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Diffusion in turbulence

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Summary. We prove long time diffusive behavior (homogenization) for convection-diffusion in a turbulent flow that it incompressible and has a stationary and square integrable stream matrix. Simple shear flow examples show that this result is sharp for flows that have stationary stream matrices.

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Contents

1 Introduction

Let $\mathbf{u}(\mathbf{x})$, with $\nabla \cdot \mathbf{u}(\mathbf{x}) = 0$, be an incompressible velocity field in R^d, $d \geq 2$, and let $\rho(t, \mathbf{x})$ be the density of an additive carried by the flow and dispersing diffusively. It satisfies the convection-diffusion equation

$$\frac{\partial \rho}{\partial t} + \mathbf{u} \cdot \nabla \rho = \sigma \Delta \rho, \tag{1.1}$$

with $\rho(0, \mathbf{x}) = \rho_0(\mathbf{x})$ and where σ is the molecular diffusivity. The density ρ is non negative and we may assume that $\int \rho_0(\mathbf{x}) d\mathbf{x} = 1$ in which case (1.1) is the

Fokker-Plank equation for a diffusing particle satisfying the stochastic differential equation

$$d\mathbf{x}(t) = \mathbf{u}(\mathbf{x}(t))dt + \sqrt{2\sigma}d\mathbf{w}(t), \tag{1.2}$$

with $\mathbf{x}(0) = \mathbf{x}_0$ and where $\mathbf{w}(t)$ is the d-dimensional Brownian motion process. A natural question to ask, and the one we consider here, is what happens to the density ρ, or the process $\mathbf{x}(t)$, after a long time. This is particularly interesting when the velocity field has a repetitive structure as, for example, when it is a periodic, an almost periodic or a stationary random function with mean zero. We expect then an overall diffusive behavior with an *effective* diffusion constant. In this paper we give sharp conditions on $\mathbf{u}$ for this to be the case.

To state our main result we introduce the stream matrix $\Psi(\mathbf{x})$ such that

$$\nabla \cdot \Psi(\mathbf{x}) = -\mathbf{u}(\mathbf{x}), \tag{1.3}$$

which is a skew symmetric matrix and always exists, because $\mathbf{u}$ is incompressible and has mean zero, but may not be stationary. We assume throughout this paper that the velocity field comes from a *stationary* stream matrix Ψ which is *square integrable* and (1.3) is meant in the weak sense. In the periodic case there always exists a periodic stream matrix. In the almost periodic or stationary random case the stream matrix exists but may not be almost periodic or stationary, respectively. In two dimensions the matrix Ψ has the form

$$\Psi(\mathbf{x}) = \begin{pmatrix} 0 & -\psi(\mathbf{x}) \\ \psi(\mathbf{x}) & 0 \end{pmatrix} \tag{1.4}$$

where $\psi(\mathbf{x})$ is the usual stream function. In three dimensions, Ψ has the form

$$\Psi(\mathbf{x}) = \begin{pmatrix} 0 & -\psi_3 & \psi_2 \\ \psi_3 & 0 & -\psi_1 \\ -\psi_2 & \psi_1 & 0 \end{pmatrix}. \tag{1.5}$$

where $\boldsymbol{\psi}(\mathbf{x}) = (\psi_1(\mathbf{x}), \psi_2(\mathbf{x}), \psi_3(\mathbf{x}))$ is the vector potential of the flow $\mathbf{u}$ so that $\nabla \cdot \Psi = \nabla \times \boldsymbol{\psi} = -\mathbf{u}$. In terms of the stream matrix Ψ, the convection diffusion equation (1.1) can be put into divergence form

$$\frac{\partial \rho(t, \mathbf{x})}{\partial t} = \nabla \cdot [(\sigma I + \Psi(\mathbf{x}))\nabla \rho(t, \mathbf{x})], \tag{1.6}$$

where I is the identity matrix. Note that the coefficient matrix $\sigma I + \Psi$ of this parabolic equation is not symmetric. Since we are interested in long time behavior we rescale space and time and let

$$\rho_n(t, \mathbf{x}) = \rho(n^2 t, n\mathbf{x})$$

with n a large parameter tending to infinity. The scaled density ρ_n satisfies the diffusion equation

$$\frac{\partial \rho_n(t, \mathbf{x})}{\partial t} = \nabla \cdot [(\sigma I + \Psi(n\mathbf{x}))\nabla \rho_n(t, \mathbf{x})], \tag{1.7}$$

whose coefficients are rapidly oscillating. The initial condition is $\rho_n(0,\mathbf{x}) = \rho_0(x) \in L^2(R^d)$, which is assumed to be independent of the parameter n.

The main result in homogenization (periodic, almost periodic or random) [20] tells us that if the stationary stream matrix $\Psi(\mathbf{x})$ is *uniformly bounded and ergodic* then there exists a constant effective diffusivity matrix σ^{eff} such that if $\bar{\rho}$ satisfies the effective diffusion equation

$$\frac{\partial \bar{\rho}(t,\mathbf{x})}{\partial t} = \sum_{i,j=1}^{d} \sigma_{ij}^{eff} \frac{\partial^2 \bar{\rho}(t,\mathbf{x})}{\partial x_i \partial x_j} \tag{1.8}$$

with $\bar{\rho}(0,\mathbf{x}) = \rho_0(\mathbf{x})$ then $\rho_n \to \bar{\rho}$ as n tends to infinity

$$\lim_{n\to\infty} \sup_{0\le t\le T} \int_{R^d} |\rho_n(t,\mathbf{x}) - \bar{\rho}(t,\mathbf{x})|^2 d\mathbf{x} = 0 \tag{1.9}$$

for any $T < \infty$ and with probability one. However, when the stream matrix $\Psi(\mathbf{x})$ is stationary and ergodic but unbounded then it is not clear that a diffusion approximation holds. The purpose of this paper is to prove the following theorem.

Theorem 1 *Suppose that the stream matrix $\Psi(\mathbf{x})$ is stationary and ergodic, that the diffusivity σ is positive and that ρ_0 is in $L^2(R^d)$. Then there exists a constant effective diffusivity matrix σ^{eff} and the random density ρ_n converges in the sense of (1.9) to $\bar{\rho}$ satisfying (1.8) if and only if*

$$\langle |\Psi(\mathbf{x})|^2 \rangle < \infty \tag{1.10}$$

where $\langle\cdot\rangle$ denotes expectation.

The effective diffusivity matrix σ^{eff} is determined from the solution of a *cell* problem, as in the case of periodic coefficients [4], which is described in detail in Sect. 3.4. It is not symmetric in general but in the above theorem only its symmetric part enters. It follows from the variational principles described in detail in appendix A of [9] that the symmetric part of the effective diffusivity matrix is always greater than or equal to σI. This means that convection always enhances the effective diffusivity and for some flows this enhancement can be very large.

To put this theorem in its proper context and to explain its significance we provide several remarks. First, the diffusion equation (1.7) is not well defined when the stream function is unbounded so part of the theorem is to make sense of (1.7). We work entirely with time independent problems through the Laplace transform of (1.7)

$$\hat{\rho}_n(\mathbf{x},\lambda) = \int_0^\infty e^{-\lambda t} \rho_n(t,\mathbf{x}) dt \ , \quad \lambda > 0 \tag{1.11}$$

which satisfies

$$-\nabla \cdot [(\sigma I + \Psi(n\mathbf{x}))\nabla \hat{\rho}_n(\mathbf{x},\lambda)] + \lambda \hat{\rho}_n(\mathbf{x},\lambda) = \rho_0(\mathbf{x}), \tag{1.12}$$

for $\mathbf{x} \in R^d$. Convergence of $\hat{\rho}_n$ in L^2, with probability one, to the Laplace transform of $\bar{\rho}$ for each $\lambda > 0$ implies (1.9). In this paper we will actually work with (1.12) over a bounded open set $\mathcal{O}$ in R^d with Dirichlet boundary conditions and $\lambda = 0$. All essential calculations are the same[1] for these two problems. Dropping the hat, the Dirichlet problem has the weak form

$$\int_{\mathcal{O}} (\sigma I + \Psi(n\mathbf{x}))\nabla \rho_n(\mathbf{x}) \cdot \nabla \phi(\mathbf{x}) d\mathbf{x} = \int_{\mathcal{O}} \rho_0(\mathbf{x})\phi(\mathbf{x}) d\mathbf{x} \tag{1.13}$$

for every test function ϕ in $C_0^\infty(\mathcal{O})$. One of the first steps in our proof is to define (1.13) when Ψ is not in L^∞ but in L^2 in the sense of (1.10). The case of uniformly bounded coefficients that are also uniformly elliptic is covered by the usual homogenization results [20], whether they are symmetric or not. The case with bounded coefficients in the discrete setting(namely, random walks in random environments) was obtained by Kozlov [14] using martingale central limit theorems. In the discrete setting, the boundedness assumption is reflected in the uniform ellipticity of the transition probabilities.

Why is the L^2 condition (1.10) necessary and sufficient for diffusive behavior? There are shear-flow examples in two dimensions for which condition (1.10) is clearly necessary and sufficient as can be seen from explicit computations. The examples are due to Matheron and De Marsily [16], who noted the significance of condition (1.10), and were studied extensively by Avellaneda and Majda [1]. This is all in the context of stationary stream matrices. In general, the stream matrix will have stationary increments (since the flow $\mathbf{u}$ is stationary) but will not be stationary. For nonstationary Ψ nondiffusive behavior is to be expected although there are no mathematical results to substantiate such behavior. Given the shear flow examples, and in the context of stationary stream matrices, it is therefore enough to show that (1.10) implies diffusive behavior. Previous attempts to extend the L^∞ homogenization results to unbounded coefficients required conditions like $\langle|\Psi|^p\rangle < \infty$ with $p = 2 + \epsilon$, $\epsilon > 0$ for $d = 2$ or $p = d$ for $d \geq 3$ which are not sharp, [2], [3], or certain additional regularity and growth conditions that are hard to verify [18]. The sharp result proven here relies essentially on the minimax variational principles used in [9] for the small σ (large Peclet number) analysis of the effective diffusivity. Similar variational principles were used to obtain bounds for complex dielectrics by Gibianski and Cherkaev [6] and by Milton [17]. A special form of the variational principles was also noted by Avellaneda and Majda [2] but it was not used.

Before reviewing the shear flow examples we note that along with the basic Theorem 1 we have a convergence theorem for the Dirichlet problem (1.13), as already mentioned, and the following.

Theorem 2 *Let $Q_{\mathbf{x}}^{(n)}$ be the probability measures on continuous paths starting at $\mathbf{x}$ for the process generated by the stochastic differential equation (1.2) with the scaling $\mathbf{x}(t) \to n\mathbf{x}(n^2 t)$. Under the hypotheses of Theorem 1 the measures $Q_{\mathbf{x}}^{(n)}$*

[1] They are more involved for the boundary value problem because of the singular boundary layers in the large n limit.

converge weakly to the Brownian motion measure with infinitesimal covariance matrix $2\sigma^{eff}$, *in measure relative to the law of the stationary flow field* $\mathbf{u}$ *and for each finite* $\mathbf{x} \in R^d$.

The convergence of the finite dimensional distributions follows immediately from Theorem 1. The tightness of the measures is proved in Sect. 7 (cf. Theorem 7.1).

Let us briefly review the shear flow examples [16] which show that the L^2 condition (1.10) is sharp.

In two dimensions let $\mathbf{x} = (x, y)$ and $\mathbf{u}(\mathbf{x}) = (u(y), 0)$. Then the convection diffusion equation (1.1) becomes

$$\frac{\partial \rho}{\partial t} + u(y)\frac{\partial \rho}{\partial x} = \frac{1}{2}\left(\frac{\partial^2 \rho}{\partial x^2} + \frac{\partial^2 \rho}{\partial y^2}\right), \tag{1.14}$$

where we have set $\sigma = 1/2$ for simplicity, and the stochastic differential equation (1.2) becomes

$$\begin{aligned} dx(t) &= -u(y(t))dt + dw_1(t) \\ dy(t) &= dw_2(t) \end{aligned} \tag{1.15}$$

where $w_1(t)$ and $w_2(t)$ are independent Brownian motions on R^1, independent also of the random horizontal, zero mean velocity $u(y)$. Assuming that $x(0) = 0$ and $y(0) = 0$ and letting

$$\langle u(y)u(0)\rangle = R(y) = \int_{-\infty}^{\infty} e^{iky}\hat{R}(k)dk \tag{1.16}$$

be the covariance R and power spectral density $\hat{R}$ of u we have

$$\begin{aligned} \langle E\{x^2(t)\}\rangle &= t + \int_0^t \int_0^t \int_{-\infty}^{\infty} E\{e^{ik(y(s_1)-y(s_2))}\}\hat{R}(k)dkds_1ds_2 \qquad (1.17) \\ &= t + \int_0^t \int_0^t \int_{-\infty}^{\infty} e^{-\frac{k^2}{2}|s_1-s_2|}\hat{R}(k)dkds_1ds_2 \\ E\{y^2(t)\} &= t \end{aligned}$$

Here $E\{\}$ denotes expectation with respect to the Brownian motions and we have assumed for simplicity that there exists a continuous power spectral density $\hat{R}$. From (1.17) we find easily that

$$\langle E\{x^2(t)\}\rangle = t + \int_{-\infty}^{\infty} 4\hat{R}(k)\left[\frac{t}{k^2} - \frac{2}{k^4}(1 - e^{-k^2 t/2})\right] dk \tag{1.18}$$

so that

$$\frac{1}{t}\langle E\{x^2(t)\}\rangle \to 1 + \int_{-\infty}^{\infty} \frac{4\hat{R}(k)}{k^2}dk \tag{1.19}$$

at t tends to infinity, provided the integral is finite. It is also easy to see that the integral in (1.19) is finite if and only if the process $\int^y u(s)ds$ is stationary and square integrable. This is the shear flow version of condition (1.10). If on the other hand the integral in (1.19) is not finite, when typically $\hat{R}(0) \neq 0$, then after a simple computation we have

$$\frac{1}{t^{3/2}}\langle E\{x^2(t)\}\rangle \to \frac{8\sqrt{2\pi}}{3}\hat{R}(0) \tag{1.20}$$

This means that we do not have diffusive behavior in the horizontal direction since the mean square displacement behaves like $t^{3/2}$ for t large. Note that $\hat{R}(0) \neq 0$ means that there will be no stationary stream function for the shear flow. The large scale (k small) fluctuations in the horizontal velocity are strong enough to produce superdiffusive behavior in the mean square horizontal particle displacement.

