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Existence and uniqueness criteria for conservative uni-instantaneous denumerable Markov processes

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Summary. Until now, denumerable Markov processes with instantaneous states have not been extensively considered, and so in this paper we present a detailed examination of the conservative uni-instantaneous (CUI) case. We determine criteria for the existence and uniqueness of a specific CUI pregenerator, and consider the general problem of constructing CUI processes.

1 Introduction

We consider the construction of a continuous-time homogeneous Markov process on a countable set E. A general reference for this research area is the new edition (and English translation) of Yang (1990); whilst a short description is given in Sect. 7 of Chen and Renshaw (1990) whose definitions, terminology and notation we shall use throughout this paper. In particular, we note that a pregenerator is a matrix $Q = (q_{ij};\ i, j \in E)$ which satisfies the conditions

$$0 \leqq q_{ij} < +\infty \quad (i \neq j;\ i, j \in E)\,, \tag{1.1}$$

$$-\infty \leqq q_{ii} \leqq 0 \quad (i \in E), \text{ and} \tag{1.2}$$

$$\sum_{j \neq i} q_{ij} \leqq -q_{ii} \quad (i \in E); \tag{1.3}$$

whilst a generator is a matrix $Q = (q_{ij})$ defined on $E \times E$ such that

$$\lim_{t \to 0^+} \frac{P(t) - I}{t} = Q \tag{1.4}$$

where $P(t)$ is a standard (though not necessarily honest) transition function. Following Reuter we call a transition function $P(t)$ (and also its resolvent, i.e. its Laplace transform) a "Q-process" if (1.4) holds, and we call a state $i \in E$ stable if

$q_i \equiv -q_{ii} < +\infty$ and instantaneous if $q_i = +\infty$. If all states are stable then Q is said to be totally stable.

It is well-known that a generator must be a pregenerator, but the converse is not always true. The following three basic and important questions therefore arise.

(i) Existence – Under what conditions does a given pregenerator become a generator?

(ii) Uniqueness – If a given Q is a generator, under what conditions does there exist only one corresponding Q-process?

(iii) Construction – How do we construct the Q-process via a given generator Q? Specifically, how do we construct all the corresponding Q-processes when the Q-process is not unique?

These questions are of particular significance since in many problems we know only the infinitesimal behaviour, i.e. the pregenerator.

Many results have been obtained for the case in which the pregenerator Q is totally stable. Perhaps the most important is the one which provides the answer to the existence problem. Feller (1940) proved that if Q is a totally stable pregenerator, then it must be a generator; i.e. there always exists a Q-process. Thus for totally stable Q we have

$$\text{pregenerator} \Leftrightarrow \text{generator}.$$

Feller also provided a construction method to yield a standard transition function – called the Feller minimal process and denoted by $F(t) = \{f_{i,j}(t)\}$. He showed that the minimal process $F(t)$ satisfies both the "backward equation"

$$df_{ij}(t)/dt = \sum_{k\in E} q_{ik} f_{kj}(t) \quad (\forall i,\ j \in E) \tag{1.5}$$

and the "forward equation"

$$df_{ij}(t)/dt = \sum_{k\in E} f_{ik}(t) q_{kj} \quad (\forall i,\ j \in E)\,, \tag{1.6}$$

and has the "minimal property" in the sense that for any Q-process $P(t) = \{p_{ij}(t)\}$

$$p_{ij}(t) \geqq f_{ij}(t) \quad (\forall i, j \in E, \forall t \geqq 0)\,. \tag{1.7}$$

In this paper, for a totally stable pregenerator (and hence generator) Q we shall always use $F(t)$ and $\Phi(\lambda) = \{\varphi_{ij}(\lambda)\}$ to denote this minimal transition function and its resolvent, respectively: both are called the Feller minimal Q-process.

Although there always exists at least one Q-process for a totally stable pregenerator Q, there may be more than one Q-process even if we confine ourselves to those Q-processes which satisfy the backward and/or forward equation. The uniqueness problem for a totally stable generator has therefore attracted a lot of attention. It was first partly solved by Doob (1945) and Reuter (1957), and then completely solved by Hou (1974). Later, Reuter (1976) gave a new proof for Hou's theorem. It turns out that the uniqueness criteria is closely linked with the equations

$$\begin{cases} (\lambda I - Q)U(\lambda) = 0 \\ 0 \leqq U(\lambda) \leqq 1 \end{cases} \tag{1.8}$$

and

$$\begin{cases} \nu(\lambda)(\lambda I - Q) = 0 \\ 0 \leqq \nu(\lambda) \in \ell\,, \end{cases} \tag{1.9}$$

where ℓ denotes the space of absolutely summable vectors.

We shall use $\mathscr{M}_\lambda^+(Q)$, or simply $\mathscr{M}_\lambda^+$, to denote the solution space of (1.8). Similarly, $\mathscr{L}_\lambda^+(Q)$ (or $\mathscr{L}_\lambda^+$) denotes the solution space of (1.9). It is well-known (see Reuter 1957) that the dimensions of the solution spaces of (1.8) and (1.9) are both independent of $\lambda > 0$. We shall therefore use $m^+(Q)$ (or m^+) and $n^+(Q)$ (or n^+) to denote the dimensions of $\mathscr{M}_\lambda^+$ and $\mathscr{L}_\lambda^+$, respectively. Moreover, we know that for Eq. (1.8) there exists a maximal solution which we shall denote by $\bar{X}(\lambda; Q)$ (or $\bar{X}(\lambda)$). Using these symbols the uniqueness criterion can be stated quite briefly (see Hou and Guo 1988).

For convenience, a totally stable Q-process $P(t)$ which satisfies the backward equation (1.5), or in matrix form

$$dP(t)/dt = QP(t) \quad (t \geqq 0)\,, \tag{1.10}$$

is called a B-type Q-process. Similarly, we call $P(t)$ an F-type Q-process if $P(t)$ satisfies the forward Eq. (1.6), i.e. in matrix form,

$$dP(t)/dt = P(t)Q \quad (t \geqq 0)\,. \tag{1.11}$$

Hence a $\mathrm{B} \cap \mathrm{F}$-type Q-process $P(t)$ means that $P(t)$ satisfies both (1.10) and (1.11), whence the Feller minimal process $F(t)$ is a $\mathrm{B} \cap \mathrm{F}$-type Q-process. Note that the equivalent (resolvent) forms of (1.10) and (1.11) are

$$(\lambda I - Q)\Psi(\lambda) = I \quad (\lambda > 0) \tag{1.12}$$

and

$$\Psi(\lambda)(\lambda I - Q) = I \quad (\lambda > 0)\,, \tag{1.13}$$

respectively, where $\Psi(\lambda)$ denotes the resolvent of $P(t)$.

Many results have also been obtained for the third (i.e. construction) problem for totally stable generators. See, for example, Chung (1962), Williams (1964), Reuter (1959, 1962) and Yang (1990). Moreover a great many results exist for some specific totally stable Q-processes, e.g. birth–death processes. In contrast, however, few results have been obtained for the non-totally-stable pregenerator scenario.

Perhaps surprisingly, a very difficult problem, namely the existence problem for totally instantaneous Q-processes (i.e. all states are instantaneous) has been solved, due to an elegant result of Williams (1976); see also Rogers and Williams (1986). Analysis of some examples of the totally instantaneous case can be seen in Blackwell (1958) and Kendall (1958).

With regard to the so-called mixing case, i.e. both stable and instantaneous states exist, to our knowledge only several examples have been studied. The first mixing pregenerator considered (letting the state space E be non-negative integers) was

$$Q_1 = \begin{bmatrix} -\infty & 1 & 1 & 1 & \cdots \\ q_1 & -q_1 & 0 & 0 & \cdots \\ q_2 & 0 & -q_2 & 0 & \cdots \\ q_3 & 0 & 0 & -q_3 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{bmatrix} \tag{1.14}$$

where

$$q_j > 0 \quad (j \geqq 1)\,. \tag{1.15}$$

This example was provided by Kolmogorov (1951), and analysed by Kendall and Reuter (1954) (see also the discussion of Chung 1967). Kendall and Reuter proved that if

$$\sum_{j=1}^{\infty} (1/q_j) < +\infty\,, \tag{1.16}$$

then Q_1 is a generator, and they constructed an honest process for when (1.16) holds. We shall call the pregenerator (1.14) a K-pregenerator.

Later, Reuter (1969) considered the more general pregenerator

$$Q_2 = \begin{bmatrix} -\infty & b_1 & b_2 & b_3 & \cdots \\ q_1 & -q_1 & 0 & 0 & \cdots \\ q_2 & 0 & -q_2 & 0 & \cdots \\ q_3 & 0 & 0 & -q_3 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{bmatrix} \tag{1.17}$$

where

$$q_j > 0 \quad (j \geqq 1) \tag{1.18}$$

and

$$b_j \geqq 0 \quad \text{and} \quad \sum_{j=1}^{\infty} b_j = +\infty\,. \tag{1.19}$$

He pointed out that if

$$\sum_{j=1}^{\infty} (b_j/q_j) < +\infty \tag{1.20}$$

then Q_2 is a generator, and that when (1.20) holds true there exists only one honest process but infinitely many non-honest processes. He gave the construction of all these processes. The pregenerator (1.17) satisfying (1.18) and (1.19) will hence be called an R-pregenerator.

Note that the above two examples pose the interesting question as to whether conditions (1.16) and (1.20) are necessary.

Another example, considered by Williams (1967), comprised a pregenerator $Q = \{q_{ij};\, i, j \in E\}$ for which there exists a state $b \in E$ such that

$$\liminf_{j \to \infty} q_{bj} > 0\,. \tag{1.21}$$

He proved that for this type of pregenerator there exists an honest Q-process iff

$$\sum_{j \in E \backslash i} q_{ij} = q_i < +\infty \quad (\forall i \in E \backslash b), \tag{1.22}$$

and

$$\sum_{j,\, k \in E \backslash b} q_{bj} \varphi_{jk}(\lambda) < +\infty \quad (\lambda > 0), \tag{1.23}$$

where $\Phi(\lambda) = \{\varphi_{jk}(\lambda); j, k \in E \backslash b\}$ is the Feller minimal process related to $Q_b = \{q_{ij}; i, j \in E \backslash b\}$. A pregenerator which satisfies both (1.21) and (1.22) will be called a W-pregenerator.

Note also that the uniqueness and construction problems remain open for W-pregenerators.

In this paper the mixing case will be discussed. More specifically, we shall consider the case where the given pregenerator $Q = \{q_{ij}; i, j \in E\}$ satisfies the following three assumptions.

Assumption I. There exists a state $b \in E$ such that $q_b = +\infty$ and $q_i < +\infty$ $(\forall i \in E \backslash b)$.

Assumption II. $\sum_{j \in E \backslash b} q_{bj} = +\infty$.

Assumption III. $\sum_{j \in E \backslash i} q_{ij} = q_i \quad (\forall i \in E \backslash b)$.

Let us call a pregenerator which satisfies these assumptions a Conservative Uni-Instantaneous (CUI) pregenerator. The corresponding Q-process (transition function or resolvent) is also called a CUI Q-process. Some known results are presented in Sect. 2, and Sect. 3 provides results for general uni-instantaneous processes. Existence and uniqueness criteria are given in Sects. 4 and 5, respectively, whilst the construction problem is discussed in Sect. 6. Lastly, examples are discussed in Sect. 7. Occasionally we shall omit Assumption II and/or Assumption III, though not Assumption I. A pregenerator (generator, process) which satisfies only Assumption I is called uni-instantaneous, or simply UI.

CUI is an important mixing type of pregenerator, since in many applications involving uni-instantaneous processes (such as branching processes with instantaneous immigration) the specified pregenerator is usually CUI. Thus the analysis of CUI Q-processes has considerable significance both in theory and application. Moreover, examples of the mixing case previously discussed in the literature, such as K-, R- and W-pregenerators, are all CUI pregenerators. Thus our discussion will include most of the known results for the mixing case.

