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The Brownian snake and solutions of $\Delta u = u^2$ in a domain

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Summary. We investigate the connections between the path-valued process called the Brownian snake and nonnegative solutions of the partial differential equation $\Delta u = u^2$ in a domain of $\mathbb{R}^d$. In particular, we prove two conjectures recently formulated by Dynkin. The first one gives a complete characterization of the boundary polar sets, which correspond to boundary removable singularities for the equation $\Delta u = u^2$. The second one establishes a one-to-one correspondence between nonnegative solutions that are bounded above by a harmonic function, and finite measures on the boundary that do not charge polar sets. This correspondence can be made explicit by a probabilistic formula involving a special class of additive functionals of the Brownian snake. Our proofs combine probabilistic and analytic arguments. An important role is played by a new version of the special Markov property, which is of independent interest.

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1 Introduction

The main goal of this work is to prove two conjectures recently formulated by Dynkin [7], which are related to super-Brownian motion and to the nonnegative solutions of the partial differential equation $\Delta u = u^2$ in a domain of $\mathbb{R}^d$. Our methods rely on the probabilistic analysis of the path-valued process introduced in [11] and studied in greater detail in [12, 13], which was recently called the Brownian snake by Dynkin and Kuznetsov [8]. As was already pointed out in [14], properties of solutions of $\Delta u = u^2$ in a domain are closely related to the behavior of the Brownian snake near the boundary of the domain. As a key tool, we establish in terms of this path-valued process a version of the special Markov property stated by Dynkin for superprocesses. This result has other important applications, see in particular [16]. On the other hand, we also use purely analytic techniques inspired from the work of Baras and Pierre [2] on removable singularities for semilinear partial differential equations.

Let us start with a brief presentation of the Brownian snake. This process takes values in the set of all stopped paths in $\mathbb{R}^d$. A stopped path in $\mathbb{R}^d$ is a pair (w, ζ) where $\zeta \in \mathbb{R}_+$ and w is a continuous mapping from $\mathbb{R}_+$ into $\mathbb{R}^d$ that is constant over $[\zeta, \infty)$. We will systematically write w instead of (w, ζ) although ζ is not determined by w. The point $w(\zeta)$ (the final position of the path) is denoted by $\hat{w}$. Fix a starting point $x \in \mathbb{R}^d$ and denote $\mathscr{W}_x$ the set of all stopped paths with initial point x. The Brownian snake is the continuous strong Markov process with values in $\mathscr{W}_x$, denoted by $(W_s, s \geqq 0)$, whose distribution is characterized by the following two properties.

(i) If ζ_s denotes the lifetime of W_s, the process $(\zeta_s, s \geqq 0)$ is a reflecting Brownian motion in $\mathbb{R}_+$ (i.e. is distributed as the modulus of a linear Brownian motion)

(ii) Conditionally on $(\zeta_s, s \geqq 0)$, the process $(W_s, s \geqq 0)$ is a time-inhomogeneous Markov process, whose transition probabilities can be described as follows. Let $s < s'$ and set $m(s, s') = \inf_{[s,s']} \zeta_r$. Then,

(a) $W_{s'}(t) = W_s(t)$, for every $t \in [0, m(s, s')]$;

(b) $(W_{s'}(m(s, s') + t) - W_s(m(s, s')), t \geqq 0)$ is a standard Brownian motion in $\mathbb{R}^d$ stopped at time $\zeta_{s'} - m(s, s')$, independent of W_s.

Notice that properties (a), (b) completely describe the conditional law of $W_{s'}$ knowing W_s (under the conditional distribution given $(\zeta_r, r \geqq 0)$), so that (i) and (ii) provide a complete characterization of the process $(W_s, s \geqq 0)$. We refer to [11] for the detailed construction of this process.

Heuristically one can think of W_s as a Brownian path in $\mathbb{R}^d$ started at x and with a random lifetime ζ_s. This lifetime evolves according to the law of linear Brownian motion reflected at the origin (a lifetime cannot be negative.) When ζ_s decreases, the path W_s is erased from its final point, and when ζ_s increases, the path W_s is extended, independently of the past of the process W.

The connection between the Brownian snake (W_s) and super-Brownian motion can be stated informally as follows. Let α be the first time when the local time at 0 of $(\zeta_s, s \geqq 0)$ becomes greater than 1. The paths W_s, $0 \leqq s \leqq \alpha$ are exactly the historical paths of a super-Brownian motion in $\mathbb{R}^d$ started at δ_x (see [11]).

A basic object for most applications is the excursion measure of (W_s) away from the trivial path with lifetime 0 in $\mathscr{W}_x$ (i.e. the path that consists only of the starting point x). This excursion measure, denoted by $\mathbb{N}_x$, can be described by properties similar to (i) and (ii). More precisely, (ii) remains valid under $\mathbb{N}_x$, and the law of reflected Brownian motion in (i) is replaced by the Itô measure of positive excursions of linear Brownian motion [12, Proposition 2.2]. We assume that $\mathbb{N}_x$, or equivalently the Itô measure, is normalized so that, for every $\varepsilon > 0$,

$$\mathbb{N}_x \left(\sup_{s \geqq 0} \zeta_s \geqq \varepsilon \right) = \frac{1}{2\varepsilon} .$$

Notice that $\mathbb{N}_x$ is an infinite measure.

Let D be a bounded domain in $\mathbb{R}^d$, $d \geqq 2$. We assume that D is sufficiently smooth, namely of class C^5 (see Sect. 3). Suppose that $x \in D$. We are interested in giving necessary and sufficient conditions on a compact subset K of ∂D in order that, with positive $\mathbb{N}_x$-measure, one of the paths W_s will exit D at a point of K. In terms of super-Brownian motion started at δ_x, this means that one of the historical paths exits the domain D at a point of K. To be specific, we set

$$\mathscr{R}^D = \{W_s(t) : \; s \geqq 0, \; 0 \leqq t \leqq \tau(W_s) \wedge \zeta_s\} ,$$

where $\tau(w) = \inf\{t \geqq 0; \; w(t) \notin D\}$ ($\inf \emptyset = \infty$). In other words, the set $\mathscr{R}^D$ is the union of the ranges of the paths W_s stopped at their respective exit times from D (in the case when they do exit D). A compact subset K of ∂D is called ∂-polar if

$$\mathbb{N}_x(\mathscr{R}^D \cap K \neq \emptyset) = 0$$

for some (or equivalently for every) $x \in D$.

We say that K has positive capacity if $K \neq \emptyset$ when $d = 2$, and when $d \geqq 3$ if K supports a nontrivial measure ν such that

$$\int\int \nu(\mathrm{d}y)\nu(\mathrm{d}z)|y - z|^{3-d} < \infty \quad \text{if } d \geqq 4 ,$$

$$\int\int \nu(\mathrm{d}y)\nu(\mathrm{d}z)\log\frac{1}{|y - z|} < \infty \quad \text{if } d = 3 .$$

Otherwise, we say that K has zero capacity.

The next result was conjectured by Dynkin [7].

Theorem 1.1 *A compact subset K of ∂D is ∂-polar if and only if it has zero capacity.*

The "only if" part of Theorem 1.1 was already derived in [13]. Let us sketch the main ideas of this proof, because similar ideas play an important role in the present work. Assuming that K has positive capacity, let ν be a nontrivial finite measure supported on K such that the previous condition holds (when $d = 2$, ν can be any measure supported on K). Let h be the harmonic function in D associated with the measure ν on the boundary. The law of the h-transform of Brownian motion started at x, stopped at its exit time from D, is a probability measure on the set of all paths that exit D at a point of K. It can be checked that this probability measure has finite energy with respect to the Brownian snake. By standard results of probabilistic potential theory, this implies that the set of all paths that exit D at a point of K is not polar for (W_s), which is the same as saying that K is not ∂-polar.

The proof of the converse statement, which is given in Sect. 3 below, requires certain analytic estimates. For every $x \in D$, set

$$u(x) = 4\,\mathbb{N}_x(\mathscr{R}^D \cap K \neq \emptyset) .$$

The function u solves the problem

$$\Delta u = u^2 \quad \text{in } D\,, \qquad u_{|\partial D\backslash K} = 0\,, \tag{1}$$

where the condition $u_{|\partial D\backslash K} = 0$ means that, for every $x \in \partial D\backslash K$,

$$\lim_{y\to x, y\in D} u(y) = 0\,.$$

More precisely, u is the maximal nonnegative solution of the problem (1) [13, Proposition 4.4]. We use analytic tools inspired from [2] to verify that if K has zero capacity then any nonnegative solution of (1) must satisfy

$$\int_D u^2(y) r(y)\,\mathrm{d}y < \infty\,,$$

where $r(y) = \mathrm{dist}(y, \partial D)$. The proof of Theorem 1.1 is then completed by an argument already used in [13] in the different setting of interior polar sets.

The next analytic corollary follows from Theorem 1.1 and the previous observations.

Corollary 1.2 *The problem* (1) *has a nontrivial nonnegative solution if and only if K has positive capacity.*

The problem of finding sufficient conditions for the existence or nonexistence of nontrivial solutions of (1) had been studied previously by analytic methods (see [10, 18]), but only partial answers were obtained. In particular, Sheu [18] gives conditions involving the Hausdorff dimension of K, for the more general equation $\Delta u = u^\alpha$, $\alpha > 1$.

In Sect. 4, we study the nonnegative solutions of $\Delta u = u^2$ in D that are bounded above by a harmonic function. If u is such a solution, then there exists a minimal harmonic function h that dominates u, which is given by the equation

$$u(x) = h(x) - \frac{1}{2}\int_D G(x, y) u^2(y)\,\mathrm{d}y\,, \tag{2}$$

where G stands for the Green function of Brownian motion in D (see [7] and Proposition 4.1 below). Let $P(x, y)$, $x \in D$, $y \in \partial D$ denote the Poisson kernel of D. Then positive harmonic functions in D are in one-to-one correspondence with finite measures on ∂D by the formula

$$h(x) = \int_{\partial D} P(x, y)\nu(\mathrm{d}y)\,. \tag{3}$$

The next result was also conjectured by Dynkin [7].

Theorem 1.3 *Nonnegative solutions of $\Delta u = u^2$ in D that are bounded above by a harmonic function are in one-to-one correspondence with finite measures on ∂D that do not charge sets of zero capacity. The correspondence is given by formulas* (2), (3).

The difficult part of Theorem 1.3 is to show that the measure associated to a solution does not charge sets of zero capacity. Our method is based on a probabilistic argument. We use the function u to construct an additive functional of the Brownian snake, we then observe that the characteristic measure of this additive functional does not charge polar sets and we apply Theorem 1.1. Following these ideas, one can reexpress the correspondence of Theorem 1.3 in a more probabilistic way. The function u is given by

$$u(x) = \mathbb{N}_x(1 - \exp(-A^{\nu}_{\infty})) , \tag{4}$$

where $(A^{\nu}_s,\ s \geqq 0)$ is the continuous additive functional whose characteristic measure is ($h(x)$ times) the law of the h-transform of Brownian motion started at x.

Formula (4) extends the probabilistic representation of the solution of the Dirichlet problem associated with $\Delta u = u^2$ that was given by Dynkin [6] (see [12, Corollary 4.3] for the formulation in terms of (W_s)). The previous results should also be compared with the paper [14], which gives a probabilistic representation theorem for *all* nonnegative solutions of $\Delta u = u^2$ in the unit disk of the plane. The relationship between the results of [14] and those of the present work is discussed in Sect. 4.

The proof of Theorem 1.3 makes a heavy use of a new version of the special Markov property stated by Dynkin [4, 5] for general superprocesses. This new version is stated and proved in detail in Sect. 2. Roughly speaking, the special Markov property gives the conditional distribution of the paths W_s after their exit time from the domain D conditionally on the σ-field $\mathscr{E}^D$ that represents the information given by the paths before their exit time. More precisely, we define the "excursions" of the Brownian snake outside the domain D, and as in the classical setting we consider the point measure associated with these excursions. Then, conditionally on the σ-field $\mathscr{E}^D$, this point measure is a Poisson measure with intensity

$$\int_{\partial D} X^D(\mathrm{d}y)\mathbb{N}_y(\,\cdot\,) ,$$

where X^D denotes the exit measure from D (defined in [4, 5] for general superprocesses, and in [12] for the Brownian snake (W_s)). Although our applications are concerned with the case of a space domain, we have chosen to present the special Markov property for the Brownian snake in its full generality, that is for a space–time domain. This generality is useful for other applications. In particular, the special Markov property for the process (W_s), in its general form, is needed in [16] to get an exact Hausdorff measure function for the support of two-dimensional super-Brownian motion at a fixed time.

Let us finally comment on the connections between the present work and the related results, mainly due to Dynkin, for superprocesses. As is suggested above in a couple of examples, it is generally easy to translate our probabilistic results for the Brownian snake into equivalent statements concerning (historical) super-Brownian motion. We think that the formulation in terms of the Brownian snake is often more tractable. For instance, the process A^{ν} in (4) is naturally interpreted as an additive functional of the Brownian snake, when the definition of the corresponding object for super-Brownian motion would be more involved. Our formulation of the special Markov property is somewhat more precise and more "trajectorial" (but also less general) than the one given in [4, 5]. A major drawback of our approach is that it only applies to the "finite variance" branching mechanism, or equivalently to the equation $\Delta u = u^2$ (see however [15] for snakes associated with a general branching mechanism).

2 On the special Markov property

2.1 Time-inhomogeneous notation

We use the notation of [12], which has been briefly recalled in Sect. 1. However, as we will work in a time-inhomogeneous setting, it will be necessary to extend the definitions of [12] in the following way. We will consider stopped paths started at time t for any $t \geqq 0$ (only the case $t = 0$ was considered in [12, 13]). A stopped path started at time t is a pair (w, ζ), where $\zeta \in [t, \infty)$ and w is a continuous mapping from $[t, \infty)$ into $\mathbb{R}^d$, that is constant on $[\zeta, \infty)$. We will systematically write w instead of (w, ζ), as the value of ζ is usually clear from the context (the "lifetime" ζ will then be denoted by $\zeta_{(w)}$, and similarly the number t is written as $t_{(w)}$). We also write $\hat{w} = w(\zeta)$, which represents the final position of the path w. We denote by $\mathscr{W}^{(t)}$ the set of all stopped paths started at time t (in the notation of [12], $\mathscr{W}^{(0)} = \mathscr{W}$) and by $\mathscr{W}$ the set of all stopped paths. The set $\mathscr{W}$ is a Polish space for the metric

$$d(w, w') = |t_{(w)} - t'_{(w')}| + \sup_{r \geqq 0} |w(r \vee t) - w'(r \vee t')| + |\zeta_{(w)} - \zeta_{(w')}| .$$

For $x \in \mathbb{R}^d$, $\mathscr{W}_x^{(t)}$ denotes the set of all stopped-paths (w, ζ) started at time t, such that $w(t) = x$. In agreement with the notation of [12], we write $\mathscr{W}_x = \mathscr{W}_x^{(0)}$. Without risk of confusion, we simply denote by (t, x) the trivial element of $\mathscr{W}_x^{(t)}$ such that $\zeta = t$ and $w(t) = x$.

Denote by $C(\mathbb{R}_+, \mathscr{W}_x^{(t)})$ the space of all continuous functions from $\mathbb{R}_+$ into $\mathscr{W}_x^{(t)}$. The canonical process on $C(\mathbb{R}_+, \mathscr{W}_x^{(t)})$ is denoted by $(W_s,\ s \geqq 0)$ and ζ_s denotes the lifetime of W_s. For every $w \in \mathscr{W}_x$, the distribution of the Brownian snake started at w is a probability measure on $C(\mathbb{R}_+, \mathscr{W}_x)$ denoted by $\mathbb{P}_w$ (see [11, 12]). We also consider the distribution of the same path-valued process stopped at time $\sigma = \inf\{s > 0,\ \zeta_s = 0\}$, which is a probability measure $\mathbb{P}^*_w$ on the subset $C_0(\mathbb{R}_+, \mathscr{W}_x) = \{W \in C(\mathbb{R}_+, \mathscr{W}_x);\ \zeta_s = 0 \text{ for } s \text{ large}\}$. Finally, the excursion measure $\mathbb{N}_x$ is a σ-finite measure on $C_0(\mathbb{R}_+, \mathscr{W}_x)$.