In several dimensions the square integrability condition can be made more explicit by using the spectral representation of the flow $\mathbf{u}$, which is stationary, divergence free and square integrable. There exists a process $\hat{\mathbf{u}}(\kappa)$ with orthogonal increments such that with probability one

$$\mathbf{u}(\mathbf{x}) = \int_{R^d} e^{i\kappa\cdot\mathbf{x}} d\hat{\mathbf{u}}(\kappa) \tag{1.21}$$

where $\overline{d\hat{\mathbf{u}}(\kappa)} = d\hat{\mathbf{u}}(-\kappa)$, since $\mathbf{u}$ is real, and

$$\begin{aligned} \langle d\hat{u}_p(\kappa)\overline{d\hat{u}_q(\kappa)}\rangle &= \hat{R}_{pq}(\kappa)d\kappa \\ \langle u_p(\mathbf{x}+\mathbf{y})u_q(\mathbf{y})\rangle &= R_{pq}(\mathbf{x}) = \int_{R^d} e^{i\kappa\cdot\mathbf{x}}\hat{R}_{pq}(\kappa)d\kappa \\ \kappa\cdot d\hat{\mathbf{u}}(\kappa) &= 0 \quad \text{(by incompressibility)} \\ R_{pq}(\mathbf{x}) &= R_{qp}(-\mathbf{x})\ , \quad \hat{R}_{pq}(\kappa) = \hat{R}_{qp}(-\kappa)\ , \quad p,q = 1,...,d. \end{aligned} \tag{1.22}$$

We assume here that the spectral measure of the covariance has a continuous density $\hat{R}_{pq}(\kappa)$ with respect to Lebesgue measure. The stream matrix Ψ satisfying (1.3) has the spectral representation

$$\Psi_{pq}(\mathbf{x}) = \int_{R^d} e^{i\kappa\cdot\mathbf{x}} \frac{1}{|\kappa|^2}\left[-i\kappa_q d\hat{u}_p(\kappa) + i\kappa_p d\hat{u}_q(\kappa)\right] \tag{1.23}$$

provided it is square integrable

$$\langle|\Psi(\mathbf{x})|^2\rangle = \sum_{pq}\langle\Psi_{pq}(\mathbf{x})\Psi_{pq}(\mathbf{x})\rangle = 2\sum_p \int_{R^d} \frac{\hat{R}_{pp}(\kappa)}{|\kappa|^2}d\kappa < \infty. \tag{1.24}$$

Since the flow $\mathbf{u}$ is square integrable we have

$$\langle|\mathbf{u}(\mathbf{x})|^2\rangle = \sum_p \int_{R^d} \hat{R}_{pp}(\kappa)d\kappa < \infty. \tag{1.25}$$

If, for example, the flow $\mathbf{u}$ has a spectral density that satisfies for some constant C

$$\sum_p \hat{R}_{pp}(\kappa) \leq \frac{C}{|\kappa|^\alpha} \tag{1.26}$$

in the neighborhood of the origin and $\alpha < d - 2$ then Theorem 1 tells us that we have diffusive behavior. In three dimensions having a bounded power spectral density at the origin will suffice but in two and one dimensions it will not. Note that in dimensions three or more Theorem 1 is natural and what one wants physically. Note also that the L^∞ condition on the stream matrix Ψ that the usual homogenization results demand is quite unnatural. In two dimensions the power spectral density must vanish at the origin if we are to have diffusive behavior and for shear flows, as we saw above, nondiffusive behavior is more typical. This is reminiscent of wave localization which occurs typically in low dimensions, if the random fluctuations are not large.

When the diffusion equation (1.1) is put into divergence form (1.6) the large time or homogenization asymptotic analysis is not sensitive to the dimension of the underlying space because we do not use Sobolev inequalities or other dimension-sensitive tools. In Theorems 1 and 2 dimension dependence enters only through the passage from the flow $\mathbf{u}$ to the stream matrix Ψ. The most natural way to relate these two quantities is the spectral representation (1.23) for which L^2 is the natural setting. That is another reason why it is important to have homogenization valid with just the L^2 condition (1.10). However, the main reason that we have looked at homogenization with unbounded coefficients so carefully is the minimax variational principles that we use and the mathematical technology around them. They are a powerful tool that may well be the key to unraveling multidimensional non-diffusive behavior (cf. [8]). They have already proven to be invaluable in the large Peclet number (small σ) analysis of the effective diffusivity for two dimensional periodic and random flows with bounded stream functions [9], [10]. It is important to note also that diffusion in random media is a big subject where many diverse issues arise. For example, we do not discuss here flows with nonzero mean or flows that are not incompressible. If the random fluctuations about the nonzero mean are small and in addition to stationarity we have some mixing then a few results are known [13], even with $\sigma = 0$. If the mean velocity is zero and the fluctuations are neither incompressible nor gradient fields then diffusive behavior has been proven for dimensions $d \geq 3$ and for small fluctuations (small Peclet number)[5]. The analytical methods for both of these cases differ substantially from those used in homogenization, and in this paper.

Since homogenization with unbounded coefficients is considered here for random incompressible flows it is natural to ask about problems with *symmetric* coefficient matrix in (1.7) that is unbounded. This is considered in detail in another paper. It illustrates nicely the use of variational principles, which in the symmetric case are well known.

2 Outline of the paper

The idea of the proof is simple but requires many steps that are technical but interesting and in turn shed light on the mathematical structure of the problem which is variational in nature. In this section we provide a guide to these issues, why and how they are addressed, and give an overall picture of the analysis. However, this outline is somewhat extensive too and perhaps hard to follow without seeing the calculations. We suggest that the reader browse over this section periodically while reading the paper.

When the stream matrix $\widetilde{\Psi}$ is bounded, both $\Gamma_0\Psi_n$ and $\widetilde{\Gamma}\widetilde{\Psi}$ in the integral form of the equations (3.10) and (3.39), respectively, are bounded operators in the Hilbert spaces of L^2 gradient fields. For unbounded $\widetilde{\Psi}$, they are unbounded operators. The first important ingredient of our approach is to remain in the Hilbert spaces of L^2 gradient fields and then to seek to define both $\Gamma_0\Psi_n$ and $\widetilde{\Gamma}\widetilde{\Psi}$ *densely*. This can be accomplished if $\widetilde{\Psi}$ is square integrable since their domains obviously contain all the *bounded* gradient fields which are dense. By means of ithe Friedrichs extension of certain quadratic forms given in terms of $\Gamma_0\Psi_n$ and $\widetilde{\Gamma}\widetilde{\Psi}$, we introduce the natural Hilbert spaces, $H_0(\Psi_n, \mathcal{O})$ (3.12) or $\mathcal{H}_g(\Psi)$ (3.41), for the solutions of the convection-diffusion problem and its cell problems. With the symmetrization procedure and the variational principles following from it, the existence and uniqueness questions become standard in the new spaces (Theorem 5.1, 5.2 and 5.4).

The symmetrization procedure is motivated by a simple observation: Since the convection-diffusion equation (1.12) (or (1.13)) has a symmetric part $(\nabla \cdot \sigma\nabla)$ and a skew symmetric part $(\nabla \cdot \Psi_n\nabla)$, where $\Psi_n(\mathbf{x}) = \Psi(n\mathbf{x})$, it is natural to separate the symmetric part ((4.5) for the cell problem, (4.30) for the Dirichlet problem) and the skew symmetric part ((4.6) for the cell problem, (4.31) for the Dirichlet problem) of the solution by adding and subtracting to it the solution of the adjoint problem (4.28)-(4.29 or (4.2) for the cell problem). This way, the equations can be written as a symmetric but non-definite system which are the Euler equations of a min-max variational principle (4.22), (4.40). Once the Euler equation corresponding to the min or the max is solved, the min-max principle is turned into a maximum (4.44), (4.50) or minimum principle (4.43), (4.47). This is done in Sect. 4, following a brief review in Sect. 3 of the analytical framework for stationary processes that was used in [20] and elsewhere.

In Sect. 5 we address the $n \to \infty$ or homogenization limit. We would like to represent the exact solutions approximately in $H_0^1(\mathcal{O})$ by functions of the form (6.10) and (6.11) suggested by the multiple scales expansions (cf. Theorem 6.1, 6.1). The idea of the proof is to first show the attainability, within arbitrary error, of the min-max principle by trial functions of a specific form ((6.12) for the minimum principle, (6.13) for the maximum principle). The gap between the upper bound and the lower bound provided by the minimum and maximum principles respectively is closed by the approximation lemma 6.3. This is basically the content of Theorem 6.2 in Sect. 6.1.

Because of the ellipticity ($\sigma = 1 > 0$), the approximation within arbitrarily small error of the exact solutions by (6.12)-(6.13) in $H_0^1(\mathcal{O})$ follows from the preceding convergence of functionals (Theorem 6.2). Since the limiting form of the approximations (6.12)-(6.13) is (6.10)-(6.11), again by the approximation lemma (6.3) Theorems 6.1 and Corollary 6.1 6.2 are natural consequences of Theorem 6.2 thanks to the variational structure.

Since Ψ is unbounded there are no Nash estimates (cf. [19]) available. We have to obtain the tightness of the probability measures from sharp L^∞ resolvent estimates. This is done in Sect. 7 by noting that the L^2 estimates of Corollary 6.1 can be strengthened to L^∞ in appropriate domains by averaging over the ensemble of fluid flows.

Let us also comment on some of the technical issues in this approach, which comprise much of Sect. 6 and 8.

To use the minimum and the maximum principles, which are nonlocal, we have to evaluate accurately the projection operator Γ_0 (3.11) acting on a fast oscillatory function. This amounts to solving in terms of approximate correctors in $H_0^1(\mathcal{O})$ the Poisson equations with large and rapidly oscillating source terms. This is the content of Lemmas 8.4 and 8.6 which are technical but straight forward energy estimates. An additional difficulty has to do with the boundary layers of the Dirichlet problem for large n and is handled by choosing the cut-off functions $\alpha_n(\mathbf{x})$ carefully. The resolvent estimates needed to show the tightness of the convection-diffusion process are obtained by further averaging the variational estimates over the ensemble of velocity fields (Theorem 2, 7.1).

It is natural to ask why we do not use (6.10)-(6.11) directly as trial functions? The answer is, as explained in Sect. 6.1, that they may not be admissible (that is, belong to $H_0(\Psi_n, \mathcal{O})$) unless Ψ is uniformly bounded. Therefore, it is essential to use the trial functions with bounded derivatives and since only the minimal L^2 assumption is imposed on Ψ, some additional strong sublinear growth estimates (Lemma 8.2 and 6.1) for the trial functions are necessary for the proofs of Lemmas 8.4 and 8.6. Even when (6.10)-(6.11) do belong to $H_0(\Psi_n, \mathcal{O})$, the arguments of the proofs of Lemma 8.4 and 8.6 would not work because of the lack of strong sublinear growth estimates for the exact correctors. This illustrates the natural complementarity between the kind of estimates needed for the trial functions to make the variational framework work and the kind of assumptions imposed on Ψ: if the latter is uniformly bounded, then the former can be square integrable; if the latter is only square integrable, then the former has to be uniformly bounded.

Going from bounded $\widetilde{\Psi}$ to unbounded but square integrable $\widetilde{\Psi}$ amounts to the transition from bounded operators $\Gamma_0\Psi_n$, $\widetilde{\Gamma}\widetilde{\Psi}$ to unbounded but densely defined operators. To deal with this transition effectively, one needs to work with a nice space of trial fields such as bounded gradient fields. This is tractable and accomplished in this paper using the variational methods.