For a CUI pregenerator $Q = \{q_{ij}; i, j \in E\}$, let $N = E \backslash b$, and $\alpha_b = \{q_{bj}; j \in N\}$ and $\beta_b = \{q_{jb}; j \in N\}$ be the row and column vectors on N. Write $Q_b = \{q_{ij}; i, j \in N\}$. Then Q can be written in the form

$$Q = \begin{bmatrix} -q_b & \alpha_b \\ \beta_b & Q_b \end{bmatrix} = \begin{bmatrix} -\infty & \alpha_b \\ \beta_b & Q_b \end{bmatrix}. \tag{1.24}$$

It is easy to see that Q_b is a totally stable pregenerator (and hence a generator) by Assumption I. Furthermore, Assumptions II and III give

$$\alpha_b \mathbf{1} = +\infty , \tag{1.25}$$

and

$$\beta_b + Q_b \mathbf{1} = \mathbf{0} , \tag{1.26}$$

respectively.

2 Preliminary results

The fundamental tool used in our analysis is the Resolvent Decomposition Theorem. This is detailed, for example, in Sect. 7 of Chen and Renshaw (1990). For convenience, the three main results, i.e. Theorems 7.7, 7.8 and 7.10, of that paper are listed below. We stress that these theorems can be applied to any generator by slightly changing the statements and proofs, i.e. none of Assumptions I, II and III are mandatory.

Theorem 2.1 *Suppose* $R(\lambda) = \{r_{ij}(\lambda);\, i, j \in E\}$ *is a* Q*-process defined on* $E \times E$, *where the generator* $Q = \begin{bmatrix} -q_b & \alpha_b \\ \beta_b & Q_b \end{bmatrix}$, b *is a state of* E, *and* $N = E \backslash \{b\}$.
Then $R(\lambda)$ *can be uniquely decomposed into*

$$R(\lambda) = \begin{bmatrix} 0 & \mathbf{0} \\ \mathbf{0} & \psi(\lambda) \end{bmatrix} + r_{bb}(\lambda) \begin{bmatrix} 1 \\ \xi(\lambda) \end{bmatrix} [1, \eta(\lambda)] \tag{2.1}$$

where

$$\psi(\lambda) \text{ is a } Q_b\text{-process} , \tag{2.2}$$

$$\eta(\lambda) \in \mathbf{H}_\psi \quad \text{and} \quad \xi(\lambda) \in \mathbf{K}_\psi , \tag{2.3}$$

$$\xi(\lambda) \leqq \mathbf{1} - \lambda\psi(\lambda)\mathbf{1} , \tag{2.4}$$

$$\lim_{\lambda\to\infty} \lambda\eta(\lambda) = \alpha_b \quad \text{and} \quad \lim_{\lambda\to\infty} \lambda\xi(\lambda) = \beta_b , \tag{2.5}$$

$$r_{bb}(\lambda) = (C + \lambda + \lambda\langle \eta(\lambda), \xi \rangle)^{-1} \tag{2.6}$$

where $\xi = \lim_{\lambda\to 0} \xi(\lambda)$ *and* C *is a finite constant such that*

$$C \geqq \lim_{\lambda\to\infty} \lambda\langle \eta(\lambda), \mathbf{1} - \xi \rangle, \quad \text{and} \tag{2.7}$$

$$C + \lim_{\lambda\to\infty} \lambda\langle \eta(\lambda), \xi \rangle = q_b . \tag{2.8}$$

So if $q_b = +\infty$, *it follows that*

$$\lim_{\lambda\to\infty} \lambda\langle \eta(\lambda), \xi \rangle = +\infty \quad \text{or, equivalently}, \tag{2.9}$$

$$\lim_{\lambda\to\infty} \lambda\langle \eta(\lambda), \mathbf{1} \rangle = +\infty . \tag{2.10}$$

If $R(\lambda)$ is honest then we have further that

$$\xi(\lambda) = \mathbf{1} - \lambda\psi(\lambda)\mathbf{1}\,, \tag{2.11}$$

$$r_{bb}(\lambda) = (\lambda + \lambda\langle\eta(\lambda), \mathbf{1}\rangle)^{-1}, \quad \textit{and} \tag{2.12}$$

$$C \equiv \lambda\langle\eta(\lambda), \mathbf{1} - \xi\rangle, \tag{2.13}$$

i.e. $\lambda\langle\eta(\lambda), \mathbf{1} - \xi\rangle$ *is independent of* λ.

In the following two theorems we shall confine ourselves to the case of $q_b = +\infty$.

Theorem 2.2 *Suppose $Q = \{q_{ij}; i, j \in E\}$ is a given pre-generator defined on $E \times E$ where*

$$Q = \begin{bmatrix} -q_b & \alpha_b \\ \beta_b & Q_b \end{bmatrix} \quad \textit{and} \quad q_b \equiv -q_{bb} = +\infty\,.$$

Then Q is a generator if there exists a Q_b-process $\psi(\lambda)$, together with a pair of $\eta(\lambda)$ and $\xi(\lambda)$ which satisfy

$$\eta(\lambda) \in \mathbf{H}_\psi \quad \textit{and} \quad \xi(\lambda) \in \mathbf{K}_\psi\,, \tag{2.14}$$

$$\xi(\lambda) \leqq \mathbf{1} - \lambda\psi(\lambda)\mathbf{1}\,, \tag{2.15}$$

$$\lim_{\lambda\to\infty} \lambda\eta(\lambda) = \alpha_b \quad \textit{and} \quad \lim_{\lambda\to\infty} \lambda\xi(\lambda) = \beta_b \tag{2.16}$$

where $\alpha_b = \{q_{bj}; j \in N\}$ and $\beta_b = \{q_{jb}; j \in N\}$,

$$\lim_{\lambda\to\infty} \lambda\langle\eta(\lambda), \mathbf{1}\rangle = +\infty\,, \quad \textit{and} \tag{2.17}$$

$$\lim_{\lambda\to\infty} \lambda\langle\eta(\lambda), \mathbf{1} - \xi\rangle < +\infty \tag{2.18}$$

where $\xi = \lim_{\lambda\to 0} \xi(\lambda)$. Furthermore, if the above conditions hold true then choose a constant C such that

$$C \geqq \lim_{\lambda\to\infty} \lambda\langle\eta(\lambda), \mathbf{1} - \xi\rangle\,, \tag{2.19}$$

and let

$$r_{bb}(\lambda) = (C + \lambda + \lambda\langle\eta(\lambda), \xi\rangle)^{-1} \quad (\lambda > 0) \quad \textit{and} \tag{2.20}$$

$$R(\lambda) = \begin{bmatrix} 0 & \mathbf{0} \\ \mathbf{0} & \psi(\lambda) \end{bmatrix} + r_{bb}(\lambda)\begin{bmatrix} 1 \\ \xi(\lambda) \end{bmatrix}[1, \eta(\lambda)] \quad (\lambda > 0)\,, \tag{2.21}$$

where $\psi(\lambda)$, $\xi(\lambda)$ and $\eta(\lambda)$ satisfy the above conditions (2.14)–(2.18). *Then $R(\lambda)$ as defined in* (2.21) *is a Q-process.*

If we are interested only in honest processes then the following theorem applies.

Theorem 2.3 *Suppose Q is a given pregenerator defined as above. If there exists a Q_b-process $\psi(\lambda)$ and a row-vector $\eta(\lambda)$ $(\lambda > 0)$ which satisfy*

$$\eta(\lambda) \in \mathbf{H}_\psi\,, \tag{2.22}$$

$$\lim_{\lambda\to\infty} \lambda\eta(\lambda) = \alpha_b \quad \textit{where} \quad \alpha_b = (q_{bj}\,; j \in N)\,, \tag{2.23}$$

$$\lim_{\lambda\to\infty} \lambda(\mathbf{1} - \lambda\psi(\lambda)\mathbf{1}) = \beta_b \quad \textit{where} \quad \beta_b = (q_{jb}\,; j \in N)\,, \quad \textit{and} \tag{2.24}$$

$$\lim_{\lambda\to\infty} \lambda\eta(\lambda)\mathbf{1} = +\infty\,, \tag{2.25}$$

then Q is a generator and there exists an honest Q-process. This honest Q-process can be constructed simply by letting

$$r_{bb}(\lambda) = (\lambda + \lambda\langle\eta(\lambda), \mathbf{1}\rangle)^{-1} \quad (\lambda > 0)$$

and

$$R(\lambda) = \begin{bmatrix} 0 & \mathbf{0} \\ \mathbf{0} & \psi(\lambda) \end{bmatrix} + r_{bb}(\lambda)\begin{bmatrix} 1 \\ \mathbf{1} - \lambda\psi(\lambda)\mathbf{1} \end{bmatrix}[1, \eta(\lambda)]\,.$$

Recall that $\mathbf{H}_\psi$ and $\mathbf{K}_\psi$ which appear in the above three theorems are defined (see Chen and Renshaw 1990) by

$$\mathbf{H}_\psi = \{\eta(\lambda);\, \mathbf{0} \leqq \eta(\lambda) \in \ell,\, \eta(\lambda) - \eta(\mu) = (\mu - \lambda)\eta(\lambda)\psi(\mu);\, \lambda, \mu > 0\} \tag{2.26}$$

and

$$\mathbf{K}_\psi = \{\xi(\lambda);\, \mathbf{0} \leqq \xi(\lambda) \leqq \mathbf{1},\, \xi(\lambda) - \xi(\mu) = (\mu - \lambda)\psi(\lambda)\xi(\mu);\, \lambda, \mu > 0\}\,, \tag{2.27}$$

where $\psi(\lambda)$ is a process (not necessarily totally stable). In particular, if $\psi(\lambda)$ is the totally stable Feller minimal process $\Phi(\lambda)$ with totally stable generator Q, then the elements $\mathbf{H}_\Phi$ and $\mathbf{K}_\Phi$ have the following representational form.

Lemma 2.4 (Reuter 1959, 1962) *Let $\Phi(\lambda)$ be the Feller minimal Q-process with the totally stable generator Q. Then*

(i) $\eta(\lambda) \in \mathbf{H}_\Phi$ iff

$$\eta(\lambda) = \alpha\Phi(\lambda) + \bar{\eta}(\lambda) \tag{2.28}$$

where $\alpha \geqq 0$, $\alpha\Phi(\lambda) \in l\ (\lambda > 0)$, $\bar{\eta}(\lambda) \in \mathbf{H}_\Phi$ and $\bar{\eta}(\lambda) \in \mathscr{L}^+_\lambda$. Furthermore, we have

$$\eta(\lambda)\,(\lambda I - Q) = \alpha, \quad \textit{with}$$

$$\eta(\lambda) \downarrow 0 \quad \textit{and} \quad \lambda\eta(\lambda) \to \alpha \quad (\lambda \uparrow \infty)\,.$$

(ii) $\xi(\lambda) \in \mathbf{K}_\Phi$ iff

$$\xi(\lambda) = \Phi(\lambda)\beta + \bar{\xi}(\lambda) \tag{2.29}$$

where $\beta \geqq 0$, $\Phi(\lambda)\beta \leqq \mathbf{1}\ (\lambda > 0)$, $\bar{\xi}(\lambda) \in \mathbf{K}_\Phi$ and $\bar{\xi}(\lambda) \in \mathscr{M}^+_\lambda$. Furthermore, we have

$$(\lambda I - Q)\xi(\lambda) = \beta, \quad \textit{with}$$

$$\xi(\lambda) \downarrow 0 \quad \textit{and} \quad \lambda\xi(\lambda) \to \beta \quad (\lambda \uparrow \infty)\,.$$

3 Some results on general uni-instantaneous Q-processes

Before giving existence criteria for CUI Q-processes, we shall first derive some results for general UI Q-processes. We assume that the pregenerator Q satisfies

Assumption I only (i.e. not necessarily Assumptions II and III). Then the pregenerator Q can be written as

$$Q = \begin{bmatrix} -q_b & \alpha_b \\ \beta_b & Q_b \end{bmatrix}$$

where $b \in E$, $N = E \backslash b$, $q_b = +\infty$, $\alpha_b = \{q_{bj}; j \in N\}$, $\beta_b = \{q_{jb}; j \in N\}$ and Q_b is a totally stable pregenerator on $N \times N$.