We will make use of the strong Markov property under $\mathbb{N}_x$ (see [12], Sect. 2). Let $(\mathscr{F}_t^\circ)$ denote the canonical filtration on $C_0(\mathbb{R}_+, \mathscr{W}_x)$, and let T be a stopping time of the filtration $(\mathscr{F}_{t+}^\circ)$ such that $T > 0$, $\mathbb{N}_x$ a.e. Denote by θ_t the usual shift on $C_0(\mathbb{R}_+, \mathscr{W}_x)$. Then if F, G are two nonnegative measurable functionals on $C_0(\mathbb{R}_+, \mathscr{W}_x)$ such that F is $\mathscr{F}_{T+}^\circ$ measurable,

$$\mathbb{N}_x(F \cdot G \circ \theta_T) = \mathbb{N}_x(F \mathbb{E}^*_{W_T}(G)) .$$

It will be useful to generalize the previous definitions to our time-inhomogeneous setting. If $w \in \mathscr{W}_x^{(t)}$, the stopped path $w(t + \cdot\,)$, whose lifetime is $\zeta_{(w)} - t$, belongs to $\mathscr{W}_x$. Then $\mathbb{P}_w$ is defined as the unique probability measure on $C(\mathbb{R}_+, \mathscr{W}_x^{(t)})$ under which the process $(W_s(t + \cdot\,),\ s \geqq 0)$ has distribution $\mathbb{P}_{w(t+\cdot)}$. The probability measure $\mathbb{P}^*_w$ is the law under $\mathbb{P}_w$ of the process (W_s) stopped at $\inf\{s, \zeta_s = t\}$. Finally, the measure $\mathbb{N}_{t,x}$ is the excursion measure of (W_s) outside the trivial path (t, x) (the law of $(W_s(t + \cdot\,), s \geqq 0)$ under $\mathbb{N}_{t,x}$ is $\mathbb{N}_x$). Both $\mathbb{P}^*_w$ and $\mathbb{N}_{t,x}$ are measures on the subset $C_0(\mathbb{R}_+, \mathscr{W}_x^{(t)})$ of

$C(\mathbb{R}_+, \mathscr{W}_x^{(t)})$ determined by the condition $\{\zeta_s = t$ for s large$\}$. We set

$$C_0(\mathbb{R}_+, \mathscr{W}) = \bigcup_{t,x} C_0(\mathbb{R}_+, \mathscr{W}_x^{(t)}) \,.$$

The set $C_0(\mathbb{R}_+, \mathscr{W})$ is equipped with the topology of uniform convergence (with respect to the metric d) and the associated Borel σ-field.

It will be convenient to reformulate Proposition 2.5 of [12] in our time-inhomogeneous notation. Let $x \in \mathbb{R}^d$ and $w \in \mathscr{W}_x$. Under $\mathbb{P}_w^*$, we can consider the excursion intervals (α_i, β_i), $i \in J$ of the lifetime process (ζ_s) above its minimum process. This means that (α_i, β_i) are all open subintervals of $[0, \sigma]$ such that $\zeta_{\alpha_i} = \zeta_{\beta_i} = \inf_{[0,\alpha_i]} \zeta_s$ and $\zeta_s > \zeta_{\alpha_i}$ for every $s \in (\alpha_i, \beta_i)$. Define $W^{(i)} \in C_0(\mathbb{R}_+, \mathscr{W})$ by

$$W_s^{(i)}(t) = W_{(\alpha_i+s)\wedge\beta_i}(t), \quad t \geqq \zeta_{\alpha_i} \,,$$

so that $W_s^{(i)} \in \mathscr{W}_{w(\alpha_i)}^{(\zeta_{\alpha_i})}$. Then, Proposition 2.5 of [12] is equivalent to the following statement.

Proposition 2.1 *The point measure*

$$\sum_{i \in J} \delta_{W^{(i)}}(\,\cdot\,)$$

is under $\mathbb{P}_w^*$ a Poisson point measure with intensity

$$2 \int_0^{\zeta_{(w)}} dt\, \mathbb{N}_{t,w(t)}(\,\cdot\,) \,.$$

Finally, we will use the following simple fact. Let $w \in \mathscr{W}_x$ and $r \in [0, \zeta_{(w)})$. Then $(w(t), t \geqq r)$ is an element of $\mathscr{W}^{(r)}$, which we denote by $w^{(r)}$. Similarly, if $\sigma_r = \inf\{s, \zeta_s = r\}$, then for every $s \in [0, \sigma_r]$, $(W_s(t), t \geqq r)$ can be viewed as an element of $\mathscr{W}^{(r)}$, denoted by $W_s^{(r)}$. Then, the law under $\mathbb{P}_w^*$ of the process $(W_{s\wedge\sigma_r}^{(r)}, s \geqq 0)$ is $\mathbb{P}_{w^{(r)}}^*$. This follows easily from the properties of the Brownian snake (or from Proposition 2.1).

2.2 The exit measure

In this section, we introduce the exit measure from a space–time domain and state its main properties. Most of the arguments are similar to the case of a space domain, which is treated in [12], and will only be sketched. We consider a domain Ω in $\mathbb{R}_+ \times \mathbb{R}^d$ and a point $x \in \mathbb{R}^d$ such that $(0, x) \in \Omega$. We denote by B a Brownian motion in $\mathbb{R}^d$ that starts at x under the probability P_x, and we set $\tau = \inf\{t \geqq 0,\ (t, B_t) \notin \Omega\}$, where $\inf \emptyset = \infty$. We assume that $P_x(\tau < \infty) > 0$. For $w \in \mathscr{W}^{(t)}$, we also set

$$\tau(w) = \inf\{r \geqq t, (r, w(r)) \notin \Omega\} \,.$$

We first work under the probability measure $\mathbb{P}_w$, for some fixed $w \in \mathscr{W}_x$. For every $s \geqq 0$, we set

$$\gamma_s = (\zeta_s - \tau(W_s))_+$$

($\gamma_s = 0$ if $\tau(W_s) = \infty$). Our assumption easily implies that

$$\int_0^\infty ds\ 1_{\{\gamma_s > 0\}} = \infty, \quad \mathbb{P}_w \text{ a.s.}$$

(see the beginning of the proof of Proposition 3.1 in [12]). We may therefore define for every $s > 0$

$$\alpha_s = \inf\left\{ r \geqq 0, \int_0^r du\ 1_{\{\gamma_u > 0\}} > s \right\}.$$

Proposition 2.2 *The process* $(\Gamma_s, s \geqq 0)$ *defined under* $\mathbb{P}_w$ *by the formula* $\Gamma_s = \gamma_{\alpha_s}$ *is a reflecting Brownian motion on* $\mathbb{R}_+$, *started at* $(\zeta_{(w)} - \tau(w))_+$.

Proof. This is exactly similar to the proof of Proposition 3.1 in [12]. For every $\varepsilon > 0$, we introduce the stopping times

$$\begin{aligned} T_0^\varepsilon &= 0\,, \\ S_1^\varepsilon &= \inf\{s \geqq 0, \zeta_s \geqq \tau(W_s) + \varepsilon\}\,, \\ T_k^\varepsilon &= \inf\{s \geqq S_k^\varepsilon, \zeta_s = \tau(W_s)\}\,, \\ S_{k+1}^\varepsilon &= \inf\{s \geqq T_k^\varepsilon, \zeta_s \geqq \tau(W_s) + \varepsilon\}\,. \end{aligned}$$

The processes

$$(\gamma_{(S_n^\varepsilon + s)\wedge T_s^\varepsilon}, s \geqq 0), \quad n = 1, 2, \ldots$$

are then independent and distributed according to the law of linear Brownian motion started at ε (at $\varepsilon \vee (\zeta_{(w)} - \tau(w))_+$ for $n = 1$) and stopped when it hits 0. The proof is completed as in [12]. □

Denote by $(L^0(s), s \geqq 0)$ the local time at 0 of the reflecting Brownian motion $(\Gamma_s, s \geqq 0)$ (our local time is right-continuous in the space variable). The exit local time of (W_s), from the domain Ω, is then defined by the formula

$$L_s^\Omega = L^0\left(\int_0^s dr\ 1_{\{\gamma_r > 0\}}\right).$$

The process L_s^Ω is a continuous increasing process, that increases only on $\{s, \zeta_s = \tau(W_x)\}$. The usual approximation of Brownian local time by occupation times shows that, for every $s \geqq 0$,

$$L_s^\Omega = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \int_0^s dr\ 1_{\{\tau(W_r) < \zeta_r < \tau(W_r) + \varepsilon\}}\,,$$

$\mathbb{P}_w$ a.s. (see Proposition 3.2 of [12]). It is then clear that $(L_s^\Omega, s \geqq 0)$ is a (continuous) additive functional of the process $(W_s, \mathbb{P}_w)$, viewed as a Markov process in $\mathscr{W}_x$.

As in [12], we can use an obvious stopping argument to define the process (L_s^Ω) under the measures $\mathbb{P}_w^*$ or $\mathbb{N}_x$.

Definition. *The exit measure from Ω, denoted by X^Ω, is the random measure on $\mathbb{R}_+ \times \mathbb{R}^d$ defined under $\mathbb{N}_x$ or under $\mathbb{P}^*_w$ by the formula*

$$\langle X^\Omega, \Phi\rangle = \int_0^\sigma \mathrm{d}L^\Omega_s \ \Phi(\zeta_s, \hat{W}_s)\,,$$

where $\sigma = \inf\{s > 0,\ \zeta_s = 0\}$.

Because of the properties of L^Ω, the measure X^Ω is supported on $\{(\zeta_s, \hat{W}_s), s \geqq 0, \zeta_s = \tau(W_s)\} \subset \partial\Omega$. By the same argument as in Proposition 3.3 of [12], for any nonnegative measurable function Φ on $\mathbb{R}_+ \times \mathbb{R}^d$,

$$\mathbb{N}_x(\langle X^\Omega, \Phi\rangle) = E_x(1_{\{\tau<\infty\}}\Phi(\tau, B_\tau))\,.$$

A simple translation in time will allow us to consider also the exit measure under the measures $\mathbb{N}_{t,x}$. If $(t, x) \notin \Omega$, or if $(\mathbb{R}_+ \times \mathbb{R}^d)\backslash\Omega$ is polar for space–time Brownian motion started at (t, x), we take $X^\Omega = 0$ under $\mathbb{N}_{t,x}$. Otherwise, we may apply the previous definition to the process $(W_s(t + \ \cdot\ ), s \geqq 0)$ and the domain $\Omega_t = \{(r, y) \in \mathbb{R}_+ \times \mathbb{R}^d, (t + r, y) \in \Omega\}$. We get a measure $X^{(t),\Omega_t}$ supported on $\partial\Omega_t$, and the measure X^Ω is under $\mathbb{N}_{t,x}$ the random measure on $[t, \infty) \times \mathbb{R}^d$ defined by the formula $\langle X^\Omega, \Phi\rangle = \int X^{(t),\Omega_t}(\mathrm{d}r\ \mathrm{d}y)\Phi(t + r, y)$. We have again, for $(t,x) \in \Omega$,

$$\mathbb{N}_{t,x}(\langle X^\Omega, \Phi\rangle) = E_{t,x}(1_{\{\tau<\infty\}}\Phi(\tau, B_\tau))\,,$$

where the Brownian motion B now starts at x at time t under the probability measure $P_{t,x}$.

Let us fix $(r,x) \in \Omega$ and work under the measure $\mathbb{N}_{r,x}$. In heuristic terms, we aim to prove that the paths W_s considered after their first exit time from the domain Ω are conditionally independent of what happens before the exit time from the domain, given the exit measure which is supported on the boundary. To give a precise meaning to this assertion, we first introduce the σ-field that contains the information given by the paths before they exit Ω. We let

$$\eta_s = \inf\left\{t, \int_0^t \mathrm{d}u\ 1_{\{\zeta_u \leqq \tau(W_u)\}} > s\right\}$$

and we define a (continuous) process W'_s by $W'_s = W_{\eta_s}$ (this makes sense $\mathbb{N}_{r,x}$ a.e.). By definition, the σ-field $\mathscr{E}^\Omega$ on $C_0(\mathbb{R}_+, \mathscr{W}^{(r)}_x)$ is generated by W' and the collection of all $\mathbb{N}_{r,x}$-negligible sets of $C_0(\mathbb{R}_+, \mathscr{W}^{(r)}_x)$.

Proposition 2.3 *The random measure X^Ω is $\mathscr{E}^\Omega$-measurable.*

Proof. Again by a suitable translation in time, we can restrict our attention to the case $r = 0$. It then suffices to treat the case when Ω is bounded. If not the case, we may find an increasing sequence of bounded domains $\Omega_{(n)}$ containing $(0, x)$, with $\Omega = \lim \uparrow \Omega_{(n)}$, in such a way that $X^\Omega = X^{\Omega_{(n)}}$ for all n sufficiently large, $\mathbb{N}_x$ a.e., and $\mathscr{E}^{\Omega_{(n)}} \subset \mathscr{E}^\Omega$ for every n.

Assuming that Ω is bounded, we have $E_{t,x}(\tau) < \infty$ for every t,x. We will use a suitable approximation of the exit local time. We let Ω_n be an increasing sequence of subdomains of Ω containing $(0,x)$, such that $\Omega = \bigcup \Omega_n$ and for every n the closure $\bar{\Omega}_n$ of Ω_n is contained in Ω. Let $\tau_n(w)$ denote the exit time from Ω_n. Then, for every n and every continuous function φ on $\partial\Omega_n$,

$$\begin{aligned}
\langle X^{\Omega_n}, \varphi\rangle &= \lim_{\varepsilon\to 0} \frac{1}{\varepsilon}\int_0^\sigma ds\ 1_{\{\tau_n(W_s)<\zeta_s<\tau_n(W_s)+\varepsilon\}}\varphi(\zeta_s, \hat{W}_s) \\
&= \lim_{\varepsilon\to 0} \frac{1}{\varepsilon}\int_0^\sigma ds\ 1_{\{\tau_n(W_s)<\zeta_s<\tau_n(W_s)+\varepsilon\}} 1_{\{\tau(W_s)\geqq\zeta_s\}}\varphi(\zeta_s, \hat{W}_s) \\
&= \lim_{\varepsilon\to 0} \frac{1}{\varepsilon}\int_0^{\sigma'} du\ 1_{\{\tau_n(W'_u)<\zeta'_u<\tau_n(W'_u)+\varepsilon\}}\varphi(\zeta'_u, \hat{W}'_u),
\end{aligned}$$

where $\sigma' = \inf\{s > 0,\ \zeta'_s = 0\}$ and ζ'_s denotes the lifetime of W'_s. In the second equality, we used our assumption $\bar{\Omega}_n \subset \Omega$. It follows that X^{Ω_n} is $\mathscr{E}^\Omega$-measurable. To complete the proof, it suffices to check that the exit local times $L_s^{\Omega_n}$ converge towards L_s^Ω, $\mathbb{N}_x$ a.e., at least along some subsequence.

Using Proposition 2.1, we can easily compute the potential of the additive functional L^Ω.

$$f(w) = \mathbb{E}_w^*(L_\sigma^\Omega) = 2\int_0^{\zeta_{(w)}\wedge\tau(w)} dt\ \mathbb{N}_{t,w(t)}(L_\sigma^\Omega) = 2(\zeta_{(w)}\wedge\tau(w))$$

by the previous formula for $\mathbb{N}_{t,x}(\langle X^\Omega, \Phi\rangle)$. Similarly, the potential of L^{Ω_n} is $f_n(w) = 2(\zeta_{(w)}\wedge\tau_n(w))$, so that $f_n(w)\uparrow f(w)$ as $n\to\infty$. Furthermore,

$$\begin{aligned}
\mathbb{N}_x((L_\sigma^\Omega)^2) &= 2\mathbb{N}_x\left(\int_0^\sigma dL_s^\Omega\ \mathbb{E}_{W_s}^*(L_\sigma^\Omega)\right) \\
&= 4\mathbb{N}_x\left(\int_0^\sigma dL_s^\Omega\,(\zeta_s\wedge\tau(W_s))\right) \\
&= 4E_x(\tau).
\end{aligned}$$

using again the formula for $\mathbb{N}_{t,x}(\langle X^\Omega, \Phi\rangle)$. Similarly, one computes $\mathbb{N}_x(L_\sigma^\Omega L_\sigma^{\Omega_n}) = \mathbb{N}_x((L_\sigma^{\Omega_n})^2) = 4E_x(\tau_n)$, and it follows that $L_\sigma^{\Omega_n}$ converges to L_σ^Ω in $L^2(\mathbb{N}_x)$.

