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**Probability Theory and Related Fields**



# Stable mixed moving averages

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**Summary.** The class of (non-Gaussian) stable moving average processes is extended by introducing an appropriate joint randomization of the filter function and of the stable noise, leading to stable mixed moving averages. Their distribution determines a certain combination of the filter function and the mixing measure, leading to a generalization of a theorem of Kanter (1973) for usual moving averages. Stable mixed moving averages contain sums of independent stable moving averages, are ergodic and are not harmonizable. Also a class of stable mixed moving averages is constructed with the reflection positivity property.

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## 1 Introduction

Stationary symmetric $\alpha$-stable ($S\alpha S$) processes have been fully described in [4], a key ingredient being a group of isometries on an $L^\alpha$ space. In the Gaussian case $\alpha = 2$ this leads to the spectral representation of all stationary Gaussian processes (which are continuous in probability). However the non-Gaussian stable case $0 < \alpha < 2$ is subtler and no explicit representation is known for all stationary stable processes (which are continuous in probability). The main two subclasses studied have explicit representations motivated by the Gaussian case: the harmonizable processes, which are superpositions of harmonics with (complex) $S\alpha S$ amplitudes, and the moving average processes, which are filtered $S\alpha S$ stationarily and independently scattered noise. In the Gaussian case $\alpha = 2$, the latter is a subclass of the

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former. However, for $0 < \alpha < 2$ the two classes are disjoint [8]. An important property of $S\alpha S$ moving average processes is that for $0 < \alpha < 2$, the distribution of the process determines essentially the filter function of the moving average (except for a translation and sign).

The purpose of this work is to expand the class of non-Gaussian $S\alpha S$ moving average processes by the introduction of an appropriate joint randomization of the filter function and of the $S\alpha S$ noise (Sect. 2). We term them $S\alpha S$ mixed moving averages. In a far reaching generalization of the theorem of Kanter [5], it is shown in Theorem 1 that the distribution of a $S\alpha S$ mixed moving average determines a certain measure $\nu$ on symmetric (under translation and sign change) sets of the unit sphere of $L^\alpha$, which combines the filter function and the mixing measure.

In Sect. 3, we show that sums of independent $S\alpha S$ moving averages, and in particular stationary $S\alpha S$ processes with finite multiplicity, are $S\alpha S$ mixed moving averages. We also show that $S\alpha S$ mixed moving averages are mixing, but not harmonizable. Finally, in Sect. 4, we characterize the reflection positivity of $S\alpha S$ mixed moving averages and provide a rich class of examples. As there are very few Markov $S\alpha S$ processes [1], the existence of reflection positive mixed moving averages may be of more general interest.

## 2 Characterization

A stable mixed moving average field is defined by

(2.1) $$X(t) = \int_{\mathscr{X}\times\mathbf{R}^d} f(x, t-s)\, M(dx, ds), \quad t \in \mathbf{R}^d,$$

where $M$ is independently scattered symmetric $\alpha$-stable ($S\alpha S$) random measure on $\mathscr{X}\times\mathbf{R}^d$ with control measure $Q \otimes \mathrm{Leb}$ and $0 < \alpha \leqq 2$, $(\mathscr{X}, Q)$ is a $\sigma$-finite complete measure space, $f\colon \mathscr{X}\times\mathbf{R}^d \to \mathbf{R}$ is measurable and such that

(2.2) $$0 < \int_{\mathscr{X}\times\mathbf{R}^d} |f(x,t)|^\alpha Q(dx)dt < \infty .$$

The usual $S\alpha S$ moving averages correspond to $\mathscr{X}$ consisting of one point, i.e. $\mathscr{X} = \{1\}$, $Q = \delta_{\{1\}}$ and $f(1,\cdot) \equiv f(\cdot) \in L^\alpha(\mathbf{R}^d)$. The distribution of the process $\{X(t)\colon t \in \mathbf{R}^d\}$ is determined by its finite dimensional characteristic functions

(2.3) $$\phi_X(a) = E\exp\left\{i \sum_{t\in\mathbf{R}^d} a(t)X(t)\right\}$$
$$= \exp\left\{-\int_{\mathscr{X}\times\mathbf{R}^d} \left|\sum_{t\in\mathbf{R}^d} a(t)f(x,t-s)\right|^\alpha Q(dx)ds\right\},$$

where $a$ is a real function on $\mathbf{R}^d$ such that $a(t) = 0$ for all but finitely many $t$, via its parameters $f$ and $Q$. It is clear from (2.3) that $X$ is stationary. The first question we address is when two stable mixed moving averages $X_1$ and $X_2$, with parameters $(f_1, Q_1)$ and $(f_2, Q_2)$, have the same distribution. For usual stable moving averages Kanter [5] has shown that $X_1$ and $X_2$ have the same distribution if and only if $f_1$ and $f_2$ differ by a global sign and a shift: $f_2(\cdot) = \varepsilon f_1(\cdot - s)$ for some $s$ and $\varepsilon = \pm 1$. For mixed stable moving averages the characterization is more complex and is given in Theorem 1.

It follows from (2.2) that for $Q-$ a.a. $x$, $f(x,\cdot)\in L^\alpha(\mathbf{R}^d)$. Without loss of generality we may further assume this holds for all $x\in\mathscr{X}$. Define $\mathbf{f}:\mathscr{X}\to L^\alpha(\mathbf{R}^d)$ by $[\mathbf{f}(x)](t)=f(x,t)$ and note that

$$v(A)=\int_{L^\alpha(\mathbf{R}^d)} 1_A\left(\frac{y(\cdot)}{\|y\|_\alpha}\right)\|y\|_\alpha^\alpha(Q\circ\mathbf{f}^{-1})(dy) \tag{2.4}$$

is a finite measure by (2.2), supported by the unit sphere $S_\alpha$ of $L^\alpha(\mathbf{R}^d)$, and by (2.3),

$$\phi_X(a)=\exp\left\{-\int_{L^\alpha(\mathbf{R}^d)}\left\|\sum_t a(t)h(t-\cdot)\right\|_\alpha^\alpha v(dh)\right\}. \tag{2.5}$$

Let $\{U_s\colon s\in\mathbf{R}^d\}$ denote the group of shift operators on $L^\alpha(\mathbf{R}^d)$, $(U_s g)(t)=g(t-s)$. Define now an equivalence relation on $L^\alpha(\mathbf{R}^d)$ by

$$g\sim h \Leftrightarrow \exists\,\varepsilon=\pm 1,\quad s\in\mathbf{R}^d\colon \varepsilon U_s g=h\,.$$

Let $\pi\colon L^\alpha(\mathbf{R}^d)\to L^\alpha(\mathbf{R}^d)/_\sim$ be the canonical mapping.

**Theorem 1** *There is a one-to-one correspondence between the distribution of the process $\{X(t)\colon t\in\mathbf{R}^d\}$ defined by* (2.1) *with $0<\alpha<2$, and the measure $v\circ\pi^{-1}$ on the quotient space $L^\alpha(\mathbf{R}^d)/_\sim$.*

The proof of Theorem 1 is preceded by two lemmas. A Borel subset $A$ of $L^\alpha(\mathbf{R}^d)$ is symmetric if $-A=A$ and shift-invariant if $U_tA=A$ for every $t\in\mathbf{R}^d$. The proof of Lemma 1 is routine and thus omitted.

**Lemma I** *Let $v_1$ and $v_2$ be two Borel measures on $L^\alpha(\mathbf{R}^d)$. Then $v_1\circ\pi^{-1}=v_2\circ\pi^{-1}$ if and only if $v_1(A)=v_2(A)$ for any open symmetric shift-invariant set $A$.*

Lemma 2 can be proven using the (time-reversed) Martingale Convergence Theorem.