According to Theorem 2.1, for each UI Q-process $R(\lambda)$ there exists a unique Q_b-process $\Psi(\lambda)$ for which Theorem 2.1 holds. So we may view $R(\lambda) \Rightarrow \Psi(\lambda)$ as a map. We shall call $\Psi(\lambda)$ the restriction process of $R(\lambda)$, and $R(\lambda)$ an expansion process of $\Psi(\lambda)$. Note, however, that expansion processes are not usually unique.

We first give a necessary condition for the existence of a UI generator. This establishes an interesting relationship between the existence of a process with instantaneous states and the uniqueness of a process without instantaneous states.

Theorem 3.1 *Suppose Q is a* UI *generator as above. Then the totally stable Q_b-process is not unique.*

Proof. We assume that the Q_b-process is unique, and then derive a contradiction. Since Q is a UI generator, there exists a Q-process $R(\lambda)$, say. Let its restriction process on $N \times N$ be $\Psi(\lambda)$. Then $\Psi(\lambda)$ is a Q_b-process. But the Q_b-process is unique, and so any Q_b-process must be the Feller minimal Q_b-process $\Phi(\lambda)$. Thus $\Psi(\lambda) = \Phi(\lambda)$. Now by Theorem 2.1 the Q-process $R(\lambda)$ can be decomposed into

$$R(\lambda) = \begin{bmatrix} 0 & \mathbf{0} \\ \mathbf{0} & \Phi(\lambda) \end{bmatrix} + r_{bb}(\lambda) \begin{bmatrix} 1 \\ \xi(\lambda) \end{bmatrix} [1, \eta(\lambda)] \,. \tag{3.1}$$

Moreover, the following results hold true

$$\eta(\lambda) \in \mathbf{H}_\Phi \quad \text{and} \quad \xi(\lambda) \in \mathbf{K}_\Phi \,, \tag{3.2}$$

$$\lim_{\lambda \to \infty} \lambda \eta(\lambda) = \alpha_b \quad \text{where} \quad \alpha_b = \{q_{bj}; j \in N\} \,, \tag{3.3}$$

$$\mathbf{0} \leqq \xi(\lambda) \leqq \mathbf{1} - \lambda \Phi(\lambda) \mathbf{1} \,, \tag{3.4}$$

$$\lim_{\lambda \to \infty} \lambda \langle \eta(\lambda), \xi \rangle = \lim_{\lambda \to \infty} \lambda \langle \eta(\lambda), \mathbf{1} \rangle = +\infty \quad \text{where } \xi = \lim_{\lambda \to 0} \xi(\lambda) \,. \tag{3.5}$$

Now by Lemma 2.4, (3.2) yields the representation

$$\eta(\lambda) = \alpha \Phi(\lambda) + \bar{\eta}(\lambda) \tag{3.6}$$

where $\lim_{\lambda \to \infty} \lambda \eta(\lambda) = \alpha$ and $\bar{\eta}(\lambda) \in \mathscr{L}_\lambda^+(Q_b)$. Comparison with (3.3) then gives $\alpha = \alpha_b$, whence (3.6) can be written as

$$\eta(\lambda) = \alpha_b \Phi(\lambda) + \bar{\eta}(\lambda) \,. \tag{3.7}$$

Since the Q_b-process is unique, it follows from Reuter (1957) or Hou (1974) that either the minimal Feller Q_b-process is honest or else $\mathscr{n}^+(Q_b) = 0$. If the former, then by (3.4) we have $\xi(\lambda) \equiv \mathbf{0}$, and thus $\xi \equiv \mathbf{0}$, in contradiction to (3.5). If the latter, then $\bar{\eta}(\lambda) = \mathbf{0}$ and so (3.7) can be further written as

$$\eta(\lambda) = \alpha_b \Phi(\lambda) \,. \tag{3.8}$$

Now if $\alpha_b \mathbf{1} < +\infty$, then by (3.8) we obtain $\lim_{\lambda \to \infty} \lambda \langle \eta(\lambda), \mathbf{1} \rangle < +\infty$, which contradicts (3.5); whilst if $\alpha_b \mathbf{1} = +\infty$ then by (3.2) we obtain $\alpha_b \Phi(\lambda) \in \ell$, whence by

Lemma 12.2.4 and Theorem 12.1.1 of Hou and Guo (1988) the Q_b-process is not unique which is again a contradiction. □

Since the uniqueness criterion for totally stable processes is well-known, we can extract information on the existence of uni-instantaneous processes through Theorem 3.1. For example, we can immediately obtain the following simple corollary.

Corollary 3.2 *Suppose* Q *is a* UI *pregenerator. If* $\sup_{i \in N} q_i < +\infty$, *then* Q *is not a generator.*

In particular, for Markov processes with a finite number of states no state is instantaneous – see Proposition 2.17 of Cinlar (1975).

Another necessary condition for the existence of the UI process is as follows.

Theorem 3.3 *Suppose* Q *is a* UI *generator. Then we have*

$$\alpha_b \Phi(\lambda) \in \ell \quad (\forall \lambda > 0), \tag{3.9}$$

i.e.

$$\sum_{j,\, k \in N} q_{bj} \varphi_{jk}(\lambda) < +\infty \quad (\forall \lambda > 0) \tag{3.10}$$

where $\Phi(\lambda) = \{\varphi_{jk}(\lambda);\, j, k \in N\}$ *is the Feller minimal* Q_b*-process.*

Proof. Since Q is a UI generator there must exist a Q-process $R(\lambda)$. By Theorem 2.1, this $R(\lambda)$ can be written in the form of (2.1), where (2.2)–(2.13) hold true. In particular, there exists a Q_b-process $\Psi(\lambda)$ and a row vector $\eta(\lambda)$ such that

$$\eta(\lambda) \in \mathbf{H}_{\Psi} \quad \text{and} \tag{3.11}$$

$$\lim_{\lambda \to \infty} \lambda \eta(\lambda) = \alpha_b \,. \tag{3.12}$$

Now by definition, (3.11) means that

$$\eta(\lambda) - \eta(\mu) = (\mu - \lambda)\eta(\lambda)\Psi(\mu) \quad (\lambda, \mu > 0), \quad \text{and} \tag{3.13}$$

$$\eta(\lambda) \in \ell \quad (\forall \lambda > 0) \,. \tag{3.14}$$

Letting $\lambda \to \infty$ in (3.13), and noting (3.12) and that $\eta(\lambda) \to 0$ when $\lambda \to \infty$, we obtain by Fatou's Lemma that

$$\eta(\lambda) \geqq \alpha_b \Psi(\lambda) \,. \tag{3.15}$$

But $\Psi(\lambda)$ is a Q_b-process and $\Phi(\lambda)$ is the minimal Q_b-process. Hence $\Psi(\lambda) \geqq \Phi(\lambda)$, and so by using (3.14) and (3.15) we obtain

$$\alpha_b \Phi(\lambda) \leqq \alpha_b \Psi(\lambda) \leqq \eta(\lambda) \in \ell \,. \tag{3.16}$$

Result (3.9) now follows. □

Remark. Although condition (3.9) is trivial when $\alpha_b \mathbf{1} < +\infty$, it is a key condition when $\alpha_b \mathbf{1} = +\infty$; see the existence criterion for CUI processes in the next section.

We shall call a UI Q-process $R(\lambda) = \{r_{ij}(\lambda);\, i, j \in E\}$ an almost B-type Q-process if

$$\lambda r_{ij}(\lambda) - \delta_{ij} = \sum_{k \in E} q_{ik} r_{kj}(\lambda) \quad (\forall i \in N, \forall j \in E) \,. \tag{3.17}$$

Similarly, we shall call a UI Q-process $R(\lambda)=\{r_{ij}(\lambda); i,j\in E\}$ an almost F-type Q-process if

$$\lambda r_{ij}(\lambda)-\delta_{ij}=\sum_{k\in E} r_{ik}(\lambda)q_{kj} \quad (\forall i\in E,\ \forall j\in N)\,. \tag{3.18}$$

An almost B $\cap$ F-type UI process $R(\lambda)$ then means that $R(\lambda)$ satisfies both (3.17) and (3.18). Note that in (3.17) we require only $i\in N$, and not $i\in E$. Similarly, in (3.18) we require only $j\in N$. Note that (3.17) and (3.18) are equivalent to the (transition function) forms

$$dp_{ij}(t)/dt=\sum_{k\in E} q_{ik}p_{kj}(t) \quad (\forall i\in N,\ \forall j\in E) \tag{3.19}$$

and

$$dp_{ij}(t)/dt=\sum_{k\in E} p_{ik}(t)q_{kj} \quad (\forall i\in E,\ \forall j\in N)\,, \tag{3.20}$$

respectively.

The following simple lemma will play an important role in what follows.

Lemma 3.4 *Suppose $R(\lambda)=\{r_{ij}(\lambda); i,j\in E\}$ is a* UI *process with the generator Q as defined above. Then*

(i) *$R(\lambda)$ is an almost* B-*type Q-process* iff *its restriction process $\Psi(\lambda)$ is a* B-*type Q_b-process,*

(ii) *$R(\lambda)$ is an almost* F-*type Q-process* iff *its restriction process $\Psi(\lambda)$ is a* F-*type Q_b-process,*

(iii) *$R(\lambda)$ is an almost* B $\cap$ F-*type Q-process* iff *its restriction process $\Psi(\lambda)$ is a* B $\cap$ F-*type Q_b-process.*

Proof. We need only prove (i), since the proof of (ii) is similar, whilst (iii) is a consequence of (i) and (ii). Suppose $R(\lambda)=\{r_{ij}(\lambda); i,j\in E\}$ is a UI process with generator Q. We need to prove that it satisfies (3.17) iff its restriction $\Psi(\lambda)$ satisfies (1.12), i.e.

$$\lambda\psi_{ij}(\lambda)-\delta_{ij}=\sum_{k\in N} q_{ik}\psi_{kj}(\lambda) \quad (\forall i,j\in N)\,. \tag{3.21}$$

Suppose $R(\lambda)$ satisfies (3.17), i.e.

$$\begin{aligned}\lambda r_{ib}(\lambda)&=q_{ib}r_{bb}(\lambda)+\sum_{k\in N} q_{ik}r_{kb}(\lambda) \quad (\forall i\in N)\\ \lambda r_{ij}(\lambda)-\delta_{ij}&=q_{ib}r_{bj}(\lambda)+\sum_{k\in N} q_{ik}r_{kj}(\lambda) \quad (\forall i,j\in N)\,.\end{aligned} \tag{3.22}$$

Substituting the decomposition form (2.1) into (3.22), and noting that $r_{bb}(\lambda)>0$, then shows that (3.22) is true iff both

$$\lambda\xi_i(\lambda)=q_{ib}+\sum_{k\in N} q_{ik}\xi_k(\lambda) \quad (\forall i\in N) \tag{3.23}$$

and

$$\lambda\psi_{ij}(\lambda)+\lambda\xi_i(\lambda)r_{bb}(\lambda)\eta_j(\lambda)-\delta_{ij}$$
$$=q_{ib}r_{bb}(\lambda)\eta_j(\lambda)+\sum_{k\in N} q_{ik}\psi_{kj}(\lambda)+\sum_{k\in N} q_{ik}\xi_k(\lambda)r_{bb}(\lambda)\eta_j(\lambda) \quad (\forall i,j\in N) \tag{3.24}$$

hold true. Substituting (3.23) into (3.24) now shows that (3.21) is true.