Standard techniques can then be used to verify that $L_s^{\Omega_n}$ converges to L_s^Ω for every $s\geqq 0$. For every $s > 0$, we set

$$M_s = L_s^\Omega + f(W_s),\quad M_s^n = L_s^{\Omega_n} + f_n(W_s),$$

so that M_s, M_s^n are continuous martingales under $\mathbb{N}_x$ (indexed by $s\in(0,\infty)$). Notice that $M_\infty = L_\sigma^\Omega$, $M_\infty^n = L_\sigma^{\Omega_n}$. By applying Doob's maximal inequality to the martingales M_{a+s}, M_{a+s}^n $(a > 0)$ and then letting a tend to 0, we get, for every $\delta > 0$,

$$\mathbb{N}_x\left(\sup_{s>0}|M_s - M_s^n| > \delta\right) \leqq \delta^{-2}\mathbb{N}_x((L_\sigma^\Omega - L_\sigma^{\Omega_n})^2).$$

Hence there exists a subsequence n_k such that

$$\lim_{k\to\infty}\sup_{s>0}|M_s - M_s^{n_k}| = 0,\quad \mathbb{N}_x\text{ a.e.}$$

As we have already noticed that $f_n(w)$ increases to $f(w)$, we conclude that, $\mathbb{N}_x$ a.e. for every $s > 0$,

$$\lim_{k\to\infty} L_s^{\Omega_{n_k}} = L_s^{\Omega} .$$

This completes the proof of Proposition 2.3. □

2.3. *The special Markov property*

We first define the excursions of (W_s) outside the domain Ω. The terminology is a little misleading, since these excursions consist of paths that start on the boundary of Ω but may come back into Ω. From now on, we work under the probability measure $\mathbb{N}_{r,x}$, where $(r, x) \in \Omega$. The random open set

$$\{s \in [0, \sigma],\ \tau(W_s) < \zeta_s\}$$

may be written as a countable union of disjoint open intervals (a_i, b_i), $i \in I$. Here I is a suitable set of indices, which may be empty, for instance if none of the paths W_s reaches the boundary of Ω. For every fixed $i \in I$, we have $\tau(W_s) = \tau(W_{a_i}) = \zeta_{a_i}$ for every $s \in [a_i, b_i]$, and the paths W_s, $s \in [a_i, b_i]$ all coincide up to their exit time of Ω. This follows from the properties of the Brownian snake, using arguments similar to those of the beginning of the proof of Proposition 3.1 in [12]. We let τ^i denote the common value of $\tau(W_s)$ for $s \in [a_i, b_i]$ and set $y^i = \hat{W}_{a_i} = W_s(\tau(W_s))$ for every $s \in [a_i, b_i]$. We then define a random element W^i of $C_0(\mathbb{R}_+, \mathscr{W}_{y^i}^{(\tau_i)})$ by the formula

$$W_s^i(t) = W_{(a_i+s)\wedge b_i}(t), \quad t \geqq \tau^i ,$$

so that W_s^i is an element of $\mathscr{W}_{y^i}^{(\tau_i)}$ with lifetime $\zeta_s^i = \zeta_{(a_i+s)\wedge b_i}$.
The processes W^i, $i \in I$ are the excursions of W "outside" Ω. As in the classical situation, the labelling of excursions is irrelevant, and one is interested in

$$\sum_{i\in I} \delta_{W^i} ,$$

which is a point measure on $C_0(\mathbb{R}_+, \mathscr{W})$.

Theorem 2.4 *For every nonnegative measurable function Φ on $C_0(\mathbb{R}_+, \mathscr{W})$,*

$$\mathbb{N}_{r,x}\left(\exp\left(-\sum_{i\in I}\Phi(W^i)\right)\Big|\mathscr{E}^{\Omega}\right) = \exp\left(-\int X^{\Omega}(\mathrm{d}t\,\mathrm{d}y)\mathbb{N}_{t,y}(1-\exp(-\Phi))\right) .$$

In other words, conditionally given $\mathscr{E}^{\Omega}$, the point measure $\sum_{i\in I}\delta_{W^i}$ is a Poisson measure with intensity $\int X^{\Omega}(\mathrm{d}t\,\mathrm{d}y)\mathbb{N}_{t,y}(\,\cdot\,)$.

Proof. First step. Without loss of generality, we treat only the case $r = 0$. For $W \in C_0(\mathbb{R}_+, \mathscr{W}_y^{(t)})$, set $h(W) = \sup_{\{s\geqq 0\}} \zeta_s - t$, so that $h(W) = 0$ if and only if $W_s = (t, y)$ for every $s \geqq 0$. By construction, $h(W^i) > 0$ for every $i \in I$. Using monotonicity arguments, we may restrict our attention to the case when $\Phi(W) = 0$ unless $h(W) > \eta$, for some fixed $\eta > 0$. There is then only a finite number of terms in the sum

$$\sum_{i \in I} \Phi(W^i)$$

(notice that $h(W^i) = \sup_{[a_i, b_i]}(\zeta_s - \tau(W_s))$). Moreover, by the monotone class theorem, it is enough to consider the case when Φ is bounded and Lipschitz continuous on $C_0(\mathbb{R}_+, \mathcal{W})$ (recall that this set is equipped with the distance of uniform convergence with respect to d).

Recall the definition of the stopping times S_k^ε, T_k^ε in Sect. 2. Under $\mathbb{N}_x$, we slightly modify this definition by using the special convention $\inf \emptyset = \sigma$. Then, $T_k^\varepsilon < \sigma$ if and only if $k \leqq N_\varepsilon$, where N_ε is a nonnegative integer. Note that $\zeta_{T_k^\varepsilon} = \tau(W_{T_k^\varepsilon})$ by construction, and that the paths W_s, $s \in [S_k^\varepsilon, T_k^\varepsilon]$ coincide up to time $\zeta_{T_k^\varepsilon}$. For $k \geqq 1$, we set

$$W_s^{\varepsilon, k}(t) = W_{(S_k^\varepsilon + s) \wedge T_k^\varepsilon}(t), \quad t \geqq \zeta_{T_k^\varepsilon}$$

so that $W_s^{\varepsilon, k}$ is a stopped path started at time $\zeta_{T_k^\varepsilon}$ at $\hat{W}_{T_k^\varepsilon}$, with lifetime $\zeta_{(S_k^\varepsilon + s) \wedge T_k^\varepsilon}$. In particular, $W^{\varepsilon, k} \in C_0(\mathbb{R}_+, \mathcal{W}_y^{(r)})$ for $r = \zeta_{T_k^\varepsilon}$ and $y = \hat{W}_{T_k^\varepsilon}$.

Lemma 2.5 *Under the previous assumptions on Φ,*

$$\lim_{\varepsilon \to 0} \sum_{k=1}^{N_\varepsilon} \Phi(W^{\varepsilon, k}) = \sum_{i \in I} \Phi(W^i), \quad \mathbb{N}_x \ a.e.$$

Proof. We may suppose $\varepsilon < \eta$. There is only a finite number of indices, denoted by $i_1, \ldots, i_l$ such that $h(W^{i_j}) \geqq \eta$. For every $j \in \{1, \ldots, l\}$, there is exactly one value $k_j = k_j(\varepsilon) \in \{1, \ldots, N_\varepsilon\}$ such that $S_{k_j}^\varepsilon \in (a_{i_j}, b_{i_j})$, $T_{k_j}^\varepsilon = b_{i_j}$. Furthermore,

$$W_s^{\varepsilon, k_j} = W^{i_j}_{(S_{k_j}^\varepsilon - a_{i_j}) + s} \,.$$

Since for every $j \in \{1, \ldots, l\}$,

$$\lim_{\varepsilon \to 0} (S_{k_j(\varepsilon)}^\varepsilon - a_{i_j}) = 0 \,,$$

it follows easily that W^{ε, k_j} converges to W^{i_j} in the topology of $C_0(\mathbb{R}_+, \mathcal{W})$. Since we have also $h(W^{\varepsilon, k}) < \eta$ if $k \notin \{k_1, \ldots, k_l\}$, the lemma follows. □

Lemma 2.6 (Second step) *For any nonnegative $\mathscr{E}^\Omega$-measurable random variable Z,*

$$\mathbb{N}_x \left(Z \exp \left(- \sum_{k=1}^{N_\varepsilon} \Phi(W^{\varepsilon, k}) \right) \right) = \mathbb{N}_x \left(Z \prod_{k=1}^{N_\varepsilon} \mathbb{E}^*_{W_0^{\varepsilon, k}}(\exp(-\Phi)) \right) .$$

Proof. We denote by $\mathscr{E}_\varepsilon^\Omega$ the σ-field generated by the processes $(W_{(T_n^\varepsilon + s) \wedge S_{n+1}^\varepsilon}$, $s \geqq 0)$, $n = 0, 1, 2, \ldots$, augmented with the $\mathbb{N}_x$-negligible sets. Clearly, $\mathscr{E}^\Omega \subset \mathscr{E}_\varepsilon^\Omega$. Let $p \geqq 1$ and let Z be of the form

$$Z = Z_0 \ldots Z_{p-1} Y \,,$$

where, for $j \in \{0, 1, \ldots, p-1\}$, Z_j is a measurable function of $(W_{(T_j^\varepsilon+s)\wedge S_{n+1}^\varepsilon}, s \geqq 0)$, and similarly $Y = G(W_{T_p^\varepsilon+s}, s \geqq 0)$ for some measurable function G. In the following calculation, we use first the strong Markov property at T_p^ε, and then the strong Markov property at S_p^ε, observing that $W_{T_p^\varepsilon}$ is a measurable function of $W_{S_p^\varepsilon}$ (indeed, the path $W_{T_p^\varepsilon}$ coincides with the path $W_{S_p^\varepsilon}$ stopped at its exit time from Ω). We have

$$\begin{aligned}
&\mathbb{N}_x\left(Z \exp\left(-\sum_{k=1}^{p} \Phi(W^{\varepsilon,k})\right)\right)\\
&\quad = \mathbb{N}_x\left(Z_0 \ldots Z_{p-1} \exp\left(-\sum_{k=1}^{p} \Phi(W^{\varepsilon,k})\right) \mathbb{E}^*_{W_{T_p^\varepsilon}}(G)\right)\\
&\quad = \mathbb{N}_x\left(Z_0 \ldots Z_{p-1} \exp\left(-\sum_{k=1}^{p-1} \Phi(W^{\varepsilon,k})\right) \mathbb{E}^*_{W_0^{\varepsilon,p}}(\exp(-\Phi))\mathbb{E}^*_{W_{T_p^\varepsilon}}(G)\right)\\
&\quad = \mathbb{N}_x\left(Z_0 \ldots Z_{p-2}\tilde{Y} \exp\left(-\sum_{k=1}^{p-1} \Phi(W^{\varepsilon,k})\right)\right),
\end{aligned}$$

where

$$\tilde{Y} = Z_{p-1}\mathbb{E}^*_{W_0^{\varepsilon,p}}(\exp(-\Phi))Y$$

is a measurable function of $(W_{T_{p-1}^\varepsilon+s}, s \geqq 0)$. In the second equality, we used the remark at the end of Sect. 2.1. We can then repeat the argument, replacing p by $p-1$ and Y by $\tilde{Y}$. By induction, we get

$$\mathbb{N}_x\left(Z \exp\left(-\sum_{k=1}^{p} \Phi(W^{\varepsilon,k})\right)\right) = \mathbb{N}_x\left(Z \prod_{k=1}^{p} \mathbb{E}^*_{W_0^{\varepsilon,k}}(\exp(-\Phi))\right).$$

By the monotone class theorem, this formula holds for any nonnegative $\mathscr{E}_\varepsilon^\Omega$-measurable variable Z. Letting p tend to ∞ gives the desired result. □

Remark. The previous proof gives the more precise following statement. Conditionally given $(W_0^{\varepsilon,1}, W_0^{\varepsilon,2}, \ldots)$, the variables $W^{\varepsilon,j}$ are independent, and also independent of $\mathscr{E}_\varepsilon^\Omega$. Furthermore, the conditional distribution of $W^{\varepsilon,j}$ is $\mathbb{P}^*_{W^{\varepsilon,j}}$.

Lemma 2.7 (*Third step*) *Let Ψ be Lipschitz continuous and bounded on $C_0(\mathbb{R}_+, \mathscr{W})$. Then,*

$$\lim_{\delta\downarrow 0} \sup_{t,y} \sup_{w\in\mathscr{W}_y^{(t)},\, 0<d(w,(t,y))<\delta} |\mathbb{E}_w^*(\Psi|h(W) \geqq \eta) - \mathbb{N}_{t,y}(\Psi|h(W) \geqq \eta)| = 0\,.$$

Proof. By an obvious translation, we may restrict our attention to the case $t = 0$. Let us fix $y \in \mathbb{R}^d$. For every $a > 0$, we define a random variable T_a on $C_0(\mathbb{R}_+, \mathscr{W}_y)$ by

$$T_a(W) = \inf\{s \geqq 0,\ \zeta_s = a\}$$

with the special convention $\inf \emptyset = \sigma$. We then consider the shift operator θ_{T_a} defined on $C_0(\mathbb{R}_+, \mathscr{W}_y)$ by

$$\theta_{T_a}(W)_s = W_{T_a+s} .$$

Obviously,

$$\lim_{a \to 0} \theta_{T_a}(W) = W, \ \mathbb{N}_y \text{ a.e.},$$

in the topology of $C_0(\mathbb{R}_+, \mathscr{W}_y)$. Therefore,

$$\lim_{a \to 0} |\mathbb{N}_y(\Psi | h(W) \geqq \eta) - \mathbb{N}_y(\Psi \circ \theta_{T_a} | h(W) \geqq \eta)| = 0 ,$$

and a translation argument shows that this convergence is uniform in y (the rate of convergence only depends on the Lipschitz constant of Ψ).

However, by the strong Markov property at T_a, if $a < \eta$,

$$\mathbb{N}_y(\Psi \circ \theta_{T_a} | h(W) \geqq \eta) = \int P_y^{(a)}(\mathrm{d}w')\mathbb{E}_{w'}^*(\Psi | h(W) \geqq \eta) ,$$

where $P_y^{(a)}(\mathrm{d}w')$ denotes the law of Brownian motion started at y (at time 0) and stopped at time a, viewed as a probability measure on $\mathscr{W}_y$. Now let $w \in \mathscr{W}_y$ and take $a = \zeta_{(w)}$. Assuming that $\zeta_{(w)} \in (0, \eta)$, we get

$$\begin{aligned}
&|\mathbb{E}_w^*(\Psi | h(W) \geqq \eta) - \mathbb{N}_y(\Psi \circ \theta_{T_a} | h(W) \geqq \eta)| \\
&\qquad \leqq \int P_y^{(a)}(\mathrm{d}w')|\mathbb{E}_w^*(\Psi | h(W) \geqq \eta) - \mathbb{E}_{w'}^*(\Psi | h(W) \geqq \eta)| .
\end{aligned}$$

We will use a coupling argument to verify that, if Ψ has Lipschitz constant K, and if $w' \in \mathscr{W}_y$ is such that $\zeta_{(w')} = \zeta_{(w)}$, then

$$|\mathbb{E}_w^*(\Psi | h(W) \geqq \eta) - \mathbb{E}_{w'}^*(\Psi | h(W) \geqq \eta)| \leqq K\mathrm{d}(w, w') . \tag{5}$$

It then follows that, for $a = \zeta_{(w)}$,

$$\begin{aligned}
&|\mathbb{E}_w^*(\Psi | h(W) \geqq \eta) - \mathbb{N}_y(\Psi | h(W) \geqq \eta)| \\
&\qquad \leqq |\mathbb{N}_y(\Psi | h(W) \geqq \eta) - \mathbb{N}_y(\Psi \circ \theta_{T_a} | h(W) \geqq \eta)| + K\int P_y^{(a)}(\mathrm{d}w')\mathrm{d}(w, w') \\
&\qquad \leqq |\mathbb{N}_y(\Psi | h(W) \geqq \eta) - \mathbb{N}_y(\Psi \circ \theta_{T_a} | h(W) \geqq \eta)| \\
&\qquad\quad + K(d(w, (0, y)) + \int P_y^{(a)}(\mathrm{d}w')\mathrm{d}((0, y), w'))
\end{aligned}$$

which tends to 0 as $d((0, y), w) \to 0$, uniformly in y.

It remains to derive the bound (5). We use the notation of Proposition 2.1. Remark that, outside a $\mathbb{P}_w^*$-negligible set, the whole process (W_s) can be reconstructed from the Poisson measure $\sum_{i \in J} W^{(i)}$ and the initial path w. We then define a new process $\tilde{W}$, whose lifetime process $\tilde{\zeta}_s$ coincides with ζ_s, by the following conditions.

(i) For $s \in (\alpha_i, \beta_i)$,

$$\tilde{W}_s(t) = \begin{cases} w'(t) & \text{if } t \leqq \zeta_{\alpha_i} , \\ w'(\zeta_{\alpha_i}) - w(\zeta_{\alpha_i}) + W^{(i)}_{s-\alpha_i}(t) & \text{if } t > \zeta_{\alpha_i} . \end{cases}$$

(ii) For $s \in \mathbb{R}_+ \setminus \bigcup(\alpha_i, \beta_i)$, $\tilde{W}_s(t) = w'(t \wedge \zeta_s)$.