3 Abstract framework

We begin with a brief review of the framework of stationary processes that is used in homogenization [20].

3.1 Random stationary stream matrix

Let $(\Omega, \mathcal{F}, P)$ be a probability space and let $\Psi(\mathbf{x}, \omega)$ be a *strictly stationary* random skew-symmetric matrix of $\mathbf{x} \in R^d$ such that each element Ψ_{ij} is a L^2 function

$$\langle |\Psi_{ij}(\mathbf{x}, \cdot)|^2 \rangle < \infty, \quad \forall i, j, \tag{3.1}$$

where $\langle \cdot \rangle$ denotes the average or integral with respect to the measure P. By strict stationarity we mean that the joint distribution of $\Psi_{ij}(\mathbf{x}_1, \omega), \Psi_{ij}(\mathbf{x}_2, \omega), \ldots, \Psi_{ij}(\mathbf{x}_n, \omega)$ for any points $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n \in R^d$ and that of $\Psi_{ij}(\mathbf{x}_1 + \ell, \omega), \Psi_{ij}(\mathbf{x}_2 + \ell, \omega), \ldots, \Psi_{ij}(\mathbf{x}_n + \ell, \omega)$ for any $\ell \in R^d$ is the same, so the averages in (3.1) are independent of $\mathbf{x}$. Without loss of generality (see Doob [7]), we may assume that there is a group of transformations $\tau_{\mathbf{x}}$, $\mathbf{x} \in R^d$ from Ω into Ω that is one to one and preserves the measure P. That is, $\tau_{\mathbf{x}} \tau_{\mathbf{y}} = \tau_{\mathbf{x}+\mathbf{y}}$ and $P(\tau_{\mathbf{x}} A) = P(A)$ for any $A \in \mathcal{F}$. We may also suppose that there is a square integrable (w.r.t. P) matrix function $\widetilde{\Psi}(\omega)$ such that

$$\Psi(\mathbf{x}, \omega) = \widetilde{\Psi}(\tau_{-\mathbf{x}} \omega), \qquad \mathbf{x} \in R^d, \qquad \omega \in \Omega .$$

We assume that the group of transformations $\tau_{\mathbf{x}}$ is ergodic with respect to the probability measure P.

The random stationary divergence free velocity $\mathbf{u}$ which we consider in this paper is given by

$$-\mathbf{u}(\mathbf{x}, \omega) = \nabla \cdot \Psi(\mathbf{x}, \omega). \tag{3.2}$$

In dimension two and three, the stream matrix Ψ has the familiar form such as (1.4) and (1.5) respectively.

3.2 Hilbert spaces of stationary functions

The group of transformations $\tau_{\mathbf{x}}$ acting on Ω induces a group of operators on the Hilbert space of real-valued functions $\mathcal{H} = L^2(\Omega, \mathcal{F}, P)$ with inner product

$$(\tilde{f}, \tilde{g}) \equiv \langle \tilde{f} \tilde{g} \rangle \equiv \int_\Omega P(d\omega) \tilde{f}(\omega) \tilde{g}(\omega) , \qquad \tilde{f}, \tilde{g} \in \mathcal{H}$$

Here $\langle \cdot \rangle$ stands for integration over Ω with respect to P, $\int_\Omega P(d\omega) \cdot$. The group of operators $T_{\mathbf{x}}$ on $\mathcal{H}$ is given by

$$(T_{\mathbf{x}} \tilde{f})(\omega) = \tilde{f}(\tau_{-\mathbf{x}} \omega) , \qquad \mathbf{x} \in R^d \qquad \omega \in \Omega.$$

Since $\tau_{\mathbf{x}}$ is measure preserving, the operators $T_{\mathbf{x}}$ form a unitary group. Therefore they have closed densely defined infinitesimal generators $\tilde{\nabla}_i$ in each direction $i = 1, 2, \ldots, d$ with domains $\mathscr{D}_i \subset \mathscr{H}$. Then,

$$\tilde{\nabla}_i = \frac{\partial}{\partial x_i} T_{\mathbf{x}}|_{\mathbf{x}=0} , \qquad i = 1, \ldots, d ,$$

with differentiation defined in the sense of convergence in $\mathscr{H}$ for elements of $\mathscr{D}_i$. The closed subset of $\mathscr{H}$

$$\mathscr{H}^1 = \bigcap_{i=1}^{d} \mathscr{D}_i$$

becomes a Hilbert space with the inner product

$$\begin{aligned}(\tilde{f}, \tilde{g})_1 &\equiv \langle \tilde{f} \tilde{g} \rangle + \langle \tilde{\nabla} \tilde{f} \cdot \tilde{\nabla} \tilde{g} \rangle \\ &\equiv \int_\Omega P(d\omega) \tilde{f}(\omega)\, \tilde{g}(\omega) + \sum_{i=1}^{a} \int_\Omega P(d\omega)\, \tilde{\nabla}_i \tilde{f}(\omega)\, \tilde{\nabla}_i \tilde{g}(\omega)\end{aligned}$$

The hypothesis that the action of the translation group $\tau_{\mathbf{x}}$ is ergodic on Ω takes the following form in $\mathscr{H}$: the only functions in $\mathscr{H}$ that are invariant under $T_{\mathbf{x}}$ are the constant functions.

Let $H_s(R^d; \mathscr{H})$ be the space of all stationary random processes $f(\mathbf{x}, \omega)$ on R^d, such that $\int_\Omega P(d\omega) f^2(\mathbf{x}, \omega) = \text{const.} < \infty$. Clearly $H_s(R^d; \mathscr{H})$ is in one-to-one correspondence with $\mathscr{H}$ since it is simply the space of all translates of $\mathscr{H}$, that is, $f(\mathbf{x}, \omega) \in H_s(R^d; \mathscr{H})$ iff $f(\mathbf{x}, \omega) = T_{\mathbf{x}} \tilde{f}(\omega), \tilde{f}(\omega) \in \mathscr{H}$. Similarly, we may identify $\mathscr{H}^1$ with the set of mean square differentiable, stationary processes $H_s^1(R^d; \mathscr{H})$. In particular, if $f \in H_s^1$, then its derivatives are also a stationary processes and

$$\nabla_i f(\mathbf{x}, \omega) = \frac{\partial f(\mathbf{x}, \omega)}{\partial x_i} = \tilde{\nabla}_i f(\mathbf{x}, \omega)$$

with equality holding $d\mathbf{x} \times P$ almost everywhere. Thus, we have $H_s^1(R^d; \mathscr{H}) = H_s(R^d; \mathscr{H}^1)$.

We define also the Hilbert spaces $\mathscr{H}_g$ and $\mathscr{H}_c$ which correspond to gradient fields and curl fields, respectively,

$$\mathscr{H}_g = \left\{ \tilde{F}_i(\omega) \in \mathscr{H} \ , \ i = 1, \ldots, d \mid \tilde{\nabla}_i \tilde{F}_j = \tilde{\nabla}_j \tilde{F}_i, \forall i, j \quad \text{weakly} \right\} \tag{3.3}$$

$$\mathscr{H}_c = \left\{ \tilde{G}_i(\omega) \in \mathscr{H} \ , \ i = 1, \ldots, d \mid \sum_i \tilde{\nabla}_i \tilde{G}_i = 0 \quad \text{weakly} \right\} \tag{3.4}$$

3.3 Weak formulation of the boundary value problem

Consider the inhomogeneous boundary value problem (1.13) with the fast oscillatory stream matrix $\Psi_n(\mathbf{x},\omega) = \Psi(n\mathbf{x},\omega)$:

$$\nabla \cdot (I + \Psi_n)\nabla \rho_n = s + \nabla \cdot \mathbf{S}, \quad \text{in} \quad \mathcal{O}, \tag{3.5}$$

$$\rho_n = 0, \quad \text{on} \quad \partial\mathcal{O}, \tag{3.6}$$

for inhomogeneous terms $s \in L^2(\mathcal{O}), \mathbf{S} \in (L^2(\mathcal{O}))^d$, where $\mathcal{O}$ is a bounded, smooth domain in R^d. This is a little more general than (1.13) and, as before, n is a large parameter that eventually we let tend to infinity.

If the stream matrix is bounded

$$\text{ess-sup}_{\omega\in\Omega}|\widetilde{\Psi}| < \infty, \tag{3.7}$$

then there exists a unique $\rho_n \in H_0^1(\mathcal{O})$ such that

$$\int_{\mathcal{O}} d\mathbf{x}\ (I + \Psi_n)\nabla\rho_n \cdot \nabla\phi + \int_{\mathcal{O}} d\mathbf{x}\ (s\phi - \mathbf{S}\cdot\nabla\phi) = 0 \tag{3.8}$$

for all $\phi \in H_0^1(\mathcal{O})$. The proof follows immediately from the the Lax-Milgram Lemma since the first integral in (3.8) defines a bounded coercive quadratic form on $H_0^1(\mathcal{O})$. Letting $\phi = \rho_n$ in (3.8), noting the skew symmetry of Ψ_n and using the Poincare inequality gives the energy bound

$$\int_{\mathcal{O}} d\mathbf{x}\ \nabla\rho_n \cdot \nabla\rho_n \le C(|s|^2_{L^2(\mathcal{O})} + |\mathbf{S}|^2_{L^2(\mathcal{O})}) \tag{3.9}$$

for some constant C.

For unbounded stream matrices, the matrix $I + \Psi_n$ defines an unbounded bilinear form, so the Lax-Milgram Lemma does not work on $H_0^1(\mathcal{O})$ right away. To motivate the introduction of the right spaces for this problem we first write (3.5) in the integral form

$$\nabla\rho_n + \Gamma_0\Psi_n\nabla\rho_n = \nabla(\Delta_0)^{-1}s + \Gamma_0\mathbf{S} \tag{3.10}$$

where the projection operator

$$\Gamma_0 = \nabla(\Delta_0)^{-1}\nabla\cdot \tag{3.11}$$

projects square integrable fields to the gradients of $H_0^1(\mathcal{O})$. The orthogonal projection Γ_0 is a bounded operator in the Hilbert space of the gradients of $H_0^1(\mathcal{O})$. Here $(\Delta_0)^{-1}$ is the inverse of the Laplacian with zero Dirichlet data on $\partial\mathcal{O}$.

The main purpose of the square integrability of $\widetilde{\Psi}$ is to define the operator $\Gamma_0\Psi_n$ *densely* in the space of the gradients of $H_0^1(\mathcal{O})$ for all $n > 0$ (see Lemma 3.1). Once this is done, the standard Hilbert space theory provides naturally the Hilbert spaces $H_0(\Psi_n,\mathcal{O})$ and $H_g(\Psi_n,\mathcal{O})$ which are defined as follows.

$$H_0(\Psi_n,\mathcal{O}) = \overline{\{g \in H_0^1(\mathcal{O}); \Gamma_0\Psi_n\nabla g \in (L^2(\mathcal{O}))^d\}} \tag{3.12}$$

completed with respect to the norm

$$||g||^2_{\Psi_n} = |\nabla g|^2_{L^2(\mathcal{O})} + |\Gamma_0 \Psi_n \nabla g|^2_{L^2(\mathcal{O})} \tag{3.13}$$

Clearly, $H_0(\Psi_n, \mathcal{O}) \subseteq H^1_0(\mathcal{O})$ and $H_0(\Psi_n, \mathcal{O}) = H^1_0(\mathcal{O})$, if $\widetilde{\Psi}$ is bounded.

We also introduce the Hilbert space of the gradients of $H_0(\Psi_n, \mathcal{O})$

$$H_g(\Psi_n, \mathcal{O}) = \{\nabla g | g \in H_0(\Psi_n, \mathcal{O})\} \tag{3.14}$$

endowed with symmetric quadratic form

$$\overline{\mathcal{Q}}_n(\nabla f, \nabla g) = \int_{\mathcal{O}} \nabla f \cdot \nabla g + \int_{\mathcal{O}} (\Gamma_0 \Psi_n)^* \Gamma_0 \Psi_n \nabla f \cdot \nabla g \tag{3.15}$$

for $\nabla f, \nabla g \in H_g(\Psi_n, \mathcal{O})$. Here $*$ denotes the adjoint.

The spaces $H_0(\Psi_n, \mathcal{O})$ and $H_g(\Psi_n, \mathcal{O})$ are obtained by the Friedrichs' extension of the symmetric form

$$\mathcal{Q}_n(\nabla f, \nabla g) = \int_{\mathcal{O}} \nabla f \cdot \nabla g + \int_{\mathcal{O}} -\Gamma_0 \Psi_n \Gamma_0 \Psi_n \nabla f \cdot \nabla g, \tag{3.16}$$

namely, $\overline{\mathcal{Q}}_n$.