Conversely, suppose (3.21) is true. In order to prove (3.17) we need only prove (3.23), since (3.17) then follows from the equivalent relations

$$\text{both (3.21) and (3.23)} \Leftrightarrow \text{both (3.23) and (3.24)} \Leftrightarrow \text{(3.22)} \Leftrightarrow \text{(3.17)}.$$

However (3.23) can be proved as follows, in a manner similar to Reuter (1957).

Since $\psi(\lambda)$ is a B-type Q_b-process

$$\lambda\psi(\lambda) - I = Q_b\psi(\lambda)\,, \tag{3.25}$$

whilst $\xi(\lambda) \in \mathbf{K}_\psi$, i.e. $\xi(\lambda) - \xi(\mu) = (\mu - \lambda)\psi(\lambda)\xi(\mu)$. Thus on using (3.25) we see that

$$(\lambda I - Q_b)\xi(\lambda) = (\lambda I - Q_b)[I + (\mu - \lambda)\psi(\lambda)]\xi(\mu) = (\mu I - Q_b)\xi(\mu) \quad (\lambda, \mu > 0)\,, \tag{3.26}$$

which means that $(\lambda I - Q_b)\xi(\lambda)$ is a constant column vector, i.e. independent of $\lambda > 0$. Permissibility of using the associative law in (3.26) can be easily verified. Noting that $\lambda\xi(\lambda) \to \beta_b$ and $\xi(\lambda) \downarrow \mathbf{0}$ when $\lambda \to \infty$ (see (2.5)), then shows this constant vector to be β_b, or

$$(\lambda I - Q_b)\xi(\lambda) = \beta_b \text{ where } \quad \beta_b = \{q_{ib}; i \in N\}\,. \tag{3.27}$$

The component form of (3.27) is just (3.23). □

4 Existence criterion for conservative uni-instantaneous Q-processes

Suppose Q is a CUI pregenerator as defined in (1.24)–(1.26). Then one of the basic results of this paper is the following existence criterion.

Theorem 4.1 *The following statements are equivalent*:

(i) *Q is a generator, i.e. there exists a Q-process*;
(ii) *there exists an honest almost* $\mathrm{B} \cap \mathrm{F}$*-type Q-process*;

(iii) $\forall\lambda > 0$, $\alpha_b\Phi(\lambda) \in \ell$, *i.e.*

$$\sum_{j,\,k \in N} q_{bj}\varphi_{jk}(\lambda) < +\infty \quad (\forall\lambda > 0)\,; \tag{4.1}$$

(iv) $\exists\lambda_0 > 0$, $\alpha_b\Phi(\lambda_0) \in \ell$, *i.e.*

$$\sum_{j,\,k \in N} q_{bj}\varphi_{jk}(\lambda_0) < +\infty\,. \tag{4.2}$$

Proof. (iii)$\Leftrightarrow$(iv) follows from the resolvent equation, (ii)$\Rightarrow$(i) is obvious, whilst (i)$\Rightarrow$(iii) follows from Theorem 3.3. Thus we need only prove that (iii)$\Rightarrow$(ii).

Suppose (iii) holds true. We shall show that the conditions of Theorem 2.3 are satisfied. Indeed, the required Q_b-process will be chosen as the Feller minimal Q_b-process $\Phi(\lambda)$, whilst the required row vector $\eta(\lambda)$ will be chosen as

$$\eta(\lambda) = \alpha_b\Phi(\lambda)\,. \tag{4.3}$$

It is easy to see that such a chosen $\eta(\lambda)$ satisfies

$$\eta(\lambda) - \eta(\mu) = (\mu - \lambda)\eta(\lambda)\Phi(\mu) \quad (\lambda, \mu > 0)\,.$$

Combining this with condition (iii), i.e. $\eta(\lambda) \in \ell$, then shows that requirement (2.22) holds true. Requirement (2.23) is obvious from (4.3), whilst (2.24) becomes

$\lim_{\lambda\to\infty} \lambda[\mathbf{1} - \lambda\Phi(\lambda)\mathbf{1}] = \beta_b$. The latter follows from (1.26) and the well-known fact that $\lim_{\lambda\to\infty} \lambda[\mathbf{1} - \lambda\Phi(\lambda)\mathbf{1}] = -Q_b\mathbf{1}$. From (4.3) it is easy to see that $\lim_{\lambda\to\infty} \lambda\eta(\lambda) = \alpha_b$, which on using Fatou's Lemma and (1.25) yields

$$\lim_{\lambda\to\infty} \lambda\eta(\lambda)\mathbf{1} \geqq \alpha_b\mathbf{1} = +\infty ,$$

which is precisely requirement (2.25). Since all the requirements of Theorem 2.3 are verified true, by this theorem there must exist an honest Q-process. Moreover, (4.3) and Lemma 3.4 show that this process is almost $\mathrm{B} \cap \mathrm{F}$ type. Statement (ii) is now proved. □

Remark. Note that (ii) in Theorem 4.1 takes the strongest form, i.e. it includes all types of existence. Thus condition (4.2) (or (4.1)) guarantees not only a Q-process but also an honest Q-process which satisfies both equations (3.17) and (3.18).

Although the existence criterion has been given in Theorem 4.1, verifying the existence condition (4.1), which involves the Feller minimal process rather than the pregenerator Q itself, is not easy in all cases. We shall therefore provide some easy-to-check conditions (necessary, as well as sufficient) which involve only the elements of the pregenerator Q. First we provide a simple necessary condition.

Corollary 4.2 *Suppose Q is a* CUI *generator. Then*

$$\sum_{j\in N} [q_{bj}/(1 + q_j)] < +\infty . \tag{4.4}$$

Proof. Use (4.2) with $\lambda_0 = 1$, together with the well-known inequality

$$\sum_{j\in N} \varphi_{ij}(\lambda) \geqq \varphi_{ii}(\lambda) \geqq (\lambda + q_i)^{-1} \quad (\forall i \in N) . \quad \square$$

Now we provide an easy-to-check sufficient condition.

Theorem 4.3 *Suppose Q is a* CUI *pregenerator. If both*

$$\sum_{j\in N} [q_{bj}/(1 + q_j)] < +\infty \tag{4.5}$$

and

$$\sum_{j\in N} [q_{bj}(q_j - q_{jb})/(1 + q_j)] < +\infty , \tag{4.6}$$

then Q is a generator. That is there exists a Q-process as well as an honest Q-process.

Proof. By Theorem 2.10.5 of Yang (1990)

$$\lambda \sum_{j\in N} \phi_{ij}(\lambda) = 1 - \bar{X}_i(\lambda) - \sum_{k\in N} \phi_{ik}(\lambda)d_k \leqq 1 - \phi_{ii}(\lambda)d_i$$
$$\leqq 1 - [d_i/(\lambda + q_i)] = (\lambda + q_i - q_{ib})/(\lambda + q_i) \quad (\forall i \in N) .$$

Thus if

$$\sum_{j\in N} [q_{bj}(\lambda + q_j - q_{jb})/(\lambda + q_j)] < +\infty , \tag{4.7}$$

then (4.1) holds true. However, it is easy to see that (4.7) holds iff both (4.5) and (4.6) hold true. □

Note that condition (4.5) in Theorem 4.3 is actually necessary by Corollary 4.2. Thus the "essential" sufficient condition is (4.6) which is, of course, not necessary. However, in some cases (4.6) will be automatically satisfied.

Corollary 4.4 *Suppose Q is a* CUI *pregenerator satisfying*

$$\sup_{j \in N} (q_j - q_{jb}) < +\infty \ . \tag{4.8}$$

Then Q is a generator iff

$$\sum_{j \in N} [q_{bj}/(1 + q_j)] < +\infty \ . \tag{4.9}$$

Proof. Since condition (4.8) guarantees that (4.5) ⇒ (4.6), the conclusion follows from Theorem 4.3 and Corollary 4.2. □

Although (4.8) is a strong assumption, it does apply to some pregenerators. For example, both K- and R-pregenerators (see Sect. 1) satisfy (4.8). Indeed, in these two cases

$$q_j \equiv q_{jb}, \quad \textit{whence} \quad \sup_{j \in N} (q_j - q_{jb}) = 0 \ .$$

Thus (4.9) is an "iff" condition for K- and R-pregenerators to be generators.

In order to get better necessary, as well as sufficient, conditions, we shall rewrite Theorem 4.1 in an equivalent form. To do this we first note (Feller 1940) that for the totally stable generator Q_b the Feller minimal process $\Phi(\lambda)$ can be obtained by the following iterative procedure. Let

$$\Phi^{(1)}(\lambda) = \operatorname{diag}\{\lambda + q_i; i \in N\}$$

$$\Phi^{(n+1)}(\lambda) = \Phi^{(1)}(\lambda) + \Phi^{(n)}(\lambda)\Pi(\lambda) \quad (n \geqq 1), \quad \text{and}$$

$$\Phi^{(n)}(\lambda) \uparrow \Phi(\lambda) \quad (n \uparrow +\infty) \ , \tag{4.10}$$

where $\Pi(\lambda) = \{\Pi_{ij}(\lambda); i, j \in N\}$ has elements

$$\Pi_{ij}(\lambda) = \begin{cases} 0 & \text{if } i = j \\ q_{ij}/(\lambda + q_j) & \text{if } i \neq j \ . \end{cases} \tag{4.11}$$

We shall call $\{\Phi^{(n)}(\lambda); n \geqq 1\}$ the Feller Asymptotic Sequence associated with the specified totally stable generator. We are now in a position to state

Theorem 4.5 *Suppose Q is a* CUI *pregenerator. Then it is a generator* iff

$$\sup_{n \geqq 1} \alpha_b \Phi^{(n)}(\lambda)\mathbf{1} < +\infty \ , \tag{4.12}$$

where $\Phi^{(n)}(\lambda)$ $(n \geqq 1)$ is the Feller Asymptotic Sequence associated with Q_b.

Proof. By (4.10), it is easy to see that

$$\sup_{n \geqq 1} \alpha_b \Phi^{(n)}(\lambda)\mathbf{1} = \lim_{n \to \infty} \alpha_b \Phi^{(n)}(\lambda)\mathbf{1} = \alpha_b \Phi(\lambda)\mathbf{1} \ , \tag{4.13}$$

and so the result follows directly from Theorem 4.1. □

Remark. By (4.2) and (4.13) we see that (4.12) holds true for all $\lambda > 0$ iff (4.12) holds true for some $\lambda_0 > 0$. We therefore often take $\lambda = 1$.

Corollary 4.6 *Suppose Q is a* CUI *generator. Then*

$$\alpha_b \Phi^{(n)}(\lambda)\mathbf{1} < +\infty \quad (\forall n \geqq 1, \forall \lambda \geqq 0). \tag{4.14}$$

In particular,

$$\sum_{i\in N}\left[[q_{bi}/(\lambda+q_i)] + \sum_{j\in N\backslash i} q_{bi}q_{ij}/[(\lambda+q_i)(\lambda+q_j)]\right] < +\infty . \tag{4.15}$$

Proof. Result (4.14) is obvious from (4.12), whilst (4.15) is just the $n = 2$ case of (4.14). □

Remark. Since $\Phi^{(n)}(\lambda) \uparrow (n \to \infty)$, we can generally obtain better and better necessary conditions through (4.14) as we use larger and larger n. For example, it is easy to construct a CUI pregenerator which satisfies (4.4) but not (4.15).

Theorem 4.7 *Let Q be a* CUI *pregenerator. If there exists* $n \geqq 1$ *such that*

$$\sum_{i\in N} q_{bi}\left[1 - \sum_{k\in N} \Phi_{ik}^{(n)}(\lambda)q_{kb}\right] < +\infty , \tag{4.16}$$

then Q is a generator. In particular, if

$$\sum_{i\in N} q_{bi}\left[1 - [q_{ib}/(\lambda+q_i)] - \sum_{k\in N\backslash i}\{q_{ik}q_{kb}/[(\lambda+q_i)(\lambda+q_k)]\}\right] < +\infty , \tag{4.17}$$

then Q is a generator.