Proposition 2.1 shows that the distribution of $\tilde{W}$ under $\mathbb{P}^*_w$ is $\mathbb{P}^*_{w'}$. Moreover, by construction, $h(\tilde{W}) = h(W)$ and $d(W_s, \tilde{W}_s) \leqq d(w, w')$ for every $s \geqq 0$, $\mathbb{P}^*_w$ a.e. The bound (5) then follows easily. □

We can now complete the proof of Theorem 2.4. Recall that $\Phi(W) = 0$ if $h(W) < \eta$. We have then, for $\varepsilon < \eta$,

$$\prod_{k=1}^{N_\varepsilon} \mathbb{E}^*_{W_0^{\varepsilon,k}}(\exp(-\Phi))$$

$$= \exp\left(\sum_{k=1}^{N_\varepsilon} \log\left(1 - \frac{\varepsilon}{\eta}(1 - \mathbb{E}^*_{W_0^{\varepsilon,k}}(\exp(-\Phi)|h \geqq \eta))\right)\right)$$

since $\mathbb{P}^*_w(h \geqq \eta) = \varepsilon/\eta$ if $\zeta_{(w)} - t_{(w)} = \varepsilon$. Then,

$$\sum_{k=1}^{N_\varepsilon} \log\left(1 - \frac{\varepsilon}{\eta}(1 - \mathbb{E}^*_{W_0^{\varepsilon,k}}(\exp(-\Phi)|h \geqq \eta))\right)$$

$$= -\frac{\varepsilon}{\eta}\sum_{k=1}^{N_\varepsilon} \mathbb{E}^*_{W_0^{\varepsilon,k}}(1 - \exp(-\Phi)|h \geqq \eta) + O(\varepsilon^2 N_\varepsilon) .$$

Write $t^{\varepsilon,k} = \zeta_{T_k^\varepsilon}$, $y^{\varepsilon,k} = \hat{W}_{T_k^\varepsilon}$ for simplicity. From the continuity of the mapping $s \to W_s$, it easily follows that

$$\lim_{\varepsilon \to 0} \sup_{k \in \{1,\dots,N_\varepsilon\}} d(W_0^{\varepsilon,k}, (t^{\varepsilon,k}, y^{\varepsilon,k})) = 0, \quad \mathbb{N}_x \text{ a.e.}$$

Then, by Lemma 2.7,

$$\left|\varepsilon\sum_{k=1}^{N_\varepsilon} \mathbb{E}^*_{W_0^{\varepsilon,k}}(1 - \exp(-\Phi)|\, h \geqq \eta) - \varepsilon\sum_{k=1}^{N_\varepsilon} \mathbb{N}_{(t^{\varepsilon,k}, y^{\varepsilon,k})}(1 - \exp(-\Phi)|h \geqq \eta)\right|$$

$$= o(\varepsilon N_\varepsilon) ,$$

$\mathbb{N}_x$ a.e. Now note that $\varepsilon N_\varepsilon$ converges a.e. to $\frac{1}{2}L^\Omega_\sigma$, by the usual approximation of local time by upcrossing numbers. Also note that $\mathbb{N}_{t,y}(h \geqq \eta) = 1/(2\eta)$. By combining the previous estimates, it follows that, $\mathbb{N}_x$ a.e.,

$$\lim_{\varepsilon \to 0}\left|\prod_{k=1}^{N_\varepsilon} E^*_{W_0^{\varepsilon,k}}(\exp(-\Phi)) - \exp\left(-2\varepsilon\sum_{k=1}^{N_\varepsilon} \mathbb{N}_{(t^{\varepsilon,k}, y^{\varepsilon,k})}(1 - \exp(-\Phi))\right)\right| = 0 .$$

The upcrossings approximation of local time gives more precisely, $\mathbb{N}_x$ a.e.,

$$\lim_{\varepsilon \to 0} \varepsilon \sum_{k=1}^{N_\varepsilon} \delta_{T_k^\varepsilon}(ds) = \tfrac{1}{2} dL^\Omega_s$$

for the weak convergence of measures. Set $\kappa(t, y) = \mathbb{N}_{t,y}(1 - \exp(-\Phi))$. Then κ is bounded and continuous, and by the previous observation,

$$2\varepsilon \sum_{k=1}^{N_\varepsilon} \mathbb{N}_{(t^{\varepsilon,k},\, y^{\varepsilon,k})}(1 - \exp(-\Phi)) = 2\varepsilon \sum_{k=1}^{N_\varepsilon} \kappa(\zeta_{T_k^\varepsilon}, \hat{W}_{T_k^\varepsilon}) \to \int_0^\sigma dL_s^\Omega \; \kappa(\zeta_s, \hat{W}_s)$$
$$= \langle X^\Omega, \kappa \rangle$$

by definition.

Let Z be bounded and $\mathscr{E}^\Omega$-measurable. Using Lemmas 2.5 and 2.6 we finally obtain

$$\mathbb{N}_x\left(Z \exp\left(-\sum_{i\in I} \Phi(W^i)\right)\right) = \lim_{\varepsilon\to 0} \mathbb{N}_x\left(Z \exp\left(-\sum_{k=1}^{N_\varepsilon} \Phi(W^{\varepsilon,k})\right)\right)$$
$$= \lim_{\varepsilon\to 0} \mathbb{N}_x\left(Z \prod_{k=1}^{N_\varepsilon} \mathbb{E}^*_{W_0^{\varepsilon,k}}(\exp(-\Phi))\right)$$
$$= \mathbb{N}_x(Z \exp(-\int X^\Omega(dt\,dy)\mathbb{N}_{t,y}(1 - e^{-\Phi}))) ,$$

which completes the proof. □

2.4 The time-homogeneous case

In the applications developed below in Sect. 4, we will consider the case of a space domain $D \subset \mathbb{R}^d$. We can then of course apply the previous results to the space–time domain $\Omega = \mathbb{R}_+ \times D$. We will however use the special Markov property in a slightly different form. We keep the notation introduced in the beginning of Sect. 2.3, with the only difference that the excursions $(W^i,\ i \in I)$ are now defined by

$$W_s^i(t) = W_{(a_i+s)\wedge b_i}(\tau_i + t), \quad t \geqq 0 ,$$

in such a way that $W^i \in C_0(\mathbb{R}_+, \mathscr{W}_{y^i})$ (recall that $y^i = W_s(\tau(W_s))$ for every $s \in [a_i, b_i]$). The exit local time will be denoted by $L_s^D = L_s^\Omega$. We also consider the "spatial" exit measure from D, denoted by X^D, which is simply the image measure of X^Ω by the mapping $(t, y) \to y$. Clearly, X^D is supported on ∂D. Finally we write $\mathscr{E}^D$ instead of $\mathscr{E}^{\mathbb{R}_+\times D}$. With this notation, the following result is then an immediate consequence of Theorem 2.4.

Corollary 2.8 *Conditionally on $\mathscr{E}^D$, the point measure $\sum_{i\in I} \delta_{W^i}$ is a Poisson measure with intensity $\int X^D(dy)\mathbb{N}_y(\,\cdot\,)$.*

3 The characterization of boundary polar sets

In this section, we prove Theorem 1.1. The arguments do not depend on Sect. 2. The first two subsections are devoted to analytic estimates concerning solutions of the problem (1).

3.1 Analytic estimates in the case of a half-space

In this subsection, we consider the domain $D := (0, \infty) \times \mathbb{R}^m$, where $m = d - 1 \geqq 2$. We shall always identify the boundary ∂D with $\mathbb{R}^m$. If K is a compact subset of ∂D, we denote by cap K the Newtonian (or logarithmic if $m = 2$) capacity of K viewed as a subset of $\mathbb{R}^m$.

Proposition 3.1 *Let u be a nonnegative solution of the equation $\Delta u = u^2$ in D. Assume that there exists a compact subset K of ∂D such that* cap $K = 0$ *and, for every $x \in \partial D \backslash K$,*

$$\lim_{y \to x, y \in D} u(y) = 0 .$$

Then, for every bounded subset H of $[0, \infty) \times \mathbb{R}^m$,

$$\int_0^\infty \int_{\mathbb{R}^m} 1_H(r, z) u^2(r, z) r \, dr \, dz < \infty .$$

Before proceeding to the proof of the proposition, we need to establish a number of preliminary estimates. To motivate these estimates, recall that in the notation of [2], our assumption cap $K = 0$ is equivalent to $c_{1,2}(K) = 0$. By Lemma 2.1 of [2] we may find a sequence of functions $f_n \in C_c^\infty(\mathbb{R}^m)$ such that $0 \leqq f_n \leqq 1$, $f_n = 1$ on a neighborhood of K, and

$$\lim_{n \to \infty} \|f_n\|_{1,2} = 0 ,$$

where $\|f\|_{1,2}$ denotes the usual Sobolev norm

$$\|f\|_{1,2} = \int_{\mathbb{R}^m} (1 + |\xi|^2) |\hat{f}(\xi)|^2 \, d\xi .$$

The outline of the proof of Proposition 3.1 is similar to that of the proof of Theorem 2.1 of [2]. We must however extend the functions f_n to $[0, \infty) \times \mathbb{R}^m$ and derive certain properties of this extension.

To begin with, we consider a function $g \in C_c^\infty(\mathbb{R}^m)$, which is nonnegative, radial (i.e. $g(z) = g(z')$ if $|z| = |z'|$) and such that

$$\int_{\mathbb{R}^m} g(z) \, dz = 1 .$$

Let f be another fixed function in $C_c^\infty(\mathbb{R}^m)$. For every $r \in \mathbb{R}$, $z \in \mathbb{R}^m$, we set

$$\varphi(r, z) = \int_{\mathbb{R}^m} f(z + rz') g(z') \, dz' .$$

Then $\varphi \in C^\infty(\mathbb{R} \times \mathbb{R}^m)$, and, for $r > 0$, $\varphi(r, \cdot) = g_r * f(\cdot)$, where $g_r(z) = r^{-m} g(z/r)$. Also observe that $\varphi(0, z) = f(z)$.

The function φ is in general not compactly supported. For this reason, we also consider a function $\rho : \mathbb{R} \to [0, 1]$ of class C^∞, such that $\rho(r) = 1$ if

$|r| \leqq 1$, $\rho(r) = 0$ if $|r| \geqq 2$ and we set

$$\psi(r, z) = \rho(r)\varphi(r, z) ,$$

so that $\psi \in C_c^\infty(\mathbb{R} \times \mathbb{R}^m)$.

Lemma 3.2 *There exist four positive constants $C_1, \ldots, C_4$ that may depend on g and ρ but not on f, such that*

$$\int_D \psi(r, z)^2 \,\mathrm{d}r\,\mathrm{d}z \leqq C_1 \|f\|_2^2 , \tag{6}$$

$$\int_D \left(|\nabla_z \psi|^2 + |\nabla_z^2 \psi|^2 + \left(\frac{\partial^2 \psi}{\partial r^2} \right)^2 \right) r \,\mathrm{d}r\,\mathrm{d}z \leqq C_2 \|f\|_{1,2}^2 \tag{7}$$

$$\int_D \left(\frac{\partial \psi}{\partial r} \right)^2 \frac{1}{r} \mathrm{d}r\,\mathrm{d}z \leqq C_3 \|f\|_{1,2}^2 \tag{8}$$

$$\int_D |\nabla \psi|^4 r \,\mathrm{d}r\,\mathrm{d}z \leqq C_4 \|f\|_{1,2}^2 \|f\|_\infty^2 . \tag{9}$$

Proof. The bound (6) is immediate. We simply observe that, for every fixed $r > 0$,

$$\int_{\mathbb{R}^m} \varphi(r, z)^2 \,\mathrm{d}z = \int_{\mathbb{R}^m} (g_r * f)(z)^2 \,\mathrm{d}z \leqq \int_{\mathbb{R}^m} f(z)^2 \,\mathrm{d}z . \tag{10}$$

We then turn to the proof of (7) and (8). We denote by $\hat{\varphi}(r, \xi)$ the Fourier transform in the variable z of $\varphi(r, z)$. Then, for $r > 0$,

$$\hat{\varphi}(r, \xi) = \hat{g}_r(\xi)\hat{f}(\xi) = \hat{g}(r\xi)\hat{f}(\xi) .$$

Notice that $\hat{g}$ is radial, so that we can write $\hat{g}(u) = \hat{g}(\xi)$ for $|\xi| = u$. Then,

$$\begin{aligned} \int_0^\infty \mathrm{d}r\, r \int_{\mathbb{R}^m} \mathrm{d}\xi (|\xi|^2 |\hat{\varphi}(r, \xi)|)^2 &= \int_{\mathbb{R}^m} \mathrm{d}\xi |\xi|^4 |\hat{f}(\xi)|^2 \int_0^\infty \mathrm{d}r\, r |\hat{g}(r\xi)|^2 \\ &= C \int_{\mathbb{R}^m} \mathrm{d}\xi |\xi|^2 |\hat{f}(\xi)|^2 , \end{aligned}$$

where

$$C = \int_0^\infty \mathrm{d}r\, r |\hat{g}(r)|^2 < \infty .$$

The same calculation gives

$$\int_0^\infty \mathrm{d}r\, r \int_{\mathbb{R}^m} \mathrm{d}\xi (|\xi|\, |\hat{\varphi}(r, \xi)|)^2 = C \int_{\mathbb{R}^m} \mathrm{d}\xi |\hat{f}(\xi)|^2 ,$$

with the same constant C. We conclude that

$$\int_0^\infty \mathrm{d}r\, r \int_{\mathbb{R}^m} \mathrm{d}z (|\nabla_z \varphi|^2 + |\nabla_z^2 \varphi|^2) \leqq C \|f\|_{1,2}^2 .$$

Clearly, the same bound holds if we replace φ by ψ.

We then consider, for $r > 0$,

$$\frac{\partial \varphi}{\partial r}(r, z) = \frac{\partial g_r}{\partial r} * f(z) .$$

We have

$$\widehat{\frac{\partial \varphi}{\partial r}}(r, \xi) = \widehat{\frac{\partial g_r}{\partial r}}(\xi)\hat{f}(\xi) = \frac{\partial}{\partial r}\hat{g}_r(\xi)\hat{f}(\xi) = |\xi|\hat{g}'(r|\xi|)\hat{f}(\xi) ,$$

where $\hat{g}'(r)$ stands for the derivative of $r \to \hat{g}(r)$. Hence,

$$\begin{aligned}\int_0^\infty \mathrm{d}r \ \frac{1}{r} \int_{\mathbb{R}^m} \mathrm{d}z \left(\frac{\partial \varphi}{\partial r}(r, z)\right)^2 &= \int_0^\infty \mathrm{d}r \ \frac{1}{r} \int_{\mathbb{R}^m} \mathrm{d}\xi \ |\xi|^2|\hat{g}'(r|\xi|)|^2|\hat{f}(\xi)|^2 \\ &= \int_{\mathbb{R}^m} \mathrm{d}\xi \ |\xi|^2|\hat{f}(\xi)|^2 \int_0^\infty \frac{\mathrm{d}r}{r} |\hat{g}'(r|\xi|)|^2 \\ &= C' \int_{\mathbb{R}^m} \mathrm{d}\xi \ |\xi|^2|\hat{f}(\xi)|^2 ,\end{aligned}$$

where $C' = \int_0^\infty \mathrm{d}r \ r^{-1}|\hat{g}'(r)|^2 < \infty$ because $\hat{g}'(0) = 0$. We have thus obtained the inequality (8) with φ instead of ψ. However, since

$$\frac{\partial \psi}{\partial r}(r, z) = \rho'(r)\varphi(r, z) + \rho(r)\frac{\partial \varphi}{\partial r}(r, z) ,$$

and $\rho'(r) = 0$ for $r \leqq 1$, the bound (8) follows immediately using (10).

By similar calculations,

$$\begin{aligned}\int_0^\infty \mathrm{d}r \, r \int_{\mathbb{R}^m} \mathrm{d}z \left(\frac{\partial^2 \varphi}{\partial r^2}(r, z)\right)^2 &= \int_0^\infty \mathrm{d}r \, r \int_{\mathbb{R}^m} \mathrm{d}\xi \ |\xi|^4|\hat{g}''(r|\xi|)|^2|\hat{f}(\xi)|^2 \\ &= \int_{\mathbb{R}^m} \mathrm{d}\xi \ |\xi|^4|\hat{f}(\xi)|^2 \int_0^\infty \mathrm{d}r \ r|\hat{g}''(r|\xi|)|^2 \\ &= C'' \int_{\mathbb{R}^m} \mathrm{d}\xi \ |\xi|^2|\hat{f}(\xi)|^2 ,\end{aligned}$$

where $C'' = \int_0^\infty \mathrm{d}r \, r|\hat{g}''(r)|^2 < \infty$. If we combine the latter bound with the previous estimate for $|\nabla_z \varphi|^2 + |\nabla_z^2 \varphi|^2$, we get the bound (7), with ψ replaced by φ. Again, since

$$\frac{\partial^2 \psi}{\partial r^2}(r, z) = \rho''(r)\varphi(r, z) + 2\rho'(r)\frac{\partial \varphi}{\partial r}(r, z) + \rho(r)\frac{\partial^2 \varphi}{\partial r^2}(r, z) ,$$

the bound (7) follows using (10) and (8).

