paths and long paths

For $n, m \geqq 0$, let $U_{n,m}$ be the summation of $\exp(-\beta_c L(w))$ for all $w \in W_0$ such that $D(w) \leqq n$ and $L(w) \geqq \lambda^{n+m/2}$, and let $V_{n,m}$ be the summation of $\exp(-\beta_c L(w))$ for all $w \in W_0$ such that $D(w) = n+1$ and $L(w) \leqq \lambda^{n-m}$.

The purpose of this section is to prove the following.

(3.1) **Proposition.** (1) *There are constants C_6 and $\gamma_6 > 0$ such that $U_{n,m} \leqq C_6 \cdot A_0^n \cdot \exp(-\gamma_6 \cdot \lambda^{m/2})$, $n, m \geqq 0$.*

(2) *There are constants C_7 and $\gamma_7 > 0$ such that*

$$V_{n,m} \leqq C_7 \cdot A_0^n \cdot \exp(-\gamma_7 \cdot 2^m), \qquad n, m \geqq 0.$$

Proof. For any $n, m \geqq 0$ and $t > 0$, let $S_{n,m}(t)$ (resp. $\tilde{S}_{n,m}(t)$) be the summation of $\exp(-\beta_c L(w))$ for all $w \in W^{(n)}$ (resp. $\tilde{W}^{(n)}$) such that $L(w) \geqq \lambda^{n+m/2} t$. Also, for any $0 \leqq m \leqq n$, let $T_{n,m}$ (resp. $\tilde{T}_{n,m}$) be the summation of $\exp(-\beta_c L(w))$ for all $w \in W^{(n)}$ (resp. $\tilde{W}^{(n)}$) such that $L(w) \leqq \lambda^{n-m}$.

(1) Note that $\lambda^{n+1+m/2} - \lambda^{n+(m+1)/2} = \lambda^{n+m/2}(\lambda - \lambda^{1/2})$. Let $t = (\lambda - \lambda^{1/2})/3$. Then we have

$$\begin{aligned}(3.2) \quad U_{n+1,m} \leqq\ & \{1 + 2(Z_n(\beta_c) + \tilde{Z}_n(\beta_c)) + 2(Z_n(\beta_c) + \tilde{Z}_n(\beta_c))^2 \\ & + 2Z_n(\beta_c)(Z_n(\beta_c) + \tilde{Z}_n(\beta_c))^2\}\, U_{n,m+1} + \{2(S_{n,m}(t) + \tilde{S}_{n,m}(t)) \\ & + 4(Z_n(\beta_c) + \tilde{Z}_n(\beta_c))(S_{n,m}(t) + \tilde{S}_{n,m}(t)) \\ & + 6(Z_n(\beta_c) + \tilde{Z}_n(\beta_c))^2 (S_{n,m}(t) + \tilde{S}_{n,m}(t)\}\, M_n\end{aligned}$$

for any $n, m \geqq 0$. Let us explain how one obtains the inequality (3.2).

The summation of $\exp(-\beta_c L(w))$ for $w \in W_0$ such that $D(w) \leqq n$ and $L(w) \geqq \lambda^{n+1+m/2}$ is dominated by $U_{n,m+1}$. This is the first term of the righthand side of (3.2). Remember that the path $w \in W$ with $D(w) = n+1$ can be classified into 3 types (Fig. 1). Since the other cases are similar, we only observe the path $w \in W_0$ of type 3 with $D(w) = n+1$. This path w consists of four parts w_1, w_2, w_3 and w_4 (see Fig. 2). The case that the length $L(w)$ of w is greater than or equal to $\lambda^{n+1+m/2}$ is covered by the following two cases: Case 1, the length of w_4 is greater than or equal to $\lambda^{n+(m+1)/2}$, and Case 2, one of the lengths of w_1, w_2, w_3 is greater than or equal to $\lambda^{n+m/2}t$.

The summation of $\exp(-\beta_c L(w))$ for w in Case 1 (resp. Case 2) is dominated by $2Z_n(\beta_c)(Z_n(\beta_c)+\tilde{Z}_n(\beta_c))^2)\,U_{n,m+1}$ (resp. $6(Z_n(\beta_c)+\tilde{Z}_n(\beta_c))^2(S_{n,m}(t)+\tilde{S}_{n,m}(t))\,M_n$). These are the fourth term and the seventh term of the righthand side of (3.2). The other terms come from the paths of types 1 and 2.