**Lemma 2** *Let $g\in L^1(\mathbf{R}^d)$. Define*

$$g_n(s)=2^{-dn}\sum_{t\in L_n} g(t+s),\quad s\in\mathbf{R}^d,$$

*if the series converges absolutely and $g_n(s)=0$ otherwise, where $L_n=\{(k_1 2^{-n},\dots,k_d 2^{-n})\colon k_1,\dots,k_d\in\mathbf{Z}\}$ is a lattice in $\mathbf{R}^d$, $n=0,1,\dots$. Then, as $n\to\infty$,*

$$g^n(s)\to\int_{\mathbf{R}^d} g(u)du \quad \text{a.e. [Leb]}\,.$$

*Proof of Theorem 1* The process $X$, regarded as a random vector in $\mathbf{R}^{\mathbf{R}^d}$ equipped with the cylindrical $\sigma$-algebra, is infinitely divisible and its distribution is uniquely determined by its Lévy measure given by

$$\Delta(C)=\int_{\mathscr{X}\times\mathbf{R}^d\times\mathbf{R}_0} 1_C(\{wf(x,\cdot-s)\})\,Q(dx)ds\frac{c_\alpha dw}{|w|^{1+\alpha}}, \tag{2.6}$$

where $c_\alpha$ is a numerical constant, $\mathbf{R}_0=\mathbf{R}\backslash\{0\}$.

Let

$$A=\{h\in L^\alpha(\mathbf{R}^d)\colon \|h-h_i\|_\alpha<r_i, i=1,\dots,m\}$$

be a *symmetric* open set in $L^\alpha(\mathbf{R}^d)$, where $h_i\in L^\alpha(\mathbf{R}^d)$ are continuous functions with compact support and $r_i>0$ (we may assume $m=2k$, $h_i=-h_{i+k}$, $r_i=r_{i+k}$, $i=1,\dots k$). Note that such $A$ generate the $\sigma$-algebra of symmetric Borel subsets of

$L^\alpha(\mathbf{R}^d)$. Consider a function $q\colon \mathbf{R}^{\mathbf{R}^d} \to [0, \infty]$ given by

$$q^\alpha(e) = \limsup_{n\to\infty} 2^{-dn} \sum_{t\in L_n} |e(t)|^\alpha, \quad e \in \mathbf{R}^{\mathbf{R}^d},$$

where $L_n$ is as in Lemma 2. We now relate with the set $A$ a cylindrical set $C$ of the form

$$C = \left\{ e \in \mathbf{R}^{\mathbf{R}^d}\colon q(e) \geqq 1,\ q\left(\frac{e}{q(e)} - h_i\right) < r_i, i = 1, \ldots, m \right\}.$$

By Lemma 2, $q(h(\cdot - s)) = \|h\|_\alpha$ $ds$ – a.e. for each $h \in L^\alpha(\mathbf{R}^d)$. Moreover, for every $h \in L^\alpha(\mathbf{R}^d)$ and $i = 1, \ldots, m$,

$$q(h(\cdot - s) - h_i(\cdot)) = \|h(\cdot - s) - h_i(\cdot)\|_\alpha \quad ds - \text{a.e.}$$

Indeed, fix $i$ and consider $h_{i,t}(u) := h_i(u + t)$, $u \in \mathbf{R}^d$, $t \in L_n$. By Lemma 2, for each $t \in L_n$, the set

$$V_t := \{s \in \mathbf{R}^d\colon q(h(\cdot - s) - h_{i,t}(\cdot - s)) = \|h - h_{i,t}\|_\alpha\}$$

is of full Lebesgue measure; hence $V = \bigcap_{n\geqq 0, t\in L_n} V_t$ is of full Lebesgue measure. Let $s \in V$ be fixed. Since $h_i$ is continuous and with compact support, both $q(h_{i,t}(\cdot - s) - h_i \cdot)$ and $\|h_{i,t}(\cdot - s) - h_i(\cdot)\|_\alpha$ can be made arbitrarily small by choosing sufficiently large $n$ and some $t \in L_n$ that is close to $s$. Using then triangle inequality and the definition of $V_t$ we infer that $q(h(\cdot - s) - h_i(\cdot)) = \|h(\cdot - s) - h_i(\cdot)\|_\alpha$.

Therefore we have

$$\begin{aligned} \Delta(C) &= \int_{\mathscr{X}\times\mathbf{R}^d\times\mathbf{R}_0} 1\left(\left\{(w, s, x)\colon q(wf(x, \cdot - s)) \geqq 1,\ q\left(\frac{f(x, \cdot - s)}{\|f(x,\cdot)\|_\alpha} - h_i\right) < r_i,\right.\right. \\ &\qquad\qquad \left.\left. i = 1, \ldots, m\right\}\right) Q(dx)ds \frac{c_\alpha dw}{|w|^{1+\alpha}} \\ &= \frac{2c_\alpha}{\alpha} \int_{\mathscr{X}\times\mathbf{R}^d} 1\left(\left\{(s, x)\colon \left\|\frac{f(x, \cdot - s)}{\|f(x,\cdot)\|_\alpha} - h_i\right\|_\alpha < r_i, \quad i = 1, \ldots, m\right\}\right) \|f(x,\cdot)\|_\alpha^\alpha Q(dx)ds \\ &= \frac{2c_\alpha}{\alpha} \int_{\mathbf{R}^d} \nu(U_s^{-1}A)ds. \end{aligned}$$

Thus the distribution of $X$ determines the measure

(2.7) $$N_\nu(A) := \int_{\mathbf{R}^d} \nu(U_s^{-1}A)ds$$

on the $\sigma$-algebra of all symmetric Borel subsets of $L^\alpha(\mathbf{R}^d)$.

Let us show that $N_\nu$ determines the measure $\nu\circ\pi^{-1}$. To this end we will construct a symmetric Borel function $t^*\colon L^\alpha(\mathbf{R}^d) \to \mathbf{R}^d$ such that

(2.8) $$t^*(U_s h) = t^*(h) + s, \quad \forall h \in L^\alpha(\mathbf{R}^d),\ s \in \mathbf{R}^d.$$

For each $h \in L^\alpha(\mathbf{R}^d)$, $h \neq 0$, consider a function $\psi_h(t) = \int_t^{t+1} |h(s)|^\alpha ds$, $1 = (1, \ldots, 1)$, $t = (t_1, \ldots, t_d) \in \mathbf{R}^d$. Clearly $\psi_h$ is continuous and vanishes as $|t| \to \infty$. Therefore the set

$$F_h = \{t \in \mathbf{R}^d\colon \psi_h(t) = \sup_{s\in\mathbf{R}^d} \psi_h(s)\}$$

is compact nonempty. The point $t^*(h) = (t_1^*, \ldots, t_d^*)$ is chosen from $F_h$ as follows: $t_1^* = \min\{t_1 \colon t \in F_h\}$, $t_2^* = \min\{t_2 \colon t \in F_h,\ t_1 = t_1^*\}, \ldots, t_d^* = \min\{t_d \colon t \in F_h,\ t_1 = t_1^*, \ldots, t_{d-1} = t_{d-1}^*\}$. We put $t^*(h) = 0$ if $h = 0$. Clearly $t^*(-h) = t^*(h)$. It is also easy to show that $t^*$ is a Borel function (just consider $\{h\colon t_1^*(h) \leqq a_1, \ldots, t_d^*(h) \leqq a_d\}$). Since $F_{U_s h} = F_h + s$, (2.8) follows from the definition of $t^*$. Put

$$B_s = \{h \in L^\alpha(\mathbf{R}^d) \colon t^*(h) \in [s, s+1)\}, \quad s \in \mathbf{R}^d.$$

Using (2.7) and (2.8) we obtain, for every symmetric shift-invariant set $A$,

$$N_\nu(A \cap B_0) = \int_{\mathbf{R}^d} \nu(U_s^{-1}(A \cap B_0)\, ds = \int_{\mathbf{R}^d} \nu(A \cap B_s) ds = \int_{[0,1)^d} \sum_{t \in L_0} \nu(A \cap B_{t+s})\, ds = \nu(A), \tag{2.9}$$

because, for each fixed $s$, the sets $B_{t+s}$, $t \in L_0$, are disjoint and $\bigcup_{t \in L_0} B_{t+s} = L^\alpha(\mathbf{R}^d)$. Since $t^*$ is symmetric , $B_0$ is symmetric, so that $N_\nu(A \cap B_0)$ is uniquely determined by the distribution of $X$. In view of (2.9) and Lemma 1 the distribution of $X$ uniquely determines $\nu \circ \pi^{-1}$.

