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Martingale and stationary solutions for stochastic Navier–Stokes equations

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Summary. We prove the existence of martingale solutions and of stationary solutions of stochastic Navier–Stokes equations under very general hypotheses on the diffusion term. The stationary martingale solutions yield the existence of invariant measures, when the transition semigroup is well defined. The results are obtained by a new method of compactness.

Mathematics Subject Classfication: 60H15, 76D05, 35Q10

1 Introduction

We are concerned with very general classes of stochastic Navier–Stokes equations. The generality, in comparison with the existing literature, lies in the assumptions on the diffusion term. Precise assumptions and references to the literature are given in Sects. 3 and 4; there we distinguish three classes, depending on the diffusion operator and the space dimension. The aim of the paper is to prove the existence of martingale solutions to these evolution equations over finite time interval, and the existence of stationary solutions, understood in the martingale sense (as defined in Sect. 4). When the transition semigroup is well defined, the law at any given time of a stationary solution is an invariant measure; further comments on the existence of invariant measures for such equations are given below.

One of the main novelties of this paper is the compactness method used to prove the existence results mentioned above. This method is well suited for stochastic Navier–Stokes equations, but it is also of interest in itself and we believe it is applicable to other classes of equations.

The existence of martingale solutions of stochastic evolution equations, further called martingale solutions, by compactness methods requires, in the usual approaches, either non-trivial estimates on the modulus of continuity of the approximating solutions (cf. [23, 18]), or the use of factorization of stochastic integrals (cf. [14, 15, 10]). In the deterministic case (cf. [17]) the compactness method for nonlinear partial differential equations is somewhat easier: when L^p bounds on the approximating solutions have been proved, the approximating equations readily give us estimates on the derivatives, and this implies strong convergence of some subsequence. This strategy does not extend to the stochastic case since the solutions are not differentiable. We propose here a method based on fractional Sobolev spaces that allows us to treat stochastic equations in a way completely similar to the deterministic case. However, we emphasize that this method is easier than the previous approaches based on estimates of the modulus of continuity, but is close to the factorization method, also because the latter is related to fractional derivatives in some sense (cf. [25]). Anyway, the present exposition in completely independent of the factorization method.

As to invariant measures, a main difficulty to obtain their existence in certain examples is to prove that the family of the laws at different times of a solution is tight. A second difficulty is that for certain equations, like Navier–Stokes equations in dimension greater than 2, or other equations with just continuous and bounded nonlinearities, where we have some lack of uniqueness and continuous dependence on initial conditions, it is not clear how to define the transition semigroup and prove that it is Feller; thus the usual Krylov–Bogolyubov approach to the construction of invariant measure cannot be employed a priori, and even the concept of invariant measure becomes ambiguous. We overcome these two problems by showing that a stationary martingale solution can be constructed as the limit of stationary solutions of approximating finite-dimensional problems. With this approach it is sufficient to show that the family of laws

$$\{\mathscr{L}(u_n(t)) : t \geqq 0, n \geqq 1\}$$

is bounded in probability (or that the random variables are uniformly bounded in some $L^p(\Omega; H)$ for a suitable p). See Sects. 3 and 4 for the notations. When the transition semigroup is well defined, we readily have existence of invariant measures. Related ideas were presented in [24, 9], but our proofs are different.

A further novelty is contained in the proof of existence of stationary martingale solutions when the correlated noise is of cylindrical covariance. See also [11].

2 Preliminaries

2.1 A lemma on Ito integrals

Let H be a separable Hilbert space (norm $|.|$, inner product $\langle ., . \rangle$). Given $p > 1, \alpha \in (0, 1)$, let $W^{\alpha,p}(0, T; H)$ be the Sobolev space of all $u \in L^p(0, T; H)$ such that

$$\int_0^T \int_0^T \frac{|u(t) - u(s)|^p}{|t - s|^{1+\alpha p}} dt\, ds < \infty$$

endowed with the norm

$$\|u\|^p_{W^{\alpha,p}(0,T;H)} = \int_0^T |u(t)|^p\, dt + \int_0^T\int_0^T \frac{|u(t)-u(s)|^p}{|t-s|^{1+\alpha p}} dt\, ds\,.$$

Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geqq 0}, P)$ be a stochastic basis (with expectation E), let K be a second separable Hilbert space, and let $w(t)$ be a cylindrical Wiener process with values in K, defined on the stochastic basis. Denote by $L_2(K,H)$ the set of Hilbert–Schmidt operators from K to H.
For any progressively measurable process $f \in L^2(\Omega \times [0,T]; L_2(K,H))$ denote by $I(f)$ the Ito integral defined as

$$I(f)(t) = \int_0^t f(s)\, dw(s) \quad t \in [0,T]\,.$$

Clearly $I(f)$ is a progressively measurable process in $L^2(\Omega \times [0,T]; H)$.

Lemma 2.1 *Let $p \geqq 2$, $\alpha < \frac{1}{2}$, be given. Then, for any progressively measurable process $f \in L^p(\Omega \times [0,T]; L_2(K,H))$, we have*

$$I(f) \in L^p(\Omega; W^{\alpha,p}(0,T;H))$$

and there exists a constant $C(p,\alpha) > 0$ independent of f such that

$$E\|I(f)\|^p_{W^{\alpha,p}(0,T;H)} \leqq C(p,\alpha) E \int_0^T \|f(t)\|^p_{L_2(K,H)} dt\,.$$

Proof. We shall denote by C a generic positive constant independent of f, but depending on p and α. The quantity $E\|I(f)\|^p_{W^{\alpha,p}(0,T;H)}$ is the sum of two terms. The first one, $E\|I(f)\|^p_{L^p(0,T;H)}$, is bounded by $CE\|f\|^p_{L^p(0,T;L_2(K,H))}$ by Burkholder–Davis–Gundy inequality. As to the second term, we have

$$E\int_0^T\int_0^T \frac{|I(f)(t)-I(f)(s)|^p}{|t-s|^{1+\alpha p}} dt\, ds = \int_0^T\int_0^T \frac{E\left(|\int_{s\wedge t}^{s\vee t} f(\sigma)\, dw(\sigma)|^p\right)}{|t-s|^{1+\alpha p}} dtds$$

$$\leqq C\int_0^T\int_0^T \frac{E\left(\left(\int_{s\wedge t}^{s\vee t} \|f(\sigma)\|^2_{L_2(K,H)} d\sigma\right)^{\frac{p}{2}}\right)}{|t-s|^{1+\alpha p}} dtds$$

$$\leqq CE\int_0^T\int_0^T \frac{|\phi(t)-\phi(s)|^{\frac{p}{2}}}{|t-s|^{1+\alpha p}} dt\, ds$$

(we have used Burkholder–Davis–Gundy inequality at the second step) where

$$\phi(t) = \int_0^t \|f(\sigma)\|^2_{L^2(K,H)} d\sigma\,.$$

Let us now distinguish two cases. If $p > 2$, we can continue the previous inequalities by

$$\leqq CE\|\phi\|^{\frac{p}{2}}_{W^{2\alpha,\frac{p}{2}}(0,T;\mathbb{R})}$$

$$\leqq CE\|\phi\|^{\frac{p}{2}}_{W^{1,\frac{p}{2}}(0,T;\mathbb{R})}$$

$$\leqq CE\left\|\frac{d\phi}{dt}\right\|^{\frac{p}{2}}_{L^{\frac{p}{2}}(0,T;\mathbb{R})} + CE\|\phi\|^{\frac{p}{2}}_{L^{\frac{p}{2}}(0,T;\mathbb{R})}$$

$$= CE\int_0^T \left(\|f(\sigma)\|^2_{L^2(K,H)}\right)^{\frac{p}{2}} d\sigma + CE\int_0^T \left(\int_0^t \|f(\sigma)\|^2_{L^2(K,H)} d\sigma\right)^{\frac{p}{2}} dt$$

$$\leqq CE\int_0^T \|f(\sigma)\|^p_{L^2(K,H)} d\sigma .$$

The proof is complete for $p > 2$. If $p = 2$ we continue the inequalities above by

$$= C\int_0^T\int_0^T \frac{E\int_{s\wedge t}^{s\vee t} \|f(\sigma)\|^2_{L_2(K,H)} d\sigma}{|t-s|^{1+2\alpha}} dt\, ds$$

$$= 2C\int_0^T\int_s^T\int_s^t \frac{E\|f(\sigma)\|^2_{L_2(K,H)}}{|t-s|^{1+2\alpha}} d\sigma\, dt\, ds$$

and by elementary application of Fubini theorem this term is bounded by

$$CE\int_0^T \|f(\sigma)\|^2_{L_2(K,H)} d\sigma .$$

The proof is complete.