Let A_0 be as in Sect. 2. Then there is an $n_2 \geqq 1$ such that

$$1+2(Z_n(\beta_c)+\tilde{Z}_n(\beta_c))+2(Z_n(\beta_c)+\tilde{Z}_n(\beta_c))^2$$
$$+2Z_n(\beta_c)(Z_n(\beta_c)+\tilde{Z}_n(\beta_c))^2 \leqq \mathrm{A}_0$$

for any $n \geqq n_2$. Therefore by (2.2) we see that there is a constant $C<\infty$ such that

$$A_0^{-(n+1)}\,U_{n+1,m} \leqq A_0^{-n}\cdot U_{n,m+1}+C(S_{n,m}(t)+\tilde{S}_{n,m}(t))$$

for any $n \geqq n_2$ and $m \geqq 0$. Note that $U_{n,m}=0$ if $m \geqq 4n$. Therefore we see that

$$A_0^{-n}\cdot U_{n,m} \leqq C A_0^{-1}\cdot \sum_{k=0}^{[4n/5]} (S_{n-k-1,m+k}(t)+\tilde{S}_{n-k-1,m+k}(t))$$

for any $n \geqq 6n_2$ and $m \geqq 0$.

On the other hand, we have

$$S_{n,m}(t) \leqq \exp(-t\cdot\lambda^{m/2}) \sum_{w\in W^{(n)}} \exp(-(\beta_c-\lambda^{-n})\,L(w))$$
$$=\exp(-t\cdot\lambda^{m/2})\cdot Z_n(\beta_c-\lambda^{-n}), \qquad n \geqq 1.$$

Similarly we have

$$\tilde{S}_{n,m}(t) \leqq \exp(-t\cdot\lambda^{m/2})\cdot \tilde{Z}_n(\beta_c-\lambda^{-n}).$$

So there is a constant $C'<\infty$ such that

$$A_0^{-n}\cdot U_{n,m} \leqq C'\cdot \sum_{k=0}^{\infty} \exp(-t\cdot\lambda^{(m+k)/2})$$

for any $n \geqq n_2$ and $m \geqq 0$. This implies our assertion (1).

(2) Note that

$$V_{n,m} \leqq \{2(T_{n,m}+\tilde{T}_{n,m})+2(T_{n,m}+\tilde{T}_{n,m})^2+2(T_{n,m}+\tilde{T}_{n,m})^3\}\,M_n \tag{3.3}$$

for any $n, m \geqq 0$. Observe that

$$T_{n,m} \leqq e\cdot \sum_{w\in W^{(n)}} \exp(-(\beta_c+\lambda^{m-n})\,L(w))=e\cdot Z_n(\beta_c+\lambda^{m-n}).$$

Similarly we have $\tilde{T}_{n,m} \leqq e \cdot \tilde{Z}_n(\beta_c + \lambda^{m-n})$. So by Proposition 1.6 there are $n_0 \in \mathbb{N}$, $\gamma_2 > 0$ and $C_2 < \infty$ such that

$$T_{n,m} + \tilde{T}_{n,m} \leqq e \cdot C_2 \cdot \exp(-\gamma_2 \cdot 2^m)$$

for any $n, m \geqq 0$ with $n \geqq n_0 + m$. Combining this with (2.2) and (3.3), we have our assertion (2).

4 Proof of Theorem

First, we prove the following.

(4.1) **Proposition.** (1) *There is an* $\alpha > 0$ *such that*

$$P_n[\|w\| \leqq (\log n)^{-\alpha} \cdot n^{\nu}] \cdot \exp((\log n)^2) \to 0 \quad as \quad n \to \infty.$$

(2) *There is a* $\beta > 0$ *such that*

$$P_n[\|w\| \geqq (\log n)^{\beta} \cdot n^{\nu}] \cdot \exp((\log n)^2) \to 0 \quad as \quad n \to \infty.$$

Here $\nu = \log 2/\log \lambda$ *as in theorem.*

Proof. (1) Let $K(\ell) = \left[\frac{\log \ell}{\log \lambda}\right]$, $\ell = 1, 2, \ldots$, where $[x]$ stands for the maximum integer less than or equal to x. Then we see that $\lambda^{K(\ell)} \leqq \ell < \lambda^{K(\ell)+1}$ Also, it is easy to see that $2^{D(w)-1} \leqq \|w\| \leqq 2^{D(w)}$, $w \in W_0$.