Note that since $\Gamma_0 \Psi_n$ is a densely defined, skew symmetric operator in the Hilbert space of the gradients of $H^1_0(\mathcal{O})$, the operator $(\Gamma_0 \Psi_n)^* \Gamma_0 \Psi_n$ is a posititve definite, self-adjoint operator in the Hilbert space of the gradients of $H^1_0(\mathcal{O})$, by Von Neumann's theorem (see [12]). Moreover,

$$(\Gamma_0 \Psi_n)^* = \Gamma_0 \Psi^*_n, \quad (\Gamma_0 \Psi_n)^* \Gamma_0 \Psi_n = \Gamma_0 \Psi^*_n \Gamma_0 \Psi_n \tag{3.17}$$

in the space of the gradients of $H^1_0(\mathcal{O})$, for Γ_0 is bounded and symmetric. The adjoint matrix $\Psi^*_n = -\Psi_n$.

The definition of $H_0(\Psi_n, \mathcal{O})$ incorporates only partial information of $\Psi_n \nabla g$. For example, we have no knowledge about the square integrability of $\Psi_n \nabla g$ for $g \in H_0(\Psi_n, \mathcal{O})$.

The problem now is to seek $\rho_n \in H_0(\Psi_n, \mathcal{O})$, rather than $H^1_0(\mathcal{O})$, such that

$$\int_{\mathcal{O}} d\mathbf{x}\, (\nabla \rho_n \cdot \nabla \phi + \Gamma_0 \Psi_n \nabla \rho_n \cdot \nabla \phi) + \int_{\mathcal{O}} d\mathbf{x}\, (s\phi - \mathbf{S} \cdot \nabla \phi) = 0 \tag{3.18}$$

for all $\phi \in H^1_0(\mathcal{O})$. At this stage, the integrals in (3.18) at least make sense for $\phi \in H^1_0(\mathcal{O})$ and $\rho_n \in H_0(\Psi_n, \mathcal{O})$. But there is no energy estimate that puts ρ_n in $H_0(\Psi_n, \mathcal{O})$ since the term with Ψ_n drops out of the energy identity. We will address the questions of existence and uniqueness in Sect. 4 using the variational methods developed in [9].

Before ending this section, let us state and prove as a lemma that $H_0(\Psi_n, \mathcal{O})$ contains $C^\infty_0(\mathcal{O})$ if $\widetilde{\Psi}$ is square integrable.

Lemma 3.1 *If* $|\tilde{\Psi}|_{L^2(\Omega)} < \infty$, *then*

$$C^\infty_0(\mathcal{O}) \subset H_0(\Psi_n, \mathcal{O}), \ \forall n \tag{3.19}$$

for almost all $\omega \in \Omega$.

Proof. Since $|\tilde{\Psi}|^2$ is integrable with respect to $P(d\omega)$, we have

$$\frac{1}{|\mathcal{O}|}\int_{\mathcal{O}} d\mathbf{x}\ |\Psi_n|^2(\mathbf{x},\omega) = \frac{1}{n^d|\mathcal{O}|}\int_{n\mathcal{O}} dy|\Psi|^2(\mathbf{x},\omega) \overset{n\to\infty}{\longrightarrow} |\tilde{\Psi}|^2_{L^2(\Omega)} \tag{3.20}$$

for almost all ω, by ergodicity of P. That is, for almost all fixed $\omega \in \Omega$, given $\delta > 0$, there exists $n_0(\omega,\delta)$ such that, for $n > n_0(\omega,\delta)$

$$\int_{\mathcal{O}} d\mathbf{x}\ |\Psi_n|^2 < |\mathcal{O}||\tilde{\Psi}|^2_{L^2(\Omega)} + \delta \tag{3.21}$$

and this estimate for $|\Psi_n|^2_{L^2(\mathcal{O})}$ is uniform in n, for almost all fixed $\omega \in \Omega$. Clearly $\Psi_n\nabla\varphi \in (L^2(\mathcal{O}))^d$, for all $\varphi \in C_0^\infty(\mathcal{O})$, hence $\Gamma_0\Psi_n\nabla\varphi \in (L^2(\mathcal{O}))^d$. A simple L^2 estimate shows that

$$|\Gamma_0\Psi_n\nabla\varphi|_{L^2(\mathcal{O})} \le |\Psi_n\nabla\varphi|_{L^2(\mathcal{O})} \le \sup_{\mathbf{x}\in\mathcal{O}} |\nabla\varphi||\Psi_n|_{L^2(\mathcal{O})} \tag{3.22}$$

So $C_0^\infty(\mathcal{O}) \subset H_0(\Psi_n,\mathcal{O}) \subset H_0^1(\mathcal{O})$ and the Lemma is proved.

3.4 Abstract cell problem and the effective diffusivity

Assuming that $\Psi(\mathbf{x},\omega)$ is uniformly bounded and stochastically continuous, Papanicolaou and Varadhan [20] showed that

$$\langle\rho_n(\mathbf{x},\cdot)\rangle \to \bar{\rho}, \quad H_0^1(\mathcal{O}) \quad \text{weakly} \tag{3.23}$$

in the limit $n \to \infty$. Here $\bar{\rho}$ is the solution of a deterministic variational problem with constant coefficients σ^{eff}

$$\int_{\mathcal{O}} d\mathbf{x}\ \sigma^{eff}\nabla\bar{\rho}\cdot\nabla\phi + \int_{\mathcal{O}} d\mathbf{x}\ (s\phi - \mathbf{S}\cdot\nabla\phi) = 0 \tag{3.24}$$

for $\forall\phi \in H_0^1(\mathcal{O})$. The matrix $\sigma^{eff} = [\sigma^{eff}_{ij}]$ is called the effective diffusivity and is determined by solving the abstract cell problem: Find two stationary random fields $\mathbf{E}_i(\mathbf{x},\omega)$ and $\mathbf{D}_i(\mathbf{x},\omega) \in (H_s^1(R^d;\mathcal{H}))^d, i = 1,...,d,$ such that

$$\mathbf{D}_i(\mathbf{x},\omega) = (I + \Psi(\mathbf{x},\omega))(\mathbf{E}_i(\mathbf{x},\omega) + \mathbf{e}^i) \tag{3.25}$$

$$\nabla\times\mathbf{E}_i(\mathbf{x},\omega) = 0 \tag{3.26}$$

$$\nabla\cdot\mathbf{D}_i(\mathbf{x},\omega) = 0 \tag{3.27}$$

$$\langle\mathbf{E}_i(\mathbf{x},\cdot)\rangle = 0 \tag{3.28}$$

where $\{\mathbf{e}^i\}$ is a set of orthonormal vectors in R^d and

$$\sigma^{eff}_{ij} = \langle\mathbf{D}_i(\mathbf{x},\cdot)\cdot\mathbf{e}^j\rangle, \quad i,j = 1,...,d. \tag{3.29}$$

Equations (3.25)-(3.28) translate into the following problem: To find (non-stationary) functions $\chi_i(\mathbf{x},\omega)$ with stationary square integrable gradients such that

$$\nabla \cdot \Big((I + \Psi(\mathbf{x}, \omega))(\nabla \chi_i + \mathbf{e}^i) \Big) = 0 \tag{3.30}$$

$$\langle \nabla \chi_i \rangle = 0. \tag{3.31}$$

The effective diffusivity is then given by

$$\sigma_{ij}^{eff} = \delta_{ij} + \langle (I + \Psi) \nabla \chi_i \cdot \mathbf{e}^j \rangle, \quad i, j = 1, ..., d. \tag{3.32}$$

We are thus seeking solutions χ_i, called *correctors*, to the convection-diffusion equation (3.30) with the normalization condition (3.31). The condition (3.31) can be shown to imply a sublinear growth condition on χ_i(cf. [20]). The sublinear growth of χ_i is crucial in the homogenization theory for bounded stream matrices [20]. Various sublinear growth conditions will again play an important role in the homogenization theory for unbounded stream matrices in this paper(cf. Sect. 8.1, 8.2).

The connection between the cell problem of this form (3.30)-(3.31) and homogenization as stated in Theorem 1 comes about by the usual multiple scale arguments and is formally the same in the random as in the periodic case [4, 20]. On physical grounds, (3.30)-(3.31) can be understood as macroscopic concentration gradients $\mathbf{e}^i$ that induce through the flow microscopic concentration fluctuations χ_i which in turn lead to enhanced fluxes $(I + \Psi)\nabla \chi_i$ by Fourier's law. The average of the enhanced flux is the macroscopic diffusivity (3.32).

When $\Psi(\mathbf{x}, \omega)$ is strictly stationary as defined in Sect. 3.1, the abstract cell problem (3.25)-(3.28) becomes

$$\widetilde{\mathbf{D}}_i(\omega) = (I + \widetilde{\Psi}(\omega))(\widetilde{\mathbf{E}}_i(\omega) + \mathbf{e}^i) \tag{3.33}$$

$$\widetilde{\nabla} \times \widetilde{\mathbf{E}}_i(\omega) = 0 \tag{3.34}$$

$$\widetilde{\nabla} \cdot \widetilde{\mathbf{D}}_i(\omega) = 0 \tag{3.35}$$

$$\langle \widetilde{\mathbf{E}}_i(\cdot) \rangle = 0 \tag{3.36}$$

whose variational form is to find $\widetilde{\mathbf{E}}_i \in \mathscr{H}_g, i = 1, ..., d$, such that

$$\int_\Omega P(d\omega) \Big(I + \widetilde{\Psi}(\omega) \Big) \Big(\widetilde{\mathbf{E}}_i(\omega) + \mathbf{e}^i \Big) \cdot \widetilde{\mathbf{F}}(\omega) = 0 \tag{3.37}$$

for all $\widetilde{\mathbf{F}}(\omega) \in \mathscr{H}_g$, and

$$\sigma_{ij}^{eff} = \int_\Omega P(d\omega) \Big(I + \widetilde{\Psi}(\omega) \Big) \Big(\widetilde{\mathbf{E}}_i(\omega) + \mathbf{e}^i \Big) \cdot \mathbf{e}^j, \quad i, j = 1, ..., d. \tag{3.38}$$

By the Lax-Milgram lemma (3.37) has a unique solution for bounded $\widetilde{\Psi}$.

For unbounded stream matrices $\widetilde{\Psi}$, the abstract cell problem can be put into a form parallel to (3.18), namely, to find $\widetilde{\mathbf{E}}_i \in \mathscr{H}_g(\widetilde{\Psi})$, $i = 1, ..., d$, such that

$$\int_\Omega P(d\omega) \Big(I + \widetilde{\Gamma} \widetilde{\Psi}(\omega) \Big) \Big(\widetilde{\mathbf{E}}_i(\omega) + \mathbf{e}^i \Big) \cdot \widetilde{\mathbf{F}}(\omega) = 0 \tag{3.39}$$

for all $\widetilde{\mathbf{F}}(\omega) \in \mathscr{H}_g$. Here

$$\widetilde{\Gamma} = \widetilde{\nabla}\widetilde{\Delta}^{-1}\widetilde{\nabla}\cdot \tag{3.40}$$

with $\widetilde{\nabla} = (\widetilde{\nabla}_1, \ldots, \widetilde{\nabla}_d)$ and $\widetilde{\Delta} = \widetilde{\nabla} \cdot \widetilde{\nabla}$, is the orthogonal projection operator that takes vector fields in $\mathscr{H}^d$ to curl free fields in $\mathscr{H}$. The space $\mathscr{H}_g(\widetilde{\Psi})$ is similar to the space $H_0(\Psi_n, \mathscr{O})$ in (3.12) and is defined by

$$\mathscr{H}_g(\widetilde{\Psi}) = \overline{\{\widetilde{\mathbf{G}} \in \mathscr{H}_g; \widetilde{\Gamma}\widetilde{\Psi}\widetilde{\mathbf{G}} \in \left(\mathscr{H}\right)^d\}}. \tag{3.41}$$

completed with respect to the norm

$$||\widetilde{\mathbf{G}}||^2_{\widetilde{\Psi}} = |\widetilde{\mathbf{G}}|^2_{L^2} + |\widetilde{\Gamma}\widetilde{\Psi}\widetilde{\mathbf{G}}|^2_{L^2}. \tag{3.42}$$

This is done through the Friedrichs' extension of the symmetric quadratic form

$$\widetilde{\mathscr{Q}}(\widetilde{\mathbf{E}}, \widetilde{\mathbf{F}}) = \langle \widetilde{\mathbf{E}} \cdot \widetilde{\mathbf{F}} \rangle - \langle \widetilde{\Gamma}\widetilde{\Psi}\widetilde{\Gamma}\widetilde{\Psi}\widetilde{\mathbf{E}} \cdot \widetilde{\mathbf{F}} \rangle. \tag{3.43}$$

The operator

$$(\widetilde{\Gamma}\widetilde{\Psi})^* \widetilde{\Gamma}\widetilde{\Psi} \tag{3.44}$$

associated with the extended quadratic form is self-adjoint and positive definite in $\mathscr{H}_g$ by Von Neumann's theorem (see [12]). Since $\widetilde{\Gamma}$ is bounded and symmetric in $\mathscr{H}_g$, we also have

$$(\widetilde{\Gamma}\widetilde{\Psi})^* = \widetilde{\Gamma}\widetilde{\Psi}^*, \quad (\widetilde{\Gamma}\widetilde{\Psi})^*\widetilde{\Gamma}\widetilde{\Psi} = \widetilde{\Gamma}\widetilde{\Psi}^*\widetilde{\Gamma}\widetilde{\Psi}. \tag{3.45}$$

The adjoint matrix $\widetilde{\Psi}^* = -\widetilde{\Psi}$.