Now we shall prove the converse part of the theorem, that $\nu \circ \pi^{-1}$ uniquely determines the distribution of $X$. Using (2.5) and noting that in the integrand, $h$ may be replaced by $\pi h$, we obtain

$$-\log \phi_X(a) = \int_{L^\alpha(\mathbf{R}^d)} \left\| \sum_{t \in \mathbf{R}^d} a(t) h(t - \cdot) \right\|_\alpha^\alpha \nu(dh) = \int_{L^\alpha(\mathbf{R}^d)/\sim} \left\| \sum_{t \in \mathbf{R}^d} a(t) h(t - \cdot) \right\|_\alpha^\alpha (\nu \circ \pi^{-1})(dh),$$

where in the last integral, $\|\cdot\|_\alpha$ is not a norm in $L^\alpha(\mathbf{R})/_\sim$ but is well-defined as a continuous function on it. Hence $\nu \circ \pi^{-1}$ determines $\phi_X(a)$. The proof of the theorem is complete. □

*Discrete case*

A discrete analog of (2.1) is obtained by replacing $\mathbf{R}^d$ by $\mathbf{Z}^d$ and Lebesgue measure by the counting measure on $\mathbf{Z}^d$. Therefore we have

$$X(n) = \int_{\mathscr{X}} \sum_{k \in \mathbf{Z}^d} f(x, n-k)\, M(dx, k), \quad n \in \mathbf{Z}^d, \tag{2.10}$$

where $M(\cdot, k)$, $k \in \mathbf{Z}^d$, are independent $S\alpha S$-stable random measures on $(\mathscr{X}, Q)$ with control measure $Q$. In order that (2.10) be well defined we clearly must assume that

$$0 < \sum_{k \in \mathbf{Z}^d} \int_{\mathscr{X}} |f(x,k)|^\alpha Q(dx) < \infty\ . \tag{2.11}$$

A measure $\nu$ is defined in an analogous way as in the continuous case; it is now supported by the unit sphere of $l^\alpha(\mathbf{Z}^d)$, and the relation $\sim$ is defined on $l^\alpha(\mathbf{Z}^d)$. An analog to Theorem 1 holds in the discrete case and its proof is similar to but simpler than the proof of Theorem 1.

**Theorem 2** *There is a one-to-one correspondence between the distribution of the process* $\{X(n)\colon n \in \mathbf{Z}^d\}$ *defined by* (2.10) *with* $0 < \alpha < 2$, *and the measure* $v \circ \pi^{-1}$ *on the quotient space* $l^\alpha(\mathbf{Z}^d)/_\sim$.

*Proof.* First we obtain, by the uniqueness of Lévy measures, that the distribution of $X$ determines the measure

$$N_v(A) = \sum_{k \in \mathbf{Z}^d} v(U_k^{-1} A)$$

uniquely on the $\sigma$-algebra of symmetric Borel subsets of $l^\alpha(\mathbf{Z}^d)$ (this is straightforward since Borel and cylindrical $\sigma$-algebras coincide in this case). Then we define $\psi_h(k) = |h(k)|$, $h \in l^\alpha(\mathbf{Z}^d)$, $k \in \mathbf{Z}^d$. Having $\psi_h$, $t^*(h)$ is defined exactly in the same way as in the proof of Theorem 1. Put $B_k = \{h \in l^\alpha(\mathbf{Z}^d)\colon t^*(h) = k\}$, $k \in \mathbf{Z}^d$. We have, for every symmetric shift-invariant set $A$,

$$N_v(A \cap B_0) = \sum_{k \in \mathbf{Z}^d} v(U_k^{-1}(A \cap B_0)) = \sum_{k \in \mathbf{Z}^d} v(A \cap B_k) = v(A).$$

By Lemma 1, which holds in the discrete case as well, the distribution of $X$ determines $v \circ \pi^{-1}$. The converse follows by the same arguments as in the proof of Theorem 1. □

We should notice that all results in this paper, given for continuous time, clearly have discrete counterparts with proofs essentially the same or simpler than in the continuous time case. The following result, in the case $d = 1$, is due to Kanter [5].

**Corollary 1** *Let* $\{X_i(t)\colon t \in \mathbf{R}^d\}$, $i = 1, 2$, *be usual* $S\alpha S$ *moving averages, with* $0 < \alpha < 2$,

$$X_i(t) = \int_{\mathbf{R}^d} f_i(t - s)\, M_i(ds), \quad t \in \mathbf{R}^d,$$

*where* $M_i$ *are independently scattered* $S\alpha S$ *random measures on* $\mathbf{R}^d$ *with Lebesgue control measure and* $f_i \in L^\alpha(\mathbf{R}^d)$. *Then* $X_1$ *and* $X_2$ *have equal distributions if and only if* $f_1 \sim f_2$.

*Proof.* The measure $v_i$, corresponding to $X_i$, is a one-point measure given by $v_i = \|f_i\|_\alpha^\alpha \delta_{\{\|f_i\|_\alpha^{-1} f_i\}}$. Therefore $v_1 \circ \pi^{-1} = v_2 \circ \pi^{-1}$ if and only if $f_1 \sim f_2$. □

## 3 Examples, ergodicity and harmonizability

In the Gaussian case $\alpha = 2$, the mixed moving averages coincide with the usual moving averages. Indeed one has

$$\begin{aligned}
\frac{1}{2} EX(t)X(t') &= \int_{\mathscr{X}} \left\{ \int_{\mathbf{R}^d} f(x, t - s) f(x, t' - s) ds \right\} Q(dx) \\
&= \int_{\mathscr{X}} \left\{ \frac{1}{(2\pi)^{d/2}} \int_{\mathbf{R}^d} e^{i(t - t', \lambda)} |F(x, \lambda)|^2 d\lambda \right\} Q(dx) \\
&= \frac{1}{(2\pi)^{d/2}} \int_{\mathbf{R}^d} e^{i(t - t', \lambda)} \phi(\lambda) d\lambda = \int_{\mathbf{R}^d} g(t - s) g(t' - s) ds,
\end{aligned}$$

where $F(x, \lambda)$ is the $L^2$-Fourier transform of $f(x, \cdot)$, $\phi(\lambda) = \int_{\mathscr{X}} |F(x, \lambda)|^2 Q(dx) \in L^1(\text{Leb})$ since $\int_{\mathbf{R}^d} \phi(\lambda) d\lambda = (2\pi)^{d/2} \int_{\mathscr{X} \times \mathbf{R}^d} |f(x, t)|^2 dt Q(dx) < \infty$, and $g \in L^2$

(Leb) is the $L^2$-Fourier transform of $\phi^{1/2} \in L^2$(Leb). Therefore

$$\{X(t)\colon t \in \mathbf{R}^d\} \stackrel{\mathcal{L}}{=} \{2^{1/2} \int_{\mathbf{R}^d} g(t-s) B(ds)\colon t \in \mathbf{R}^d\}$$

where $B$ is Gaussian white noise.

We shall see now that in the non-Gaussian stable case $0 < \alpha < 2$ the mixed moving averages form a larger class than the usual moving averages.