Proof. Result (4.16) follows from Theorem 4.1, Theorem 2.10.5 of Yang (1990) and

$$\bar{X}(\lambda) + \sum_{k\in N} \varphi_{ik}(\lambda)q_{kb} \geqq \sum_{k\in N} \Phi_{ik}^{(n)}(\lambda)q_{kb} \quad (\forall n \geqq 1),$$

whilst (4.17) is just condition (4.16) when $n = 2$. □

As in Corollary 4.6, we can obtain successively better sufficient conditions if we use successively larger n in (4.16).

Using Theorem 4.5 we can obtain another simple sufficient condition.

Theorem 4.8 *If Q is a* CUI *pregenerator satisfying*

$$\sum_{j\in N} [q_{bj}/(1+q_j)] < +\infty , \tag{4.18}$$

and if there exists a $\lambda_0 > 0$ *such that*

$$\sup_{i\in N} \sum_{j\in N\backslash i} [q_{ij}/(\lambda_0+q_j)] < 1 , \tag{4.19}$$

then Q is a generator.

Proof. Note that

$$\alpha_b \Phi^{(n+1)}(\lambda)\mathbf{1} = \alpha_b \Phi^{(1)}(\lambda)\mathbf{1} + \alpha_b \Phi^{(n)}(\lambda)\Pi(\lambda)\mathbf{1} . \tag{4.20}$$

Let

$$\Theta(\lambda) = \sup_{i\in N} \sum_{j\in N\backslash i} [q_{ij}/(\lambda+q_j)] . \tag{4.21}$$

Then by (4.11) we obtain $\Pi(\lambda)\mathbf{1} \leqq \Theta(\lambda)\mathbf{1}$. Whence use of (4.20) yields

$$\alpha_b \Phi^{(n+1)}(\lambda)\mathbf{1} \leqq \alpha_b \Phi^{(1)}(\lambda)\mathbf{1} + \Theta(\lambda)\alpha_b \Phi^{(n)}(\lambda)\mathbf{1} \quad (\forall n \geqq 1) . \tag{4.22}$$

Note that condition (4.18) guarantees that $\alpha_b \Phi^{(1)}(\lambda)\mathbf{1} < +\infty$ ($\forall\lambda > 0$). So (4.22) together with condition (4.19), i.e. $\Theta(\lambda_0) < 1$, yields

$$\alpha_b \Phi^{(n)}(\lambda_0)\mathbf{1} \leqq \left[\sum_{k=0}^{n-1} \Theta^k(\lambda_0)\right] \alpha_b \Phi^{(1)}(\lambda_0)\mathbf{1} \quad (\forall n \geqq 1) .$$

Thus $\forall n$, $\alpha_b \Phi^{(n)}(\lambda_0)\mathbf{1} < +\infty$. Whence as $\Theta(\lambda_0) < 1$, we obtain

$$\sup_{n\geqq 1}\{\alpha_b \Phi^{(n)}(\lambda_0)\mathbf{1}\} \leqq \left[\sum_{k=0}^{\infty} \Theta^k(\lambda_0)\right] \alpha_b \Phi^{(1)}(\lambda)\mathbf{1} = [1-\Theta(\lambda_0)]^{-1} \alpha_b \Phi^{(1)}(\lambda_0)\mathbf{1} < +\infty .$$

The theorem now follows from Theorem 4.5 (see the remark which follows it). □

Note that condition (4.18) is necessary. So if condition (4.19) is satisfied for some CUI pregenerator, then condition (4.18) becomes the "iff" condition.

Let us consider one particular kind of CUI pregenerator for which Theorem 4.8 applies. A CUI pregenerator Q is called bias-bounded if there exists a constant M such that

$$q_{ij} \leqq M \quad (\forall i \neq j;\ i, j \in N) . \tag{4.23}$$

Moreover, if it further satisfies (1.21), i.e. it is also a W-pregenerator, then we shall call it W-bias-bounded. Note that R- and K-pregenerators are trivially bias-bounded, whilst K-pregenerators are also trivially W-bias-bounded.

Lemma 4.9 *Suppose Q is a bias-bounded* CUI *pregenerator satisfying*

$$\sum_{j\in N} (1+q_j)^{-1} < +\infty . \tag{4.24}$$

Then there exists $\lambda_0 > 0$ such that $\Theta(\lambda_0) < 1$ where $\Theta(\lambda_0)$ is defined in (4.21).

Proof. Condition (4.24) yields $\sum_{j\in N}(\lambda + q_j)^{-1} < +\infty$ ($\forall\lambda > 0$), and hence

$$\inf_{j\in N} (\lambda + q_j)^{-1} = 0 .$$

Combining this with (4.23) we obtain

$$\Theta(\lambda) = \sup_{i\in N} \sum_{j\in N\setminus i} [q_{ij}/(\lambda+q_j)] \leqq M \sup_{i\in N} \sum_{j\in N\setminus i} (\lambda+q_j)^{-1}$$

$$= M \sup_{i\in N} \left\{\left[\sum_{j\in N} (\lambda+q_j)^{-1}\right] - (\lambda+q_i)^{-1}\right\}$$

$$= M \left\{\left[\sum_{j\in N} (\lambda+q_j)^{-1}\right] - \inf_{i\in N} (\lambda+q_i)^{-1}\right\} = M \sum_{j\in N} (\lambda+q_j)^{-1} .$$

Thus if we can show that there exists $\lambda_0 > 0$ such that

$$M \sum_{j\in N} (\lambda_0 + q_j)^{-1} < 1 , \tag{4.25}$$

then $\Theta(\lambda_0) < 1$ as required.

However, expression (4.25) is obviously true because of condition (4.24). Indeed, if $M = 0$, then (4.25) is trivial. Whilst if $M > 0$, then (4.25) is equivalent to

$$\sum_{j \in N} (\lambda_0 + q_j)^{-1} < M^{-1} . \tag{4.26}$$

But condition (4.24) guarantees that

$$\sum_{j \in N} (\lambda + q_j)^{-1} < +\infty \quad (\forall \lambda > 0) .$$

Whence on noting that $(\lambda + q_j)^{-1}$ is a monotone function of $\lambda > 0$, the Monotone Convergence Theorem yields

$$\lim_{\lambda \to \infty} \sum_{j \in N} (\lambda + q_j)^{-1} = \sum_{j \in N} \lim_{\lambda \to \infty} (\lambda + q_j)^{-1} = 0 .$$

So as $M^{-1} > 0$, it is easy to find $\lambda_0 > 0$ such that (4.26) is true. □

Theorem 4.10 *Suppose Q is a bias-bounded* CUI *pregenerator which satisfies both*

$$\sum_{j \in N} [q_{bj}/(1 + q_j)^{-1}] < +\infty , \quad \text{and} \tag{4.27}$$

$$\sum_{j \in N} (1 + q_j)^{-1} < +\infty . \tag{4.28}$$

Then Q is a generator and (4.27) *is actually necessary. In particular, if Q is a* W*-bias-bounded* CUI *pregenerator, then it is a generator* iff (4.27) *holds true.*

Proof. The first part of the theorem follows from Theorem 4.8 and Lemma 4.9; whilst the latter part follows from the fact that, for a W-pregenerator, (4.27) ⇒ (4.28) and that (4.27) is necessary. □

5 Uniqueness criterion for conservative uni-instantaneous Q-processes

In this section we provide the criterion for the uniqueness of CUI Q-processes: this is the second main result of this paper. When we discuss the uniqueness problem, we shall, of course, assume that the existence condition (4.1) is satisfied, i.e. we assume that the given Q is a generator.

First we point out that in contrast to the totally stable case, non-totally-stable generators (not necessarily uni-instantaneous state generators) always relate to infinitely many Q-processes. Indeed, we may make the following claim.

Proposition. *Suppose Q is a non-totally-stable generator. Then there always exist infinitely many Q-processes.*

Since this proposition is well-known, there is no need to give a proof here. [A proof may be obtained by using Theorem 2.1 (see Corollary 7.9 of Chen and Renshaw 1990)]. However, although there always exist infinitely many Q-processes for a non-totally stable generator, the honest one may be unique. Thus there arises a non-trivial and interesting uniqueness problem for honest non-totally-stable Q-processes. We shall discuss this problem for the CUI generator.

Recall that a CUI pregenerator Q can be written in the form

$$Q = \begin{bmatrix} -q_b & \alpha_b \\ \beta_b & Q_b \end{bmatrix} = \begin{bmatrix} -\infty & \alpha_b \\ \beta_b & Q_b \end{bmatrix} , \tag{5.1}$$

where Q_b is a totally stable generator which satisfies both

$$\alpha_b \mathbf{1} = +\infty ,$$

and

$$\beta_b + Q_b \mathbf{1} = \mathbf{0} .$$

Since Q_b is totally stable, we may define $\mathcal{M}_\lambda^+(Q_b)$, $m^+(Q_b)$, $\mathcal{L}_\lambda^+(Q_b)$ and $n^+(Q_b)$ (as in Sect. 1). We repeat that $\mathcal{M}_\lambda^+(Q_b)$ denotes the solution space of the equation

$$\begin{cases}(\lambda I - Q_b)U(\lambda) = \mathbf{0} \\ \mathbf{0} \leqq U(\lambda) \leqq \mathbf{1}\end{cases} \tag{5.2}$$

and $m^+(Q_b)$ denotes the dimension of $\mathcal{M}_\lambda^+(Q_b)$, whilst $\mathcal{L}_\lambda^+(Q_b)$ denotes the solution space of the equation

$$\begin{cases}v(\lambda)(\lambda I - Q_b) = \mathbf{0} \\ \mathbf{0} \leqq v(\lambda) \in \ell ,\end{cases} \tag{5.3}$$

and $n^+(Q_b)$ is the dimension of $\mathcal{L}_\lambda^+(Q_b)$. As noted previously, both $m^+(Q_b)$ and $n^+(Q_b)$ are independent of $\lambda > 0$.

In order to construct the uniqueness criterion we require a concept of the sub-space of $\mathcal{M}_\lambda^+(Q_b)$. Let α be any row vector on N. Then the solution space of the equation

$$\begin{cases}(\lambda I - Q_b)U(\lambda) = \mathbf{0} \\ \alpha U(\lambda) < +\infty \\ \mathbf{0} \leqq U(\lambda) \leqq \mathbf{1}\end{cases} \tag{5.4}$$

is denoted by $\mathcal{M}_\lambda^+(Q_b;\alpha)$. It is easy to see that $\mathcal{M}_\lambda^+(Q_b;\alpha)$ is indeed a subspace of $\mathcal{M}_\lambda^+(Q_b)$ (in the sense of linear space), and that the dimension of this sub-space is also independent of $\lambda > 0$. We may call it an α-conditional sub-space since it is usually dependent on the vector α. We use $m^+(Q_b;\alpha)$ to denote the dimension of this α-conditional sub-space, (which is independent of $\lambda > 0$). If $\alpha\mathbf{1} < +\infty$, then, of course, the α-conditional sub-space coincides with the original space $\mathcal{M}_\lambda^+(Q_b)$. However, if $\alpha\mathbf{1} = +\infty$, then this α-conditional sub-space may be a proper sub-space. We shall just encounter the latter case when we discuss the uniqueness problem for the CUI generator.

We are now ready to provide the following uniqueness criterion.