The existence and uniqueness of (3.39), as well as existence of the effective diffusivity, is addressed in Sect. 4.

4 Variational principles

The main step in the derivation of variational principles is the symmetrization procedure that transforms the original problem and its adjoint into a symmetric but nondefinite system which are the Euler equations of a min-max variational principle. For the derivations in this section we assume that the stream matrix Ψ is uniformly bounded so all the calculations make sense in the usual way. For unbounded but square integrable Ψ we take the symmetrized system as starting point of the analysis and establish existence and uniqueness in appropriate spaces, then work our way back to the original problems. This is done in Sect. 5.

4.1 Symmetrization and min-max variational principle

4.1.1 Symmetrization of the abstract cell problem. Following closely [9], we denote the intensity and flux fields of the abstract cell problem (3.33)-(3.36) with the superscript + and those of the adjoint problem with the superscript −. Thus

$$\widetilde{\mathbf{D}}^+_{\mathbf{e}^i} = (I + \widetilde{\Psi}(\omega))\,\widetilde{\mathbf{E}}^+_{\mathbf{e}^i} \tag{4.1}$$

$$\widetilde{\mathbf{D}}^-_{\mathbf{e}^i} = (I - \widetilde{\Psi}(\omega))\,\widetilde{\mathbf{E}}^-_{\mathbf{e}^i} \tag{4.2}$$

$$\widetilde{\nabla} \times \widetilde{\mathbf{E}}^+_{\mathbf{e}^i} = \widetilde{\nabla} \times \widetilde{\mathbf{E}}^-_{\mathbf{e}^i} = 0 \tag{4.3}$$

$$\widetilde{\nabla} \cdot \widetilde{\mathbf{D}}^+_{\mathbf{e}^i} = \widetilde{\nabla} \cdot \widetilde{\mathbf{D}}^-_{\mathbf{e}^i} = 0. \tag{4.4}$$

Define now the sum and difference fields

$$\widetilde{\mathbf{E}}_{ij} = \frac{1}{2}(\widetilde{\mathbf{E}}^+_{\mathbf{e}^i} + \widetilde{\mathbf{E}}^-_{\mathbf{e}^j}), \tag{4.5}$$

$$\widetilde{\mathbf{E}}'_{ij} = \frac{1}{2}(\widetilde{\mathbf{E}}^+_{\mathbf{e}^i} - \widetilde{\mathbf{E}}^-_{\mathbf{e}^j}), \tag{4.6}$$

$$\widetilde{\mathbf{D}}_{ij} = \frac{1}{2}(\widetilde{\mathbf{D}}^+_{\mathbf{e}^i} + \widetilde{\mathbf{D}}^-_{\mathbf{e}^j}), \tag{4.7}$$

$$\widetilde{\mathbf{D}}'_{ij} = \frac{1}{2}(\widetilde{\mathbf{D}}^+_{\mathbf{e}^i} - \widetilde{\mathbf{D}}^-_{\mathbf{e}^j}), \tag{4.8}$$

which are related to each other by

$$\widetilde{\mathbf{D}}_{ij} = \widetilde{\mathbf{E}}_{ij} + \widetilde{\Psi}\widetilde{\mathbf{E}}'_{ij} \tag{4.9}$$

$$\widetilde{\mathbf{D}}'_{ij} = \widetilde{\mathbf{E}}'_{ij} + \widetilde{\Psi}\widetilde{\mathbf{E}}_{ij} \tag{4.10}$$

$$\widetilde{\nabla} \times \widetilde{\mathbf{E}}_{ij} = \widetilde{\nabla} \times \widetilde{\mathbf{E}}'_{ij} = 0 \tag{4.11}$$

$$\widetilde{\nabla} \cdot \widetilde{\mathbf{D}}_{ij} = \widetilde{\nabla} \cdot \widetilde{\mathbf{D}}'_{ij} = 0 \tag{4.12}$$

The effective diffusivity is defined by

$$\sigma^{eff}_{+}(\mathbf{e}^i, \mathbf{e}^j) = \sigma^{eff}(\mathbf{e}^i, \mathbf{e}^j) = \int_\Omega P(d\omega)\,\widetilde{\mathbf{D}}^+_{\mathbf{e}^i} \cdot \mathbf{e}^j \tag{4.13}$$

and we define also

$$\sigma^{eff}_{-}(\mathbf{e}^j, \mathbf{e}^i) = \int_\Omega P(d\omega)\,\widetilde{\mathbf{D}}^-_{\mathbf{e}^j} \cdot \mathbf{e}^i \tag{4.14}$$

with the mean field conditions

$$\int_\Omega P(d\omega)\,\widetilde{\mathbf{E}}^+_{\mathbf{e}^i} = \mathbf{e}^i \;, \quad \int_\Omega P(d\omega)\,\widetilde{\mathbf{E}}^-_{\mathbf{e}^j} = \mathbf{e}^j \;. \tag{4.15}$$

It is easy to see that

$$\sigma^{eff}_{-}(\mathbf{e}^j, \mathbf{e}^i) = \sigma^{eff}_{+}(\mathbf{e}^i, \mathbf{e}^j) = \sigma^{eff}(\mathbf{e}^i, \mathbf{e}^j) \tag{4.16}$$

because

$$
\begin{aligned}
&\sigma_-^{eff}(\mathbf{e}^j, \mathbf{e}^i)\\
&= \int_\Omega P(d\omega)\, \left(I - \tilde{\Psi}\right) \widetilde{\mathbf{E}}_{\mathbf{e}^j}^- \cdot \mathbf{e}^i\\
&= \int_\Omega P(d\omega)\, \left(I - \tilde{\Psi}\right) \widetilde{\mathbf{E}}_{\mathbf{e}^j}^- \cdot \widetilde{\mathbf{E}}_{\mathbf{e}^i}^+\\
&= \int_\Omega P(d\omega) \widetilde{\mathbf{E}}_{\mathbf{e}^j}^- \cdot \left(I + \tilde{\Psi}\right) \widetilde{\mathbf{E}}_{\mathbf{e}^i}^+\\
&= \int_\Omega P(d\omega) \mathbf{e}^j \cdot \widetilde{\mathbf{D}}_{\mathbf{e}^i}^+\\
&= \sigma_+^{eff}(\mathbf{e}^i, \mathbf{e}^j).
\end{aligned}
\tag{4.17}
$$

The first equaltity in (4.17) is simply the definition of the effective diffusivity for $-\widetilde{\Psi}$. The second equality in (4.17) is due to the adjoint cell problem similar to (3.37) but with change of sign in Ψ.

In other words, σ_-^{eff} is the adjoint of σ_+^{eff}. Thus,

$$
\sigma^{eff}(\mathbf{e}^i, \mathbf{e}^j) = \frac{1}{2}\sigma_+^{eff}(\mathbf{e}^i, \mathbf{e}^j) + \frac{1}{2}\sigma_-^{eff}(\mathbf{e}^j, \mathbf{e}^i) \tag{4.18}
$$

which in turn equals

$$
\begin{aligned}
&\frac{1}{2}\int_\Omega P(d\omega)\widetilde{\mathbf{D}}_{\mathbf{e}^i}^+ \cdot \widetilde{\mathbf{E}}_{\mathbf{e}^j}^- + \frac{1}{2}\int_\Omega P(d\omega)\widetilde{\mathbf{D}}_{\mathbf{e}^j}^- \cdot \widetilde{\mathbf{E}}_{\mathbf{e}^i}^+\\
&= \frac{1}{4}\int_\Omega P(d\omega) \left(\widetilde{\mathbf{D}}_{\mathbf{e}^i}^+ + \widetilde{\mathbf{D}}_{\mathbf{e}^j}^-\right) \cdot \left(\widetilde{\mathbf{E}}_{\mathbf{e}^i}^+ + \widetilde{\mathbf{E}}_{\mathbf{e}^j}^-\right)\\
&\quad -\frac{1}{4}\int_\Omega P(d\omega) \left(\widetilde{\mathbf{D}}_{\mathbf{e}^i}^+ - \widetilde{\mathbf{D}}_{\mathbf{e}^j}^-\right) \cdot \left(\widetilde{\mathbf{E}}_{\mathbf{e}^i}^+ - \widetilde{\mathbf{E}}_{\mathbf{e}^j}^-\right)\\
&= \int_\Omega P(d\omega)\widetilde{\mathbf{D}}_{ij} \cdot \widetilde{\mathbf{E}}_{ij} - \int_\Omega P(d\omega)\widetilde{\mathbf{D}}'_{ij} \cdot \widetilde{\mathbf{E}}'_{ij}
\end{aligned}
\tag{4.19}
$$

and the mean field conditions (4.15) become

$$
\begin{aligned}
\int_\Omega P(d\omega)\widetilde{\mathbf{E}}'_{ij} &= \frac{\mathbf{e}^i - \mathbf{e}^j}{2},\\
\int_\Omega P(d\omega)\widetilde{\mathbf{E}}_{ij} &= \frac{\mathbf{e}^i + \mathbf{e}^j}{2}
\end{aligned}
\tag{4.20}
$$

In view of (4.9) and (4.10), (4.19) is equivalent to

$$
\sigma^{eff}(\mathbf{e}^i, \mathbf{e}^j) = \int_\Omega P(d\omega) \begin{pmatrix} -I & -\tilde{\Psi} \\ \tilde{\Psi} & I \end{pmatrix} \begin{pmatrix} \widetilde{\mathbf{E}}'_{ij} \\ \widetilde{\mathbf{E}}_{ij} \end{pmatrix} \cdot \begin{pmatrix} \widetilde{\mathbf{E}}'_{ij} \\ \widetilde{\mathbf{E}}_{ij} \end{pmatrix} \tag{4.21}
$$

which admits a min-max variational characterization

$$
\sigma^{eff}(\mathbf{e}^i, \mathbf{e}^j) = \inf_{\substack{\tilde{\nabla}\times\widetilde{\mathbf{F}}=0 \\ \langle\widetilde{\mathbf{F}}\rangle=(\mathbf{e}^i+\mathbf{e}^j)/2}} \; \sup_{\substack{\tilde{\nabla}\times\widetilde{\mathbf{F}}'=0 \\ \langle\widetilde{\mathbf{F}}'\rangle=(\mathbf{e}^i-\mathbf{e}^j)/2}} \int_\Omega P(d\omega) \begin{pmatrix} -I & -\tilde{\Psi} \\ \tilde{\Psi} & I \end{pmatrix} \begin{pmatrix} \widetilde{\mathbf{F}}' \\ \widetilde{\mathbf{F}} \end{pmatrix} \cdot \begin{pmatrix} \widetilde{\mathbf{F}}' \\ \widetilde{\mathbf{F}} \end{pmatrix} \tag{4.22}
$$

since (4.12) are the Euler equations of (4.21). This is the min-max variational principle for the symmetrized cell problem that we will be using to extend the theory to unbounded coefficients.

The effective diffusivity σ^{eff} is not symmetric in general. But if, for example, the probability distribution $P(d\omega)$ is invariant under the transformation $\widetilde{\Psi} \to -\widetilde{\Psi}$, then σ^{eff} can be shown to be symmetric from (4.16) since $\sigma_{+}^{eff}(\mathbf{e}^i,\mathbf{e}^j) = \sigma_{-}^{eff}(\mathbf{e}^j,\mathbf{e}^i) = \sigma_{+}^{eff}(\mathbf{e}^j,\mathbf{e}^i)$. The last equality is due to the invariance of P with respect to change in sign of $\widetilde{\Psi}$, [9].