*Sums of independent usual moving averages*

Let

$$X(t) = \sum_{k=1}^{n} \int_{\mathbf{R}^d} f_k(t-s) M_k(ds), \quad t \in \mathbf{R}^d, \tag{3.1}$$

where $M_1, \ldots, M_n$ are mutually independent, independently scattered, $S\alpha S$ random measures on $\mathbf{R}^d$ with Lebesgue control measure and $0 < \alpha < 2$. We observe that if, for some $k \neq j$, $f_k \sim c_{k,j} f_j$ where $c_{k,j} \in \mathbf{R}$, then the process

$$\int_{\mathbf{R}^d} f_k(t-s) M_k(ds) + \int_{\mathbf{R}^d} f_j(t-s) M_j(ds), \quad t \in \mathbf{R}^d,$$

has the same distribution as

$$(1 + |c_{k,j}|^\alpha)^{1/\alpha} \int_{\mathbf{R}^d} f_j(t-s) M_j(ds), \quad t \in \mathbf{R}^d.$$

Therefore one can always choose a *minimal* set of functions $\{f_1, \ldots, f_n\}$ in (3.1) to represent in distribution the process $X$, minimal in the sense that for no $k \neq j$, and no $c \in \mathbf{R}^1$, $f_k \sim cf_j$. The sum of independent moving averages (3.1) is a special case of (2.1), with $Q$ being the counting measure on $\mathcal{X} = \{1, \ldots, n\}$ and $f(x,t) = f_x(t)$, $x \in \mathcal{X}$. In such a case, when the set $\{f_1, \ldots, f_n\}$ is minimal, the measure $\nu \circ \pi^{-1}$ is given by $\nu \circ \pi^{-1} = \sum_{k=1}^{n} \|f_k\|_\alpha^\alpha \delta_{\{\pi(\|f_k\|_\alpha^{-1} f_k)\}}$ and has exactly $n$-point support. Hence, applying Theorem 1, we obtain

**Corollary 2** *Let $\{X^{(i)}(t)\colon t \in \mathbf{R}^d\}$ be given by* (3.1), *with $f_k$ replaced by $f_k^{(i)}$, $M_k$ by $M_k^{(i)}$, and $n$ by $n^{(i)}$, $i = 1, 2$. Assume that the representing sets of functions $\{f_1^{(1)}, \ldots, f_{n^{(1)}}^{(1)}\}$ and $\{f_1^{(2)}, \ldots, f_{n^{(2)}}^{(1)}\}$ are minimal. Then $X^{(1)}$ and $X^{(2)}$ have equal distributions if and only if $n^{(1)} = n^{(2)} = n$ and there exists a permutation $p$ of $\{1, \ldots, n\}$ such that $f_k^{(1)} \sim f_{pk}^{(2)}$, $k = 1, \ldots, n$.*

Corollary 2 shows that when $0 < \alpha < 2$ there are (regular) stationary $S\alpha S$ processes with any given multiplicity $n \geqq 1$, whereas all regular stationary Gaussian processes have multiplicity $n = 1$. Also note that *finite sums of independent usual $S\alpha S$ moving averages are dense in the class of mixed moving averages,* as every $\sigma$-finite measure $Q$ on a Borel space can be approximated by discrete measures supported by finite sets of points.

*Mixed memory moving averages*

We will now consider mixed moving averages (2.1) defined by a Radon measure $Q$ on $\mathcal{X} = \mathbf{R}_+^d$ and

$$f(x,t) = 1_{[0,x]}(t) := \begin{cases} 1, & \text{if } 0 \leqq t_i \leqq x_i, \quad i = 1, \ldots, d, \\ 0, & \text{otherwise,} \end{cases}$$

$x = (x_1, \ldots, x_d) \in \mathscr{X}$, $t = (t_1, \ldots, t_d) \in \mathbf{R}^d$. Since $\|f(x, \cdot)\|_\alpha^\alpha = x_1 \ldots x_d$, assumption (2.2) translates in this case to

$$0 < \int_{\mathbf{R}_+^d} x_1 \ldots x_d Q(dx) < \infty .$$

Using Theorem 1 we can show a one-to-one correspondence between the distribution of the corresponding $\alpha$-stable process $X$ and the measure $Q$. In fact, because of the special form of $X$, we can establish an even stronger result, in a simpler way, by evaluating the bivariate characteristic functions of $X$.

**Proposition 1** *There is a one-to-one correspondence between the bivariate distributions of a mixed moving average defined above and the measure $Q$ on $\mathbf{R}_+^d$.*

*Proof.* Write $t \leqq s$ if $t, s \in \mathbf{R}^d$ and $t_1 \leqq s_1, \ldots, t_d \leqq s_d$, and denote $[t, s] = \{u \in \mathbf{R}^d\colon t \leqq u \leqq s\}$, $|[t,s]| = \mathrm{Leb}[t,s]$. Set also $[t, \infty] = \{u \in \mathbf{R}^d\colon t \leqq u\}$, $[t, \infty)^c = \mathbf{R}_+^d \setminus [t, \infty)$. Then for $t \in \mathbf{R}_+^d$ noting that $[-t, x-t] \cap [0, x] = [0, x-t]$, $x \in \mathbf{R}_+^d$, we have

$$- \log E \exp\{i(aX(0) + bX(t))\} = \int_{\mathbf{R}_+^d \times \mathbf{R}^d} |a 1_{[0,x]}(u) + b 1_{[0,x]}(t+u)|^\alpha Q(dx) du$$

$$= \int_{[t,\infty)} Q(dx) \{|a+b|^\alpha |[0, x-t]| + |a|^\alpha |[0,x] \setminus [0, x-t]|$$

$$+ |b|^\alpha |[-t, x-t] \setminus [0, x-t]|\}$$

$$+ \int_{[t,\infty)^c} Q(dx) \{|a|^\alpha |[0,x]| + |b|^\alpha |[-t, x-t]|\}$$

$$= (|a|^\alpha + |b|^\alpha) m_Q(0) + (|a+b|^\alpha - |a|^\alpha - |b|^\alpha) m_Q(t),$$

where $m_Q(t) = \int_{[t,\infty)} |[0, x-t]| Q(dx)$, $t \in \mathbf{R}_+^d$. It follows that the univariate distribution of $X$ determines $m_Q(0) = \int_{\mathbf{R}_+^d} x_1 \ldots x_d Q(dx)$ and then the bivariate distributions determine $m_Q(t)$ for $t \in \mathbf{R}_+^d$. Integrating by parts, one has $m_Q(t) = \int_{[t,\infty)} Q([x, \infty)) dx$, $t \in \mathbf{R}_+^d$, hence $m_Q(t)$ determines the measure $Q$. Conversely, clearly the measure $Q$ determines all multivariate distributions of $X$. □

*Ergodicity*

We notice that stable mixed moving average processes in $\mathbf{R}^1$ are mixing. A real stationary process $\{X(t)\colon t \in \mathbf{R}\}$ is mixing if and only if

$$\lim_{T \to \infty} E(\xi \eta_T) = E(\xi) E(\eta)$$

for all $\xi$: $(X(t)\colon t \leqq 0)$-measurable and $\eta$: $(X(t)\colon t \geqq 0)$-measurable with $E\xi^2 < \infty$, $E\eta^2 < \infty$, where $\eta_T$ is $\eta$ shifted by $T$. The following result can be proven essentially like Theorem 2 in [2].