It remains to prove (9). Note that $\|\psi\|_\infty \leqq \|f\|_\infty$. For $j = 1,\dots,m$, we integrate by parts to obtain

$$\int_D \mathrm{d}r\,\mathrm{d}z\, r\left(\frac{\partial\psi}{\partial z_j}\right)^4 = 3\left|\int_D \mathrm{d}r\,\mathrm{d}z\, r\,\psi\left(\frac{\partial\psi}{\partial z_j}\right)^2\frac{\partial^2\psi}{\partial z_j^2}\right|$$
$$\leqq 3\|f\|_\infty\left(\int_D \mathrm{d}r\,\mathrm{d}z\, r\left(\frac{\partial\psi}{\partial z_j}\right)^4\right)^{1/2}\times\left(\int_D \mathrm{d}r\,\mathrm{d}z\, r\left(\frac{\partial^2\psi}{\partial z_j^2}\right)^2\right)^{1/2},$$

which gives the bound

$$\int_D \mathrm{d}r\,\mathrm{d}z\, r\left(\frac{\partial\psi}{\partial z_j}\right)^4 \leqq 9\|f\|_\infty^2\int_D \mathrm{d}r\,\mathrm{d}z\, r\left(\frac{\partial^2\psi}{\partial z_j^2}\right)^2 \leqq 9C\|f\|_\infty^2\|f\|_{1,2}^2,$$

by (7). Similarly,

$$\int_D \mathrm{d}r\,\mathrm{d}z\, r\left(\frac{\partial\psi}{\partial r}\right)^4 = -\int_D \mathrm{d}r\,\mathrm{d}z\,\psi\left(\frac{\partial\psi}{\partial r}\right)^3 - 3\int_D \mathrm{d}r\,\mathrm{d}z\, r\,\psi\left(\frac{\partial\psi}{\partial r}\right)^2\frac{\partial^2\psi}{\partial r^2}$$
$$\leqq \|f\|_\infty\left(\left(\int_D \mathrm{d}r\,\mathrm{d}z\, r\left(\frac{\partial\psi}{\partial r}\right)^4\right)^{1/2}\left(\int_D \mathrm{d}r\,\mathrm{d}z\,\frac{1}{r}\left(\frac{\partial\psi}{\partial r}\right)^2\right)^{1/2}\right.$$
$$\left.+3\left(\int_D \mathrm{d}r\,\mathrm{d}z\, r\left(\frac{\partial\psi}{\partial r}\right)^4\right)^{1/2}\left(\int_D \mathrm{d}r\,\mathrm{d}z\, r\left(\frac{\partial^2\psi}{\partial r^2}\right)^2\right)^{1/2}\right),$$

and the desired result follows from (7) and (8). □

We need another lemma before proving Proposition 3.1.

Lemma 3.3 *Let u and K satisfy the assumptions of Proposition 3.1. Let F be a bounded subset of D and let V be a neighborhood of K in $\mathbb{R}\times\mathbb{R}^m$. There exists a constant C such that, for every $(r,z)\in F\backslash V$,*

$$u(r,z)\leqq Cr,\quad |\nabla u(r,z)|\leqq C,\quad |\nabla^2 u(r,z)|\leqq \frac{C}{r}.$$

Proof. Without loss of generality, we may assume that $V=(-\alpha,\alpha)\times O$, where $\alpha>0$ and O is a neighborhood of K in $\mathbb{R}^m$. For $z\in\mathbb{R}^m$ and $r>0$, denote by B_r^z the open ball with center $(-1,z)$ and radius r, and by $\bar{B}_r^z$ the closed ball. We may then choose $\delta<\alpha$ so small that for every $z\in\mathbb{R}^m\backslash O$, the closed ball $\bar{B}_{1+\delta}^z$ does not intersect K. We set $U_z=B_{1+\delta}^z\backslash\bar{B}_1^z$. Let v_z be the unique nonnegative solution of the problem

$$\Delta v_z=v_z^2\quad\text{in } U_z,\quad v_{z|\partial B_{1+\delta}^z}=+\infty,\quad v_{z|\partial B_1^z}=0$$

(the existence and uniqueness of v_z follows for instance from Theorem 7.1 of [1]). The maximum principle (see e.g. [6], appendix) ensures that $u\leqq v_z$ on $D\cap U_z$.

By rotational and translation invariance, one may write $v_z(x) = \theta(|x - (-1, z)|)$ for $x \in U_z$, where the function θ solves the differential equation

$$\theta'' + \frac{m-1}{r}\theta' = \theta^2$$

on $(1, 1+\delta)$, with the boundary conditions $\theta(1) = 0, \theta(1+\delta) = \infty$. Observing that $r^{m-1}\theta'$ is then nondecreasing, one gets the bound $\theta(1+r) \leqq Cr$ for $r \in (1, 1+(\delta/2))$, for some constant C. It readily follows that $u(r,z) \leqq Cr$ for $r \in (0, \delta/2)$, for every $z \in \mathbb{R}^m \backslash O$. This gives the first bound of Lemma 3.3, since u is also bounded on any compact subset of D.

The second bound follow from the first one by using the boundary gradient estimates of [9], p. 40. The same argument can be used for the third bound, observing that $v = \partial u / \partial z_j$ (or $\partial u / \partial r$) solves $\Delta v = 2uv$ in D. □

Proof of Proposition 3.1 As explained after the statement of Proposition 3.1, we may find a sequence of functions $f_n \in C_c^\infty(\mathbb{R}^m)$ such that $0 \leqq f_n \leqq 1$, $f_n = 1$ on a neighborhood of K, and

$$\lim_{n\to\infty} \| f_n \|_{1,2} = 0 \, .$$

With each function f_n, we associate a function ψ_n as described before Lemma 3.2. By construction, $\psi_n = 1$ on a neighborhood of K in $\mathbb{R} \times \mathbb{R}^m$. Furthermore, the functions ψ_n satisfy the bounds of Lemma 3.2.

Now let $\gamma \in C_c^\infty(\mathbb{R} \times \mathbb{R}^m)$ taking values in [0,1] and such that $\gamma(r,z) = \gamma(0,z)$ when $|r| \leqq 1$. We set $\gamma_n = \gamma(1 - \psi_n)$, so that γ_n vanishes on a neighborhood of K in $\mathbb{R} \times \mathbb{R}^m$. By the bound (6) applied to ψ_n, γ_n converges to γ in $L^2(\mathbb{R} \times \mathbb{R}^m, \mathrm{d}r\,\mathrm{d}z)$. We will verify that

$$\sup_n \int_D u^2 \gamma_n^4 r \,\mathrm{d}r\,\mathrm{d}z < \infty \, . \tag{11}$$

By Fatou's lemma, this implies

$$\int_D u^2 \gamma^4 r \,\mathrm{d}r\,\mathrm{d}z < \infty \, ,$$

and the result of Proposition 3.1 follows.

It remains to prove (11). We start from the equality

$$\int_D u^2 \gamma_n^4 r \,\mathrm{d}r\,\mathrm{d}z = \int_D \Delta u \, \gamma_n^4 r \,\mathrm{d}r\,\mathrm{d}z \, .$$

We integrate by parts in the right side. To this end, we observe that the integrals

$$\int_D \mathrm{d}r\,\mathrm{d}z \, r \frac{\partial^2 u}{\partial z_j^2} \gamma_n^4, \quad j = 1, \ldots, d$$

converge absolutely, by Lemma 3.3 and the fact that γ_n vanishes on a neighborhood of K in $\mathbb{R} \times \mathbb{R}^m$. This justifies the following calculation

$$\int_D \mathrm{d}r\,\mathrm{d}z \, r \frac{\partial^2 u}{\partial z_j^2} \gamma_n^4 = \int_0^\infty \mathrm{d}r \, r \int_{\mathbb{R}^m} \mathrm{d}z \frac{\partial^2 u}{\partial z_j^2} \gamma_n^4 = \int_0^\infty \mathrm{d}r \, r \int_{\mathbb{R}^m} \mathrm{d}z \, u \frac{\partial^2}{\partial z_j^2} (\gamma_n^4) \, .$$

Similarly,

$$\int_D \mathrm{d}r\,\mathrm{d}z\, r\frac{\partial^2 u}{\partial r^2}\gamma_n^4 = -\int_{\mathbb{R}^m} \mathrm{d}z \int_0^\infty \mathrm{d}r \frac{\partial u}{\partial r}\frac{\partial}{\partial r}(r\gamma_n^4) = \int_{\mathbb{R}^m} \mathrm{d}z \int_0^\infty \mathrm{d}r\, u \frac{\partial^2}{\partial r^2}(r\gamma_n^4)\,.$$

In the first integration by parts, we use the bound on $\partial u/\partial r$ provided by Lemma 3.3, together again with the fact that γ_n vanishes on a neighborhood of K. In the second integration by parts, we also use the boundary condition satisfied by u.

Summarizing, we get

$$\int_D \mathrm{d}r\,\mathrm{d}z\, r\Delta u\,\gamma_n^4 = \int_D \mathrm{d}r\,\mathrm{d}z \left(r u \Delta \left(\gamma_n^4\right) + 2u\frac{\partial}{\partial r}\left(\gamma_n^4\right)\right)\,.$$

Since $\Delta(\gamma_n^4) = 4\gamma_n^3\Delta\gamma_n + 12\gamma_n^2|\nabla\gamma_n|^2$, we have

$$\begin{aligned}\int_D \mathrm{d}r\,\mathrm{d}z\, r u\Delta\left(\gamma_n^4\right) \leqq\; & 4\left(\int_D \mathrm{d}r\,\mathrm{d}z\, r\, u^2\gamma_n^6\right)^{1/2}\left(\int_D \mathrm{d}r\,\mathrm{d}z\, r(\Delta\gamma_n)^2\right)^{1/2}\\ & + 12\left(\int_D \mathrm{d}r\,\mathrm{d}z\, r\, u^2\gamma_n^4\right)^{1/2}\left(\int_D \mathrm{d}r\,\mathrm{d}z\, r|\nabla\gamma_n|^4\right)^{1/2}\,,\end{aligned}$$

and similarly

$$\int_D \mathrm{d}r\,\mathrm{d}z\, u\frac{\partial}{\partial r}\left(\gamma_n^4\right) \leqq 4\left(\int_D \mathrm{d}r\,\mathrm{d}z\, r\, u^2\gamma_n^6\right)^{1/2}\left(\int_D \mathrm{d}r\,\mathrm{d}z\frac{1}{r}\left(\frac{\partial\gamma_n}{\partial r}\right)^2\right)^{1/2}\,.$$

Now observe that

$$\begin{aligned}\nabla\gamma_n &= (1-\psi_n)\nabla\gamma - \gamma\nabla\psi_n\,,\\ \Delta\gamma_n &= (1-\psi_n)\Delta\gamma - 2\nabla\psi_n\cdot\nabla\gamma - \gamma\Delta\psi_n\,.\end{aligned}$$

Thanks to the estimates (7)–(9) applies to ψ_n, and observing that $\partial\gamma/\partial r = 0$ for $|r| \leqq 1$, we get that

$$\sup\left(\int_D \mathrm{d}r\,\mathrm{d}z\, r\,(\Delta\gamma_n)^2, \int_D \mathrm{d}r\,\mathrm{d}z\, r|\nabla\gamma_n|^4, \int_D \mathrm{d}r\,\mathrm{d}z\frac{1}{r}\left(\frac{\partial\gamma_n}{\partial r}\right)^2\right) \leqq C_0\,,$$

for a certain constant C_0 independent of n. Finally, using the trivial bound $\gamma_n^6 \leqq \gamma_n^4$, we have

$$\int_D u^2\gamma_n^4 r\,\mathrm{d}r\,\mathrm{d}z = \int_D \Delta u\,\gamma_n^4 r\,\mathrm{d}r\,\mathrm{d}z \leqq 24C_0^{1/2}\left(\int_D u^2\gamma_n^4 r\,\mathrm{d}r\,\mathrm{d}z\right)^{1/2}\,,$$

and thus

$$\sup_n \int_D u^2\gamma_n^4 r\,\mathrm{d}r\,\mathrm{d}z \leqq 24^2 C_0\,,$$

which completes the proof of (11) and of Proposition 3.1. □

3.2. The case of a general domain

We now consider a domain D in $\mathbb{R}^d$, $d \geqq 3$, such that $D \neq \mathbb{R}^d$. We assume that D is of class C^5. This means that, for every $x_0 \in \partial D$, there exists a neighborhood U of x_0 in $\mathbb{R}^d$, a neighborhood V of 0 in $\mathbb{R}^{d-1}$, a mapping $h: V \to \mathbb{R}$ of class C^5 such that $h(0) = 0, \nabla h(0) = 0$, and an orthonormal system $(e_1, \ldots, e_d)$ in $\mathbb{R}^d$ so that

$$U \cap D = U \cap \left\{ x = x_0 + \sum_{j=1}^{d} y_j e_j ; (y_1, \ldots, y_{d-1}) \in V, y_d > h(y_1, \ldots, y_{d-1}) \right\} .$$

For $y \in D$, we set $r(y) = \mathrm{dist}(y, \partial D) = \inf_{(z \in \partial D)} |y - z|$.

Recall from the introduction the definition of a set of zero capacity.

Proposition 3.4 *Let u be a nonnegative solution of the equation $\Delta u = u^2$ in D. Assume that there exists a compact subset K of ∂D such that* $\mathrm{cap}\, K = 0$ *and, for every $x \in \partial D \backslash K$,*

$$\lim_{y \to x, y \in D} u(y) = 0 .$$

Then, for every bounded subset H of D,

$$\int_H u^2(y) r(y) \mathrm{d}y < \infty .$$

Proof. It is enough to check that, for every $x_0 \in \partial D$, there exists a neighborhood O of x_0 such that

$$\int_{O \cap D} u^2(y) r(y) \, \mathrm{d}y < \infty .$$

We fix $x_0 \in \partial D$ and choose $U, V, h, (e_1, \ldots, e_d)$ as explained above in the definition of a domain of class C^5. We can then find $\varepsilon > 0$ and a neighborhood V_1 of 0 in $\mathbb{R}^{d-1}$, with $V_1 \subset V$, so that

$$U_1 := \left\{ x = x_0 + \sum_{j=1}^{d} y_j e_j ; |y_d - h(y_1, \ldots, y_{d-1})| < \varepsilon, (y_1, \ldots, y_{d-1}) \in V_1 \right\}$$

is contained in U. Then, if $x = x_0 + \sum_{j=1}^{d} y_j e_j \in U_1$, we have $x \in D$ if and only if $y_d > h(y_1, \ldots, y_{d-1})$, $x \in \partial D$ if and only if $y_d = h(y_1, \ldots, y_{d-1})$.

Let $x = x_0 + \sum_{j=1}^{d} y_j e_j \in \partial D \cap U_1$. The inward-pointing unit vector normal to ∂D at x is

$$N_x = N_{(y_1, \ldots, y_{d-1})} = \left(1 + \sum_{j=1}^{d-1} \left(\frac{\partial h}{\partial y_j}(y_1, \ldots, y_{d-1}) \right)^2 \right)^{-1/2}$$
$$\times \left(- \sum_{j=1}^{d-1} \left(\frac{\partial h}{\partial y_j}(y_1, \ldots, y_{d-1}) \right) e_j + e_d \right) .$$

We consider the mapping η defined from $\mathbb{R} \times V_1$ into $\mathbb{R}^d$ by

$$\eta(r, s_1, \ldots, s_{d-1}) = x_0 + \sum_{j=1}^{d-1} s_j e_j + h(s_1, \ldots, s_{d-1}) e_d + r N_{(s_1, \ldots, s_{d-1})} .$$

Then η is of class $C^4, \eta(0) = x_0$, and the Jacobian matrix of η at 0 is inversible. By the inverse function theorem, there exists a neighborhood Γ of 0 in $\mathbb{R} \times V_1$, a neighborhood O of x_0, which can be supposed to be contained in U_1, such that η is a C^4 diffeomorphism from Γ onto O. We may and will assume in addition that η is also a diffeomorphism from a neighborhood of $\bar{\Gamma}$ onto a neighborhood of $\bar{O}$ ($\bar{\Gamma}$ denotes the closure of Γ). Furthermore, we can suppose that Γ is of the form

$$\Gamma = (-\delta, \delta) \times B_{(d-1)}(0, \delta) ,$$

where $\delta > 0$ and $B_{(d-1)}(0, \delta)$ denotes the ball of radius δ centered at the origin in $\mathbb{R}^{d-1}$. Taking δ smaller if necessary, we can finally assume that for $(r, s) \in \Gamma, \eta(r, s)$ belongs to D if and only if $r > 0$ and in that case $r = r(\eta(r, s))$, in the notation introduced before Proposition 3.4. We set $\psi = \eta^{-1}$ and write $\psi(y) = (r(y), s_1(y), \ldots, s_{d-1}(y))$, in agreement with the previous notation. Notice that $|\nabla r(y)| = 1$ and $\nabla r \cdot \nabla s_j = 0$ for $j = 1, \ldots, d-1$.