Note that only the symmetric part of the effective tensor σ^{eff} appears in the final homogenized equation in Theorem 1. There is an identity for the symmetric part which will be useful later.

$$\frac{1}{2}\left\{\sigma^{eff}(\mathbf{e}^i,\mathbf{e}^j)+\sigma^{eff}(\mathbf{e}^j,\mathbf{e}^i)\right\} = \langle \widetilde{\mathbf{E}}_{ii}\cdot\widetilde{\mathbf{E}}_{jj} - 2\widetilde{\Psi}\widetilde{\mathbf{E}}_{ii}\widetilde{\mathbf{E}}'_{jj} - \widetilde{\mathbf{E}}'_{ii}\cdot\widetilde{\mathbf{E}}'_{jj}\rangle,\ \forall i,j. \tag{4.23}$$

Its derivation is straight-forward. Using the definitions (4.5)-(4.6) and multiplying out the expressions the right hand side is equal to

$$\begin{aligned}&\frac{1}{2}\langle \widetilde{\mathbf{E}}_{\mathbf{e}^i}^{-}\cdot\widetilde{\mathbf{E}}_{\mathbf{e}^j}^{+}+\widetilde{\mathbf{E}}_{\mathbf{e}^i}^{+}\cdot\widetilde{\mathbf{E}}_{\mathbf{e}^j}^{-}-\widetilde{\Psi}\left(\widetilde{\mathbf{E}}_{\mathbf{e}^i}^{-}\cdot\widetilde{\mathbf{E}}_{\mathbf{e}^j}^{+}-\widetilde{\mathbf{E}}_{\mathbf{e}^i}^{+}\cdot\widetilde{\mathbf{E}}_{\mathbf{e}^j}^{-}+\widetilde{\mathbf{E}}_{\mathbf{e}^i}^{+}\cdot\widetilde{\mathbf{E}}_{\mathbf{e}^j}^{+}-\widetilde{\mathbf{E}}_{\mathbf{e}^i}^{-}\cdot\widetilde{\mathbf{E}}_{\mathbf{e}^j}^{-}\right)\rangle\\&=\langle\left(\widetilde{\mathbf{E}}_{\mathbf{e}^i}^{+}+\widetilde{\Psi}\widetilde{\mathbf{E}}_{\mathbf{e}^i}^{+}\right)\cdot\widetilde{\mathbf{E}}_{\mathbf{e}^j}^{-}+\left(\widetilde{\mathbf{E}}_{\mathbf{e}^i}^{-}-\widetilde{\Psi}\widetilde{\mathbf{E}}_{\mathbf{e}^i}^{-}\right)\cdot\widetilde{\mathbf{E}}_{\mathbf{e}^j}^{+}-\widetilde{\Psi}\left(\widetilde{\mathbf{E}}_{\mathbf{e}^i}^{+}\cdot\widetilde{\mathbf{E}}_{\mathbf{e}^j}^{+}-\widetilde{\mathbf{E}}_{\mathbf{e}^i}^{-}\cdot\widetilde{\mathbf{E}}_{\mathbf{e}^i}^{-}\right)\rangle\end{aligned} \tag{4.24}$$

after cancelling terms like $\widetilde{\mathbf{E}}_{\mathbf{e}^i}^{+}\cdot\widetilde{\mathbf{E}}_{\mathbf{e}^j}^{+}$ and $\widetilde{\mathbf{E}}_{\mathbf{e}^i}^{-}\cdot\widetilde{\mathbf{E}}_{\mathbf{e}^i}^{-}$. This reduces further to

$$\frac{1}{2}\left\{\sigma^{eff}(\mathbf{e}^i,\mathbf{e}^j)+\sigma^{eff}(\mathbf{e}^j,\mathbf{e}^i)\right\} = \frac{1}{2}\langle\left(\widetilde{\mathbf{E}}_{\mathbf{e}^i}^{+}+\widetilde{\Psi}\widetilde{\mathbf{E}}_{\mathbf{e}^i}^{+}\right)\cdot\mathbf{e}^j\rangle+\frac{1}{2}\langle\left(\widetilde{\mathbf{E}}_{\mathbf{e}^i}^{-}-\widetilde{\Psi}\widetilde{\mathbf{E}}_{\mathbf{e}^i}^{-}\right)\cdot\mathbf{e}^j\rangle \tag{4.25}$$

because of (4.1)-(4.4) and the skew symmetry of $\widetilde{\Psi}$. Now observe that the first term of (4.25) is just $\sigma_{+}^{eff}(\mathbf{e}^i,\mathbf{e}^j)$ and the second term $\sigma_{-}^{eff}(\mathbf{e}^i,\mathbf{e}^j)$. The identity (4.23) then follows immediately from (4.16).

4.1.2 Symmetrization of the boundary value problem. Consider the inhomogeneous boundary value problem (3.5)-(3.6) and its adjoint, denoted with superscripts +, −, respectively:

$$\nabla\cdot(I+\Psi_n)\nabla\rho_n^{+} = s+\nabla\cdot\mathbf{S},\quad \text{in}\ \mathcal{O}, \tag{4.26}$$

$$\rho_n^{+} = 0,\quad \text{on}\ \partial\mathcal{O}, \tag{4.27}$$

$$\nabla\cdot(I-\Psi_n)\nabla\rho_n^{-} = s+\nabla\cdot\mathbf{S},\quad \text{in}\ \mathcal{O}, \tag{4.28}$$

$$\rho_n^{-} = 0,\quad \text{on}\ \partial\mathcal{O}. \tag{4.29}$$

in a bounded domain $\mathcal{O}$. Let ρ_n, ρ_n' be the sum and difference

$$\rho_n = \frac{1}{2}(\rho_n^{+}+\rho_n^{-}) \tag{4.30}$$

$$\rho_n' = \frac{1}{2}(\rho_n^{+}-\rho_n^{-}). \tag{4.31}$$

In terms of ρ_n, ρ_n' , we put (4.26)-(4.29) into symmetrized form by adding and subtracting (4.26) and (4.28)

$$\nabla \cdot \nabla \rho_n + \nabla \cdot \Psi_n \nabla \rho_n' = s + \nabla \cdot \mathbf{S}, \quad \text{in} \quad \mathcal{O}, \tag{4.32}$$

$$\nabla \cdot \nabla \rho_n' + \nabla \cdot \Psi_n \nabla \rho_n = 0, \quad \text{in} \quad \mathcal{O}, \tag{4.33}$$

$$\rho_n = \rho_n' = 0, \quad \text{on} \quad \partial\mathcal{O} \tag{4.34}$$

or, equivalently

$$(\nabla, \nabla) \cdot \begin{pmatrix} -I & -\Psi_n \\ \Psi_n & I \end{pmatrix} \begin{pmatrix} \nabla \rho_n' \\ \nabla \rho_n \end{pmatrix} = \begin{pmatrix} 0 \\ s + \nabla \cdot \mathbf{S} \end{pmatrix} \text{ in } \mathcal{O}, \tag{4.35}$$

$$\rho_n = \rho_n' = 0, \quad \text{on } \partial\mathcal{O}. \tag{4.36}$$

Equations (4.32)-(4.33) are formal and should be understood in the weak sense

$$\int_{\mathcal{O}} d\mathbf{x}\, \nabla \rho_n \cdot \nabla \phi + \int_{\mathcal{O}} d\mathbf{x}\, \Psi_n \nabla \rho_n' \cdot \nabla \phi = \int_{\mathcal{O}} d\mathbf{x}\, (\mathbf{S} \cdot \nabla \phi - s\phi) \tag{4.37}$$

$$\int_{\mathcal{O}} d\mathbf{x}\, \nabla \rho_n' \cdot \nabla \phi + \int_{\mathcal{O}} d\mathbf{x}\, \Psi_n \nabla \rho_n \cdot \nabla \phi = 0 \tag{4.38}$$

for all $\phi \in H_0^1(\mathcal{O})$ (recall that Ψ is assumed to be bounded in this section).

Clearly (4.35)-(4.36) are the Euler equations of the quadratic functional

$$\begin{aligned} &J_n(s + \nabla \cdot \mathbf{S}) \\ &= \frac{1}{|\mathcal{O}|} \left\{ \int_{\mathcal{O}} d\mathbf{x} \begin{pmatrix} -I & -\Psi_n \\ \Psi_n & I \end{pmatrix} \begin{pmatrix} \nabla \rho_n' \\ \nabla \rho_n \end{pmatrix} \cdot \begin{pmatrix} \nabla \rho_n' \\ \nabla \rho_n \end{pmatrix} \right. \\ &\left. + \int_{\mathcal{O}} d\mathbf{x}\, 2(s\rho_n - \mathbf{S} \cdot \nabla \rho_n) \right\}, \end{aligned} \tag{4.39}$$

that is, ρ_n and ρ_n' are a critical point of the min-max variational principle

$$\begin{aligned} &J_n(s + \nabla \cdot \mathbf{S}) \\ &= \inf_{\phi|_{\partial\mathcal{O}}=0} \sup_{\phi'|_{\partial\mathcal{O}}=0} \frac{1}{|\mathcal{O}|} \left\{ \int_{\mathcal{O}} d\mathbf{x} \begin{pmatrix} -I & -\Psi_n \\ \Psi_n & I \end{pmatrix} \begin{pmatrix} \nabla \phi' \\ \nabla \phi \end{pmatrix} \cdot \begin{pmatrix} \nabla \phi' \\ \nabla \phi \end{pmatrix} \right. \\ &\left. + \int_{\mathcal{O}} d\mathbf{x}\, 2(s\phi - \mathbf{S} \cdot \nabla \phi) \right\}. \end{aligned} \tag{4.40}$$

This is the variational characterization that we will use to extend the theory to unbounded coefficients.

4.2 Nonlocal (minimum and maximum) variational principles

We can get minimum (maximum) principles by eliminating the supremum (infimum) from the min-max variational principles (4.22) and (4.40). The resulting variational principles are nonlocal in nature because the solutions of the supremum (infimum) involve projection operators.

4.2.1 Abstract cell problem. The Euler equation for the supremum in (4.22) is

$$\widetilde{\nabla} \cdot \widetilde{\mathbf{F}}' + \widetilde{\nabla} \cdot \tilde{\Psi}\widetilde{\mathbf{F}} = 0 \,. \tag{4.41}$$

Using the projection operator

$$\tilde{\Gamma} = \widetilde{\nabla}\tilde{\Delta}^{-1}\widetilde{\nabla}\cdot \tag{4.42}$$

that projects square integrable vector fields to curl free ones in $\mathscr{H}_g$, writing the solution of (4.41) in the form $\widetilde{\mathbf{F}}' = -\tilde{\Gamma}\tilde{\Psi}\widetilde{\mathbf{F}} + (\mathbf{e}^i - \mathbf{e}^j)/2$ and substituting it in (4.22), we get

$$\begin{aligned}
&\sigma^{eff}(\mathbf{e}^i, \mathbf{e}^j) \\
&= \inf_{\substack{\tilde{\nabla}\times\tilde{\mathbf{F}}=0 \\ \langle\tilde{\mathbf{F}}\rangle=(\mathbf{e}^i+\mathbf{e}^j)/2}} \int_\Omega P(d\omega)\left(\widetilde{\mathbf{F}}\cdot\widetilde{\mathbf{F}} - 2\tilde{\Psi}\widetilde{\mathbf{F}}\cdot\widetilde{\mathbf{F}}' - \widetilde{\mathbf{F}}'\cdot\widetilde{\mathbf{F}}'\right) \\
&= \inf_{\substack{\tilde{\nabla}\times\tilde{\mathbf{F}}=0 \\ \langle\tilde{\mathbf{F}}\rangle=(\mathbf{e}^i+\mathbf{e}^j)/2}} \int_\Omega P(d\omega)(\widetilde{\mathbf{F}}\cdot\widetilde{\mathbf{F}} + \tilde{\Gamma}\tilde{\Psi}\widetilde{\mathbf{F}}\cdot\tilde{\Gamma}\tilde{\Psi}\widetilde{\mathbf{F}} - \tilde{\Psi}\widetilde{\mathbf{F}} \\
&\cdot(\mathbf{e}^i - \mathbf{e}^j) - |\frac{\mathbf{e}^i - \mathbf{e}^j}{2}|^2)
\end{aligned} \tag{4.43}$$

Note that (4.43) is nonlocal because of the projection operator $\tilde{\Gamma}$.