**Theorem 3** *Every S$\alpha$S mixed moving average process with $0 < \alpha \leqq 2$ is mixing.*

The stable generalized moving average fields are actually doubly stationary in the sense (introduced in [2. Sect. 6]) that their representing kernel $f(\cdot, t - \cdot)$ is stationary with respect to the control measure. Indeed for all $B \in \mathscr{B}(\mathbf{R}^n)$, $t_1, \ldots, t_n$, $\tau \in \mathbf{R}^d$, $n \geqq 1$,

$$\begin{aligned}(Q \otimes \mathrm{Leb})\{(x,s)\colon (f(x,t_1+\tau-s), \ldots, f(x,t_n+\tau-s)) \in B\} \\ = \int_{\mathscr{X}} \mathrm{Leb}\{s\colon (f(x,t_1+\tau-s), \ldots, f(x,t_n+\tau-s)) \in B\}\, Q(dx) \\ = \int_{\mathscr{X}} \mathrm{Leb}\{u\colon (f(x,t_1-u), \ldots, f(x,t_n-u)) \in B\}\, Q(dx) \\ = (Q \otimes \mathrm{Leb})\{(x,s)\colon (f(x,t_1-s), \ldots, f(x,t_n-s)) \in B\}\,.\end{aligned}$$

*Non-harmonizability*

Generally speaking, a $S\alpha S$ process is called harmonizable if it is the Fourier transform of a $S\alpha S$ random measure

$$\int_{\mathbf{R}^d} e^{i\langle t,\lambda\rangle} Z(d\lambda), \quad t \in \mathbf{R}^d, \tag{3.2}$$

where the integral is appropriately defined. The random measure $Z$ is necessarily complex valued. (A complex random variable is $S\alpha S$ if its real and imaginary parts are jointly $S\alpha S$.) For simplicity we will consider only the one dimensional case $d = 1$ here.

Let $L(X)$ be the closure in probability of the complex linear span of $\{X(t)\colon t \in \mathbf{R}\}$ defined by (2.1). Then every $Y \in L(X)$ is of the form $Y = \int g\,dM$ where the complex-valued function $g$ is in $L^\alpha(\mathscr{X} \times \mathbf{R}, Q \otimes \mathrm{Leb})$ and the norm $\|Y\|_\alpha = \|g\|_\alpha := \|g\|_{L^\alpha(Q\otimes\mathrm{Leb})}$ metrizes convergence in probability. We say that the process $X$ is *harmonizable* if there is an $L(X)$-valued strongly countably additive measure $Z$ of finite semivariation on $(\mathbf{R}, \mathscr{B}(\mathbf{R}))$ such that

$$X(t) = \int_{\mathbf{R}} e^{it\lambda} Z(d\lambda), \quad t \in \mathbf{R}, \tag{3.3}$$

where the integral is defined in the usual way (see [3, pp. 318–328]). (For independently scattered $Z$ this is the usual integral.)

**Theorem 4** *A (nondegenerate) $S\alpha S$ mixed moving average process with $1 < \alpha < 2$ is not harmonizable.*

*Proof.* The map $J\colon (L(X), \|\cdot\|_\alpha) \to L^\alpha(\mathscr{X} \times \mathbf{R}, Q \otimes \mathrm{Leb})$ defined by $J(X(t)) = f(*, t - \cdot)$ can be extended to a linear isometry on $L(X)$, denoted also by $J$. Assume $X$ is harmonizable. Then $Z(B) \in L(X)$ for each $B \in \mathscr{B}(\mathbf{R})$ and since $J$ is an isometry, $\mu(B) := J(Z(B))$ is a strongly countably additive $L^\alpha(\mathscr{X} \times \mathbf{R}, Q \otimes \mathrm{Leb})$-valued measure of finite semivariation on $\mathscr{B}(\mathbf{R})$. From (2.1) and (3.3) we obtain

$$f(*, t - \cdot) = \int_{\mathbf{R}} e^{it\lambda} \mu(d\lambda)(*, \cdot)\,. \tag{3.4}$$

The following Lemma 3 implies that for every $g \in L^{\alpha'}(\mathscr{X}, Q)$, $1/\alpha + 1/\alpha' = 1$,

$$f_g(t - \cdot) := \int_{\mathscr{X}} f(x, t - \cdot) g(x) Q(dx) = \int_{\mathbf{R}} e^{it\lambda} \mu_g(d\lambda)(\cdot)\,,$$

where $f_g \in L^\alpha(\mathbf{R}, \mathrm{Leb})$ and $\mu_g\colon \mathscr{B}(\mathbf{R}) \to L^\alpha(\mathbf{R}, \mathrm{Leb})$ defined in (b) of Lemma 3 is a strongly countably additive measure of finite semivariation.Then the arguments in the proof of Proposition 1.9 in [8] imply that $f_g = 0$ in $L^\alpha(\mathbf{R}, \mathrm{Leb})$. Hence for each

$g \in L^{\alpha'}(\mathscr{X}, Q)$ and $h \in L^{\alpha'}(\mathbf{R}, \text{Leb})$ we have

$$0 = \int_{\mathbf{R}} f_g(t)h(t)dt = \int_{\mathscr{X}\times\mathbf{R}} f(x,t)g(x)h(t)Q(dx)dt\,.$$

Since functions of the form $g(x)h(t)$ are linearly dense in $L^{\alpha'}(\mathscr{X}\times\mathbf{R}, Q\otimes\text{Leb})$, it follows that $f = 0$ in $L^{\alpha}(\mathscr{X}\times\mathbf{R}, Q\otimes\text{Leb})$ which contradicts (2.2). □

**Lemma 3** *Let* $1 < \alpha < 2$, $1/\alpha + 1/\alpha' = 1$, $g \in L^{\alpha'}(\mathscr{X}, Q)$ *and* $f$, $\mu$ *be as in* (3.4). *Then*

(a) $f_g(\cdot) := \int_{\mathscr{X}} f(x,\cdot)g(x)Q(dx) \in L^{\alpha}(\mathbf{R}, \text{Leb})$,

(b) $\mu_g(B)(\cdot) := \int_{\mathscr{X}} \mu(B)(x,\cdot)g(x)Q(dx)$ *is a strongly countably additive* $L^{\alpha}(\mathbf{R}, \text{Leb})$*-valued measure of finite semivariation on* $\mathscr{B}(\mathbf{R})$, *and*

(c) $f_g(t - \cdot) = \int_{\mathbf{R}} e^{it\lambda}\mu_g(d\lambda)(\cdot)$, $t \in \mathbf{R}$.

*Proof.* (a) This follows from (2.2) and

$$\int_{\mathbf{R}} |f_g(t)|^{\alpha}dt \leqq \int_{\mathscr{X}\times\mathbf{R}} |f(x,t)|^{\alpha}Q(dx)dt \left\{\int_{\mathscr{X}} |g(x)|^{\alpha'}Q(dx)\right\}^{\alpha/\alpha'} < \infty\,.$$

(b) As in (a) we obtain $\mu_g(B)(\cdot) \in L^{\alpha}(\mathbf{R}, \text{Leb})$ for all $B \in \mathscr{B}(\mathbf{R})$. Also for each $h \in L^{\alpha'}(\mathbf{R}, \text{Leb})$ we have

$$\int_{\mathbf{R}} \mu_g(B)(t)h(t)\,dt = \int_{\mathscr{X}\times\mathbf{R}} \mu(B)(x,t)g(x)h(t)Q(dx)dt$$

and since $g(*)h(\cdot) \in L^{\alpha'}(\mathscr{X}\times\mathbf{R}, Q\otimes\text{Leb})$ and $\mu(B)(*,\cdot)$ is strongly countably additive $L^{\alpha}(\mathscr{X}\times\mathbf{R}, Q\otimes\text{Leb})$-valued measure on $\mathscr{B}(\mathbf{R})$, it follows that $\mu_g(B)(\cdot)$ is weakly countably additive $L^{\alpha}(\mathbf{R}, \text{Leb})$-valued measure on $\mathscr{B}(\mathbf{R})$, and by Pettis' theorem [3, p. 318] is also strongly countably additive. By (a) we have

$$\left\| \sum_{n=1}^{N} a_n\mu_g(B_n) \right\|_{L^{\alpha}(\mathbf{R},\text{Leb})} \leqq \left\| \sum_{n=1}^{N} a_n\mu(B_n) \right\|_{L^{\alpha}(\mathscr{X}\times\mathbf{R}, Q\otimes\text{Leb})} \|g\|_{L^{\alpha'}(\mathbf{R},\text{Leb})}\,,$$

and since $\mu$ is of finite semivariation so is $\mu_g$.