2.2 Compact embedding of certain function spaces

The previous lemma will be used together with the following compactness results, which represents a variation of the compactness theorems of [17], Ch. I, Sect. 5, and [22], Sect. 13.3.

Theorem 2.1 *Let $B_0 \subset B \subset B_1$ be Banach spaces, B_0 and B_1 reflexive, with compact embedding of B_0 in B. Let $p \in (1,\infty)$ and $\alpha \in (0,1)$ be given. Let X be the space*

$$X = L^p(0,T;B_0) \cap W^{\alpha,p}(0,T;B_1)$$

endowed with the natural norm. Then the embedding of X in $L^p(0,T;B)$ is compact.

Proof. Step 1. First assume $B = B_1$. Similarly to [22], let us define the operators J_a in $L^p(0,T;B_i)$, $i = 0,1$, $a > 0$, by setting

$$J_a g(s) = \frac{1}{2a}\int_{s-a}^{s+a} g(t)dt = \frac{1}{2a}\int_{-a}^{a} g(t+s)dt,$$

where we set $g(t) = 0$ if t is not in $[0,T]$. Clearly, for given a, J_a is a bounded linear operator from $L^p(0,T;B_i)$ to $C([0,T];B_i)$, $i = 0,1$ (see the details in [22]).

Let $\mathcal{G} \subset X$ be a bounded set of functions. We have to prove that $\mathcal{G}$ is relatively compact in $L^p(0,T;B_1)$.
Let us prove that $J_a g \to g$ in $L^p(0,T;B_1)$, as $a \to 0$, uniformly with respect to $g \in \mathcal{G}$. Indeed,

$$\begin{aligned}|J_a g(s) - g(s)|_{B_1} &\leqq \frac{1}{2a}\int_{-a}^{a} |g(t+s) - g(s)|_{B_1} dt \\ &\leqq \frac{1}{(2a)^{\frac{1}{p}}}\left(\int_{-a}^{a} |g(t+s) - g(s)|_{B_1}^p dt\right)^{\frac{1}{p}}\end{aligned}$$

that implies

$$\int_0^T |J_a g(s) - g(s)|_{B_1}^p ds \leqq \tfrac{1}{2a}\int_0^T\int_{-a}^{a} |g(t+s) - g(s)|_{B_1}^p dt ds\,.$$

Since $|t| \leqq a$ in the integral, we have

$$\begin{aligned}\int_0^T |J_a g(s) - g(s)|_{B_1}^p ds &\leqq \frac{1}{2}a^{\alpha p}\int_0^T\int_{-a}^{a} \frac{1}{a^{1+\alpha p}}|g(t+s) - g(s)|_{B_1}^p dt ds \\ &\leqq \frac{1}{2}a^{\alpha p}\int_0^T\int_{-a}^{a} \frac{1}{|t|^{1+\alpha p}}|g(t+s) - g(s)|_{B_1}^p dt ds \\ &\leqq \frac{1}{2}a^{\alpha p}\int_0^T\int_0^T \frac{1}{|r-s|^{1+\alpha p}}|g(r) - g(s)|_{B_1}^p dr ds \\ &\leqq \frac{1}{2}a^{\alpha p}|g|_{W^{\alpha,p}(0,T;B_1)}^p\,.\end{aligned}$$

Since $\mathcal{G}$ is bounded in $W^{\alpha,p}(0,T;B_1)$, we obtain the claimed uniform convergence result.

In [22] it is easily proved that, given a, the set $J_a\mathcal{G}$ is relatively compact in $C([0,T];B_1)$. Along with the previous uniform convergence result, this implies that $\mathcal{G}$ is relatively compact in $L^p(0,T;B_1)$.

Step 2. Let us remove the assumption $B_1 = B$. In [17] the following interpolation result is proved: for each $\varepsilon > 0$ there exists a constant $C_\varepsilon > 0$ such that

$$|x|_B \leqq \varepsilon|x|_{B_0} + C_\varepsilon|x|_{B_1}$$

for all $x \in B_0$.

If u_n is a weakly convergent sequence in X, converging to some u we have to prove that u_n converges strongly to u in $L^p(0,T;B)$. The case $u=0$ is sufficient. But since

$$|u_n|_{L^p(0,T;B)} \leqq \varepsilon |u_n|_{L^p(0,T;B_0)} + C_\varepsilon |u_n|_{L^p(0,T;B_1)}$$

it is sufficient to prove that u_n converges strongly to 0 in $L^p(0,T;B_1)$. This has been proved in step 1, completing the proof of the theorem.

Theorem 2.2 *If $B_1 \subset \tilde{B}$ are two Banach spaces with compact embedding, and the real numbers $\alpha \in (0,1), p > 1$ satisfy*

$$\alpha p > 1$$

then the space $W^{\alpha,p}(0,T;B_1)$ is compactly embedded into $C([0,T];\tilde{B})$. Similarly, if the Banach spaces $B_1,\dots,B_n$ are compactly embedded into $\tilde{B}$ and the real numbers

$$\alpha_1,\dots,\alpha_n \in (0,1), \quad p_1,\dots,p_n > 1$$

satisfy

$$\alpha_i p_i > 1, \quad \forall i = 1,\dots,n$$

then the space

$$W^{\alpha_1,p_1}(0,T;B_1) + \cdots + W^{\alpha_n,p_n}(0,T;B_n)$$

is compactly embedded into $C([0,T];\tilde{B})$.

Proof. The space $W^{\alpha,p}(0,T;B_1)$ is continuously embedded into $C^\gamma([0,T];B_1)$ for all $\gamma \in (0,\alpha p - 1)$. Thus, if a set $\mathcal{G}$ is bounded in $W^{\alpha,p}(0,T;B_1)$, it is bounded in $C^\gamma([0,T];B_1)$. It follows that the functions in $\mathcal{G}$ are equiuniformly continuous in $C([0,T];B_1)$ and then in $C([0,T];\tilde{B})$; and for each $s \in [0,T]$ the set

$$\{f(s) : f \in \mathcal{G}\}$$

is bounded in B_1 and thus relatively compact in $\tilde{B}$. We can apply Ascoli–Arzelà theorem to conclude that $\mathcal{G}$ is relatively compact in $C([0,T];\tilde{B})$.

The proof of the second part is trivial.

3 Stochastic Navier–Stokes equations

3.1 Definitions

Let D be a bounded open domain of $\mathbb{R}^d$ with regular boundary ∂D. We consider the d-dimensional stochastic Navier–Stokes equation in D

$$\frac{\partial u(t,x)}{\partial t} - \Delta u(t,x) + (u(t,x)\cdot\nabla)u(t,x) = -\nabla p(t,x) + f(t,x) + G(u,\xi)(t,x) \tag{1}$$

$t \in [0,T]$, $x \in D$, with the incompressibility condition

$$\operatorname{div} u(t,x) = 0, \quad t \in [0,T],\ x \in D \tag{2}$$

the boundary condition

$$u(t,x) = 0, \quad t \in [0,T],\ x \in \partial D \tag{3}$$

and the initial condition

$$u(0,x) = u_0(x), \quad x \in D\,. \tag{4}$$

Here $\xi(t,x)$ is a Gaussian random field, white noise in time, subject to the restrictions imposed below, and G is an operator acting on noise and solution, that we shall specify in different forms in the three cases at the end of this section.

We consider the usual abstract form of equations (1)–(4). Let $\mathscr{V}$ be the space of infinitely differentiable d-dimensional vector fields $u(x)$ on D with compact support strictly contained in D, satisfying $\operatorname{div} u(x) = 0$. We denote by V_α the closure of $\mathscr{V}$ in $[H^\alpha(D)]^d$, for $\alpha \geqq 0$, and we set in particular

$$H = V_0, \quad V = V_1\,.$$

We denote by $|\,.\,|$ and $\langle .\,,.\rangle$ the norm and inner product in H. Identifying H with its dual space H', and identifying H' with a subspace of V_α' (the dual space of V_α) we have $V_\alpha \subset H \subset V_\alpha'$, and we can denote the dual pairing between V_α and V_α' by $\langle .\,,.\rangle$ when no confusion may arise.

Moreover, we set $D(A) = [H^2(D)]^d \cap V$, and define the linear operator $A : D(A) \subset H \to H$ as $Au = -P\Delta u$, where P is the projection from $[L^2(D)]^d$ to H. Since V coincides with $D(A^{1/2})$, we can endow V with the norm $\|u\| = |A^{1/2}u|$. The operator A is positive selfadjoint with compact resolvent; we denote by $0 < \lambda_1 \leqq \lambda_2 \leqq \cdots$ the eigenvalues of A, and by $e_1, e_2, \ldots$ a corresponding complete orthonormal system in H of eigenvectors of A. We remark that $\|u\|^2 \geqq \lambda_1 |x|^2$.