Similarly, we can eliminate the infimum in (4.22) and derive a nonlocal maximum principle

$$\begin{aligned}
&\sigma^{eff}(\mathbf{e}^i, \mathbf{e}^j) \\
&= \sup_{\substack{\tilde{\nabla}\times\tilde{\mathbf{F}}'=0 \\ \langle\tilde{\mathbf{F}}'\rangle=(\mathbf{e}^i-\mathbf{e}^j)/2}} \int_\Omega P(d\omega)(-\widetilde{\mathbf{F}}'\cdot\widetilde{\mathbf{F}}' - \tilde{\Gamma}\tilde{\Psi}\widetilde{\mathbf{F}}'\cdot\tilde{\Gamma}\tilde{\Psi}\widetilde{\mathbf{F}}' + \tilde{\Psi}\widetilde{\mathbf{F}}' \\
&\cdot(\mathbf{e}^i + \mathbf{e}^j) + |\frac{\mathbf{e}^i + \mathbf{e}^j}{2}|^2)
\end{aligned} \tag{4.44}$$

4.2.2 Boundary value problem. The Euler equation for the supremum in (4.40) is

$$\Delta\phi' + \nabla\cdot\Psi_n\nabla\phi = 0, \quad \text{in} \quad \mathscr{O}, \tag{4.45}$$

$$\phi' = 0, \quad \text{on} \quad \partial\mathscr{O} \tag{4.46}$$

Solving with the help of the projection operator Γ_0 defined by (3.11) we have $\nabla\phi' = -\Gamma_0\Psi_n\nabla\phi$ from (4.45), (4.46) and substituting this into the min-max principle (4.40), we obtain the minimum principle

$$J_n(s+\nabla\cdot\mathbf{S}) = \inf_{\phi|_{\partial\mathscr{O}}=0} \frac{1}{|\mathscr{O}|}\int_{\mathscr{O}} d\mathbf{x}\ (\nabla\phi\cdot\nabla\phi + \Gamma_0\Psi_n\nabla\phi\cdot\Gamma_0\Psi_n\nabla\phi + 2s\phi - 2\mathbf{S}\cdot\nabla\phi) \tag{4.47}$$

which is nonlocal. Similarly, we can eliminate the infimum in (4.40) by solving

$$\Delta\phi + \nabla\cdot\Psi_n\nabla\phi' = s + \nabla\cdot\mathbf{S}, \quad \text{in} \quad \mathscr{O}, \tag{4.48}$$

$$\phi = 0, \quad \text{on} \quad \partial\mathscr{O} \tag{4.49}$$

and obtain a nonlocal maximum principle. There is an extra difficulty in (4.48) due to the interaction of the oscillation in $\nabla \cdot \Psi_n \nabla \phi'$ and the macroscopic source term $s + \nabla \cdot \mathbf{S}$, which will be handled in Sects. 8.4 and 8.5. Therefore, it is a bit clumsy to express this maximum principle in terms of ϕ' as

$$\begin{aligned} &J_n(s + \nabla \cdot \mathbf{S}) \\ &= \sup_{\phi'|_{\partial\mathcal{O}}=0} \frac{1}{|\mathcal{O}|} \int_{\mathcal{O}} d\mathbf{x} \left(-\nabla\phi' \cdot \nabla\phi' - \Gamma_0\Psi_n\nabla\phi' \cdot \Gamma_0\Psi_n\nabla\phi'\right. \\ &\left.-2s\Delta^{-1}\nabla \cdot \Psi_n\nabla\phi' + 2\mathbf{S} \cdot \Gamma_0\Psi_n\nabla\phi' + s\Delta^{-1}s - \Gamma_0\mathbf{S} \cdot \Gamma_0\mathbf{S}\right). \quad (4.50) \end{aligned}$$

The most economic form is

$$\begin{aligned} J_n(s + \nabla \cdot \mathbf{S}) &= \sup_{\phi'|_{\partial\mathcal{O}}=0} \frac{1}{|\mathcal{O}|} \int_{\mathcal{O}} d\mathbf{x} \ \left(\nabla\phi \cdot \nabla\phi - 2\Psi_n\nabla\phi \cdot \nabla\phi' - \nabla\phi' \cdot \nabla\phi'\right. \\ &\left.+2s\phi - 2\mathbf{S} \cdot \nabla\phi\right) \quad (4.51) \end{aligned}$$

where the supremum is subject to (4.48)-(4.49).

One can explore the duality between the the gradient fields($\mathscr{H}_g$) and the curl fields ($\mathscr{H}_c$) and deduce the dual min-max principles both for the cell problem (cf. [9]) and the boundary value problem. They are not used in this paper however.

5 Existence, uniqueness and a priori estimates

5.1 Existence

For a bounded stream matrix $\widetilde{\Psi}$, the symmetrized Dirichlet problem leads to the system (4.37)-(4.38). In this section, we show that (4.37)-(4.38) are also naturally solvable once the Hilbert space $H_g(\Psi_n, \mathcal{O})$ (or $H_0(\Psi_n, \mathcal{O})$) is defined.

In terms of the norm $\|\cdot\|_\Psi$, the functional (4.47) is simply

$$J_n(s + \nabla \cdot \mathbf{S}) = \inf_{\phi \in H_0(\Psi_n, \mathcal{O})} \frac{1}{|\mathcal{O}|}\{\|\phi\|^2_{\Psi_n} + \int_{\mathcal{O}} (2s\phi - 2\mathbf{S} \cdot \nabla\phi)\}. \quad (5.1)$$

The Euler equation of (5.1) is

$$\int_{\mathcal{O}} d\mathbf{x} \ \nabla\rho_n \cdot \nabla\phi + \int_{\mathcal{O}} d\mathbf{x} \ (-\Gamma_0\Psi_n\Gamma_0\Psi_n\nabla\rho_n) \cdot \nabla\phi = \int_{\mathcal{O}} d\mathbf{x} \ (\mathbf{S} \cdot \nabla\phi - s\phi) \quad (5.2)$$

for $\forall\phi \in H_0(\Psi_n, \mathcal{O})$. Now the right hand side of (5.2) defines a bounded linear functional on $H_0(\Psi_n, \mathcal{O})$ for $\mathbf{S}, s \in L^2$ and the left hand side is the bilinear form associated with the norm $\|\cdot\|_{\Psi_n}$, so existence and uniqueness are guaranteed by the Riesz representation theorem.

We note that ρ'_n also belongs to $H_0(\Psi_n, \mathcal{O})$ since

$$\begin{aligned} \nabla\rho'_n &= -\Gamma_0\Psi_n\nabla\rho_n \in L^2(\mathcal{O}) \quad (5.3) \\ \Gamma_0\Psi_n\nabla\rho'_n &= -\nabla\rho_n + \nabla\Delta_0^{-1}(s + \nabla \cdot \mathbf{S}) \in L^2(\mathcal{O}). \quad (5.4) \end{aligned}$$

Thus we have shown

Theorem 5.1 *If $|\tilde{\Psi}|_{L^2(\Omega)} < \infty$, then there exist unique $\rho_n, \rho_n' \in H_0(\Psi_n, \mathcal{O})$ such that (4.37), (4.38) hold for all $\varphi \in H_0(\Psi_n, \mathcal{O})$, for almost all $\omega \in \Omega$.*

Next we show that the space of test functions ϕ in Theorem 3.1 can be enlarged from $H_0(\Psi_n, \mathcal{O})$ to $H_0^1(\mathcal{O})$:

Theorem 5.2 *If $|\tilde{\Psi}|_{L^2(\Omega)} < \infty$, then there exist unique $\rho_n, \rho_n' \in H_0(\Psi_n, \mathcal{O})$ such that (4.37), (4.38) hold for all $\varphi \in H_0^1(\mathcal{O})$, for almost all $\omega \in \Omega$.*

Proof. This theorem is an immediate consequence of Theorem 5.1, Lemma 5.1 and these facts:

(i) $C_0^\infty(\mathcal{O})$ is dense in $H_0^1(\mathcal{O})$

(ii)

$$\int_{\mathcal{O}} d\mathbf{x}\, \Psi_n \nabla \rho_n' \cdot \nabla \phi = \int_{\mathcal{O}} d\mathbf{x}\, \Psi_n \nabla \rho_n' \cdot \Gamma_0 \nabla \phi = \int_{\mathcal{O}} d\mathbf{x}\, \Gamma_0 \Psi_n \nabla \rho_n' \cdot \nabla \phi \tag{5.5}$$

and similarly

$$\int_{\mathcal{O}} d\mathbf{x}\, \Psi_n \nabla \rho_n \cdot \nabla \phi = \int_{\mathcal{O}} d\mathbf{x}\, \Gamma_0 \Psi_n \nabla \rho_n \cdot \nabla \phi \tag{5.6}$$

(iii) $\Gamma_0 \Psi_n \nabla \rho_n', \Gamma_0 \Psi_n \nabla \rho_n \in L^2(\mathcal{O})$. So, if (4.37), (4.38) hold for $\phi \in C_0^\infty(\mathcal{O})$, then they also hold for $\phi \in H_0^1(\mathcal{O})$.

The existence result of the original (before symmetrization) Dirichlet boundary value problem follows from Theorem 5.2:

Corollary 5.1 *Assume $\langle |\widetilde{\Psi}|^2 \rangle < \infty$. There exist unique $\rho_n^+, \rho_n^- \in H_0(\Psi_n, \mathcal{O})$ such that*

$$\int_{\mathcal{O}} d\mathbf{x}\, \nabla \rho_n^+ \cdot \nabla \phi + \int_{\mathcal{O}} d\mathbf{x}\, \Psi_n \nabla \rho_n^+ \cdot \nabla \phi = \int_{\mathcal{O}} d\mathbf{x}\, (\mathbf{S} \cdot \nabla \phi - s\phi) \tag{5.7}$$

$$\int_{\mathcal{O}} d\mathbf{x}\, \nabla \rho_n^- \cdot \nabla \phi - \int_{\mathcal{O}} d\mathbf{x}\, \Psi_n \nabla \rho_n^- \cdot \nabla \phi = \int_{\mathcal{O}} d\mathbf{x}\, (\mathbf{S} \cdot \nabla \phi - s\phi) \tag{5.8}$$

for all $\phi \in H_0^1(\mathcal{O})$, for almost all $\omega \in \Omega$.

Proof. By taking

$$\rho_n^+ = \rho_n + \rho_n' \tag{5.9}$$

$$\rho_n^- = \rho_n - \rho_n' \tag{5.10}$$

and adding and subtracting (4.37), (4.38) the Theorem follows.

5.2 *Uniform estimates*

Here we derive some $n-$uniform estimates for the solutions of (4.37)-(4.38) in Theorem 5.2. These uniform estimates come naturally as byproducts of the new Hilbert space $H_0(\Psi_n, \mathcal{O})$ formulation. We do not need them in the convergence proof and we present them here for completeness.

Theorem 5.3 *Assume $|\tilde{\Psi}|_{L^2(\Omega)} < \infty$. Let ρ_n, ρ_n' be the solution of the system (4.37), (4.38) and ρ_n^+, ρ_n^- the the solution of (5.7), (5.8), respectively. We have*

$$\|\rho_n\|_{\Psi_n} \leq C(|s|_{L^2(\mathcal{O})} + |\mathbf{S}|_{L^2(\mathcal{O})}) \tag{5.11}$$

$$\|\rho_n'\|_{\Psi_n} \leq C(|s|_{L^2(\mathcal{O})} + |\mathbf{S}|_{L^2(\mathcal{O})}) \tag{5.12}$$

$$\|\rho_n^+\|_{\Psi_n} \leq C(|s|_{L^2(\mathcal{O})} + |\mathbf{S}|_{L^2(\mathcal{O})}) \tag{5.13}$$

$$\|\rho_n^-\|_{\Psi_n} \leq C(|s|_{L^2(\mathcal{O})} + |\mathbf{S}|_{L^2(\mathcal{O})}) \tag{5.14}$$

for some constant C depending only on the domain $\mathcal{O}$.

Proof. For arbitrary $\delta > 0$, $g \in H_0^1(\mathcal{O})$

$$\left|\int_{\mathcal{O}} (s + \nabla \cdot \mathbf{S})g\right| \leq \delta|g|^2_{L^2(\mathcal{O})} + \tfrac{1}{\delta}|s|^2_{L^2(\mathcal{O})} + \delta|\nabla g|^2_{L^2(\mathcal{O})} + \tfrac{1}{\delta}|\mathbf{S}|^2_{L^2(\mathcal{O})} \tag{5.15}$$

$$\leq (c+1)\delta|\nabla g|^2_{L^2(\mathcal{O})} + \tfrac{1}{\delta}(|s|^2_{L^2(\mathcal{O})} + |\mathbf{S}|^2_{L^2(\mathcal{O})}) \tag{5.16}$$

where c is the constant associated with the Poincare inequality, depending only on the domain $\mathcal{O}$. Thus,

$$\|\rho_n\|^2_{\Psi_n} = \int_{\mathcal{O}}(\nabla\rho_n \cdot \nabla\rho_n + \Gamma_0\Psi_n\nabla\rho_n \cdot \Gamma_0\Psi_n\nabla\rho_n) \tag{5.17}$$

$$\leq |\mathcal{O}|J_n(s + \nabla \cdot \mathbf{S}) + (c+1)\delta|\nabla\rho_n|^2_{L^2(\mathcal{O})} + \tfrac{1}{\delta}(|s|^2_{L^2(\mathcal{O})} + |\mathbf{S}|^2_{L^2(\mathcal{O})}) \tag{5.18}$$

Consequently,

$$\begin{aligned}(1-(c+1)\delta)&|\nabla\rho_n|^2_{L^2(\mathcal{O})} + |\Gamma_0\Psi_n\nabla\rho_n|^2_{L^2(\mathcal{O})}\\ &\leq |\mathcal{O}|J_n(s+\nabla\cdot\mathbf{S}) + \frac{1}{\delta}((|s|^2_{L^2(\mathcal{O})} + |\mathbf{S}|^2_{L^2(\mathcal{O})})\\ &\leq \frac{1}{\delta}((|s|^2_{L^2(\mathcal{O})} + |\mathbf{S}|^2_{L^2(\mathcal{O})})\end{aligned} \tag{5.19}$$

since

$$J_n(s + \nabla \cdot \mathbf{S}) \leq 0 \tag{5.20}$$

by taking the trial function $g \equiv 0$ in (5.1). Let $\delta = \frac{1}{2(c+1)}$, so that $1-(c+1)\delta = \frac{1}{2}$. We obtain

$$\|\rho_n\|^2_{\Psi_n} \leq 4(c+1)(|s|^2_{L^2(\mathcal{O})} + |\mathbf{S}|^2_{L^2(\mathcal{O})}). \tag{5.21}$$

From the identities

$$\nabla\rho_n' = -\Gamma_0\Psi_n\nabla\rho_n, \quad \Gamma_0\Psi_n\nabla\rho_n' = -\nabla\rho_n + \nabla\Delta_0^{-1}(s + \nabla\cdot\mathbf{S}) \tag{5.22}$$

we also have

$$\|\rho_n'\|^2_{\Psi_n} \leq (8(c+1)+2)((|s|^2_{L^2(\mathcal{O})} + |\mathbf{S}|^2_{L^2(\mathcal{O})}), \tag{5.23}$$

after applying the energy estimate to $\nabla\Delta_0^{-1}(s + \nabla\cdot\mathbf{S})$. Since $\rho_n^+ = \rho_n + \rho_n', \rho_n^- = \rho_n - \rho_n'$, it follows from (5.21), (5.23) that

$$\|\rho_n^+\|^2_{\Psi_n} \leq C((|s|^2_{L^2(\mathcal{O})} + |\mathbf{S}|^2_{L^2(\mathcal{O})}) \tag{5.24}$$

$$\|\rho_n^-\|^2_{\Psi_n} \leq C((|s|^2_{L^2(\mathcal{O})} + |\mathbf{S}|^2_{L^2(\mathcal{O})}), \tag{5.25}$$

for some constant C depending only on the domain. This completes the proof.