Let f be a function of class C^2 on D and let $g = f \circ \eta$, which is defined on $\Gamma_+ := \{(r, s) \in \Gamma;\ r > 0\}$. For $y \in O$, we can compute $\Delta f(y)$ in terms of the partial derivatives of g at the point $\psi(y)$. Straightforward calculations give

$$\Delta f(y) = \frac{\partial^2 g}{\partial r^2}(\psi(y)) + (\Delta r(y)) \frac{\partial g}{\partial r}(\psi(y)) + Lg(\psi(y)) ,$$

where the operator L only involves partial derivatives with respect to the variables s_j:

$$Lg(\psi(y)) = \sum_{j=1}^{d-1} \Delta s_j(y) \frac{\partial g}{\partial s_j}(\psi(y)) + \sum_{i,j=1}^{d-1} \left(\nabla s_i \cdot \nabla s_j\right)(y) \frac{\partial^2 g}{\partial s_i \partial s_j}(\psi(y)) .$$

Note that the functions $\Delta r(y)$, $\Delta s_j(y)$, $\nabla s_j(y)$ are of class C^2 since η and ψ are of class C^4.

We then define a function ρ on Γ by

$$\rho(r, s) = \exp\left(\frac{1}{2} \int_0^r \Delta r \circ \eta(u, s)\, \mathrm{d}u\right) .$$

The function ρ is of class C^2 and is bounded below and above by positive constants. Notice that

$$\frac{\partial \rho}{\partial r} = \frac{1}{2}(\Delta r \circ \eta) \rho .$$

Now consider the compact subset K of zero capacity of ∂D. Denote by K' the closure of $\psi(K \cap O)$, which is contained in $\{0\} \times \mathbb{R}^{d-1}$. Viewed as a subset of $\mathbb{R}^{d-1}$, K' has zero Newtonian (logarithmic if $d = 3$) capacity (recall that ψ can be extended to a diffeomorphism from a neighborhood of $\bar{O}$ onto a neighborhood of $\bar{\Gamma}$). Let γ be a function in $C_c^\infty(\mathbb{R} \times \mathbb{R}^{d-1})$, whose support is contained in Γ and such that $\gamma(r,s) = \gamma(0,s)$ if $|r| \leqq \delta/2$. With the function γ and the compact set K' of zero capacity, we associate a sequence (γ_n) defined in the same way as in Sect. 3.1.

Let u satisfy the assumption of Proposition 3.4. Then, if $J(y)$ denotes the Jacobian of ψ at y,

$$\begin{aligned}\int_{O \cap D} u(y)^2 (\gamma_n \circ \psi(y))^4 r(y)\,\mathrm{d}y &\leqq C \int_{O \cap D} u(y)^2 \rho \circ \psi(y) (\gamma_n \circ \psi(y))^4 r(y) J(y)\,\mathrm{d}y \\ &= C \int_{\Gamma_+} u^2 \circ \eta(r,s) \rho(r,s) \gamma_n(r,s)^4 r\,\mathrm{d}r\,\mathrm{d}s\,,\end{aligned}$$

for a certain constant C. Set $v = u \circ \eta$. Then,

$$u^2 \circ \eta = \Delta u \circ \eta = \frac{\partial^2 v}{\partial r^2} + (\Delta r \circ \eta) \frac{\partial v}{\partial r} + Lv\,,$$

and we have thus obtained the bound

$$\int_{O \cap D} u(y)^2 (\gamma_n \circ \psi(y))^4 r(y)\,\mathrm{d}y \leqq C \int_{\Gamma_+} \left(\frac{\partial^2 v}{\partial r^2} + (\Delta r \circ \eta) \frac{\partial v}{\partial r} + Lv \right) \rho \gamma_n^4 r\,\mathrm{d}r\,\mathrm{d}s\,.$$

We will use integrations by parts to bound the right side of the previous formula. To justify these calculations, we first observe that the analogue of Lemma 3.3 remains valid, with exactly the same proof. Hence, if Λ is a neighborhood of K' in $\mathbb{R} \times \mathbb{R}^{d-1}$, there exists a constant C' such that, for every $(r,s) \in \Gamma_+ \backslash \Lambda$,

$$v(r,s) \leqq C' r, \quad |\nabla v(r,s)| \leqq C', \quad |\nabla^2 v(r,s)| \leqq \frac{C'}{r}\,.$$

We first consider

$$\begin{aligned}\int_{\Gamma_+} \left(\frac{\partial^2 v}{\partial r^2} + (\Delta r \circ \eta) \frac{\partial v}{\partial r} \right) \rho \gamma_n^4 r\,\mathrm{d}r\,\mathrm{d}s &= \int_{\Gamma_+} \left(\rho \frac{\partial^2 v}{\partial r^2} + 2 \frac{\partial \rho}{\partial r} \frac{\partial v}{\partial r} \right) \gamma_n^4 r\,\mathrm{d}r\,\mathrm{d}s \\ &= \int_{B_{(d-1)}(0,\delta)} \mathrm{d}s \int_0^\delta \mathrm{d}r\, r \frac{\partial^2}{\partial r^2} (\rho v) \gamma_n^4 \\ &\quad - \int_{\Gamma_+} \mathrm{d}r\,\mathrm{d}s\, r \frac{\partial^2 \rho}{\partial r^2} v \gamma_n^4\,,\end{aligned}$$

where we have used the definition of ρ in the first equality. On one hand,

$$\begin{aligned}\left|\int_{\Gamma_+} \mathrm{d}r\,\mathrm{d}s\, r\frac{\partial^2\rho}{\partial r^2} v\gamma_n^4\right| &\leqq c_1 \int_{\Gamma_+} \mathrm{d}r\,\mathrm{d}s\, rv\gamma_n^4 \\ &\leqq c_1\left(\int_{\Gamma_+} \mathrm{d}r\,\mathrm{d}s\, rv^2\gamma_n^4\right)^{1/2}\left(\int_{\Gamma_+} \mathrm{d}r\,\mathrm{d}s\, r\gamma_n^4\right)^{1/2} \\ &\leqq c_2\left(\int_{O\cap D} u(y)^2\,(\gamma_n\circ\psi(y))^4\, r(y)\,\mathrm{d}y\right)^{1/2}.\end{aligned}$$

On the other hand, by two integrations by parts (using the previous bounds on v and its partial derivatives),

$$\int_0^\delta \mathrm{d}r\, r\frac{\partial^2}{\partial r^2}(\rho v)\gamma_n^4 = 8\int_0^\delta \mathrm{d}r\,\rho\, v\gamma_n^3\frac{\partial\gamma_n}{\partial r} + 12\int_0^\delta \mathrm{d}r\, r\,\rho v\gamma_n^2\left(\frac{\partial\gamma_n}{\partial r}\right)^2 + 4\int_0^\delta \mathrm{d}r\, r\rho v\gamma_n^3\frac{\partial^2\gamma_n}{\partial r^2}\,.$$

Since the function ρ is bounded, we can then argue exactly as in the proof of Proposition 3.1, using the Cauchy–Schwarz inequality. Thanks to the properties of the functions γ_n, we obtain

$$\begin{aligned}\left|\int_{B_{(d-1)}(0,\delta)} \mathrm{d}s\int_0^\delta \mathrm{d}r\, r\frac{\partial^2}{\partial r^2}(\rho v)\gamma_n^4\right| &\leqq c_3\left(\int_{\Gamma_+} \mathrm{d}r\,\mathrm{d}s\, rv^2\gamma_n^4\right)^{1/2} \\ &\leqq c_4\left(\int_{O\cap D} u(y)^2(\gamma_n\circ\psi(y))^4 r(y)\,\mathrm{d}y\right)^{1/2}.\end{aligned}$$

It remains to consider the term

$$\int_{\Gamma_+} \mathrm{d}r\,\mathrm{d}s\, rLv\,\rho\gamma_n^4\,.$$

We integrate by parts to get rid of the derivatives with respect to the variables s_j (again, we use the previous bounds on the derivatives of v to observe that the integral is absolutely convergent). We get terms of the form

$$\int_{\Gamma_+} \mathrm{d}r\,\mathrm{d}s\, rv\varphi\gamma_n^4, \qquad \int_{\Gamma_+} \mathrm{d}r\,\mathrm{d}s\, rv\varphi\gamma_n^3\frac{\partial\gamma_n}{\partial s_i}\,,$$

$$\int_{\Gamma_+} \mathrm{d}r\,\mathrm{d}s\, rv\varphi\gamma_n^2\frac{\partial\gamma_n}{\partial s_i}\frac{\partial\gamma_n}{\partial s_j}, \qquad \int_{\Gamma_+} \mathrm{d}r\,\mathrm{d}s\, rv\varphi\gamma_n^3\frac{\partial^2\gamma_n}{\partial s_i\partial s_j}\,,$$

where the functions φ are bounded over Γ_+. Then, by exploiting once again the properties of the functions γ_n, as in the proof of Proposition 3.1, we obtain

$$\begin{aligned}\int_{\Gamma_+} \mathrm{d}r\,\mathrm{d}s\, rLv\rho\gamma_n^4 &\leqq c_5\left(\int_{\Gamma_+} \mathrm{d}r\,\mathrm{d}s\, rv^2\gamma_n^4\right)^{1/2} \\ &\leqq c_6\left(\int_{O\cap D} u(y)^2\,(\gamma_n\circ\psi(y))^4\, r(y)\,\mathrm{d}y\right)^{1/2}.\end{aligned}$$

By combining the previous bounds, we get

$$\int_{O\cap D} u(y)^2 (\gamma_n \circ \psi(y))^4 r(y)\,\mathrm{d}y \leqq c_7 \left(\int_{O\cap D} u(y)^2 (\gamma_n \circ \psi(y))^4 r(y)\,\mathrm{d}y \right)^{1/2} .$$

Hence, by letting n tend to ∞,

$$\int_{O\cap D} u(y)^2 (\gamma \circ \psi(y))^4 r(y)\,\mathrm{d}y < \infty ,$$

which completes the proof of Proposition 3.4. □

3.3. *The characterization of boundary polar sets*

We will now prove Theorem 1.1. As points are not ∂-polar when $d = 2$, we need only consider the case $d \geqq 3$. Our arguments follow closely the proof of Proposition 3.2 in [13]. We consider a bounded domain D of class C^5 in $\mathbb{R}^d$. We have to prove that any compact subset K of ∂D such that cap $K = 0$ (in the sense of Sect. 3.2) is ∂-polar. Recall that the converse follows from Corollary 4.2 of [13].

Let K be a compact subset of ∂D such that cap $K = 0$. We will argue by contradiction. Suppose that K is not ∂-polar. Then, if

$$u(x) = \mathbb{N}_x(\mathscr{R}^D \cap K)$$

the function $4u$ is a nontrivial nonnegative solution of $\Delta u = u^2$ in D. By Proposition 3.4,

$$\int_D u(y)^2 r(y)\,\mathrm{d}y < \infty .$$

Denote by $G(x, y)$ the Green function for Brownian motion in D (killed at the boundary of D). By a classical bound, we have $G(x, y) \leqq C(x) r(y)$, where the constant $C(x)$ depends only on x. Hence, we have also, for every $x \in D$,

$$\int_D u(y)^2 G(x, y)\,\mathrm{d}y < \infty .$$

We now fix $x \in D$ and denote by H_K the set of all stopped paths started at x that exit D at a point of K. If $\tau(w) = \inf\{t \geqq 0;\ w(t) \notin D\}$,

$$H_K = \{w \in \mathscr{W}_x;\ \tau(w) < \infty,\ w(\tau(w)) \in K\} .$$

According to Theorem 4.5 of [13], the set H_K is an equilibrium set for the process (W_s) in $\mathscr{W}_x$, killed when its lifetime vanishes. Moreover, the capacitary measure μ of H_K can be described as follows. Denote by μ_t the image measure of the restriction of μ to stopped paths with lifetime greater than t, by the mapping $w \to (w(r \wedge t), r \geqq 0)$ (μ_t is viewed as a probability measure on the set of all stopped paths with lifetime t). Also denote by P_x^t the law of Brownian motion started at x stopped at time t (viewed as a probability measure on the same set of stopped paths). Then, for every $t \geqq 0$, μ_t is absolutely continuous with respect to P_x^t, and its Radon–Nikodym derivative is

$$\frac{\mathrm{d}\mu_t}{\mathrm{d}P_x^t} = 1_{(t<\tau(w))} u(w(t)) \exp\left(-2\int_0^t u(w(r))\,\mathrm{d}r \right) .$$

Theorem 4.5 of [13] also implies that

$$\int_0^{\zeta} u(w(t))\,\mathrm{d}t = \infty, \quad \mu(\mathrm{d}w) \text{ a.e.}$$

because the measure μ has finite energy and therefore cannot give a positive mass to an M_x-polar set (see [13] for the definition of M_x-polar sets). It follows that

$$\int \mu(\mathrm{d}w) \int_0^{\zeta} u(w(t))\,\mathrm{d}t = \infty\,.$$

On the other hand, by the previous observations,

$$\begin{aligned}\int \mu(\mathrm{d}w) \int_0^{\zeta} u(w(t))\,\mathrm{d}t &= \int_0^{\infty} \mathrm{d}t\, E_x^t \left(1_{\{t<\tau(w)\}} u^2(w(t)) \exp\left(-2\int_0^t u(w(r))\,\mathrm{d}r\right)\right) \\ &\leqq E_x\left(\int_0^{\tau} \mathrm{d}t\, u^2(B_t)\right) = \int_D u(y)^2 G(x,y)\,\mathrm{d}y < \infty\,.\end{aligned}$$

We arrive at a contradiction, which completes the proof of Theorem 1.1. □

4 Solutions dominated by a harmonic function

Throughout Sect. 4, we consider a *bounded* domain D in $\mathbb{R}^d$, $d \geqq 2$, and G denotes the Green function of Brownian motion in D. We denote by X^D the spatial exit measure from the domain D (see Sect. 2.4). When D is smooth, or more generally when D is regular in the sense of the classical Dirichlet problem, we have the following properties [6, 12]. If g is a continuous nonnegative function on ∂D, the formula

$$u(x) = \mathbb{N}_x(1 - \exp(-\langle X^D, g\rangle)), \quad x \in D\,,$$

defines the unique nonnegative solution of the equation $\Delta u = 4u^2$ in D with boundary condition $u_{|\partial D} = g$. Alternatively, u can be characterized as the unique nonnegative solution of the integral equation

$$u(x) = E_x(g(B_\tau)) - 2E_x\left(\int_0^{\tau} u^2(B_t)\,\mathrm{d}t\right)\,,$$

where τ denotes as usual the first exit time from D.

4.1 Preliminaries

In this subsection, D is a general bounded domain in $\mathbb{R}^d$.

Proposition 4.1 *Let u be a nonnegative solution of the equation $\Delta u = 4u^2$ in D. Assume that u is bounded above by a function harmonic in D. Then there exists a unique harmonic function h such that, for every $x \in D$,*

$$u(x) = h(x) - 2\int_D G(x,y)u^2(y)\,\mathrm{d}y. \tag{12}$$

The function h is the smallest harmonic majorant of u. Conversely, if h is a given nonnegative harmonic function in D, Eq. (12) *is satisfied by at most one nonnegative function u.*

Proof. Let u be a nonnegative solution of $\Delta u = 4u^2$ in D, which is bounded above by a harmonic function. Then, let (D_n) be an increasing sequence of smooth subdomains of D such that $\bar{D}_n \subset D_{n+1}$ and $D = \bigcup_n D_n$. As u solves the equation in D_n with boundary condition $u_{|\partial D_n}$, we have by the previous observations, for $x \in D_n$,

$$u(x) = E_x(u(B_{\tau_n})) - 2E_x\left(\int_0^{\tau_n} u^2(B_t)\,\mathrm{d}t\right) , \tag{13}$$

with an obvious notation. This shows in particular that $E_x(u(B_{\tau_n}))$ increases as $n \uparrow \infty$. If k is a harmonic function such that $u \leqq k$,

$$E_x(u(B_{\tau_n})) \leqq E_x(k(B_{\tau_n})) = k(x) .$$

This allows us to set, for every $x \in D$

$$h(x) = \lim_{n\uparrow\infty} \uparrow E_x(u(B_{\tau_n})) ,$$

and the function h is harmonic in D. Passing to the limit in (13), we get

$$u(x) = h(x) - 2E_x\left(\int_0^{\tau} u^2(B_t)\,\mathrm{d}t\right) ,$$

which gives (12). Clearly, $h \geqq u$ and $h \leqq k$, which shows that h is the smallest harmonic majorant of u.