We define the bilinear operator $B(u,v) : V \times V \to (V \cap [L^d(D)]^d)'$ (cf. [17], Lemma 6.1, Ch. I) as

$$\langle B(u,v), z\rangle = \int_D z(x) \cdot (u(x) \cdot \nabla) v(x)\, dx$$

for all $z \in V \cap [L^d(D)]^d$ (it can be extended by continuity to other combinations of functions spaces). By the incompressibility condition we have

$$\langle B(u,v), v\rangle = 0, \quad \langle B(u,v), z\rangle = -\langle B(u,z), v\rangle\,.$$

By [24], B can be extended to a continuous operator

$$B : H \times H \to D(A^{-\alpha})$$

for certain $\alpha > 1$.

In place of Eqs. (1)–(4) we shall consider the abstract stochastic evolution equation

$$\begin{cases} du(t) + Au(t)dt + B(u(t), u(t))dt = f(t)dt + G(u(t))dw(t), & t \in [0, T] \\ u(0) = u_0 \, . \end{cases} \tag{5}$$

Here we assume that

(i) $w(t)$ is a cylindrical Wiener process in a separable Hilbert space K defined on the stochastic basis $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}_{t \geqq 0}, P)$ (expectation denoted by E)
(ii) for some Hilbert space Y, with $H \subset Y, G$ is a mapping from V to $L_2(K, Y)$
(iii) $u_0 \in H$ and $f \in L^2(0, T; V')$.

Assumption ii) has only the role of a general framework and it is not sufficient to study Eq. (5). Additional assumptions on G will be imposed in the three different cases developed below.

Remark. It is not difficult to generalize the previous equation including a non-linear operator $F(u)$ subject to usual hypotheses that lead to the existence of weak solutions, and a time dependent operator $A(t)$ in place of A subject to classical variational hypotheses. These extension are not considered here for sake of simplicity.

Definition 3.1 We say that there exists a martingale solution of the equation (5) if there exists a stochastic basis $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}_{t \in [0,T]}, P)$, a cylindrical Wiener process w on the space K and a progressively measurable process $u : [0, T] \times \Omega \to H$, with P-a.e. paths

$$u(., \omega) \in C([0, T]; D(A^{-\alpha})) \cap L^\infty(0, T; H) \cap L^2(0, T; V)$$

such that P-a.s., the identity

$$\begin{aligned} \langle u(t), v\rangle + \int_0^t \langle Au(s), v\rangle ds + \int_0^t \langle B(u(s), u(s)), v\rangle ds \\ = \langle u_0, v\rangle + \int_0^t \langle f(s), v\rangle ds + \left\langle \int_0^t G(u(s))dw(s), v \right\rangle \end{aligned} \tag{6}$$

holds true for all $t \in [0, T]$ and all $v \in D(A^\alpha)$.

Recall that a function

$$g \in L^\infty(0, T; H) \cap C([0, T]; D(A^{-\alpha}))$$

belongs to H for all $t \in [0, T]$, and is weakly continuous in H (see [21], p. 263). Thus, for a martingale solution in the previous sense we also have

$$u(., \omega) \in C([0, T]; H_w) \quad P\text{-a.s.}$$

where $C([0, T]; H_w)$ denotes the space of H-valued weakly continuous functions on $[0, T]$.

3.2 Case 1: regular diffusion coefficient

The results of this section are essentially covered by [7]. In addition to the hypotheses of the previous section, assume that

(G.1) $G : H \to L_2(K, H)$ is continuous and

$$|G(u)|^2_{L_2(K,\,H)} \leqq \lambda_0 |u|^2 + \rho, \quad u \in H$$

for some positive real numbers λ_0 and ρ.

Theorem 3.1 *Under assumption* (G.1) *there exists a martingale solution of the equation* (5).

Proof. Step 1. Let $\alpha > \frac{d}{2}$ be fixed, so that

$$D(A^{\frac{\alpha}{2}}) \subset [H^\alpha(D)]^d \subset [C(\overline{D})]^d .$$

We have

$$V_\alpha \subset D(A^{\frac{\alpha}{2}}) \subset V \subset H \subset V' \subset D(A^{-\frac{\alpha}{2}}) \subset V_{-\alpha} .$$

Moreover, B is locally Lipschitz from V to $D(A^{-\frac{\alpha}{2}})$. Let P_n be the operator from $D(A^{-\frac{\alpha}{2}})$ to $D(A^{\frac{\alpha}{2}})$ defined as

$$P_n x = \sum_{i=1}^{n} \langle x, e_i \rangle e_i \quad x \in D(A^{-\frac{\alpha}{2}})$$

(we denote by $\langle . \rangle$ also the dual pairing between $D(A^{\frac{\alpha}{2}})$ and $D(A^{-\frac{\alpha}{2}})$). Its restriction to H is the orthogonal projection onto the space spanned by $e_1, \dots, e_n$. Moreover, it satisfies.

$$\langle P_n x, y \rangle = \langle x, P_n y \rangle$$

for all $x, y \in D(A^{-\frac{\alpha}{2}})$.

Let $B_n(u,u)$ be the Lipschitz operator in $P_n H$ defined as

$$B_n(u,u) = \chi_n(u) B(u,u) \quad u \in P_n H$$

where $\chi_n : H \to \mathbb{R}$ is defined as $\chi_n(u) = \Theta_n(|u|)$, with $\Theta_n : \mathbb{R} \to [0,1]$ of class C^∞, such that $\chi_n(u) = 1$ if $|u| \leqq n$, $\chi_n(u) = 0$ if $|u| > n+1$.
Consider the classical Faedo–Galerkin approximation scheme defined by the processes $u_n(t) \in P_n H$, solutions of

$$\begin{cases} du_n + Au_n\,dt + P_n B_n(u_n, u_n)dt = P_n f(t)dt + P_n G(u_n)dw(t), & t \in [0,T] \\ u_n(0) = P_n u_0. \end{cases} \tag{7}$$

Since all the coefficients are continuous and with linear growth in $P_n H$, this equation has a martingale solution $u_n \in L^2(\Omega; C([0,T]; P_n H))$.
One can now prove that there exist two positive constants $C_1(p), C_2$, for each $p \geqq 2$, such that

$$E(\sup_{0 \leqq s \leqq T} |u_n(s)|^p) \leqq C_1(p) \tag{8}$$

$$E \int_0^T \|u_n(s)\|^2 ds \leqq C_2 \tag{9}$$

for all n. The proof is rather classical (although in the present generality we cannot refer to other works) and thus it will be postponed to Appendix 1; we do this also because in the appendix the proof of (8) and (9) will be given as a byproduct of more general estimates needed later in Sects. 3.3 and 4.1.

Step 2. Decompose now u_n as

$$u_n(t) = P_n u_0 - \int_0^t A u_n(s)\,ds - \int_0^t P_n B_n(u_n(s), u_n(s))\,ds + \int_0^t P_n f(s)\,ds + \int_0^t P_n G(u_n(s))\,dw(s)$$

$$= J_n^1 + J_n^2(t) + J_n^3(t) + J_n^4(t) + J_n^5(t)\,. \tag{10}$$

We have

$$E|J_n^1|^2 \leqq C_3$$

$$E\|J_n^2\|^2_{W^{1,2}(0,T;V')} \leqq C_4$$

by (9),

$$E\|J_n^4\|^2_{W^{1,2}(0,T;V')} \leqq C_5$$

for suitable positive constants C_3, C_4, C_5, where, for a generic Banach space B and a real number $p \geqq 1$, $W^{1,p}(0,T;B)$ denotes the space of all $u \in L^p(0,T;B)$ such that $\frac{du}{dt} \in L^p(0,T;B)$; clearly $W^{1,p}(0,T;B) \subset W^{\alpha,p}(0,T;B)$ for all $\alpha \in (0,1)$ and $p > 1$.
As to J_n^5, from Lemma 2.1, the uniform assumption (G1) on G, and (8), we have

$$E\|J_n^5\|^2_{W^{\alpha,2}(0,T;H)} \leqq C_6(\alpha)$$

for all $\alpha \in (0, \frac{1}{2})$, and for some constant $C_6(\alpha) > 0$. Finally, since $D(A^{\frac{\alpha}{2}}) \subset [L^\infty(D)]^d$, and then

$$|\langle B(u,u), v\rangle| \leqq C|u|\,\|u\|\,|A^{\frac{\alpha}{2}} v|, \quad u \in V,\ v \in D(A^{\frac{\alpha}{2}}) \tag{11}$$

for some constant $C > 0$, as to J_n^3 we have

$$\|P_n B_n(u_n, u_n)\|^2_{L^2(0,T;D(A^{-\frac{\alpha}{2}}))} \leqq C_7 \sup_{0\leqq t\leqq T} |u_n(t)|^2 \int_0^T \|u_n(s)\|^2 ds$$

whence

$$\|J_n^3\|^2_{W^{1,2}(0,T;D(A^{-\frac{\alpha}{2}}))} \leqq C_8^2 \sup_{0\leqq t\leqq T} |u_n(t)|^2 \int_0^T \|u_n(s)\|^2 ds$$

for some constants $C_7, C_8 > 0$. This implies, by (8) and (9),

$$E\|J_n^3\|_{W^{1,2}(0,T;D(A^{-\frac{\alpha}{2}}))} \leqq C_8\sqrt{C_1(2)C_2}\,.$$