Theorem 5.1 *Suppose Q is a* CUI *generator as* (5.1). *Then the honest* CUI *Q-process is unique* iff *the following two conditions hold simultaneously*:

(i) $m^+(Q_b;\alpha_b) = 0$,

(ii) $n^+(Q_b) = 0$,

namely both Eqs. (5.3) *and* (5.4) *have only a zero solution. Moreover, if the honest Q-process is not unique, i.e. if either* (i) *or* (ii) *is not satisfied, then there exist infinitely many honest Q-processes.*

Proof. Let $R(\lambda) = \{r_{ij}(\lambda); i, j \in E\}$ be an arbitrary honest CUI Q-process. Then by Theorem 2.1, $R(\lambda)$ can be uniquely decomposed into

$$R(\lambda) = \begin{bmatrix} 0 & \mathbf{0} \\ \mathbf{0} & \psi(\lambda)\end{bmatrix} + r_{bb}(\lambda)\begin{bmatrix} 1 \\ \xi(\lambda)\end{bmatrix}[1, \eta(\lambda)] \tag{5.5}$$

where:

$$\psi(\lambda) \quad \text{is a } Q_b\text{-process} ; \tag{5.6}$$

$$\eta(\lambda) \in \mathbf{H}_\psi ; \tag{5.7}$$

$$\xi(\lambda) = \mathbf{1} - \lambda\psi(\lambda)\mathbf{1} ; \tag{5.8}$$

$$\lim_{\lambda\to\infty} \lambda\eta(\lambda) = \alpha_b \quad \text{and} \quad \lim_{\lambda\to\infty} \lambda\xi(\lambda) = \beta_b \text{ ; and} \tag{5.9}$$

$$r_{bb}(\lambda) = (\lambda + \lambda\langle\eta(\lambda), \mathbf{1}\rangle)^{-1} . \tag{5.10}$$

Note that (5.8) and (5.10) follow from the honesty condition. Thus if we can prove that $\psi(\lambda)$ and $\eta(\lambda)$ in (5.5) have a unique form, then the Q-process $R(\lambda)$ is also unique.

First, we prove that $\psi(\lambda)$ must be the Feller minimal Q_b-process $\Phi(\lambda)$. Indeed, any CUI Q-process $R(\lambda)$ must be almost B-type (because CUI generators satisfy (1.26)), whence by Lemma 3.4 $\psi(\lambda)$ must be a B-type Q_b-process. So by Theorem 2.12.1 of Yang (1990) $\psi(\lambda)$ can be written as

$$\psi(\lambda) = \Phi(\lambda) + B(\lambda) \tag{5.11}$$

where

$$\lambda B(\lambda)\mathbf{1} \in \mathscr{M}_\lambda^+(Q_b) . \tag{5.12}$$

Moreover, (5.7) and (5.9), together with Fatou's Lemma, yield

$$\alpha_b\psi(\lambda) \leqq \eta(\lambda) \in \ell .$$

That is,

$$\lambda\alpha_b\psi(\lambda)\mathbf{1} < +\infty .$$

Hence by (5.11)

$$\alpha_b\lambda B(\lambda)\mathbf{1} < +\infty . \tag{5.13}$$

Combining (5.12) and (5.13) shows that $\lambda B(\lambda)\mathbf{1}$ is a solution of Eq. (5.4). So from condition (i) of our theorem, $\lambda B(\lambda)\mathbf{1} \equiv \mathbf{0}$, whence $B(\lambda) \equiv 0$ since $B(\lambda) \geqq 0$. It then follows from (5.11) that $\psi(\lambda)$ must be the Feller minimal Q_b-process $\Phi(\lambda)$.

Second, we prove that $\eta(\lambda)$ in (5.5) is also unique. Since we have proved that $\psi(\lambda)$ is the Feller minimal Q_b-process, Lemma 2.4 together with (5.7) yields

$$\eta(\lambda) = \alpha\Phi(\lambda) + \bar{\eta}(\lambda) \tag{5.14}$$

where $\alpha = \lim_{\lambda\to\infty} \lambda\eta(\lambda)$ and $\bar{\eta}(\lambda) \in \mathscr{L}_\lambda^+(Q_b)$. By (5.9) we know that $\alpha = \alpha_b$. Moreover, condition (ii) yields $\bar{\eta}(\lambda) = \mathbf{0}$. So $\eta(\lambda)$ can be written as

$$\eta(\lambda) = \alpha_b\Phi(\lambda) . \tag{5.15}$$

Thus the form of $\eta(\lambda)$ is also unique. The proof of sufficiency is now complete.

To prove necessity we first note that if condition (ii) does not hold, then Eq. (5.3) has a non-zero solution, $\bar{\eta}(\lambda)$, say. Moreover, without loss of generality, we may further require that this $\bar{\eta}(\lambda) \in \mathbf{H}_\Phi$ (see Reuter 1959, or Hou and Guo 1988). Choose any constant $k > 0$, and let $\psi(\lambda) = \Phi(\lambda)$ and $\eta(\lambda) = \alpha_b\Phi(\lambda) + k\bar{\eta}(\lambda)$. Then together with (5.8) and (5.10) we can construct $R(\lambda)$ as in (5.5). It is easy to show that such an $R(\lambda)$ is an honest Q-process, whence different k yield different honest Q-processes. Since there are infinitely many ways to choose $k > 0$, the honest Q-process is not unique; in fact there exist infinitely many of them.

If condition (i) does not hold, then Eq. (5.4) has a non-zero solution, $\bar{\xi}(\lambda)$, say. Moreover, it is easy to show, without loss of generality, that we may further assume that $\bar{\xi}(\lambda) \in \mathbf{K}_\Phi$. Then on using this $\bar{\xi}(\lambda)$, we can construct infinitely many B-type Q_b-processes (see, Reuter 1959). Each of them, $\psi(\lambda)$ say, will satisfy

$$\alpha_b \psi(\lambda)\mathbf{1} < +\infty \tag{5.16}$$

since $\alpha_b \Phi(\lambda)\mathbf{1} < +\infty$ (existence condition (4.1)) and $\alpha_b \bar{\xi}(\lambda) < +\infty$ (since $\bar{\xi}(\lambda) \in \mathcal{M}_\lambda^+(Q_b;\alpha_b)$). Now choose any such B-type Q_b-process, and let $\eta(\lambda) = \alpha_b \psi(\lambda)$. Then on using (5.8) and (5.10) again we can construct $R(\lambda)$ as in (5.5). By noting (5.16) it is easy to show that such an $R(\lambda)$ is well-defined, that it is an honest Q-process, and that different $\psi(\lambda)$ generate different Q-processes $R(\lambda)$. Thus Q-processes are not unique and there exist infinitely many of them. This concludes the proof of the necessity of both conditions (i) and (ii), together with the last part of the theorem. □

Since $\mathcal{M}_\lambda^+(Q_b;\alpha_b)$ is a sub-space of $\mathcal{M}_\lambda^+(Q_b)$, we have $m^+(Q_b;\alpha_b) \leqq m^+(Q_b)$. Thus an immediate corollary of Theorem 5.1 is the following sufficient condition.

Corollary 5.2 *Suppose Q is a* CUI *generator satisfying $m^+(Q_b) = n^+(Q_b) = 0$. Then the honest Q-process is unique.*

Note that K- and R-generators satisfy the condition of Corollary 5.2

On the other hand, even if $m^+(Q_b) > 0$ we may still have $m^+(Q_b;\alpha_b) = 0$. It is interesting to note that for W-generators the latter is always true.

Lemma 5.3 *Suppose Q is a* W-*generator. Then $m^+(Q_b;\alpha_b) = 0$.*

Proof. We must prove that Eq. (5.4) has only the zero solution for a W-generator. Suppose not, i.e. there exists a non-zero solution, $\bar{\xi}(\lambda) = \{\bar{\xi}_i(\lambda); i \in N\}$, say. Let $\sup_{i \in N} \bar{\xi}_i(\lambda) = c(\lambda)$. Then $c(\lambda) > 0$. Now let

$$\tilde{\xi}(\lambda) = \bar{\xi}(\lambda)/c(\lambda), \text{ i.e. } \tilde{\xi}_i(\lambda) = \bar{\xi}_i(\lambda)/c(\lambda) \quad (\forall i \in N) .$$

Then it is obvious that

$$\mathbf{0} \leqq \tilde{\xi}(\lambda) \leqq \mathbf{1}; \quad (\lambda I - Q)\tilde{\xi}(\lambda) = (\lambda I - Q)\bar{\xi}(\lambda)/c(\lambda) = \mathbf{0} ;$$

and that

$$\alpha_b \tilde{\xi}(\lambda) = \alpha_b \bar{\xi}(\lambda)/c(\lambda) < +\infty .$$

In short, $\tilde{\xi}(\lambda)$ is also a non-zero solution of (5.4). Note that

$$\sup_{i \in N} \tilde{\xi}_i(\lambda) = 1 . \tag{5.17}$$

Since $\alpha_b \tilde{\xi}(\lambda) < +\infty$, we have

$$\lim_{j \to \infty} q_{bj} \tilde{\xi}_j(\lambda) = 0 .$$

But Q is a W-generator, and thus satisfies

$$\lim_{j \to \infty} \inf q_{bj} > 0 .$$

So we must therefore have

$$\lim_{j \to \infty} \tilde{\xi}_j(\lambda) = 0 .$$

Thus there exists a finite subset G of N such that

$$\sup_{i\in N\setminus G} \tilde{\xi}_i(\lambda) < 1 .$$

Comparison with (5.17) then yields

$$\sup_{i\in G} \tilde{\xi}_i(\lambda) = 1 .$$

Since G is a finite set the above supremum must be attained in a state h, say. So as $\tilde{\xi}(\lambda)$ is a solution of Eq. (5.4), and hence a solution of Eq. (5.2), whilst $\bar{X}(\lambda)$ is the maximal solution of Eq. (5.2), we have

$$1 = \tilde{\xi}_h(\lambda) \leqq \bar{X}_h(\lambda) \leqq 1 .$$

Thus $\bar{X}_h(\lambda) = 1$. Now using Theorem 2.10.5 of Yang (1990) we obtain

$$0 \leqq \lambda\varphi_{hh}(\lambda) \leqq \lambda \sum_{j\in N} \varphi_{hj}(\lambda) = 1 - \sum_{k\in N} \varphi_{hk}(\lambda)d_k$$
$$- \bar{X}_h(\lambda) \leqq 1 - \bar{X}_h(\lambda) = 1 - 1 = 0 \quad (\lambda > 0).$$

Hence $\varphi_{hh}(\lambda) = 0$ which is a contradiction. □

For W-generators the uniqueness condition is therefore quite simple.

Theorem 5.4 *Suppose Q is a* W*-generator. Then the honest Q-process is unique* iff $\varkappa^+(Q_b) = 0$.

Proof. Follows from Theorem 5.1 and Lemma 5.3. □

If we are only concerned with almost F-type Q-processes for general CUI generators, then the uniqueness criterion is also quite simple.

Theorem 5.5 *Suppose Q is a* CUI *generator. Then the following statements are equivalent.*

(i) *there exists only one honest Q-process which satisfies Eq.* (3.18);
(ii) *there exists only one honest Q-process which satisfies both Eqs.* (3.17) *and* (3.18);
(iii) $\varkappa^+(Q_b) = 0$.

We shall omit the proof of Theorem 5.5 since (i)⇔(ii) is obvious, whilst (i)⇔(iii) is actually a by-product of the proof of Theorem 5.1 together with Lemma 3.4.

6 Construction of conservative uni-instantaneous Q-processes

Suppose Q is a CUI generator. Then, as before, Q_b is a totally stable generator. We shall assume that the Feller minimal Q_b-process $\Phi(\lambda)$, the entrance element $\bar{X}$ and passive element X^0, namely

$$X^0 = \lim_{\lambda\to 0} \lambda\Phi(\lambda)\mathbf{1} \quad \text{and} \quad \bar{X} = \lim_{\lambda\to 0} \bar{X}(\lambda) , \tag{6.1}$$

are all known since they can be easily constructed via Q_b. Moreover, it is well-known that for any $\eta(\lambda) \in \mathbf{H}_\Phi$

$$\sigma^0 = \lambda\langle\eta(\lambda), X^0\rangle \tag{6.2}$$

is independent of $\lambda > 0$ and finite. In particular, if $\bar{\eta}(\lambda) \in \mathscr{L}_\lambda^+$ and $\bar{\eta}(\lambda) \in \mathbf{H}_\Phi$, then

$$\bar{\sigma}^0 = \lambda\langle\bar{\eta}(\lambda), X^0\rangle \tag{6.3}$$

is finite and independent of $\lambda > 0$ (see Yang 1990).