(c) For each $h \in L^{\alpha'}(\mathbf{R}, \text{Leb})$ we have

$$\begin{aligned}
\int_{\mathbf{R}} f_g(t-s)h(s)ds &= \int_{\mathbf{R}}\int_{\mathscr{X}} f(x,t-s)g(x)h(s)Q(dx)ds \\
&= \int_{\mathbf{R}}\int_{\mathscr{X}} \left\{\int_{\mathbf{R}} e^{it\lambda}\mu(d\lambda)(x,s)\right\} g(x)h(s)Q(dx)ds \qquad \text{by (3.4)} \\
&= \int_{\mathbf{R}} \left\{\int_{\mathbf{R}} e^{it\lambda}\mu_g(d\lambda)(s)\right\} h(s)ds \qquad \text{by (b)}
\end{aligned}$$

and since $f_g(t - \cdot)$ and $\int_{\mathbf{R}} e^{it\lambda}\mu_g(d\lambda)(\cdot)$ are both in $L^{\alpha}(\mathbf{R}, \text{Leb})$, the conclusion follows. □

## 4 Reflection positivity

A real process $X = \{X(t)\colon t \in \mathbf{R}\}$ is called *reversible* if the time-reversed process $\{X(-t)\colon t \in \mathbf{R}\}$ has the same distribution as $X$. A strictly stationary reversible process $X = \{X(t)\colon t \in \mathbf{R}\}$ is said to be *reflection positive* if for any $n \geqq 1$ and $0 < t_1 < \ldots < t_n$ and any bounded measurable function $F\colon \mathbf{R}^n \to \mathbf{C}$ the following inequality is true:

$$E\{F(X(t_1),\ldots,X(t_n))\bar{F}(X(-t_1),\ldots,X(-t_n))\} \geqq 0\,. \tag{4.1}$$

The notion of reflection positivity is a generalization of the Markov property for reversible processes, and has its origin in quantum field theory, see e.g. [7]. Klein [6] characterized (vector-valued) Gaussian reflection positive processes; in particular, he showed that the covariance function of any real reflection positive process is given by the Laplace transform of a positive finite measure on $[0,\infty)$.

In this section we characterize reflection positivity for general (non-Gaussian) infinitely divisible processes and in particular also for $S\alpha S$ mixed moving averages. We also introduce a $S\alpha S$ Ornstein–Uhlenbeck type process which is reflection positive.

Let $X = \{X(t)\colon t \in \mathbf{R}\}$ be a real process such that for any collection $\tau = (t_1,\ldots,t_n)$, $t_1 < \ldots < t_n$, $n \geqq 1$, the distribution of $X^\tau := (X(t_1),\ldots,X(t_n))$ is infinitely divisible having the Lévy–Khinchine representation

$$E\exp\{i(a,X^\tau)\} = \exp\left\{\int_{\mathbf{R}_0^n} (e^{i(a,u)} - 1 - i(a,\phi(u)))\,\Delta^\tau(du)\right\}, \tag{4.2}$$

where $a \in \mathbf{R}^n$, $\phi_k(u) = 1$ if $u_k > 1$, $= u_k$ if $|u_k| \leqq 1$, $= -1$ if $u_k < -1$, $k = 1,\ldots,n$, and $\Delta^\tau$ is the (Lévy) measure on $\mathbf{R}_0^n = \mathbf{R}^n \backslash \{0\}$ such that

$$\int_{\mathbf{R}_0^n} (u,u)(1+(u,u))^{-1}\,\Delta^\tau(du) < \infty\,.$$

**Theorem 5** *Let $X = \{X(t)\colon t \in \mathbf{R}\}$ be a real stationary reversible infinitely divisible process having the Lévy–Khinchine representation* (4.2). *Then $X$ is reflection positive if and only if for any $\tau = (t_1,\ldots,t_n)$, $0 < t_1 < \ldots < t_n$, $n \geqq 1$, and any $F \in L^2(\mathbf{R}_0^n, \Delta^\tau)$ the following inequality is true*

$$\int_{\mathbf{R}_0^{2n}} F(u_+)\bar{F}(u_-)\,\Delta^{\tau\cup(-\tau)}(du) \geqq 0\,, \tag{4.3}$$

*where $\tau \cup (-\tau) = (-t_n,\ldots,-t_1,t_1,\ldots,t_n)$, $u = (u_{-n},\ldots,u_{-1},u_1,\ldots,u_n) \in \mathbf{R}_0^{2n}$, $u_+ = (u_1,\ldots,u_n)$, $u_- = (u_{-1},\ldots,u_{-n})$, and the functions $u \mapsto F(u_+)$, $u \mapsto F(u_-)\colon \mathbf{R}_0^{2n} \to \mathbf{C}$ in* (4.3) *satisfy the condition*

$$F(0,\ldots,0) = 0\,. \tag{4.4}$$

*Remark 1* The system of Lévy measures $\{\Delta^\tau\colon \tau = (t_1,\ldots,t_n), t_1 < \ldots < t_n, n \geqq 1\}$ satisfying (4.3) will be called *conditionally reflection positive*, the word 'conditionally' referring to the condition (4.4). It should be noted that, although the function $F$ in Theorem 5 is originally defined on the set $\mathbf{R}_0^n = \mathbf{R}^n \backslash \{0\}$ only, the expressions $F(u_+)$, $F(u_-)$ in (4.3) denote its extensions on $\mathbf{R}_0^{2n}$ defined with the help of (4.4), as $\{u \in \mathbf{R}_0^{2n}\colon u_+ = 0\}$ and $\{u \in \mathbf{R}_0^{2n}\colon u_- = 0\}$ are non-empty. Note also that

$$\int_{\mathbf{R}_0^{2n}} |F(u_+)|^2 \Delta^{\tau\cup(-\tau)}(du) = \int_{\mathbf{R}_0^n} |F(u)|^2 \Delta^\tau(du) < \infty$$

because of (4.4) and the consistency relation between the Lévy measures $\Delta^\tau$ and $\Delta^{\tau\cup(-\tau)}$; hence the left hand side of (4.3) is well-defined.

*Proof of Theorem 5* Let us show the sufficiency of the conditions of the theorem for $X$ to be reflection positive. Note that it suffices to prove (4.1) for $F\colon \mathbf{R}^n \to \mathbf{C}$ of the form $F(x_1, \ldots, x_n) = \sum_{k=1}^m c_k \exp\{i(\lambda^{(k)}, x)\}$, $c_k \in \mathbf{C}$, $\lambda^{(k)} \in \mathbf{R}^n$, $x = (x_1, \ldots, x_n) \in \mathbf{R}^n$, $k = 1, \ldots, m$, $m = 1, 2, \ldots$, in which case it becomes

$$\sum_{k,k'=1}^{m} c_k \overline{c_{k'}} \exp\{d_{kk'}\} \geqq 0 , \tag{4.5}$$

where

$$d_{kk'} = \int_{\mathbf{R}_0^{2n}} [\exp\{i(\lambda^{(k)}, u_+) - i(\lambda^{(k')}, u_-)\} - 1 - i(\lambda^{(k)}, \phi(u_+)) + i(\lambda^{(k')}, \phi(u_-))] \Delta^{\tau \cup (-\tau)}(du) \tag{4.6}$$

according to (4.2). As $X$ is reversible, the matrix $(d_{kk'})_{k,k'=\overline{1,m}}$ is Hermitian, and it is well-known (see e.g. [10]) that in such a case (4.5) is equivalent to the positive definitiveness of the matrix $\tilde{d}_{kk'} := d_{kk'} - d_{k1} - d_{1k'} + d_{11}$, $k, k' = 1, \ldots, m$, or the inequality

$$\sum_{k,k'=1}^{m} c_k \bar{c}_{k'} \tilde{d}_{kk'} \geqq 0 \tag{4.7}$$

for any $c_1, \ldots, c_m \in \mathbf{C}$. Without loss of generality, we may take $\lambda^{(1)} = 0$, and from (4.6) we find that

$$\tilde{d}_{kk'} = \int_{\mathbf{R}_0^{2n}} (e^{i(\lambda^{(k')}, u_+)} - 1)(e^{-i(\lambda^{(k')}, u_-)} - 1) \Delta^{\tau \cup (-\tau)}(du) ,$$

which together with (4.3) implies (4.7), or the sufficiency part of the theorem.