Collecting all the previous inequalities we have

$$E\|u_n\|_{W^{\alpha,2}(0,T;D(A^{-\frac{\alpha}{2}}))} \leqq C_9(\alpha)$$

for all $\alpha \in (0, \frac{1}{2})$, and for some constant $C_9(\alpha) > 0$. Recalling (9), this implies that the laws $\mathscr{L}(u_n)$ are bounded in probability in

$$L^2(0,T;V) \cap W^{\alpha,2}(0,T;D(A^{-\frac{\alpha}{2}}))$$

and thus that the family $\mathscr{L}(u_n)$ is tight in $L^2(0,T;H)$, by Theorem 2.1.

Arguing similarly on the term J_n^5, on the basis of the estimate (8), we apply Theorem 2.2 and have that the family $\mathscr{L}(u_n)$ is tight in $C([0,T];D(A^{-\beta/2}))$, for all given $\beta > \frac{d}{2}$ (we can choose α at the beginning such that $\alpha < \beta$). Thus we can find a subsequence, still denoted later by u_n, such that $\mathscr{L}(u_n)$ converges weakly in $L^2(0,T;H) \cap C([0,T];D(A^{-\beta/2}))$.

Step 3. Fix $\beta > d/2$. By Skorohod embedding theorem (cf. [16] p. 9), there exists a stochastic basis $(\Omega^1, \mathscr{F}^1, \{\mathscr{F}_t^1\}_{t\in[0,T]}, P^1)$ and, on this basis, $L^2(0,T;H) \cap C([0,T];D(A^{-\beta/2}))$-valued random variables u^1, u_n^1, $n \geqq 1$, such that u_n^1 has the same law of u_n on $L^2(0,T;H) \cap C([0,T];D(A^{-\beta/2}))$, and $u_n^1 \to u^1$ in $L^2(0,T;H) \cap C([0,T];D(A^{-\beta/2}))$, P-a.s. Of course, for each n,

$$\mathscr{L}(u_n^1)(C([0,T];P_nH)) = 1$$

and by (8) and (9) we have

$$E(\sup_{0\leqq s\leqq T} |u_n^1(s)|^p) \leqq C_1(p)$$

$$E\int_0^T \|u_n^1(s)\|^2 ds \leqq C_2$$

for all n, and all $p \geqq 2$. Hence, we also have

$$u^1(.,\omega) \in L^2(0,T;V) \cap L^\infty(0,T;H) \quad P\text{-a.s.} \tag{12}$$

and $u_n^1 \to u^1$ weakly in $L^2(\Omega \times [0,T];V)$ (the identification of the limit as u^1 is easy, and also the fact that the whole sequence converges).

For each $n \geqq 1$, the process $M_n^1(t)$ with trajectories in $C([0,T];H)$ defined as

$$M_n^1(t) = u_n^1(t) - P_n u^1(0) + \int_0^t Au_n^1(s)\,ds + \int_0^t P_n B_n(u_n^1(s), u_n^1(s))\,ds - \int_0^t P_n f(s)\,ds$$

is a square integrable martingale with respect to the filtration

$$(\mathscr{G}_n^1)_t = \sigma\{u_n^1(s),\ s \leqq t\}$$

with quadratic variation

$$\langle\langle M_n^1\rangle\rangle_t = \int_0^t P_n G(u_n^1(s)) G(u_n^1(s))^* P_n\,ds\,. \tag{13}$$

Indeed (see [10], Sect. 8.4), for all $s \leqq t \in [0,T]$, all bounded continuous functions on $L^2(0,s;H)$ or $C([0,s];D(A^{-\beta/2}))$, and all $v,z \in \mathscr{V}$, since u_n^1 and u_n have the same law, we have

$$E(\langle M_n^1(t) - M_n^1(s), v\rangle \phi(u_n^1|_{[0,s]})) = 0 \tag{14}$$

and

$$E((\langle M_n^1(t), v\rangle\langle M_n^1(t), z\rangle - \langle M_n^1(s), v\rangle\langle M_n^1(s), z\rangle$$

$$- \int_s^t \langle G(u_n^1(\sigma))^* P_n v, G(u_n^1(\sigma))^* P_n z\rangle \, d\sigma)\phi(u_n^1|_{[0,s]})) = 0 \,. \tag{15}$$

We can take the limit in (14) and (15). All terms in (14) and (15) are uniformly integrable in ω (see (8), (G1), (13)), and converge P-a.s. (in appendix 2 we shall prove, in particular, that

$$\left\langle \int_0^t P_n B_n(u_n^1(s), u_n^1(s))\, ds, v \right\rangle \to \left\langle \int_0^t B(u^1(s), u^1(s))\, ds, v \right\rangle \tag{16}$$

for all t and v, P-a.s.). Then we obtain that for all $s \leqq t \in [0, T]$, all bounded continuous functions on $L^2(0, s; H)$ or $C([0, s]; D(A^{-\beta/2}))$, and all $v, z \in \mathscr{V}$, we have

$$E(\langle M^1(t) - M^1(s), v\rangle \phi(u^1|_{[0,s]})) = 0 \tag{17}$$

and

$$E((\langle M^1(t), v\rangle\langle M^1(t), z\rangle - \langle M^1(s), v\rangle\langle M^1(s), z\rangle$$

$$- \int_s^t \langle G(u^1(\sigma))^* v, G(u^1(\sigma))^* z\rangle d\sigma)\phi(u^1|_{[0,s]})) = 0 \tag{18}$$

where $M^1(t)$ is defined as

$$M^1(t) = u^1(t) - u^1(0) + \int_0^t Au^1(s)\, ds + \int_0^t B(u^1(s), u^1(s))\, ds - \int_0^t f(s)\, ds \,. \tag{19}$$

This identity must be interpreted, P-a.s., as an identity in $C([0, s]; D(A^{-\beta/2}))$; note in particular that, by (11) and (12), $B(u^1, u^1) \in L^1(0, T; D(A^{-\beta/2}))$. From (17) and (18), with $v, z \in D(A^{\beta/2})$, we see that $A^{-\beta/2}M^1(t)$ is a square integrable martingale in H with respect to the filtration

$$(\mathscr{G}^1)_t = \sigma\{u^1(s),\ s \leqq t\}$$

with quadratic variation

$$\langle\langle A^{-\beta/2} M^1 \rangle\rangle_t = \int_0^t A^{-\beta/2} G(u^1(s)) G(u^1(s))^* A^{-\beta/2}\, ds \,. \tag{20}$$

The conclusion of the proof, by a representation theorem for martingales, is the same as in [10].

3.3 Case 2: coercive diffusion coefficient

The result of this section extends results of [3, 4, 6, 8], which were limited to $d = 2 (d \leqq 4$ in [6]) and operators G satisfying a skew-symmetry condition of the form $\langle G(u)k, u\rangle = 0$ for all $u \in V$, $k \in K$.

Assume

(G2) $G : V \to L_2(K,H)$ is continuous,

$$2\langle Au, u\rangle - |G(u)|^2_{L_2(K,H)} \geqq \eta\|u\|^2 - \lambda_0|u|^2 - \rho$$

for all $u \in V$, and for some real numbers $\eta \in (0,2], \lambda_0$ and ρ; for each $v \in \mathscr{V}$ the mapping $u \mapsto G(u)^*v$ extends by continuity to a continuous mapping from H to K, and there exists a constant $C(v) > 0$ such that

$$|G(u)^*v|^2_K \leqq C(v)(|u|^2 + 1), \quad u \in H .$$

For $\eta = 2$ we find the estimate of assumption (G1) (but there we assumed strong continuity of G on H).