It remains to check the converse statement. Suppose that h is a given nonnegative harmonic function in D and that u solves (12). It follows easily that u solves $\Delta u = 4u^2$ in D. Let the sequence (D_n) be as previously. For every integer n, let $w_n(x)$, $x \in D_n$ be the unique nonnegative solution of

$$w_n(x) = h(x) - 2E_x\left(\int_0^{\tau_n} w_n^2(B_t)\,\mathrm{d}t\right) , \quad x \in D_n .$$

Now observe that u solves $\Delta u = 4u^2$ in D_n with boundary condition $u_{|\partial D_n} \leqq h_{|\partial D_n}$, so that the maximum principle ([6], appendix) gives $u \leqq w_n$ in D_n. Similarly, since w_{n+1} solves $\Delta w_{n+1} = 4w_{n+1}^2$ in D_n with boundary condition $w_{n+1|\partial D_n} \leqq h_{|\partial D_n}$, we have $w_n \geqq w_{n+1}$.

We can therefore set

$$w_n(x) = \lim_{n\uparrow\infty} \downarrow w_n(x) \geqq u(x) .$$

Since $w_n \geqq u$, we have for $x \in D_n$

$$w_n(x) \leqq h(x) - 2E_x\left(\int_0^{\tau_n} u^2(B_t)\,\mathrm{d}t\right) ,$$

and by passing to the limit as $n \to \infty$,

$$w(x) \leqq h(x) - 2E_x \left(\int_0^\tau u^2(B_t)\,\mathrm{d}t \right) = u(x) .$$

We finally get $u = w$, which shows that u is uniquely determined by h. $\square$

The next two subsections will be devoted to characterizing the harmonic functions h for which (12) has a solution.

4.2 *Construction of the solution associated with a given harmonic function*

We now assume that D is a domain of class C^2 and denote by $P(x, y)$, $x \in D$, $y \in \partial D$ the Poisson kernel of D. If ν is a finite measure on ∂D, the associated harmonic function is defined by

$$h(x) = P\nu(x) = \int_{\partial D} P(x, y)\nu(\mathrm{d}y) .$$

Recall from the introduction the definition of compact subsets of ∂D of zero capacity.

Proposition 4.2 *Let ν be a finite measure on ∂D and $h = P\nu$. Assume that ν does not charge the compact subsets of ∂D of zero capacity. Then Eq.* (12) *has a* (*unique*) *nonnegative solution u.*

Proof. We first assume that ν satisfies the (stronger) assumption

$$\int\int \nu(\mathrm{d}y)\nu(\mathrm{d}z)|y - z|^{3-d} < \infty \quad \text{if } d \geqq 4 ,$$

$$\int\int \nu(\mathrm{d}y)\nu(\mathrm{d}z) \log \frac{1}{|y - z|} < \infty \quad \text{if } d = 3 . \tag{14}$$

From the known behavior of the Green function and the Poisson kernel near the boundary, this assumption is equivalent to

$$\int_D \mathrm{d}y\, G(x, y)h(y)^2 < \infty , \tag{15}$$

for every $x \in D$ (see the proof of Corollary 4.2 in [13]). We use the sequence D_n and the functions w_n of the proof of Proposition 4.1. Recall that w_n is defined on the subdomain D_n and solves

$$w_n(x) = h(x) - 2E_x \left(\int_0^{\tau_n} w_n^2(B_t)\,\mathrm{d}t \right), \quad x \in D_n .$$

The same argument as in the proof of Proposition 4.1 shows that $w_{n+1}(x) \leqq w_n(x)$ for $x \in D_n$. We can therefore set, for every $x \in D$,

$$u(x) := \lim_{n\to\infty} \downarrow w_n(x) \geqq 0 .$$

We notice that, as $n \to \infty$,

$$E_x\left(\int_0^{\tau_n} w_n^2(B_t)\,\mathrm{d}t\right) \to E_x\left(\int_0^{\tau} u^2(B_t)\,\mathrm{d}t\right) = \int_D \mathrm{d}y\, G(x,y)\, u^2(y),$$

where the passage to the limit is justified by dominated convergence, since $w_n \leqq h$ and

$$E_x\left(\int_0^{\tau} h^2(B_t)\,\mathrm{d}t\right) = \int_D \mathrm{d}y\, G(x,y) h(y)^2 < \infty ,$$

by (15). Then, by passing to the limit in the integral equation satisfied by w_n, we get that u satisfies (12).

In the general case, we use Lemma 4.2 of [2]. According to this lemma, any finite measure θ on $\mathbb{R}^m\,(m = d-1)$ that does not charge compact sets of capacity zero is the increasing limit of a sequence θ_p of measures belonging to $W^{-1,2}(\mathbb{R}^m)$. For a finite measure θ_p, the condition $\theta_p \in W^{-1,2}(\mathbb{R}^m)$ means that

$$\int_{\mathbb{R}^m} \mathrm{d}\xi \frac{|\hat{\theta}_p(\xi)|^2}{1+|\xi|^2} < \infty ,$$

or equivalently, if g_1 denotes the usual Bessel potential (see e.g. [17]), $g_1 * \theta_p \in L^2(\mathbb{R}^m)$. It is then easy to verify that this condition is in turn equivalent to

$$\int\int \theta_p(\mathrm{d}y)\theta_p(\mathrm{d}z)|y-z|^{2-m} < \infty \quad \text{if } m \geqq 3 ,$$

$$\int\int \theta_p(\mathrm{d}y)\theta_p(\mathrm{d}z) \log \frac{1}{|y-z|} < \infty \quad \text{if } m = 2 .$$

By using local charts on the boundary ∂D, we deduce from the previous observations that, under the assumption of Proposition 4.2, v is the increasing limit of a sequence v_p of measures on ∂D that satisfy (14). By the first part of the proof, we can associate with every measure v_p the unique nonnegative solution u_p of

$$u_p(x) = h_p(x) - 2\int_D G(x,y)\, u_p^2(y)\,\mathrm{d}y, \quad x \in D ,$$

where $h_p = Pv_p$. The sequence (h_p) increases towards h. It easily follows that the sequence (u_p) is also monotone increasing (we can introduce for every p a sequence $w_{n,p}$ defined as previously, and the maximum principle implies that, for every n, the sequence $(w_{n,p},\ p \in \mathbb{N})$ is increasing). We then set

$$u(x) = \lim_{p\to\infty} \uparrow u_p(x) \leqq h(x) ,$$

and by monotone convergence we get that u solves (12). □

4.3 Harmonic functions associated with solutions

From now on, we assume that D is of class C^5, in order to apply Theorem 1.1. We aim to prove the converse statement of Proposition 4.2. The following identity is a special case of a formula recalled in Sect. 2.2. If g is a nonnegative measurable function on ∂D, then, for every $x \in D$,

$$\mathbb{N}_x(\langle X^D, g\rangle) = E_x(g(B_\tau)) . \tag{16}$$

Proposition 4.3 *Let u be a nonnegative solution of $\Delta u = 4u^2$ in D. Assume that u is bounded above by a harmonic function, and let $h = P\nu$ be the smallest harmonic majorant of u. Then the measure ν does not charge the sets of zero capacity.*

Proof. In contrast to Proposition 4.2, we will use a probabilistic argument to prove Proposition 4.3. The basic idea is to use the solution u to construct a continuous additive functional of the Brownian snake. The characteristic measure of this additive functional can then be identified with the law of the h-transform of Brownian motion, where h is the smallest harmonic majorant of u. However, this characteristic measure cannot charge sets polar for the Brownian snake and by Theorem 1.1, this implies that ν does not charge sets of zero capacity. From now on, we work under the assumptions of Proposition 4.3. We fix a sequence (D_n) as previously.

Lemma 4.4 *There exists a random variable Z defined on $C_0(\mathbb{R}_+, \mathscr{W})$ such that, for every $x \in D$,*

$$Z = \lim_{n\to\infty} \langle X^{D_n}, h\rangle = \lim_{n\to\infty} \langle X^{D_n}, u\rangle, \quad \mathbb{N}_x \ a.e.$$

Furthermore, $\mathbb{N}_x(Z) = h(x)$, $\mathbb{N}_x(1 - \mathrm{e}^{-Z}) = u(x)$.

Proof. Fix $x \in D$ and choose n_0 so that $x \in D_n$ if $n \geqq n_0$. By the special Markov property (Corollary 2.8), for $n \geqq n_0$

$$\mathbb{N}_x(\langle X^{D_{n+1}}, h\rangle | \mathscr{E}^{D_n}) = \int X^{D_n}(\mathrm{d}z)\mathbb{N}_z(\langle X^{D_{n+1}}, h\rangle) = \langle X^{D_n}, h\rangle ,$$

because $\mathbb{N}_z(\langle X^{D_{n+1}}, h\rangle) = E_z(h(B_{\tau_{n+1}})) = h(z)$, by (16). Therefore, the sequence $(\langle X^{D_n}, h\rangle, n \geqq n_0)$ is an $\mathbb{N}_x$-martingale. Observe that this martingale is identically zero except on the set $\{X^{D_{n_0}} \neq 0\}$, which has finite $\mathbb{N}_x$-measure. The almost everywhere convergence of $\langle X^{D_n}, h\rangle$ then follows from well-known martingale convergence theorems. We set

$$Z = \lim_{n\to\infty} \langle X^{D_n}, h\rangle .$$

We now turn to the convergence of $\langle X^{D_n}, u\rangle$. Fix $x \in D$ and n_0 as previously. By the results recalled at the beginning of Sect. 4, we have

$$u(x) = \mathbb{N}_x(1 - \exp(-\langle X^{D_n}, u\rangle)), \quad x \in D_n .$$

We can then check that $(\exp(-\langle X^{D_n}, u\rangle), n \geqq n_0)$ is an $\mathbb{N}_x$-martingale. In fact, if $(W^i)_{i\in I}$ are the excursions of (W_s) outside D_n, we get from the special

Markov property and the exponential formula for Poisson measures that

$$\begin{aligned}\mathbb{N}_x(\exp(-\langle X^{D_{n+1}}, u\rangle)|\mathscr{E}^{D_n}) &= \mathbb{N}_x\left(\exp\left(-\sum_{i\in I}\langle X^{D_{n+1}}(W^i), u\rangle\right)\bigg|\mathscr{E}^{D_n}\right)\\ &= \exp(-\int X^{D_n}(\mathrm{d}z)\mathbb{N}_x(1-\exp(-\langle X^{D_{n+1}}, u\rangle)))\\ &= \exp(-\langle X^{D_n}, u\rangle)\,.\end{aligned}$$

Observe as previously that the martingale $(\exp(-\langle X^{D_n}, u\rangle),\ n \geqq n_0)$ is identically one except on the set $\{X^{D_{n_0}} \neq 0\}$, which has finite $\mathbb{N}_x$ measure. The martingale convergence theorems imply the almost everywhere convergence of $\exp(-\langle X^{D_n}, u\rangle)$. Hence, the sequence $\langle X^{D_n}, u\rangle$ also converges a.e. towards a variable Z' with values in $[0,\infty]$. Since $u \leqq h$, we have $Z' \leqq Z$ so that $Z' < \infty$, a.e.

Let us check that $Z = Z'$. By Eqs. (12) and (16), we have

$$\begin{aligned}\mathbb{N}_x(\langle X^{D_n}, h\rangle - \langle X^{D_n}, u\rangle) &= 2\mathbb{N}_x\left(\int X^{D_n}(\mathrm{d}z)E_z\left(\int_0^\tau u^2(B_t)\,\mathrm{d}t\right)\right)\\ &= 2E_x\left(E_{B_{\tau_n}}\left(\int_0^\tau u^2(B_t)\,\mathrm{d}t\right)\right)\\ &= 2E_x\left(\int_{\tau_n}^\tau u^2(B_t)\,\mathrm{d}t\right)\,,\end{aligned}$$

which tends to 0 as $n \to \infty$. Here dominated convergence is justified by the bound

$$E_x\left(\int_0^\tau u^2(B_t)\,\mathrm{d}t\right) \leqq h(x) < \infty$$

that follows from (12). Using Fatou's lemma we conclude that $\mathbb{N}_x(Z - Z') = 0$, so that $Z = Z', \mathbb{N}_x$ a.e.

By passing to the limit as $n \to \infty$ in the formula

$$u(x) = \mathbb{N}_x(1 - \exp(-\langle X^{D_n}, u\rangle))$$

we immediately get $u(x) = \mathbb{N}_x(1 - \mathrm{e}^{-Z})$. It remains to prove that $h(x) = \mathbb{N}_x(Z)$. This is less easy because it is not clear that $\langle X^{D_n}, h\rangle$ converges to Z in $L^1(\mathbb{N}_x)$. However Fatou's lemma gives

$$\mathbb{N}_x(Z) \leqq \liminf_{n\to\infty} \mathbb{N}_x(\langle X^{D_n}, h\rangle) = h(x)\,,$$

by (16) again and the harmonicity of h. We will verify that the function $x \to \mathbb{N}_x(Z)$ is harmonic in D. Since $u(x) = \mathbb{N}_x(1 - \mathrm{e}^{-Z}) \leqq \mathbb{N}_x(Z)$, it follows that $\mathbb{N}_x(Z)$ is a harmonic majorant of u, hence $\mathbb{N}_x(Z) \geqq h(x)$, which completes the proof.

Let us fix $x \in D$ and let O be an open ball containing x, whose closure is contained in D. Denote by $(W^j, j \in J)$ the excursions of (W_s) outside O. By construction, we have

$$Z = \sum_{j \in J} Z(W^j), \quad \mathbb{N}_x \text{ a.e.}$$

Observe that there is only a finite number of nonzero terms in the previous sum, namely the terms corresponding to excursions that reach ∂D. Then from the special Markov property and (16),

$$\mathbb{N}_x(Z) = \mathbb{N}_x(\int X^O(\mathrm{d}y)\mathbb{N}_y(Z)) = E_x(\mathbb{N}_{B_\eta}(Z)),$$

where $\eta = \inf\{t;\ B_t \notin O\}$. The harmonicity of the function $\mathbb{N}_x(Z)$ follows at once. □

Lemma 4.5 *Let $(L_s^{D_n})$ denote the exit local time of the process (W_s) from the domain D_n. Set*

$$A_s^n = \int_0^s \mathrm{d}L_u^{D_n}\, h(\hat{W}_u).$$

Then, the sequence of processes $(A_s^n,\ s \geqq 0)$ converges $\mathbb{N}_x$ a.e. uniformly on $\mathbb{R}_+$, for every $x \in D$. The limiting process is denoted by $(A_s,\ s \geqq 0)$. For every nonnegative measurable function φ on $\mathcal{W}_x$,

$$\mathbb{N}_x\left(\int_0^\infty \mathrm{d}A_s\, \varphi(W_s)\right) = h(x) E_x^h\left(\varphi(B_{\leqq\tau})\right), \tag{17}$$

where P_x^h denotes the law of the h-transform of B started at x and we use the notation $B_{\leqq\tau}$ to denote the stopped path $(B_t, 0 \leqq t \leqq \tau)$ viewed as an element of $\mathcal{W}_x$.

Remark. We have $A_\infty^n = \langle X^{D_n}, h\rangle$, so that the convergence of A_∞^n follows from Lemma 4.4.