The necessity follows from the argument above and the fact that functions $F$: $\mathbf{R}_0^n \to \mathbf{C}$ of the form $F(u) = \sum_{k=1}^m c_k (e^{i(\lambda^{(k)}, u)} - 1)$, $c_k \in \mathbf{C}$, $\lambda^{(k)} \in \mathbf{R}^n$, $k = 1, \ldots, m$, $m = 1, 2, \ldots$ are dense in $L^2(\mathbf{R}_0^n, \Delta^\tau)$. This is of course equivalent to the statement that functions $F_+$: $\mathbf{R}_0^{2n} \to \mathbf{C}$ of the form $F_+(u) = \sum_{k=1}^m c_k (e^{i(\lambda^{(k)}, u_+)} - 1)$, $c_k \in \mathbf{C}$, $\lambda^{(k)} \in \mathbf{R}^n$, $k = 1, \ldots, m$, $m = 1, 2, \ldots$ are dense in the subspace of $L^2(\mathbf{R}_0^{2n}, \Delta^{\tau \cup (-\tau)})$ consisting of functions $u \mapsto F(u_+)$ with $F_{|\mathbf{R}_0^n} \in L^2(\mathbf{R}_0^n, \Delta^\tau)$ and $F(0) = 0$; see Remark 2. □

**Corollary 3** *Let*

$$X(t) = \int_{\mathscr{X} \times \mathbf{R}} f(x, t - s) M(dx, ds), \quad t \in \mathbf{R} , \tag{4.8}$$

*be a SαS reversible mixed moving average with* $0 < \alpha < 2$ *and control measure* $Q$. *Then* $\{X(t): t \in \mathbf{R}\}$ *is reflection positive if and only if the inequality*

$$\int_{\mathscr{X} \times \mathbf{R} \times \mathbf{R}_0} F(wf(x, t_1 - s), \ldots, wf(x, t_n - s)) \times \bar{F}(wf(x, -t_1 - s), \ldots, wf(x, -t_n - s)) Q(dx) ds |w|^{-1-\alpha} dw \geqq 0 \tag{4.9}$$

*hold for any* $n \geqq 1$, $0 < t_1 < \ldots < t_n$ *and any measurable function* $F$: $\mathbf{R}^n \to \mathbf{C}$ *such that* $F(0, \ldots, 0) = 0$ *and*

$$\int_{\mathscr{X} \times \mathbf{R} \times \mathbf{R}_0} |F(wf(x, t_1 - s), \ldots, wf(x, t_n - s))|^2 Q(dx) ds |w|^{-1-\alpha} dw < \infty .$$

*Proof.* It follows from Theorem 5 and the expression (2.6) for the Lévy measure of $X$ in (4.8). □

It seems that there is no easy way to verify the inequality (4.9) for concrete $f$ and $Q$. In particular, we do not know whether there exist any reflection positive processes among the usual $S\alpha S$ moving averages, with $Q$ concentrated at one point. However, as we shall see below, there is a class of such processes with non-trivial $Q$, corresponding to reversible Markov processes killed at a constant rate $c > 0$.

Let $\mathscr{X} = \Omega \times \mathbf{R}_+$ and

$$Q(dx) = \mathbf{P}(d\omega) \otimes e^{-c\zeta} d\zeta, \quad x = (\omega, \zeta), \tag{4.10}$$

where $(\Omega, \mathbf{P})$ is a probability space with a stationary reversible Markov process $\{\xi(t, \omega): t \in \mathbf{R}\}$ defined on it, and taking values in a measurable space $(\mathscr{Y}, \mu)$. We assume that $\mu$ is the invariant measure for $\{\xi(t): t \in \mathbf{R}\}$, i.e. $\mu(A) = \mathbf{P}\{\xi(t) \in A\}$ for any measurable $A \subset \mathscr{Y}$ and any $t \in \mathbf{R}$. Let $g \in L^\alpha(\mu)$. Consider the $S\alpha S$ mixed moving average

$$X_g(t) = \int_{\Omega \times \mathbf{R}_+ \times \mathbf{R}} g(\xi(t - s, \omega)) \, 1_{[0, \zeta]}(t - s) M(d\omega, d\zeta, ds), \quad t \in \mathbf{R}, \tag{4.11}$$

with control measure $Q$ given by (4.10).

*Example.* In the simplest case $\Omega = \mathscr{Y} = \{1\}$, $\xi(t, \omega) \equiv 1, g = 1$, the process (4.11) can be written as

$$X(t) = \int_{\mathbf{R}_+ \times \mathbf{R}} 1_{[0, \zeta]}(t - s) M(d\zeta, ds), \quad t \in \mathbf{R}, \tag{4.12}$$

with control measure $Q(d\zeta) = \mu e^{-c\zeta} d\zeta$, $\zeta \in \mathbf{R}_+$, $\mu, c > 0$, i.e. (4.12) is a particular case of the mixed memory moving average processes discussed in Sect. 3. As it follows from (4.14) below, the multivariate characteristic function of $\{X(t): t \in \mathbf{R}\}$ is given by

$$E \exp\left\{ i \sum_{j=1}^n a_j X(t_j) \right\} \tag{4.13}$$

$$= \exp\left\{ -\mu c^{-2} \sum_{1 \leqq i \leqq k \leqq n} e^{-c|t_k - t_i|} (1 - e^{-c\Delta t_{i-1}})(1 - e^{-c\Delta t_k}) |a_i + \ldots + a_k|^\alpha \right\},$$

$-\infty =: t_0 < t_1 < \ldots < t_n < t_{n+1} := +\infty$, $\Delta t_i := t_{i+1} - t_i$, $i = 1, \ldots, n$. In the Gaussian case $\alpha = 2$, (4.13) becomes

$$E \exp\left\{ i \sum_{j=1}^n a_j X(t_j) \right\} = \exp\left\{ -\mu c^{-2} \sum_{i,j=1}^n e^{-c|t_i - t_j|} a_i a_j \right\},$$

i.e. (4.12) is a representation of the Ornstein–Uhlenbeck process with the covariance $2\mu c^{-2} \exp\{-c|t - s|\}$, $t, s \in \mathbf{R}$. Although for $\alpha < 2$ the process (4.12) is not Markov (unlike two other $S\alpha S$ analogs of the Ornstein–Uhlenbeck process, namely the usual $\alpha$-stable moving averages $X_+(t) = \int_{-\infty}^t e^{-c(t-s)} M(ds)$ and $X_-(t) = \int_t^\infty e^{c(t-s)} M(ds)$, which are Markov [1]), it has other important properties not shared by the processes $X_+$ and $X_-$, namely the reversibility and reflection positivity.