Remark. The typical application is to linear operators that, in the concrete formalism of Eq. (1), have the form

$$G(u,\xi)(t,x) = \sum_{i=1}^{N} \left((b^i(x) \cdot \nabla)u(t,x) + c^i(x)u(t,x)\right) \frac{d\beta^i(t)}{dt}$$

where $\beta^1,\ldots,\beta^N$ are independent standard Brownian motions, $b^1,\ldots,b^N$ are C^∞ vector fields in $\overline{D}$, $c^1,\ldots,c^N$ are C^∞ scalar fields in $\overline{D}$, and

$$\sum_{j,k=1}^{d} \left(2\delta_{i,j} - \sum_{i=1}^{N} b^i_j(x)b^i_k(x)\right) \zeta_i\zeta_j \geqq \alpha|\zeta|^2$$

for all $\zeta \in \mathbb{R}^d$. In abstract notations, $K = \mathbb{R}^N$ and

$$G(u)k = P\sum_{i=1}^{N} \left((b^i \cdot \nabla)u + c^i u\right) k^i, \quad k = (k^1,\ldots,k^N) \in K,\ u \in V .$$

The definition of martingale solution is the same as in Sect. 3.1. We have:

Theorem 3.2 *Under assumption* (G.2), *there exists a martingale solution to problem* (5).

We do not develop the proof in all details, since it is similar to the proof of Theorem 3.1. Only the way to prove the basic estimates (8) and (9) is slightly different, but the proof given in appendix 1 is in fact given under the more general bound of assumption (G2). The lengthy assumption (G2) is imposed only to take the limit in the quadratic variation of the martingale $M^1_n(t)$ defined in step 3 of the proof of Theorem 3.1.

3.4 Case 3: cylindrical noise

In contrast to the previous sections, here we develop an example where the covariance of the coloured noise is not nuclear (but even not an isomorphism, since the space dimension is 2). The result of this section extends a result of [12], relative to the case of additive noise. In addition to the hypotheses of Sect. 3.1, here we assume that

(D3) $d = 2$,

(G3) there exists $\beta_0 \in (0, \frac{1}{4})$ such that G is a bounded linear operator from H to $D(A^{-\frac{1}{4}+\beta_0})$, $A^{-\frac{1}{4}+\beta_0}G(u) \in L_2(K,H)$ for each $u \in H$, and the mapping $u \mapsto A^{-\frac{1}{4}+\beta_0}G(u)$ from H to $L_2(K,H)$ is bounded and continuous.

If $\beta_0 \geqq \frac{1}{4}$, this problem would be covered by the assumptions of Sect. 3.2.

Remark 1. If for all $u \in H$ the operator $A^{\frac{1}{4}+\beta_0+\varepsilon}G(u)$ is a bounded linear operator in H for some $\varepsilon > 0$, and the mapping $u \mapsto A^{\frac{1}{4}+\beta_0+\varepsilon}G(u)$ is bounded and continuous from H to $L(K,H)$, then assumption (G3) is satisfied (since we are in space dimension 2). Indeed, from the Hilbert–Schmidt embedding of the Sovolev space $H^{1+2\varepsilon}(D)$ into $C(\overline{D})$, $\varepsilon > 0$, it follows that $A^{-\frac{1}{2}-\varepsilon}$ is an Hilbert–Schmidt operator in H. Therefore,

$$\begin{aligned} A^{-\frac{1}{4}+\beta_0}G(u) &= A^{-\frac{1}{2}-\varepsilon}A^{\frac{1}{2}+\varepsilon}A^{-\frac{1}{4}+\beta_0}G(u) \\ &= A^{-\frac{1}{2}-\varepsilon}A^{\frac{1}{4}+\beta_0+\varepsilon}G(u) \end{aligned}$$

is Hilbert–Schmidt when $A^{\frac{1}{4}+\beta_0+\varepsilon}G$ is bounded.

Remark 2. An example of operator G satisfying condition (G3) above is the operator defined as

$$Gx = \sum_{j=1}^{\infty} \sigma_j(u)\langle x, e_j\rangle e_j$$

with the coefficients $\sigma_j(u)$ equicontinuous on H, satisfying the condition

$$\sum_{j=1}^{\infty} \frac{\sigma_j^2}{\lambda_j^{\frac{1}{2}-2\beta_0}} < \infty \tag{21}$$

for some $\beta_0 > 0$, where

$$\sigma_j^2 := \sup_{u \in H} \sigma_j^2(u) .$$

The definition of martingale solution given in Sect. 3.1 must be modified here since in general we cannot expect that u takes values in V (compare with [12]). We say that a martingale solution of Eq. (5) consists of a stochastic basis $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}_{t\in[0,T]}, P)$, a cylindrical Wiener process $w(t)$ on the space K, and a progressively measurable process $u : [0,T] \times \Omega \to H$, with P-a.e. paths

$$u(.,\omega) \in C([0,T]; D(A^{-\alpha})) \cap L^\infty(0,T;H) \cap L^2(0,T;D(A^{\frac{1}{4}}))$$

such that P-a.s. the identity

$$\begin{aligned} \langle u(t), \phi\rangle + \int_0^t \langle u(s), A\phi\rangle\, ds &= \langle u_0, \phi\rangle + \int_0^t \langle B(u(s), \phi), u(s)\rangle\, ds \\ &\quad + \int_0^t \langle f(s), \phi\rangle ds + \left\langle \int_0^t G(u(s))dw(s), \phi \right\rangle \end{aligned} \tag{22}$$

holds true for all $t \in [0,T]$ and all $\phi \in \mathscr{V}$. Note that for regular vector fields u and ϕ we have

$$\langle B(u,u),\phi\rangle = -\langle B(u,\phi),u\rangle$$

and the latter expression extends by continuity to $u \in D(A^{\frac{1}{4}})$ and $\phi \in V$, since $D(A^{\frac{1}{4}}) \subset [L^4(D)]^2$ by Sobolev embedding theorem in two dimensions. This motivates the definition. Note that $v \in \mathscr{V}$ implies $v \in D(A^{\frac{1}{4}-\beta_0})$, so that the last term in (22) is well defined since

$$\int_0^T G(u(t))\,dw(t)$$

takes values in $D(A^{-\frac{1}{4}+\beta_0})$.

Remark. We make an observation on the method of proof of the following theorem. We cannot prove uniform estimates on the approximating solutions u_n by Ito formula as in the previous cases, since the noise is not nuclear covariance. Then we proceed initially as in the additive noise case (cf. [12]) and work with the system

$$\begin{cases} dz(t) + (A+\alpha)z(t)dt = G(v(t)+z(t))dw(t), \;\; z(0)=0 \\ dv(t) + Av(t)dt + B(v+z,v+z)dt = f(t)dt + \alpha z(t)dt, \;\; v(0)=u_0 \end{cases} \tag{23}$$

with a given $\alpha \geqq 0$. For the purpose of this section it is sufficient to take $\alpha = 0$, but we develop the estimates in general so that the arguments of Sect. 4.2 will be more transparent, without repeating the same computations. Of course, the second equation in (23) can be interpreted as a deterministic differential equation depending on the random process z.

Theorem 3.3 *Under the assumptions* (D.3)–(G.3), *there exists a martingale solution to problem* (5).

Proof. Let B_n be defined as in Sect. 3.2. Equation (7), considered under the assumptions of the present theorem, has a progressively measurable global solution $u_n \in L^2(\Omega; C([0,T]; P_nH))$. Let z_n be the solution of the equation

$$dz_n + (A+\alpha)z_n\,dt = P_nG(u_n)dw(t), \quad z_n(0)=0$$

with a given $\alpha \geqq 0$. Let $v_n = u_n - z_n$. The pair (z_n, v_n) is a progressively measurable solution, with

$$(z_n, v_n) \in L^2(\Omega; C([0,T]; P_nH)) \times L^2(\Omega; C^1([0,T]; P_nH)),$$

of the system

$$\begin{cases} dz_n + (A+\alpha)z_n dt = P_nG(v_n+z_n)dw(t), \quad z_n(0)=0 \\ dv_n + Av_n dt + P_nB_n(v_n+z_n, v_n+z_n)dt \\ = P_nf dt + \alpha z_n dt, \quad v_n(0) = P_nu_0\,. \end{cases} \tag{24}$$

We can rewrite the equation for z_n in the mild form

$$z_n(t) = e^{-t(A+\alpha)}P_nz_0 + \int_0^t e^{-(t-s)(A+\alpha)}P_nG(u_n(s))dw(s)\,.$$