Proof. For every integer n, we consider the excursions of (W_s) outside D_n (cf. Sect. 2.3). More precisely, we are only interested in a finite number of these excursions, namely those that visit ∂D_{n+1}. Denote by W_i^n, $i \in I_n$ these excursions and by (a_i^n, b_i^n) the corresponding time intervals. We may take $I_n = \{1, \ldots, l_n\}$ and assume that the excursions are ordered, in the sense that $b_i^n < b_j^n$ if $i < j$. By the special Markov property, conditionally on the σ-field $\mathcal{E}^{D_n}$, the point measure

$$\sum_{i \in I_n} \delta_{W_i^n}$$

is under $\mathbb{N}_x$ a Poisson measure with intensity

$$\int X^{D_n}(\mathrm{d}y)\mathbb{N}_y(\,\cdot\, \cap \{\mathcal{R} \cap \partial D_{n+1} \neq \emptyset\}),$$

where $\mathscr{R}$ denotes the range of $\hat{W}$, $\mathscr{R} = \{\hat{W}_s, s \geqq 0\}$. Together with Lemma 4.4, this allows us to define, for every n and for every $i \in I_n$,

$$Z(W_i^n) = \lim_{m\to\infty} \langle X^{D_m}(W_i^n), h\rangle = \lim_{m\to\infty} \int_{a_i^n}^{b_i^n} \mathrm{d}L_s^{D_m}\, h(\hat{W}_s)\,,$$

where the last equality follows from the definition of the excursions W_i^n and of the exit measure X^{D_m}.

We have therefore $\mathbb{N}_x$ a.e., for every n and every $i \in I_n$

$$\lim_{m\to\infty} (A^m_{b_i^n} - A^m_{a_i^n}) = Z(W_i^n)\,.$$

Claim. We have

$$\lim_{n\to\infty} \left(\sup_{i\in I_n} Z(W_i^n)\right) = 0, \quad \mathbb{N}_x \text{ a.e.}$$

To prove the claim, recall that the smallest harmonic majorant of u is constructed as

$$h(x) = \lim_{n\to\infty} \uparrow E_x(u(B_{\tau_n}))$$

(see the proof of Proposition 4.1). Therefore,

$$\lim_{n\to\infty} \downarrow E_x(h(B_{\tau_n}) - u(B_{\tau_n})) = 0\,.$$

On the other hand, by the special Markov property and Lemma 4.4,

$$\begin{aligned}
E_x(h(B_{\tau_n}) - u(B_{\tau_n})) &= \mathbb{N}_x(\textstyle\int X^{D_n}(\mathrm{d}y)\,(h(y) - u(y))) \\
&= \mathbb{N}_x(\textstyle\int X^{D_n}(\mathrm{d}y)\mathbb{N}_y(Z - (1 - \mathrm{e}^{-Z}))) \\
&= \mathbb{N}_x\left(\sum_{i\in I_n}(Z(W_i^n) - (1 - \mathrm{e}^{-Z(W_i^n)}))\right) \\
&\geqq c\mathbb{N}_x\left(\sup_{i\in I_n}(Z(W_i^n)^2 \wedge 1)\right)\,,
\end{aligned}$$

thanks to the elementary inequality $a - 1 + \mathrm{e}^{-a} \geqq c(a^2 \wedge 1)$, valid for every $a \geqq 0$, for a certain constant $c > 0$. The claim follows, since the sequence

$$\sup_{i\in I_n}(Z(W_i^n)^2)$$

is clearly decreasing (any excursion outside D_{n+1} is "contained" in an excursion outside D_n).

The almost everywhere convergence of A_s^n will now follow easily from the claim. Notice that, for $m > n$, A_s^m does not increase on $\mathbb{R}_+ \setminus \bigcup_i (a_i^n, b_i^n)$, because of the support property of the exit local time. Then, for $t \geqq 0$, if j is the smallest integer such that $b_j^n \geqq t$ ($j = l_n$ if there are no such integers), we have

$$\sum_{i=1}^{j-1}(A^m_{b^n_i} - A^m_{a^n_i}) \leqq A^m_t \leqq \sum_{i=1}^{j}(A^m_{b^n_i} - A^m_{a^n_i}) .$$

It follows that, for every $t \geqq 0$,

$$\limsup_{m\to\infty} A^m_t - \liminf_{m\to\infty} A^m_t \leqq \sup_{i\in I_n} Z(W^n_i) .$$

Since we also know that A^m_∞ converges to a finite limit (by Lemma 4.4), we conclude from the claim that A^m_t converges to a finite limit, denoted by A_t, for every $t \geqq 0$, $\mathbb{N}_x$ a.e. By construction A_t does not increase on $\mathbb{R}_+ \setminus \bigcup_i (a^n_i, b^n_i)$ and, for every n and every $i \in I_n$,

$$A_{b^n_i} - A_{a^n_i} = Z(W^n_i) .$$

The claim then implies that A_t is continuous, $\mathbb{N}_x$ a.e. The a.e. uniform convergence of A^n_t towards A_t follows from Dini's theorem.

It remains to check formula (17). We may assume that φ is continuous and $0 \leqq \varphi \leqq 1$. We have

$$\int_0^\infty \mathrm{d}A_s\, \varphi(W_s) = \lim_{n\to\infty} \int_0^\infty \mathrm{d}A^n_s\, \varphi(W_s), \quad \mathbb{N}_x \text{ a.e.},$$

so that, by Fatou's lemma,

$$\mathbb{N}_x\left(\int_0^\infty \mathrm{d}A_s\, \varphi(W_s)\right) \leqq \liminf_{n\to\infty} \mathbb{N}_x\left(\int_0^\infty \mathrm{d}A^n_s\, \varphi(W_s)\right) ,$$

$$\mathbb{N}_x\left(\int_0^\infty \mathrm{d}A_s\, (1-\varphi(W_s))\right) \leqq \liminf_{n\to\infty} \mathbb{N}_x\left(\int_0^\infty \mathrm{d}A^n_s\, (1-\varphi(W_s))\right) .$$

By Lemma 4.4, we have also $\mathbb{N}_x(A_\infty) = h(x) = \lim_{n\to\infty} \mathbb{N}_x(A^n_\infty)$, and we can conclude that

$$\mathbb{N}_x\left(\int_0^\infty \mathrm{d}A_s\, \varphi(W_s)\right) = \lim_{n\to\infty} \mathbb{N}_x\left(\int_0^\infty \mathrm{d}A^n_s\, \varphi(W_s)\right) .$$

Therefore, using Proposition 3.3 of [12] (which is the "historical" version of (16)),

$$\begin{aligned}
\mathbb{N}_x\left(\int_0^\infty \mathrm{d}A_s\, \varphi(W_s)\right) &= \lim_{n\to\infty} \mathbb{N}_x\left(\int_0^\infty \mathrm{d}L^{D_n}_s\, h(\hat{W}_s)\varphi(W_s)\right) \\
&= \lim_{n\to\infty} E_x(h(B_{\tau_n})\varphi(B_{\leqq \tau_n})) \\
&= \lim_{n\to\infty} h(x)E^h_x(\varphi(B_{\leqq \tau_n})) \\
&= h(x)E^h_x(\varphi(B_{\leqq \tau})) .
\end{aligned}$$

This completes the proof of Lemma 4.5. $\square$

We can now complete the proof of Proposition 4.3. Note that

$$h(x)P_x^h = \int \nu(\mathrm{d}y)P(x,y)P_{xy}^D ,$$

where P_{xy}^D stands for the distribution of Brownian motion started at x and conditioned to exit D at y. In particular, the distribution of B_τ under $h(x)P_x^h$ is $P(x,y)\nu(dy)$. Also observe that $P(x,y) > 0$ for every $x \in D, y \in \partial D$ under our assumptions.

Let K be a compact subset of ∂D of zero capacity, and, for a fixed $x \in D$, let

$$H_K := \{w \in \mathscr{W}_x; \tau(w) < \infty \text{ and } w(\tau(w)) \in K\} .$$

By Theorem 1.1,

$$W_s \notin H_K \quad \text{for every } s \geqq 0, \ \mathbb{N}_x \text{ a.e.}$$

We then apply (17) with $f = 1_{H_K}$. It follows that

$$0 = \mathbb{N}_x\left(\int_0^\infty \mathrm{d}A_s\, 1_{H_K}(W_s)\right) = h(x)P_x^h(B_\tau \in K) = \int 1_K(y)P(x,y)\nu(\mathrm{d}y) ,$$

and we conclude that $\nu(K) = 0$. □

Theorem 1.3 follows from Propositions 4.1–4.3.

4.4 The probabilistic representation of solutions

As a by-product of the proof of Theorem 1.3, we will now give a probabilistic representation of the solutions that are bounded above by a harmonic function. We need to introduce a (weak) definition of an additive functional. Recall from [11] that, for every fixed $x \in \mathbb{R}^d$, the process $(W_s, \mathbb{P}_w)$ is symmetric with respect to the measure

$$M_x = \int_0^\infty \mathrm{d}t\, P_x^t ,$$

where P_x^t denotes the law of Brownian motion started at x and stopped at time t. As usual, we write

$$\mathbb{P}_{M_x}^* = \int M_x(\mathrm{d}w)\mathbb{P}_w^* .$$

We denote by $\mathscr{F}_s$ the σ-field on $C_0(\mathbb{R}_+, \mathscr{W}_x)$ generated by $(W_r, 0 \leqq r \leqq s)$, augmented with the collection of all $\mathbb{P}_{M_x}^*$-negligible sets of $\mathscr{F}_\infty$.

A (continuous) additive functional of the process $(W_s, \mathbb{P}_w^*)$ in $\mathscr{W}_x$ is a continuous increasing process (A_s) on $C_0(\mathbb{R}_+, \mathscr{W}_x)$, such that $A_0 = 0$, for every $s \geqq 0$, A_s is $\mathscr{F}_s$-measurable and for every $s, s' \geqq 0$,

$$A_{s+s'} = A_s + A_{s'} \circ \theta_s, \quad \mathbb{P}_{M_x}^* \text{ a.e. and } \mathbb{N}_x \text{ a.e.}, \tag{18}$$

where θ_s denotes the usual shift operator.

Proposition 4.6 *Let u be a nonnegative solution of $\Delta u = u^2$ in D, which is bounded above by a harmonic function, and let h be the smallest harmonic majorant of u. Then, for every $x \in D$,*

$$u(x) = \mathbb{N}_x(1 - \exp(-A^h_\infty)),$$

where $(A^h_s, s \geqq 0)$ is the continuous additive functional of (W_s) characterized up to $\mathbb{P}^*_{M_x}$-indistinguishability by

$$A^h_s = \lim_{n\to\infty} \int_0^s dL^{D_n}_s\, h(W_s) \quad \textit{for every } s \geqq 0, \ \mathbb{P}^*_{M_x} \textit{ a.e.}$$

Furthermore,

$$\mathbb{E}^*_w(A^h_\infty) = 2 \int_0^{\tau(w)\wedge\zeta(w)} h(w(t))dt, \quad M_x(dw) \textit{ a.e.} \tag{19}$$

Remark. By (19) and Proposition 1.1 of [13], we have

$$\mathbb{E}^*_w(A^h_\infty) = 2 \int_0^{\tau(w)\wedge\zeta(w)} h(w(t))\,dt = U(\mu)(w), \quad M_x(dw) \text{ a.e.}$$

where $\mu = h(x)P^h_x$ (here P^h_x is interpreted as a probability measure on $\mathscr{W}_x$), and $U(\mu)$ denotes the potential of the measure μ (see [13]). In this sense, we say that $\mu = h(x)P^h_x$ is the characteristic measure of the additive functional (A_s). The formula of Lemma 4.5

$$\mathbb{N}_x\left(\int_0^\infty dA^h_s\, \varphi(W_s)\right) = h(x)E^h_x(\varphi)$$

is then classical (see e.g. [3, Lemma 2]).

Proof. Set

$$A^n_s = \int_0^s dL^{D_n}_u\, h(W_u),$$

as in Lemma 4.5. By this result, the sequence of processes (A^n_s) converges uniformly, $\mathbb{N}_x$-a.e. Fix any $\varepsilon > 0$. Then M_x and the law of W_ε under $\mathbb{N}_x(\,\cdot\,|\sigma > \varepsilon)$ are mutually absolutely continuous. It follows that the sequence (A^n_s) also converges uniformly, $\mathbb{P}^*_{M_x}$-a.e. We may take

$$A^h_s = \lim_{n\to\infty} A^n_s,$$

on the ($\mathscr{F}_\infty$-measurable) set where (A^n_s) converges uniformly on $\mathbb{R}_+$, and otherwise $A^h_s = 0$ for every $s \geqq 0$. Clearly, A^h_s enjoys the desired measurability property. Moreover property (18) follows by passing to the limit in the analogous equation satisfied by A^n. The formula $u(x) = \mathbb{N}_x(1 - \exp(-A^h_\infty))$ holds by Lemma 4.4 (remark that $Z = A^h_\infty$ in the notation of this lemma).

It remains to check the identity (19). By combining (16) and Proposition 2.1, one easily gets

$$\mathbb{E}_w^*(A_\infty^n) = 2 \int_0^{\tau_n(w)\wedge\zeta_{(w)}} \mathrm{d}t\, h(w(t)),$$

where $\tau_n(w) = \inf\{t \geqq 0;\ w(t) \notin D_n\}$. In particular, $M_x(\mathrm{d}w)$-a.e.,

$$\mathbb{E}_w^*(A_\infty^h) \leqq \liminf_{n\to\infty} \mathbb{E}_w^*(A_\infty^n) = 2 \int_0^{\tau(w)\wedge\zeta_{(w)}} \mathrm{d}t\, h(w(t)),$$

On the other hand, we also know that $\mathbb{N}_x(A_\infty^n) = h(x) = \mathbb{N}_x(A_\infty^h)$. By an argument already used in the proof of Lemma 4.5, this implies that, for any fixed $\varepsilon > 0$,

$$\mathbb{N}_x(A_\infty^h - A_\varepsilon^h) = \lim_{n\to\infty} \mathbb{N}_x(A_\infty^n - A_\varepsilon^n).$$

Using the Markov property at ε, we get

$$\mathbb{N}_x(\mathbb{E}_{W_\varepsilon}^*(A_\infty^h)) = \lim_{n\to\infty} \mathbb{N}_x(\mathbb{E}_{W_\varepsilon}^*(A_\infty^n)) = 2\mathbb{N}_x\left(\int_0^{\tau(W_\varepsilon)\wedge\zeta_\varepsilon} \mathrm{d}t\, h(W_\varepsilon(t))\right),$$

by monotone convergence. In view of the previous bound on $\mathbb{E}_w^*(A_\infty^h)$, this implies

$$\mathbb{E}_{W_\varepsilon}^*(A_\infty^h) = 2 \int_0^{\tau(W_\varepsilon)\wedge\zeta_\varepsilon} \mathrm{d}t\, h(W_\varepsilon(t)), \quad \mathbb{N}_x \text{ a.e.}$$

and (19) follows by exploiting the fact that M_x is absolutely continuous with respect to the law of W_ε under $\mathbb{N}_x(\,\cdot\,|\sigma > \varepsilon)$. □

The paper [14] gives a probabilistic representation for *all* nonnegative solutions of $\Delta u = 4u^2$ in the unit disk of the plane (it is plausible that this result can be extended to any smooth domain in the plane). In the case when D is the unit disk, the measure X^D has a continuous density with respect to the Lebesgue measure on ∂D. Denote by $(Z(y), y \in \partial D)$ this continuous density. Recall from Sect. 1 the definition of $\mathscr{R}^D$. Then, any nonnegative solution can be represented in the form

$$u(x) = \mathbb{N}_x(\mathscr{R}^D \cap K \neq \emptyset) + \mathbb{N}_x(1_{(\mathscr{R}^D\cap K=\emptyset)}(1 - \exp(-\textstyle\int \nu(\mathrm{d}y)\, Z(y)))),$$

where K is a compact subset of ∂D and ν is a Radon measure on $\partial D\backslash K$. More precisely, the previous formula gives a one-to-one correspondence between solutions u and such pairs (K, ν).

In this correspondence, solutions bounded by a harmonic function exactly correspond to pairs (K, ν) with $K = \emptyset$ (and therefore ν is finite, because ∂D is compact). The integral

$$\int \nu(\mathrm{d}y)\, Z(y)$$

can then be easily identified with A_∞^h, where $h = P\nu$. In this special case, we recover the results of Theorem 1.3 and Proposition 4.6 (notice that there are no nonempty ∂-polar sets when $d = 2$).

The combination of the results of [14] and of the present paper suggest that in higher dimensions one could obtain a representation formula of the type

$$u(x) = \mathbb{N}_x(\mathscr{R}^D \cap K \neq \emptyset) + \mathbb{N}_x(1_{(\mathscr{R}^D \cap K = \emptyset)}(1 - \exp(-A_\infty))),$$

where K would be a compact subset of ∂D and (A_s) an additive functional of (W_s) associated with a Radon measure ν on $\partial D \backslash K$ not charging ∂-polar sets. The proof of such a representation formula remains an open problem.

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Note added in proof. Dynkin and Kuznetsov have recently extended the analytic results of the present work to more general equations of the type $Lu = u^\alpha$.

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