We are now ready to present the following construction theorem.

Theorem 6.1 *Suppose Q is a* CUI *generator. Let $\Phi(\lambda) = \{\varphi_{ij}(\lambda); i, j \in N\}$ be the Feller minimal Q_b-process. Choose any $\bar{\xi}(\lambda) \in \mathscr{M}_\lambda^+(Q_b)$ such that $\bar{\xi}(\lambda) \in \mathbf{K}_\Phi$, i.e. $\mathbf{0} \leqq \bar{\xi}(\lambda) \leqq \mathbf{1}$, which satisfies*

$$\bar{\xi}(\lambda) - \bar{\xi}(\mu) = (\mu - \lambda)\Phi(\lambda)\bar{\xi}(\mu)\,, \tag{6.4}$$

and choose $\bar{\eta}(\lambda) \in \mathscr{L}_\lambda^+(Q_b)$ which satisfies both $\bar{\eta}(\lambda) \in \mathbf{H}_\Phi$, i.e.

$$\bar{\eta}(\lambda) - \bar{\eta}(\mu) = (\mu - \lambda)\bar{\eta}(\lambda)\Phi(\mu), \quad \mathbf{0} \leqq \bar{\eta}(\lambda) \in \ell\,, \tag{6.5}$$

$$\langle\alpha_b, \bar{X} - \bar{\xi}\rangle < +\infty \tag{6.6}$$

and

$$W_\lambda = \lambda\langle\bar{\eta}(\lambda), \bar{X} - \bar{\xi}\rangle \uparrow W < +\infty\,. \tag{6.7}$$

Now choose a constant c such that

$$c \geqq \langle\alpha_b, X^0\rangle + \bar{\sigma}^0 + \langle\alpha_b, \bar{X} - \bar{\xi}\rangle + W \tag{6.8}$$

where X^0, $\bar{X}$ and $\bar{\sigma}^0$ are defined by (6.1) *and* (6.3), *respectively, and $\bar{\xi} = \lim_{\lambda\to 0}\bar{\xi}(\lambda)$. Finally, let*

$$\eta(\lambda) = \alpha_b\Phi(\lambda) + \bar{\eta}(\lambda)\,, \tag{6.9}$$

$$\xi(\lambda) = \Phi(\lambda)\beta_b + \bar{\xi}(\lambda), \quad \text{and} \tag{6.10}$$

$$r_{bb}(\lambda) = (c + \lambda + \lambda\langle\eta(\lambda), \xi\rangle)^{-1} \tag{6.11}$$

where $\xi = \lim_{\lambda\to 0}\xi(\lambda)$. Then

$$R(\lambda) = \begin{bmatrix} 0 & \mathbf{0} \\ \mathbf{0} & \Phi(\lambda) \end{bmatrix} + r_{bb}(\lambda)\begin{bmatrix} 1 \\ \xi(\lambda) \end{bmatrix}[1, \eta(\lambda)] \quad (\lambda > 0) \tag{6.12}$$

is a CUI *Q-process.*

Conversely, if $m^+(Q_b; \alpha_b) = 0$, then every CUI *Q-process may be constructed in the above manner.*

The Q-processes are honest iff *both*

$$\bar{\xi}(\lambda) = \bar{X}(\lambda) \tag{6.13}$$

and

$$c = \langle\alpha_b, X^0\rangle + \bar{\sigma}^0 \tag{6.14}$$

hold in the above construction.

Proof. Condition (6.5) and existence condition (4.1), together with Lemma 2.4, show that $\eta(\lambda)$ defined in (6.9) satisfies

$$\eta(\lambda) \in \mathbf{H}_\Phi\,. \tag{6.15}$$

Similarly, conditions (6.4) and (6.8), together with Lemma 2.4, show that $\xi(\lambda)$ defined in (6.10) satisfies

$$\xi(\lambda) \in \mathbf{K}_\Phi\,. \tag{6.16}$$

Moreover, (6.9), (6.10) and Lemma 2.4 also show that

$$\lim_{\lambda\to\infty} \lambda\eta(\lambda) = \alpha_b \quad \text{and} \quad \lim_{\lambda\to\infty} \lambda\xi(\lambda) = \beta_b\,, \tag{6.17}$$

whilst

$$\xi(\lambda) \leqq \mathbf{1} - \lambda\Phi(\lambda)\mathbf{1} \tag{6.18}$$

is obvious through $\bar{\xi}(\lambda) \leqq \bar{X}(\lambda)$ and (6.10). That

$$\lim_{\lambda\to\infty} \lambda\langle\eta(\lambda), \mathbf{1}\rangle = +\infty \tag{6.19}$$

follows directly from the fact that $\alpha_b\mathbf{1} = +\infty$. Note also that (6.6), (6.7) and (6.8) guarantee that

$$\lim_{\lambda\to\infty} \lambda\langle\eta(\lambda), \mathbf{1}-\xi\rangle < +\infty\,. \tag{6.20}$$

Results (6.15)–(6.20) show that all the requirements of Theorem 2.2 are satisfied, and so $R(\lambda)$ as constructed in (6.12) is a CUI Q-process. That (6.13) and (6.14) are the "if and only if" conditions for the constructed CUI Q-process being honest follows from Theorem 2.3.

Conversely, suppose that $m^+(Q_b;\alpha_b) = 0$ and $R(\lambda)$ is an arbitrary CUI Q-process. From the proof of Theorem 5.1 we know that $m^+(Q_b;\alpha_b) = 0$ leads to the restriction process of any CUI Q-process $R(\lambda)$ being the Feller minimal Q_b-process $\Phi(\lambda)$. Thus by Theorem 2.1, $R(\lambda)$ can be uniquely decomposed into

$$R(\lambda) = \begin{bmatrix} 0 & \mathbf{0} \\ \mathbf{0} & \Phi(\lambda) \end{bmatrix} + r_{bb}(\lambda)\begin{bmatrix} 1 \\ \xi(\lambda) \end{bmatrix}[1,\, \eta(\lambda)] \tag{6.21}$$

where

$$\eta(\lambda) \in \mathbf{H}_\Phi \quad \text{and} \quad \xi(\lambda) \in \mathbf{K}_\Phi\,. \tag{6.22}$$

In addition,

$$\lim_{\lambda\to\infty} \lambda\eta(\lambda) = \alpha_b, \quad \lim_{\lambda\to\infty} \lambda\xi(\lambda) = \beta_b\,, \tag{6.23}$$

and

$$r_{bb}(\lambda) = (c + \lambda + \lambda\langle\eta(\lambda), \xi\rangle)^{-1}\,, \tag{6.24}$$

where $\xi = \lim_{\lambda\to 0} \xi(\lambda)$ and c is a finite constant such that

$$c \geqq \lim_{\lambda\to\infty} \lambda\langle\eta(\lambda), \mathbf{1}-\xi\rangle\,. \tag{6.25}$$

Lemma 2.4, together with (6.22) and (6.23), now yields the representations of $\eta(\lambda)$ and $\xi(\lambda)$ as

$$\eta(\lambda) = \alpha_b\Phi(\lambda) + \bar{\eta}(\lambda) \tag{6.26}$$

and

$$\xi(\lambda) = \Phi(\lambda)\beta_b + \bar{\xi}(\lambda)\,, \tag{6.27}$$

where $\bar{\eta}(\lambda) \in \mathscr{L}_\lambda^+(Q_b) \cap \mathbf{H}_\Phi$ and $\bar{\xi}(\lambda) \in \mathscr{M}_\lambda^+(Q_b) \cap \mathbf{K}_\Phi$. From (6.27) we obtain

$$\xi = \Gamma\beta_b + \bar{\xi}\,, \tag{6.28}$$

where

$$\Gamma = \lim_{\lambda \to 0} \Phi(\lambda) \quad \text{and} \quad \bar{\xi} = \lim_{\lambda \to 0} \bar{\xi}(\lambda) . \tag{6.29}$$

Whilst substituting (6.26) and (6.28) into (6.25) yields

$$c \geqq \lim_{\lambda \to \infty} \lambda \langle \alpha_b \Phi(\lambda), \mathbf{1} - \Gamma\beta_b - \bar{\xi} \rangle + \lim_{\lambda \to \infty} \lambda \langle \bar{\eta}(\lambda), \mathbf{1} - \Gamma\beta_b - \bar{\xi} \rangle . \tag{6.30}$$

Now by Lemma 2.11.2 of Yang (1990) and the basic Assumption II it is easy to show that

$$1 = X^0 + \bar{X} + \Gamma\beta_b . \tag{6.31}$$

Whence substituting (6.31) into (6.30) yields

$$c \geqq \lim_{\lambda \to \infty} \lambda \langle \alpha_b \Phi(\lambda), X^0 + \bar{X} - \bar{\xi} \rangle + \lim_{\lambda \to \infty} \lambda \langle \bar{\eta}(\lambda), X^0 + \bar{X} - \bar{\xi} \rangle . \tag{6.32}$$

By (6.2) we know that both $\lambda\langle \alpha_b \Phi(\lambda), X^0 \rangle$ and $\lambda \langle \bar{\eta}(\lambda), X^0 \rangle$ are finite constants; the former is $\langle \alpha_b, X^0 \rangle$ whilst the latter is $\bar{\sigma}^0$. Thus (6.32) leads to

$$c \geqq \langle \alpha_b, X^0 \rangle + \bar{\sigma}^0 + \lim_{\lambda \to \infty} \lambda \langle \alpha_b \Phi(\lambda), \bar{X} - \bar{\xi} \rangle + \lim_{\lambda \to \infty} \lambda \langle \bar{\eta}(\lambda), \bar{X} - \bar{\xi} \rangle . \tag{6.33}$$

However c is a constant, hence both limits in the right hand side of (6.33) must be finite. Results (6.6), (6.7) and (6.8) now follow. □

Theorem 6.1 tells us that provided $\mathscr{m}^+(Q_b; \alpha_b) = 0$, the construction problem of CUI Q-processes has been completely solved. Since $\mathscr{m}^+(Q_b) = 0$ leads to $\mathscr{m}^+(Q_b; \alpha) = 0$, the construction problem for the case of $\mathscr{m}^+(Q_b) = 0$ is therefore also solved. For example, if Q_b is lower triangular (not Q itself!) then all the Q-processes can be constructed via Q by Theorem 6.1. In particular, we can easily construct all the Q-processes for K- and R-generators.

Moreover, even if $\mathscr{m}^+(Q_b)$ is non-countable, we may still have $\mathscr{m}^+(Q_b; \alpha_b) = 0$. For example, we see from Lemma 5.3 that for the W-generator we always have $\mathscr{m}^+(Q_b; \alpha) = 0$ no matter how complicated the Martin exit boundary of Q_b is. So the construction of all Q-processes for W-generators is solved as well. This is, at first sight, a surprising result since even for the familiar totally stable Q-process we can construct only for the case where the Martin exit boundary is finite.

When both $\mathscr{m}^+(Q_b; \alpha_b) = 0$ and $\mathscr{n}^+(Q_b) = 0$ (but $\mathscr{m}^+(Q_b)$ may be arbitrary), the construction is particularly simple. We present it here in order to emphasize its concise and precise nature.