Therefore, for all $p \geqq 1$, by Burkholder–Davis–Gundy inequality

$$E|A^{\frac{1}{4}-\frac{\beta_0}{2}} z_n(t)|^p$$

$$\leqq C_0(p)E\left(\left(\int_0^t |A^{\frac{1}{2}-\frac{\beta_0}{2}} e^{-(t-s)(A+\alpha)}|^2_{L(H)} |A^{-\frac{1}{4}+\beta_0} G(u_n(s))|^2_{L_2(K,H)} ds\right)^{\frac{p}{2}}\right)$$

$$\leqq C_1(p)E\left(\left(\int_0^t \frac{e^{-(t-s)\alpha}}{(t-s)^{1-\beta_0}} ds\right)^{\frac{p}{2}}\right) \leqq C_2(p,\alpha) \tag{25}$$

for all $n \geqq 1$ and $t \in [0,T]$ (cf. [19], Theorem 6.13, Ch. II).
Moreover, denoting by C a generic positive constant independent of n, since

$$\begin{aligned} -\langle B_n(v_n+z_n,z_n),v_n\rangle &= +\langle B_n(v_n+z_n,v_n),z_n\rangle \\ &\leqq C\|v_n\|(|A^{\frac{1}{4}}v_n|\,|A^{\frac{1}{4}}z_n| + |A^{\frac{1}{4}}z_n|^2) \\ &\leqq C\|v_n\|^{\frac{3}{2}}|v_n|^{\frac{1}{2}}|A^{\frac{1}{4}}z_n| + C\|v_n\|\,|A^{\frac{1}{4}}z_n|^2 \\ &\leqq \frac{1}{4}\|v_n\|^2 + C|v_n|^2|A^{\frac{1}{4}}z_n|^4 + C|A^{\frac{1}{4}}z_n|^4 \end{aligned}$$

from

$$\begin{aligned} \frac{1}{2}\frac{d|v_n|^2}{dt} + \|v_n\|^2 \leqq &- \langle B_n(v_n+z_n,z_n),v_n\rangle \\ &+ \frac{1}{4}\|v_n\|^2 + |f|^2_{V'} + \frac{1}{4}\|v_n\|^2 + C\alpha^2|z_n|^2 \end{aligned}$$

we have

$$\begin{aligned} &\frac{1}{2}\frac{d|v_n|^2}{dt} + \frac{1}{4}\|v_n\|^2 \\ &\quad\leqq C|v_n|^2|A^{\frac{1}{4}}z_n|^4 + C|A^{\frac{1}{4}}z_n|^4 + |f|^2_{V'} + C\alpha^2|z_n|^2 \end{aligned} \tag{26}$$

whence, by Gronwall lemma,

$$\begin{aligned} |v_n(t)|^2 \leqq\; & e^{\int_0^t C|A^{\frac{1}{4}}z_n(\sigma)|^4}|P_n u_0|^2 \\ &+ \int_0^t e^{\int_s^t C|A^{\frac{1}{4}}z_n(\sigma)|^4}(C|A^{\frac{1}{4}}z_n|^4 + |f|^2_{V'} + C\alpha^2|z_n|^2)\,ds \\ \leqq\; & e^{\int_0^T C|A^{\frac{1}{4}}z_n(\sigma)|^4}\left(|u_0|^2 + \int_0^T (C|A^{\frac{1}{4}}z_n|^4 + |f|^2_{V'} + C\alpha^2|z_n|^2)\,ds\right) \end{aligned}$$

and also

$$\int_0^T \|v_n(s)\|^2 ds \leqq C\int_0^T (\|v_n|^2|A^{\frac{1}{4}}z_n|^4 + |A^{\frac{1}{4}}z_n|^4 + |f|^2_{V'} + C\alpha^2|z_n|^2)\,ds\,.$$

The bound (25) implies that for all $\varepsilon > 0$ there exists $R_1 > 0$ such that

$$P\left(\int_0^T |A^{\frac{1}{4}} z_n(\sigma)|^4 d\sigma > R_1\right) < \varepsilon \quad \forall n \geqq 1\,. \tag{27}$$

Therefore we deduce that there exist $R_2, R_3 > 0$ such that

$$P(\sup_{t\in[0,T]} |v_n(t)|^2 > R_2) < \varepsilon \quad \forall n \geqq 1 \tag{28}$$

$$P\left(\int_0^T \|v_n(s)\|^2 ds > R_3\right) < \varepsilon \quad \forall n \geqq 1\,. \tag{29}$$

Finally, from the identity

$$z_n(t) = -\int_0^t Az_n(s)\,ds + \int_0^t P_n G(v_n + z_n)\,dw(s)\,,$$

the boundedness of G, and (25) with $p = 4$, we deduce

$$E\|z_n\|^4_{W^{\beta,4}(0,T;D(A^{-\frac{3}{4}}))} \leqq C(\beta)$$

for all $\beta \in (0, \frac{1}{2})$. Moreover, from the equation for v_n and the previous inequalities (27)–(29) we deduce that there exists $R_4 > 0$ such that

$$P\left(\int_0^T \left|\frac{dv_n}{dt}\right|^2_{V'} ds > R_4\right) < \varepsilon \quad \forall n \geqq 1\,. \tag{30}$$

By the compactness Theorems 2.1 and 2.2, the family of the laws of (z_n, v_n) in

$$L^4(0,T;D(A^{\frac{1}{4}})) \cap C([0,T];D(A^{-1})) \times L^2(0,T;H) \cap C([0,T];D(A^{-1}))$$

is tight (in fact, we have continuous functions with values in more regular spaces, but this is unessential). It follows that u_n is tight in

$$L^2(0,T;H) \cap C([0,T];D(A^{-1}))\,.$$

The rest of the proof is similar to step 3 of the proof of Theorem 3.1, and can be developed either on the pair (z_n, v_n), yielding a martingale solution of system (23) (which in turns gives us a martingale solution of (5)) or directly on u_n. We make some remarks on the latter approach.
Processes u_n^1 are given by Skorohod embedding theorem, and the processes $M_n^1(t)$ are introduced on H by setting

$$M_n^1(t) = u_n^1(t) - u_n^1(0) + \int_0^t Au_n^1(s)ds + \int_0^t P_n B_n(u_n^1, u_n^1)\,ds - \int_0^t P_n f(s)\,ds\,.$$

One proves that $M_n^1(t)$ is a martingale with respect to the filtration

$$(\mathscr{G}_n^1)_t = \sigma\{u_n^1(s),\ s \leqq t\}$$

with quadratic variation

$$\langle\langle M_n^1\rangle\rangle_t = \int_0^t P_n G(u_n^1)G(u_n^1)^* P_n \, ds \, .$$

To take the limit in the equations corresponding to (14) and (15), no estimates of uniform integrability on u_n^1 are necessary, since G is bounded. In the limit we have

$$M^1(t) = u^1(t) - u^1(0) + \int_0^t Au^1(s)\,ds + \int_0^t B(u^1,u^1)\,ds - \int_0^t f(s)\,ds$$

identity in $C([0,T];D(A^{-1}))$ (for instance), and $A^{-1}M^1(t)$ is an H-valued continuous martingales with quadratic variation given by

$$\langle\langle A^{-1}M^1\rangle\rangle_t = \int_0^t A^{-1}G(u^1)G(u^1)^* A^{-1} ds \, .$$

The conclusion is classical.

4 Stationary martingale solutions

In this section we prove that the equations considered in the previous cases have a martingale solution that is a stationary process in H. In the cases 1 and 2, we say that a stationary martingale solution over $[0,\infty)$ consists of a stochastic basis $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}_{t\geqq 0}, P)$, a cylindrical Wiener process $w(t)$ in the space K, and a progressively measurable process $u : [0,\infty)\times\Omega \to H$, with P-a.e. paths

$$u(.,\omega) \in C([0,T];D(A^{-\alpha})) \cap L^\infty(0,T;H) \cap L^2(0,T;V)$$

for all $T > 0$ (thus also $u(.,\omega) \in C([0,T];H_w)$), u is a *stationary process* in H, such that P-a.s. the identity

$$\langle u(t), v\rangle + \int_\tau^t \langle Au(s), v\rangle ds + \int_\tau^t \langle B(u(s),u(s)), v\rangle ds$$
$$= \langle u(\tau), v\rangle + \int_\tau^t \langle f(s), v\rangle ds + \left\langle \int_\tau^t G(u(s))dw(s), v \right\rangle \, . \tag{31}$$

holds true for all $t \geqq \tau \geqq 0$ and all $v \in \mathscr{V}$. Of course it is equivalent to take just $\tau = 0$.

The definition in the case 3 is similar, taking into account the differences between the two cases over finite time horizon.
To analyze stationary solutions we shall always assume that f is constant:

$$f \in V' \, .$$

4.1 Cases 1 and 2

Consider first case 1.

Theorem 4.1 *In addition to the hypothesis* (G.1), *assume that*

$$2\lambda_1 > \lambda_0$$

where λ_1 is the first eigenvalue of A. Then there exists a stationary martingale solution.