By Proposition 3.1 we have for $0 \leqq m \leqq K(\ell)$

$$\begin{aligned} &\#(\{w \in W_0; L(w) = \ell, D(w) \leqq K(\ell) - m\}) \\ &\leqq \exp(\beta_c \ell) \cdot U_{K(\ell)-m,2m} \\ &\leqq C \cdot \exp(\beta_c \ell + ((\log A_0)/\log \lambda) \cdot \log \ell - \gamma \cdot \lambda^m). \end{aligned}$$

So combining this with Proposition 2.5, we see that if a is large enough, there is a constant C' such that

$$P_n[D(w) \leqq K(n) - a \cdot \log(\log n)] \leqq C' \cdot \exp(-(\log n)^3), \qquad n = 1, 2, \ldots.$$

This implies our assertion (1).
(2) By Proposition 3.1 we have for $m, k \geqq 0$

$$\begin{aligned} &\#(\{w \in W_0; L(w) = \ell, D(w) = K(\ell) + m + k + 2\}) \\ &\leqq \exp(\beta_c \ell) \cdot V_{K(\ell)+m+k+1,m+k} \\ &\leqq C \cdot \exp(\beta_c \ell)\, A_0^{K(\ell)+m+k+1} \cdot \exp(-\gamma \cdot 2^{m+k}) \\ &\leqq C \cdot A_0 \cdot \exp(\beta_c \ell) + K(\ell) \cdot \log A_0 + m \cdot \log A_0 - \gamma \cdot 2^{m-1}) \cdot A_0^k \cdot \exp(-\gamma \cdot 2^{k-1}). \end{aligned}$$

Therefore if b is sufficiently large, there is a constant C' such that

$$P_n[D(w) \geqq K(n) + b \cdot \log(\log n)] \leqq C' \cdot \exp(-(\log n)^3), \qquad n = 1, 2, \ldots.$$

This implies our assertion (2).

This completes the proof.

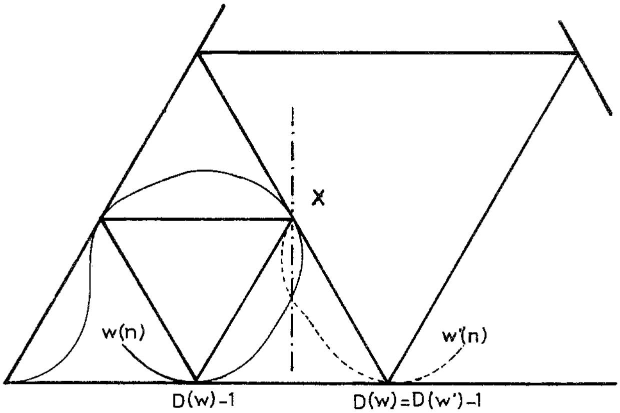

Fig. 3

(4.2) **Lemma.**

$$E^{P_n}[2^{(D(w)-1)s}, |w(n)| \leqq 2^{D(w)-1}]$$
$$\leqq E^{P_n}[2^{(D(w)-1)s}, |w(n)| \geqq 2^{D(w)-1}]$$

for any $n = 1, 2, \ldots,$ *and* $s > 0$.

Proof. This follows from a reflection principle shown in Fig. 3. Note in Fig. 3 that a self-avoiding path $w \in W_0$ satisfying $|w(n)| \leqq 2^{D(w)-1}$ must hit the pivot point x.

(4.3) **Corollary.** *For any* $s > 0$ *and* $n = 1, 2, \ldots,$

$$2^{-s-1} \cdot E^{P_n}[2^{sD(w)}] \leqq E^{P_n}[|w(n)|^s] \leqq E^{P_n}[\|w\|^s] \leqq E^{P_n}[2^{sD(w)}].$$

Proof. This follows from Lemma 4.2 and the fact that $|w(L(w))| \leqq \|w\| \leqq 2^{D(w)}$.

Now let us prove our theorem. The assertion (1) is a consequence of Propositions 2.3 and 2.5. The assertion (3) follows from Proposition 4.1 immediately. So we only have to prove the assertion (2).

In view of Proposition 4.3 it suffices to prove the assertion (2) with $\|w\|$ in place of $|w(n)|$. Let $s > 0$. Note that by Chebychev's inequality

$$E^{P_n}[\|w\|^s] \geqq \{(\log n)^{-\alpha} n^{\nu}\}^s (1 - P_n[\|w\| \leqq (\log n)^{-\alpha} n^{\nu}]).$$

So by Proposition 4.1(1) we see that

$$\varliminf_{n \to \infty} (\log n)^{s\alpha} \cdot n^{-s\nu} E^{P_n}[\|w\|^s] > 0. \tag{4.4}$$

Also, note that

$$E^{P_n}[\|w\|^s] \leqq \{(\log n)^{\beta} n^{\nu}\}^s + n^s \cdot P_n[\|w\| \geqq (\log n)^{\beta} n^{\nu}].$$

So by Proposition 4.1(2) we see that

$$\varlimsup_{n \to \infty} (\log n)^{-s\beta} \cdot n^{-s\nu} E^{P_n}[\|w\|^s] < \infty. \tag{4.5}$$

By (4.4) and (4.5), we have

$$\lim_{n \to \infty} (\log n)^{-1} \cdot \log E^{P_n}[\|w\|^s] = \nu \cdot s.$$

This implies the assertion (2).

This completes the proof of Theorem.

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