Corollary 6.2 *Suppose Q is a* CUI *generator which satisfies both $\mathscr{n}^+(Q_b) = 0$ and $\mathscr{m}^+(Q_b; \alpha_b) = 0$. Choose a constant c such that*

$$c \geqq \langle \alpha_b, X^0 \rangle, \tag{6.34}$$

and let

$$R(\lambda) = \begin{bmatrix} 0 & \mathbf{0} \\ \mathbf{0} & \Phi(\lambda) \end{bmatrix} + r_{bb}(\lambda) \begin{bmatrix} 1 \\ \mathbf{1} - \lambda\Phi(\lambda)\mathbf{1} \end{bmatrix} [1, \alpha_b \Phi(\lambda)] \tag{6.35}$$

where

$$r_{bb}(\lambda) = [c - \langle \alpha_b, X^0 \rangle + \lambda + \lambda\alpha_b \Phi(\lambda)\mathbf{1}]^{-1} . \tag{6.36}$$

Then $R(\lambda)$ is a CUI *Q-process. Conversely, every* CUI *Q-process can be constructed in the above manner. The honest* CUI *Q-process, $R^*(\lambda)$, say, is unique and can be constructed by simply taking the equality in* (6.34). *Furthermore, $R^*(\lambda)$ is maximal in the sense that for any other* CUI *Q-process $R(\lambda)$*

$$R^*(\lambda) \geqq R(\lambda) \quad (\forall \lambda > 0) . \tag{6.37}$$

Proof. All the conclusions follow from Theorem 6.1 except (6.37). However, (6.37) is obvious on noting that the construction of all the Q-processes depends only upon a constant. □

7 Examples

In this section, we present some examples to illustrate the results of the last three sections.

Example 1 – K-pregenerator (see (1.14)). Here $E = \{0, 1, 2, \ldots\}$ and $b = \{0\}$.

Theorem 7.1 *Suppose Q is a* K-*pregenerator as in* (1.14). *Then*
(i) *Q is a generator* iff

$$\sum_{j=1}^{\infty} 1/(1 + q_j) < + \infty \text{ , or equivalently} \tag{7.1}$$

$$\sum_{j=1}^{\infty} 1/q_j < + \infty \ . \tag{7.2}$$

(ii) *When Q is a generator, there exists only one honest Q-process, together with infinitely many non-honest ones. All of them can be constructed as follows. Choose a constant $c \in [0, \infty)$, let*

$$\rho(\lambda) = \left\{ c + \lambda + \lambda \sum_{j=1}^{\infty} 1/(\lambda + q_j) \right\}^{-1}$$

and then let

$$r_{ij}(\lambda) = \begin{cases} \rho(\lambda) & \text{if } i = j = 0; \\ \rho(\lambda)/(\lambda + q_j) & \text{if } i = 0,\ j > 0; \\ \rho(\lambda) q_i/(\lambda + q_i) & \text{if } i > 0,\ j = 0; \\ \delta_{ij}/(\lambda + q_i) + \rho(\lambda) q_i/[(\lambda + q_i)(\lambda + q_j)] & \text{if } i > 0,\ j > 0 \ . \end{cases} \tag{7.3}$$

Then $R(\lambda) = \{r_{ij}(\lambda); i, j \geqq 0\}$ is a Q-process. When $c = 0$, we obtain the unique honest Q-process.

Proof. See Corollaries 4.4, 5.2 and 6.2, together with the fact that for a K-pregenerator, Q_b is triangular, and so the Feller minimal Q_b-process $\Phi(\lambda) = \{\varphi_{ij}(\lambda)\}$ takes the form $\varphi_{ij}(\lambda) = \delta_{ij}/(\lambda + q_i)$. Note that (7.1)$\Leftrightarrow$(7.2) is obvious. □

Example 2 – R-pregenerator (see (1.17)–(1.19)). Here, again $E = \{0, 1, 2, \ldots\}$ and $b = \{0\}$.

Theorem 7.2 *Suppose Q is an* R-*pregenerator as in* (1.17)–(1.19). *Then*

(i) *Q is a generator* iff

$$\sum_{j=1}^{\infty} b_j/(1+q_j) < +\infty \ . \tag{7.4}$$

(ii) *When Q is a generator, there exists only one honest Q-process, together with infinitely many non-honest ones. All can be constructed as follows. Choose a constant $c \in [0, \infty)$, let*

$$\rho(\lambda) = \left\{c + \lambda + \lambda\left\{\sum_{j=1}^{\infty} b_j/(\lambda + q_j)\right\}\right\}^{-1} , \tag{7.5}$$

and then let

$$r_{ij}(\lambda) = \begin{cases} \rho(\lambda) & \text{if } i = j = 0; \\ \rho(\lambda) b_j/(\lambda + q_j) & \text{if } i = 0,\ j > 0; \\ \rho(\lambda) q_i/(\lambda + q_i) & \text{if } i > 0,\ j = 0; \\ \delta_{ij}/(\lambda + q_i) + \rho(\lambda) q_i b_j/[(\lambda + q_i)(\lambda + q_j)] & \text{if } i > 0,\ j > 0 . \end{cases} \tag{7.6}$$

Then $R(\lambda) = \{r_{ij}(\lambda); i, j \geqq 0\}$ is a Q-process. When $c = 0$, we obtain the unique honest Q-process.

Proof. The same as Theorem 7.1. □

Remark 1 From Theorems 7.1 and 7.2, we see that for both K- and R-generators construction depends only upon a non-negative constant. It is easy to see that if $c_1 \geqq c_2 \geqq 0$, then the corresponding processes, $R_1(\lambda)$ and $R_2(\lambda)$, say, satisfy

$$R_1(\lambda) \leqq R_2(\lambda) \quad (\forall \lambda > 0) .$$

Thus we can set up a natural order relationship between them. In particular, the unique honest process which corresponds to $c = 0$ is maximal. There is, of course, no minimal process. This is a remarkable feature of Q-processes with instantaneous states, and contrasts to the totally stable case for which there always exists a minimal process but no maximal one.

Remark 2 In contrast to K-pregenerators, for R-pregenerators condition (7.4) is not equivalent to condition (1.20), i.e.

$$\sum_{j=1}^{\infty} (b_j/q_j) < +\infty \ .$$

We can easily provide an R-pregenerator which satisfies (7.4) (and thus a generator) but not (1.20). For example, let

$$q_{2n} = 1/(2n); \quad q_{2n-1} = 2n - 1; \quad b_{2n} = 1/(4n^2); \quad b_{2n-1} = 1/(2n-1) \quad (n \geqslant 1) .$$

Then $\Sigma b_j = +\infty$, $\Sigma b_j/(1 + q_j) < +\infty$, but $\Sigma b_j/q_j = +\infty$. Thus for a K-pregenerator Q, condition (1.16) is an "iff" condition under which it becomes a generator, whilst for an R-generator condition (1.20) is sufficient but not necessary. This provides the answer for the necessary condition problem for K- and R-pregenerators (see Sect. 1).

Remark 3 It is interesting to ask what will happen if $\Sigma b_j < +\infty$ for the Q of (1.17) (it is, of course, no longer a CUI pregenerator). It turns out that it is never a generator – proved by using Theorem 2.1.

Example 3 – W-pregenerator (see (1.21)–(1.22)).

Theorem 7.3 *Suppose Q is a* W*-pregenerator as in* (1.21)–(1.22). *Then*

(i) *it is a generator* iff (1.23) *holds true;*
(ii) *when Q is a generator, the honest Q-process is unique* iff $n^+(Q_b) = 0$, *whilst all the Q-processes can be constructed as in Theorem* 6.1.

Proof. See Theorems 4.1, 5.4 and 6.1. □

Example 4 – CUI branching pregenerator. Let E be non-negative integer. A pregenerator $Q = \{q_{ij}\}$ defined by

$$q_{ij} = \begin{cases} -\infty & \text{if } i = j = 0 \\ \alpha_j & \text{if } i = 0 \text{ and } j \geqq 1 \\ ib_{j-i+1} & \text{if } i \geqq 1 \text{ and } j \geqq i-1 \\ 0 & \text{otherwise}, \end{cases} \tag{7.7}$$

where

$$\alpha_j \geqq 0, \quad \sum_{j=1}^{\infty} \alpha_j = +\infty, \quad \text{and} \tag{7.8}$$

$$-\infty < b_1 \leqq 0, \quad b_j \geqq 0\ (j \neq 1) \quad \text{and} \quad \sum_{j=0}^{\infty} b_j = 0, \tag{7.9}$$

is called a CUI branching pregenerator.

Theorem 7.4 *Suppose Q is a* CUI *branching pregenerator as defined in* (7.7)–(7.9). Then

(i) *Q is a generator* iff $\sum_{k=1}^{\infty} \sum_{j=1}^{\infty} \alpha_j \varphi_{jk}(\lambda) < +\infty \ (\lambda > 0)$

where $\Phi(\lambda) = \{\varphi_{ij}(\lambda)\}$ is the Feller minimal Q_b-process;
(ii) *when Q is a generator there exists only one honest Q-process* iff $m^+(Q_b; \alpha_b) = 0$; *and*
(iii) *there exists only one honest almost* F*-type Q-process.*

Proof. It is easy to show that $n^+(Q_b) = 0$ for a CUI branching pregenerator (see, for example Lemma 3.1 of Chen and Renshaw 1990); whence (ii) and (iii) follow from Theorems 5.1 and 5.5, respectively, whilst (i) follows from Theorem 4.1. □

The unique honest almost F-type Q-process in (iii) is usually called a Markov branching process with instantaneous immigration (BPII). Further discussion on this process can be seen in Chen and Renshaw (1990).

Example 5 – CUI birth–death pregenerator. Let E be non-negative integer. A pregenerator $Q = \{q_{ij}\}$ defined as

$$q_{ij} = \begin{cases} -\infty & \text{if } i = j = 0 \\ \alpha_j & \text{if } i = 0 \text{ and } j \geqq 1 \\ b_i & \text{if } i \geqslant 1 \text{ and } j = i + 1 \\ a_i & \text{if } i \geqq 1 \text{ and } j = i - 1 \\ -(a_i + b_i) & \text{if } i = j \geqq 1 \\ 0 & \text{otherwise} , \end{cases} \tag{7.10}$$

where

$$\alpha_j \geqq 0, \quad \sum_{j=1}^{\infty} \alpha_j = +\infty \quad \text{and} \tag{7.11}$$

$$0 < a_i < +\infty, \quad 0 < b_i < +\infty \quad (\forall i \geqq 1), \tag{7.12}$$

is called a CUI birth–death pregenerator.

For a CUI birth–death pregenerator Q, Q_b is an ordinary birth–death generator. So one can analyse this Q_b-birth–death process (see, for example, Feller 1959). In particular, one can define the natural scalar $\{z_i; i \geqq 1\}$, the boundary point z, i.e.

$$z = \lim_{i \to \infty} z_i , \tag{7.13}$$

and the standard measure $\{\mu_j\}$. Through these one can define the boundary point z as being regular, entrance, exit or natural. One can further define

$$m_i = (z_{i+1} - z_i) \sum_{k=0}^{i} \mu_k, \quad \text{and} \tag{7.14}$$

$$N_i = \sum_{j=i}^{\infty} m_j . \tag{7.15}$$

Note that using the results of the last three sections, together with known results on the ordinary birth–death process (see, again, Feller 1959), we can further analyse the CUI birth–death pregenerator to obtain the following theorem (detailed analysis and proof have been omitted).

Theorem 7.5 *Suppose Q is a* CUI *birth–death pregenerator as in* (7.10)–(7.12). *Then*

(i) *Q is a generator* iff *the boundary point is regular or exit and, for N_i defined in* (7.15),

$$\sum_{i=1}^{\infty} \alpha_i N_i < +\infty ; \tag{7.16}$$

(ii) *when Q is a generator and the boundary point is exit, then there exists only one honest almost* F-*type Q-process whilst the general honest Q-process is unique* iff

$$\alpha_b \bar{X}(\lambda) = +\infty ; \tag{7.17}$$

(iii) *when Q is a generator and the boundary point is regular, then there exist infinitely many honest Q-processes as well as infinitely many honest almost* F-*type* Q-*processes.*

A discussion on the existence problem for CUI birth–death pregenerators can also be seen in Tang (1987).

Generalizing the results of this paper to the case of finitely many instantaneous states, though not straightforward, involves no essential difficulty.

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