Proof. Consider the approximation scheme defined in Sect. 3.2, with $u_0 = 0$. Let G_m be Lipschitz continuous function such that $G_m \to P_n G$ uniformly on bounded sets in $P_n H$. Let u_n^m be a solution of the following equation

$$du_n^m + (Au_n^m + P_n B_n(u_n^m, u_n^m) - P_n f)\,dt = G_m(u_n^m)dw(t)\,,$$
$$u_n^m(0) = 0\,.$$

By (35) with $\eta = 2$ (since we are under the assumption (G1)), and the inequality $\|x\|^2 \geqq \lambda_1 |x|^2$, $x \in V$, we have

$$\frac{d}{dt}E(|u_n^m|^p) + \lambda_1 p(1-\varepsilon)E(|u_n^m|^p)$$
$$\leqq \left(\frac{1}{2}p(p-1)\lambda_0 + \varepsilon|f|_{V'}^2\right)E(|u_n^m|^p) + C(\varepsilon, p, \rho)(|f|_{V'}^2 + 1)\,.$$

Thus, if $2\lambda_1 > \lambda_0$, there exists $\varepsilon > 0$ and $p > 2$ such that

$$\lambda_1 p(1-\varepsilon) > \frac{1}{2}p(p-1)\lambda_0 + \varepsilon|f|_{V'}^2$$

and therefore, by Gronwall lemma, we have

$$E(|u_n^m(t)|^p) \leqq C, \quad \forall t \geqq 0, n \geqq 1 \tag{32}$$

for some constant $C > 0$. This implies that the process $u_n^m(t)$ is bounded in probability, and thus, by the Krylov–Bogoliubov method there exists an invariant measure μ_n^m for Eq. (7). Since the family $\mathscr{L}(u_n^m)$ is tight on $C([0,T]; P_n H)$, we can tend with m to infinity deriving that there exists a stationary solution u_n of Eq. (7). We can construct a stochastic basis $(\Omega', \mathscr{F}'\{\mathscr{F}'\}_{t\geqq 0}, P')$ and, on this basis, a cylindrical Wiener process $w'(t)$ with values in K, and $\mathscr{F}_0'$-measurable $P_n H$-valued random variables $u_{0,n}'$ with laws μ_n, satisfying

$$E'(|u_{0,n}'|^p) \leqq C, \quad \forall n \geqq 1$$

(E' is the expectation in the new stochastic basis). The corresponding solutions $u_n'(t)$ of Eq. (7) are stationary processes in $P_n H$. Endow $L^2_{\text{loc}}(0,\infty; H)$ by the distance

$$d_2(u,v) = \sum_{k=1}^{\infty} \frac{1}{2^k}(|u-v|_{L^2(0,k;H)} \wedge 1)$$

and, similarly, $C([0,\infty]; D(A^{-\frac{\beta}{2}}))$ by the distance

$$d_\infty(u,v) = \sum_{k=1}^{\infty} \frac{1}{2^k}(|u-v|_{C([0,k];D(A^{-\frac{\beta}{2}}))} \wedge 1)$$

($\beta > d/2$ as in the proof of theorem 3.1). We can repeat the bounds of the proof of Theorem 3.1: note that, on one side, there exists $\alpha < \frac{1}{2}$ such that $\alpha p > 2$ (to apply Theorem 2.2); on the other side, the estimates (8) and (9) depend only on $E'(|u'_{0,n}|^p)$ (see Appendix 1), which are equibounded. Then we obtain that the laws of u'_n are tight in

$$L^2_{\text{loc}}(0,\infty;H) \cap C([0,\infty];D(A^{-\beta/2}))\,.$$

To this end, note that the convergence with respect to $d_2 + d_\infty$ is equivalent to the convergence on every finite time interval; then, if a set

$$\mathcal{G} \subset L^2_{loc}(0,\infty;H) \cap C([0,\infty];D(A^{-\frac{\beta}{2}}))$$

has the property that for all k the set

$$\mathcal{G}_k = \{u|_{[0,k]} : u \in \mathcal{G}\}$$

is compact, then $\mathcal{G}$ is compact in

$$L^2_{\text{loc}}(0,\infty;H) \cap C([0,\infty];D(A^{-\frac{\beta}{2}}))\,.$$

Let u^1_n, u^1 be given by Skorohod embedding theorem, as in that proof. Since u^1_n is stationary in H, u^1 is also stationary in H. Indeed, by the a.s. convergence in $C([0,\infty];D(A^{-\beta/2}))$, u^1 is stationary in $D(A^{-\beta/2})$; but, for all t, $u^1(t)$ are H-valued random variables; this fact readily implies that u^1 is stationary in H.

The process u^1 is a martingale solution, by the same proof as in the previous section. Note that in the definition of $M^1_n(t)$ and $M^1(t)$ we have only to replace $P_n u_0$ and u_0 by $u^1_n(0)$ and $u^1(0)$ respectively, that converge one to the other in $D(A^{-\beta/2})$. The theorem is proved.

Uniqueness results for solutions of the stochastic Navie–Stokes equation considered here have been proved under certain assumptions (for non-trivial results for [20], [5]); in such cases it seems possible to define the transition semigroup and obtain, as a byproduct of our results of existence of stationary martingale solutions, the existence of invariant measures in the classical sense.

Finally, consider case 2 of the previous section.

Theorem 4.2 *In addition to the hypothesis* (G2), *assume that*

$$\eta\lambda_1 > \lambda_0\,.$$

Then there exists a stationary martingale solution.

The proof is entirely similar to the previous case, and will be omitted.

4.2 Case 3

In the proof of Theorem 4.1, we have seen that the main point is the uniform estimate (32). Since in this section we deal with noise that does not have nuclear covariance, we cannot apply successfully the Ito formula, and the proof of an estimate of the form (32) is more complicated. The proof that we present here has also been applied to Burgers equation in [11]. Restricted to additive noise, it simplifies the argument of [12], that moreover does not extend, apparently, to the coloured noise case.

Theorem 4.3 *Under assumptions* (D3)–(G3), *there exists a stationary martingale solution.*

Proof. Consider Eq. (7), as in the proof of Theorem 2.3, and choose $u_0 = 0$. It is straightforward to see, as in the previous two cases, that the family of laws $\mathscr{L}(u_n(t))$, $t \geqq 0$, is tight in H, for all given $n \geqq 1$. By Krylov–Bogolyubov method, there exists an invariant measure μ_n for Eq. (7). We shall prove below that the family μ_n is bounded in probability in H. Then, considering the stationary solutions u_n' as in Theorem 4.1, the bounds of the proof of Theorem 3.3 (with $\alpha = 0$ for instance) show that u_n' is tight in

$$L^2_{loc}(0,\infty;H) \cap C([0,\infty];D(A^{-1}))$$

and the conclusion of the proof is the same as in Theorem 4.1.
Hence we have to prove that μ_n is bounded in probability in H. It is sufficient to prove that the family

$$\{\mathscr{L}(u_n(t)) : t \geqq 0,\ n \geqq 1\}$$

is bounded in probability in H. Fix $\varepsilon > 0$. Let $\alpha = \alpha(\varepsilon)$ be given below, and let (z_n, v_n) be the solution of system (24). Note that z_n and v_n depend on α, while α_n does not, and $u_n = v_n + z_n$ for each α.
Note that estimate (25) holds true with the constant $C_2(p,\alpha)$ uniform in $t \geqq 0$ and $n \geqq 1$, and

$$C_2(p,\alpha) \to 0 \tag{33}$$

as $\alpha \to \infty$. Therefore, it is sufficient to prove that there exists $M > 0$ such that

$$P(|v_n(t)|^2 > M) < \frac{\varepsilon}{2} \tag{34}$$

for all $t \geqq 0$ and $n \geqq 1$.
Recall now (26). For all $M > 1$ we have

$$\begin{aligned}
\frac{d}{dt}\log(|v_n|^2 \vee M) &= \mathbf{1}_{\{|v_n|^2 > M\}} \frac{1}{|v_n|^2}\frac{d|v_n|^2}{dt} \\
&\leqq \mathbf{1}_{\{|v_n|^2 > M\}} \left(-\frac{1}{2}\frac{\|v_n\|^2}{|v_n|^2} + C|A^{\frac{1}{4}}z_n|^4\right) \\
&\quad + \mathbf{1}_{\{|v_n|^2 > M\}} \left(\frac{C|f|^2_{V'} + C|A^{\frac{1}{4}}z_n|^4 + C\alpha^2|z_n|^2}{|v_n|^2}\right) \\
&\leqq -\frac{\lambda_1}{2}\mathbf{1}_{\{|v_n|^2 > M\}} + C|A^{\frac{1}{4}}z_n|^4 \\
&\quad + \frac{1}{M}(C|f|^2_{V'} + C|A^{\frac{1}{4}}z_n|^4 + C\alpha^2|z_n|^2)\,.
\end{aligned}$$

Since $v_n(0) = P_n u_0 = 0$, we deduce

$$\frac{\lambda_1}{2} P(|v_n(t)|^2 > M) \leqq CE(|A^{\frac{1}{4}} z_n|^4) + \frac{1}{M}(C|f|_{V'}^2 + CE(|A^{\frac{1}{4}} z_n|^4) + C\alpha^2 E(|z_n|^2))$$

$$\leqq C_3 C_2(4\alpha) + \frac{1}{M}(C_3|f|_{V'}^2 + C_3 C_2(4\alpha) + C_3 \alpha^2 C_2(2\alpha))$$

by (26), for some constant $C_3 > 0$. Recalling (33), we can find α such that

$$\frac{2}{\lambda_1} C_3 C_2(4\alpha) < \frac{\varepsilon}{4}.$$

Then, for M large enough, (34) is satisfied. The proof is complete.