**Theorem 6** *For any $g \in L^\alpha(\mu)$ and $0 < \alpha < 2$, the process $\{X_g(t): t \in \mathbf{R}\}$ in (4.11) is well-defined and is reflection positive.*

*Proof.* By the invariance of $\mu$,

$$\int_{\Omega\times\mathbf{R}_+\times\mathbf{R}} |g(\xi(t,\omega))|^\alpha 1_{[0,\zeta]}(t)e^{-c\zeta}\mathbf{P}(d\omega)d\zeta dt$$

$$= \int_{\mathscr{Y}} |g|^\alpha d\mu \int_0^\infty \int_{-\infty}^\infty 1_{[0,\zeta]}(t)e^{-c\zeta}d\zeta dt = c^{-2}\int_{\mathscr{Y}} |g|^\alpha d\mu < \infty \;,$$

i.e. $X_g(t)$ is well-defined. Next, if $t_1 < \ldots < t_n$ and $a_1, \ldots, a_n \in \mathbf{R}$, then

$$-\log E\exp\left\{i\sum_{j=1}^n a_j X_g(t_j)\right\} = \int_0^\infty \int_{-\infty}^\infty \mathbf{E}\left|\sum_{j=1}^n a_j g(\xi(t_j - s))1_{[0,\zeta]}(t_j - s)\right|^\alpha e^{-c\zeta}d\zeta ds$$

$$= \sum_{i=1}^n \sum_{k=i}^n \int_{t_{i-1}}^{t_i} ds \int_{t_k - s}^{t_{k+1}-s} e^{-c\zeta}d\zeta\,\mathbf{E}\left|\sum_{j=i}^n a_j g(\xi(t_j - s))\right|^\alpha$$

$$= \sum_{i=1}^n \sum_{k=i}^n \int_{t_{i-1}}^{t_i} ds \int_{t_k - s}^{t_{k+1}-s} e^{-c\zeta}d\zeta\,\mathbf{E}\left|\sum_{j=i}^n a_j g(\xi(t_j))\right|^\alpha$$

$$(4.14)\qquad = \sum_{i=1}^n \sum_{k=i}^n c^{-2}(e^{ct_i} - e^{ct_{i-1}})(e^{-ct_k} - e^{-ct_{k+1}})\mathbf{E}\left|\sum_{j=1}^n a_j g(\xi(t_j))\right|^\alpha$$

by stationarity of $\{\xi(t): t\in\mathbf{R}\}$, where $t_0 := -\infty$, $t_{n+1} := +\infty$. From (4.14) and the reversibility of $\{\xi_t: t\in\mathbf{R}\}$ it follows immediately that $\{X_g(t): t\in\mathbf{R}\}$ is reversible too.

In order to show that $\{X_g(t): t\in\mathbf{R}\}$ is reflection positive, by Corollary 3, it suffices to verify the inequality

$$(4.15)\quad I := \int_{\Omega\times\mathbf{R}_+\times\mathbf{R}} F(g(\xi(t_1 - s))1_{[0,\zeta]}(t_1 - s), \ldots, g(\xi(t_n - s))1_{[0,\zeta]}(t_n - s))$$

$$\times \bar{F}(g(\xi(-t_1 - s))1_{[0,\zeta]}(-t_1 - s), \ldots, g(\xi(-t_n - s))1_{[0,\zeta]}(-t_n - s))$$

$$\times \mathbf{P}(d\omega)e^{-c\zeta}d\zeta ds \geqq 0$$

for any $0 < t_1 < \ldots < t_n$ and any bounded measurable function $F: \mathbf{R}^n \to \mathbf{C}$ such that $F(0, \ldots, 0) = 0$. Using the latter condition, as in (4.14), we obtain,

$$I = \sum_{i,k=1}^n \int_{-t_{i-1}}^{-t_i} ds \int_{t_k - s}^{t_{k+1}-s} e^{-c\zeta}d\zeta$$

$$\times \mathbf{E}\{F(g(\xi(t_1 - s)), \ldots, g(\xi(t_k - s)), 0, \ldots, 0)$$

$$\times \bar{F}(g(\xi(-t_1 - s)), \ldots, g(\xi(-t_i - s)), 0, \ldots, 0)\}$$

$$= \sum_{i,k=1}^n c^{-2}(e^{-ct_i} - e^{-ct_{i+1}})(e^{-ct_k} - e^{-ct_{k+1}})$$

$$\times \mathbf{E}\{F(g(\xi(t_1)), \ldots, g(\xi(t_k)), 0, \ldots, 0)\bar{F}(g(\xi(-t_1)), \ldots, g(\xi(-t_i)), 0, \ldots, 0)\}$$

$$= \mathbf{E}\{\Phi(\xi(t_1), \ldots, \xi(t_n))\bar{\Phi}(\xi(-t_1), \ldots, \xi(-t_n))\}$$

$$= \int_{\mathscr{Y}} |\mathbf{E}[\Phi(\xi(t_1), \ldots, \xi(t_n))|\xi(0) = y]|^2\mu(dy) \geqq 0$$

by the conditional independence of past and future of $\xi$ given $\xi(0)$ and the reversibility of the Markov process $\{\xi(t): t \in \mathbf{R}\}$, where $\Phi(y_1, \ldots, y_n) := c^{-1}\sum_{i=1}^{n}(e^{-ct_i} - e^{-ct_{i-1}})F(g(y_1), \ldots, g(y_i), 0, \ldots, 0)$, $t_{n+1} := +\infty$. $\square$

*Remark 2* Theorem 6 can be extended to reversible Markov processes $\{\xi(t): t \in \mathbf{R}\}$ with *infinite* invariant measure $\mu$; in particular, $\{\xi(t): t \in \mathbf{R}\}$ may be the Brownian motion on $(\mathbf{R}, \text{Leb})$.

*Remark 3* In a similar way one can show that the process $\{X_g(t): t \in \mathbf{R}, g \in L^\alpha(\mu)\}$ is reflection positive as a vector valued process [7], namely, for any $0 < t_1 < \ldots < t_n$, $g_1, \ldots, g_n \in L^\alpha(\mu)$ and any bounded measurable function $F$: $\mathbf{R}^n \to \mathbf{C}$ the following inequality is true:

$$E\{F(X_{g_1}(t_1), \ldots, X_{g_n}(t_n))\bar{F}(X_{g_1}(-t_1), \ldots, X_{g_n}(-t_n))\} \geqq 0 .$$

Note that for a fixed $t \in \mathbf{R}$ and any $g_1, \ldots, g_n \in L^\alpha(\mu)$ with supp $g_i \cap$ supp $g_j = \emptyset$ $(i \neq j)$, the random variables $X_{g_1}(t), \ldots, X_{g_n}(t)$ are *independent*; in particular, the set-indexed family $\{X_A(t) \equiv X_{1_A}(t): \mu(A) < \infty\}$ is a $S\alpha S$ random measure on $\mathscr{Y}$ with control measure $c^{-2}\mu$. Therefore (4.11) can be regarded as a stationary $S\alpha S$ evolution of $S\alpha S$ random measure. The Markov dynamics $\{\xi(t): t \in \mathbf{R}\}$ implicit in the stochastic integral (4.11) can be used in a more direct way to represent it as a functional of a Poisson Ornstein–Uhlenbeck process of non-interacting particle systems discussed in [11].

*Remark 4* Theorem 6 is true in the Gaussian case $\alpha = 2$ too, the distribution of (4.11) being completely determined by the covariance

$$\begin{aligned} EX_{g_1}(0)X_{g_2}(t) &= \int_0^\infty ds\mathbf{E}[g_1(\xi(s))g_2(\xi(t+s))] \int_{t+s}^\infty e^{-c\zeta}d\zeta \\ &= c^{-2}e^{-ct}\int_{\mathscr{Y}} g_1(y)T_t g_2(y)\mu(dy), \end{aligned}$$

$t > 0$, $g_1, g_2 \in L^2(\mu)$, where $T_t g(y) = \mathbf{E}[g(\xi(t))|\xi(0) = y]$ and $(T_t)_{t \geqq 0}$ is a semigroup of bounded self-adjoint operators in $L^2(\mu)$. The corresponding Gaussian Ornstein–Uhlenbeck process was studied by Meyer [9].

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