Appendix 1

We prove several bounds used in the paper. We first prove (8) and (9) under the bound in assumption (G2), which is more general than the bound in (G1). Thus we shall cover simultaneously the cases of Sects. 3.2 and 3.3. Moreover, we consider Eq. (7) with more general initial conditions $u_{0,n}$ that are $\mathscr{F}_0$-measurable random variables, to cover also the bounds required in Sect. 4.1. By Ito formula, for all $p \geqq 2$ we have

$$d|u_n(t)|^p \leqq p|u_n(t)|^{p-2}\langle u_n, du_n\rangle + \frac{1}{2}p(p-1)|u_n(t)|^{p-2}|P_n G(u_n)|_{L_2(K,H)}^2 dt.$$

Since $\langle B_n(u_n, u_n), u_n\rangle = 0$, using (G2) we have

$$d|u_n(t)|^p + p|u_n(t)|^{p-2}\|u_n\|^2 dt$$

$$\leqq p|u_n(t)|^{p-2}\langle f, u_n\rangle dt + p|u_n(t)|^{p-2}\langle G(u_n) dw(t), u_n\rangle$$

$$+ \frac{1}{2}p(p-1)|u_n(t)|^{p-2}(\lambda_0|u_n(t)|^2 + \rho + (2-\eta)\|u_n\|^2) dt.$$

Thus, for all $\varepsilon > 0$,

$$d|u_n(t)|^p + |u_n(t)|^{p-2}(p - p\varepsilon - \frac{1}{2}p(p-1)(2-\eta))\|u_n\|^2 dt$$

$$\leqq p|u_n(t)|^{p-2}\frac{1}{4\varepsilon}|f|_{V'}^2 dt + p|u_n(t)|^{p-2}\langle G(u_n) dw(t), u_n\rangle$$

$$+ \frac{1}{2}p(p-1)|u_n(t)|^{p-2}(\lambda_0|u_n(t)|^2 + \rho) dt \tag{35}$$

$$\leqq \left(\frac{1}{2}p(p-1)\lambda_0 + \varepsilon|f(t)|_{V'}^2\right)|u_n(t)|^p$$

$$+ C(\varepsilon, p, \rho)(|f(t)|_{V'}^2 + 1) + p|u_n(t)|^{p-2}\langle G(u_n) dw(t), u_n\rangle$$

for some constant $C(\varepsilon, p, \rho) > 0$ (we have used the young inequality:

$$ab \leqq \frac{\varepsilon}{r} a^r + \frac{1}{r' \varepsilon^{\frac{r'}{r}}} b^{r'}$$

$a, b, \varepsilon > 0,\ r > 1, 1/r + 1/r' = 1$). Choose $\varepsilon > 0$ such that

$$p - p\varepsilon - \frac{1}{2} p(p-1)(2-\eta) > 0 .$$

Then

$$E(|u_n(t)|^p) \leqq E(|u_{0,n}|^p) + \int_0^t \left(\frac{1}{2} p(p-1)\lambda_0 + \varepsilon |f(s)|^2_{V'}\right) E(|u_n(s)|^p)\, ds$$

$$+ C(\varepsilon, p, \rho) \int_0^t \left(|f(s)|^2_{V'} + 1\right) ds .$$

If

$$\sup_n E(|u_{0,n}|^p) < \infty$$

from Gronwall lemma there exists a constant $C > 0$ such that

$$E(|u_n(t)|^p) \leqq C, \quad \forall t \in [0, T], n \geqq 1 . \tag{36}$$

Using this bound in (35) we also obtain

$$E \int_0^T |u_n(t)|^{p-2} \|u_n(t)\|^2 dt \leqq C, \quad \forall n \geqq 1 \tag{37}$$

for a new constant $C > 0$. For $p = 2$, this completes the proof of (9).

By Burkholder–Davis–Gundy inequality, for some constant $C > 0$ we have

$$E \sup_{0 \leqq s \leqq t} \left| \int_0^s p |u_n(\sigma)|^{p-2} \langle P_n G(u_n(\sigma)) dw(\sigma), u_n(\sigma) \rangle \right|$$

$$\leqq CpE \left(\left(\int_0^t |u_n(s)|^{2p-2} |G(u_n(s))|^2_{L_2(K,H)} ds \right)^{\frac{1}{2}} \right)$$

$$\leqq CpE \left(\sup_{0 \leqq s \leqq t} |u_n(s)|^{\frac{p}{2}} \left(\int_0^t |u_n(s)|^{p-2} (\lambda_0 |u_n(s)|^2 \right.\right.$$

$$\left.\left. + \rho + (2-\eta) \|u_n(s)\|^2) \, ds \right)^{\frac{1}{2}} \right)$$

$$\leqq \frac{1}{2}E\left(\sup_{0\leqq s\leqq t}|u_n(s)|^p\right)$$

$$+\frac{1}{2}C^2p^2E\int_0^t \lambda_0 \sup_{0\leqq s\leqq \sigma}|u_n(s)|^p\,d\sigma+\frac{1}{2}C^2p^2\rho E\int_0^t |u_n(s)|^{p-2}ds$$

$$+\frac{1}{2}C^2p^2(2-\eta)E\int_0^t |u_n(s)|^{p-2}\|u_n(s)\|^2\,ds$$

$$\leqq \frac{1}{2}E(\sup_{0\leqq s\leqq t}|u_n(s)|^p)+Cp^2\lambda_0E\int_0^t \sup_{0\leqq s\leqq \sigma}|u_n(s)|^p\,d\sigma+C$$

for some constant $C>0$, in virtue of (36) and (37). Thus by (35) we have

$$E(\sup_{0\leqq s\leqq t}|u_n(s)|^p)$$

$$\leqq E(|u_{0,n}|^p)+\int_0^T\left(\frac{1}{2}p(p-1)\lambda_0+\varepsilon|f(s)|_{V'}^2\right)E\left(\sup_{0\leqq r\leqq s}|u_n(r)|^p\right)ds$$

$$+C(\varepsilon,p,\rho)\int_0^t\left(|f(s)|_{V'}^2+1\right)ds$$

$$+\frac{1}{2}E(\sup_{0\leqq s\leqq t}|u_n(s)|^p)+Cp^2\lambda_0E\int_0^t \sup_{0\leqq s\leqq \sigma}|u_n(s)|^p d\sigma+C\,.$$

By Gronwall lemma, we get (8).

Appendix 2

Here we prove (16). We have

$$\left\langle\int_0^t P_nB_n(u_n^1(s),u_n^1(s))ds,\,v\right\rangle=-\int_0^t\int_D \chi_n(u_n^1(s))(u_n^1)_i(u_n^1)_j\frac{\partial v_j}{\partial x_i}dx\,ds$$

that converges P-a.s. to

$$-\int_0^t\int_D (u^1)_i(u^1)_j\frac{\partial v_j}{\partial x_i}dx\,ds=\left\langle\int_0^t B(u^1(s),u^1(s))ds,\,v\right\rangle\,.$$

This happens because, on a set of P-measure 1,

$$\chi_n(u_n^1)(u_n^1)_i(u_n^1)_j\to(u^1)_i(u^1)_j \tag{38}$$

in $L^1(D\times[0,T])$. To prove (38), take any subsequence n_k; there exists a subsequence ν_h of n_k such that $u_{\nu_h}^1\to u^1$ a.s. on $[0,T]$ with values in $L^2(D)$. Therefore $\chi_{\nu_h}(u_{\nu_h}^1)$ converges to 1 a.s. on $[0,T]$, and the convergence is uniformly bounded by 1. Since $u_{\nu_h}^1$ converges to u^1 also in $L^2(D\times[0,T])$, it is easy by triangle inequality to see that (38) is true for the subsequence $v_{\bar{h}}$. This implies that it is true for the whole sequence.

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