5.3 Cell problem and correctors

The effective diffusivity is defined by the cell problem. In this section we study the existence of the intensity and flux fields and give bounds for their norm. The method is completely analogous to that for the Dirichlet problems in Sect. 5.1, with some minor changes, such as replacing the projection operator Γ_0 (3.11) by $\widetilde{\Gamma}$ (4.42). We work in the variational framework on the the space of $\mathscr{H}_g(\widetilde{\Psi})$ defined in (3.41) and this makes the questions of existence and uniqueness standard.

We state the existence theorem and provide a brief explanation with details omitted.

Theorem 5.4 *Assume* $\langle\tilde{\Psi}^2\rangle < \infty$. *There exists unique* $\tilde{\mathbf{E}}_{ij} - \langle\tilde{\mathbf{E}}_{ij}\rangle, \tilde{\mathbf{E}}'_{ij} - \langle\tilde{\mathbf{E}}'_{ij}\rangle \in \mathscr{H}_g(\tilde{\Psi})$ *such that*

$$\int_\Omega P(d\omega)\tilde{\mathbf{E}}_{ij}\cdot\tilde{\mathbf{F}} + \int_\Omega P(d\omega)\tilde{\Psi}\tilde{\mathbf{E}}'_{ij}\cdot\tilde{\mathbf{F}} = 0 \tag{5.26}$$

$$\int_\Omega P(d\omega)\tilde{\mathbf{E}}'_{ij}\cdot\tilde{\mathbf{F}} + \int_\Omega P(d\omega)\tilde{\Psi}\tilde{\mathbf{E}}_{ij}\cdot\tilde{\mathbf{F}} = 0 \tag{5.27}$$

$$\langle\tilde{\mathbf{E}}_{ij}\rangle = \frac{\mathbf{e}^i+\mathbf{e}^j}{2} \tag{5.28}$$

$$\langle\tilde{\mathbf{E}}'_{ij}\rangle = \frac{\mathbf{e}^i-\mathbf{e}^j}{2} \tag{5.29}$$

for all $\tilde{\mathbf{F}} \in \mathscr{H}_g(\Omega), i, j = 1, \ldots, d$.

Proof. The mean field conditions (5.28), (5.29) play the role of the inhomogeneous terms $s + \nabla\cdot\mathbf{S}$ in (5.26), (5.27), in the form

$$-\nabla\cdot\tilde{\Psi}\left(\frac{\mathbf{e}^i+\mathbf{e}^j}{2}\right) \quad \text{and} \quad -\nabla\cdot\tilde{\Psi}\left(\frac{\mathbf{e}^i-\mathbf{e}^j}{2}\right), \tag{5.30}$$

respectively. The L^2 integrability of $\widetilde{\Psi}$ then implies existence and uniqueness as in Theorem 5.2.

The system (5.26)-(5.29) are Euler equations for the min-max principle

$$\sigma^{\mathit{eff}}_{ij} = \sigma^{\mathit{eff}}(\mathbf{e}^i,\mathbf{e}^j) = \inf_{\substack{\nabla\times\tilde{\mathbf{F}}=0\\ \langle\tilde{\mathbf{F}}\rangle=\frac{\mathbf{e}^i+\mathbf{e}^j}{2}}}\ \sup_{\substack{\nabla\times\tilde{\mathbf{F}}'=0\\ \langle\tilde{\mathbf{F}}'\rangle=\frac{\mathbf{e}^i-\mathbf{e}^j}{2}}} \left\langle \begin{pmatrix} -I & -\tilde{\Psi} \\ \tilde{\Psi} & I \end{pmatrix}\begin{pmatrix}\mathbf{F}'\\ \mathbf{F}\end{pmatrix}\cdot\begin{pmatrix}\mathbf{F}'\\ \mathbf{F}\end{pmatrix}\right\rangle,$$
$$i, j = 1\cdots d \tag{5.31}$$

which defines the effective diffusivity $\sigma^{\mathit{eff}} = (\sigma^{\mathit{eff}}_{ij})$. Note that $0 < \sigma^{\mathit{eff}}_{ij} < \infty, i, j = 1, \ldots, d$ because of the integrability condition, $|\tilde{\Psi}|_{L^2(\Omega)} < \infty$ as can be seen by taking as trial fields $\tilde{\mathbf{F}} = \frac{\mathbf{e}^i+\mathbf{e}^j}{2}, \tilde{\mathbf{F}}' = \frac{\mathbf{e}^i-\mathbf{e}^j}{2}$.

The field $\tilde{\mathbf{E}}_{ij}$ can also be characterized as the minimizer of the minimum principle (4.43). The direct and adjoint intensity fields come from $\tilde{\mathbf{E}}^{\pm}_{ij} = \tilde{\mathbf{E}}_{ij} \pm \tilde{\mathbf{E}}'_{ij}, i, j = 1\cdots d$, and so we have existence and uniqueness for them.

Theorem 5.5 *The intensity fields* $\tilde{\mathbf{E}}_{ij}^{\pm} \in \mathscr{H}_g(\Psi), i,j = 1 \cdots d$ *solve uniquely the cell problems*

$$\int_\Omega P(d\omega)(I + \tilde{\Psi})\tilde{\mathbf{E}}_{ij}^{+} \cdot \tilde{\mathbf{F}} = 0 \tag{5.32}$$

$$\int_\Omega P(d\omega)(I - \tilde{\Psi})\tilde{\mathbf{E}}_{ij}^{-} \cdot \tilde{\mathbf{F}} = 0 \tag{5.33}$$

for all $\tilde{\mathbf{F}} \in \mathscr{H}_g, i,j = 1 \cdots d$.

We also have the a priori bounds

Theorem 5.6 *There is a constant* C *such that for* $i,j = 1, \ldots, d$

$$\|\tilde{\mathbf{E}}_{ij}\|_{\tilde{\Psi}} \le C|\tilde{\Psi}|_{L^2(\Omega)} \tag{5.34}$$

$$\|\tilde{\mathbf{E}}'_{ij}\|_{\tilde{\Psi}} \le C|\tilde{\Psi}|_{L^2(\Omega)} \tag{5.35}$$

$$\|\tilde{\mathbf{E}}^{+}_{ij}\|_{\tilde{\Psi}} \le C|\tilde{\Psi}|_{L^2(\Omega)} \tag{5.36}$$

$$\|\tilde{\mathbf{E}}^{-}_{ij}\|_{\tilde{\Psi}} \le C|\tilde{\Psi}|_{L^2(\Omega)} \tag{5.37}$$

In the theory of homogenization [4], [20], a prominent role is played by the *correctors* χ_j^+, χ_j^- which are defined, up to constant, by

$$\nabla\chi_j^+(\mathbf{x},\omega) = \mathbf{E}_{e^j}^+(\mathbf{x},\omega), \quad \nabla\chi_j^-(\mathbf{x},\omega) = \mathbf{E}_{e^j}^-(\mathbf{x},\omega) \tag{5.38}$$

where $\mathbf{E}_{e^j}^+(\mathbf{x},\omega) = \tilde{\mathbf{E}}_{e^j}^+(\tau_{-\mathbf{x}}\omega), \mathbf{E}_{e^j}^-(\mathbf{x},\omega) = \tilde{\mathbf{E}}_{e^j}^-(\tau_{-\mathbf{x}}\omega)$. Let us fix the constant by setting

$$\chi_j^+(0,\omega) = 0, \quad \chi_j^-(0,\omega) = 0.$$

The symmetrized correctors

$$\chi_j = \frac{1}{2}(\chi_j^+ + \chi_j^-), \quad \chi_j' = \frac{1}{2}(\chi_j^+ - \chi_j^-) \tag{5.39}$$

satisfy

$$\nabla\chi_j(\mathbf{x},\omega) = \mathbf{E}_{jj}(\mathbf{x},\omega), \quad \nabla\chi_j'(\mathbf{x},\omega) = \mathbf{E}'_{jj}(\mathbf{x},\omega) \tag{5.40}$$

where $\mathbf{E}_{jj}(\mathbf{x},\omega) = \tilde{\mathbf{E}}_{jj}(\tau_{-\mathbf{x}}\omega), \mathbf{E}'_{jj}(\mathbf{x},\omega) = \tilde{\mathbf{E}}'_{jj}(\tau_{-\mathbf{x}}\omega)$.

The correctors are square integrable but not stationary in general. However, they satisfy certain sublinear growth condition for large $|\mathbf{x}|$ which play an essential role in the convergence proof and are analyzed in detail in Sects. 8.1 and 8.2.

6 Convergence

We shall establish in this section the main result of this paper which is the strong convergence theorem of homogenization in the case of L^2 skew symmetric coefficients.

Theorem 6.1 *Assume that the stream matrix is square integrable* $\langle|\widetilde{\Psi}|^2\rangle < \infty$ *and let* $\chi_n^{+j} = \frac{1}{n}\chi_j^+(n\mathbf{x}, \omega)$, $\chi_n^{-j} = \frac{1}{n}\chi_j^-(n\mathbf{x}, \omega)$ *and similarly* $\chi_n^j, \chi_n'^j$ *be the scaled correctors with the unscaled ones defined by (5.38) and (5.39). Then*

$$\int_{\mathscr{O}} d\mathbf{x} \left(\nabla\left(\rho_n - \bar{\rho} - \sum_j \chi_n^j(\mathbf{x}, \omega)\frac{\partial\bar{\rho}(\mathbf{x})}{\partial x_j}\right)\right)^2 \to 0 \tag{6.1}$$

$$\int_{\mathscr{O}} d\mathbf{x} \left(\nabla\left(\rho_n' - \sum_j \chi_n'^j(\mathbf{x}, \omega)\frac{\partial\bar{\rho}(\mathbf{x})}{\partial x_j}\right)\right)^2 \to 0 \tag{6.2}$$

as $n \to \infty$, *for almost all* ω, *where* $\bar{\rho}$ *satisfies the homogenized problem*

$$\nabla \cdot \left(\tfrac{1}{2}\left(\sigma^{\mathit{eff}} + \sigma^{\mathit{eff}\dagger}\right)\nabla\bar{\rho}\right) = s + \nabla \cdot \mathbf{S}, \quad \textit{in} \quad \mathscr{O} \tag{6.3}$$

$$\bar{\rho} = 0, \quad \textit{on} \quad \partial\mathscr{O} \tag{6.4}$$

We also have the following corollary of Theorem 6.1 :

Corollary 6.1 *Assume that* $\langle|\widetilde{\Psi}|^2\rangle < \infty$. *Then*

$$\int_{\mathscr{O}} d\mathbf{x} \left(\nabla\left(\rho_n^+ - \bar{\rho} - \sum_j \chi_n^{+j}(\mathbf{x}, \omega)\frac{\partial\bar{\rho}(\mathbf{x})}{\partial x_j}\right)\right)^2 \to 0 \tag{6.5}$$

$$\int_{\mathscr{O}} d\mathbf{x} \left(\nabla\left(\rho_n^- - \bar{\rho} - \sum_j \chi_n^{-j}(\mathbf{x}, \omega)\frac{\partial\bar{\rho}(\mathbf{x})}{\partial x_j}\right)\right)^2 \to 0 \tag{6.6}$$

as $n \to \infty$, *for almost all* ω, *where* $\bar{\rho}$ *is again the solution of (6.3),(6.4).*

By Lemma 8.2 (Sect. 8.1), we have that

$$\int_{\mathscr{O}} d\mathbf{x}\, (\chi_n^{+j})^2 \to 0, \int_{\mathscr{O}} d\mathbf{x}\, (\chi_n^{-j})^2 \to 0, \tag{6.7}$$

$j = 1, ..., d$, with probability one. Thus we have the following corollary

Corollary 6.2 *Assume that* $\langle|\widetilde{\Psi}|^2\rangle < \infty$. *Then*

$$\int_{\mathscr{O}} d\mathbf{x}\, \left(\rho_n^+ - \bar{\rho}\right)^2 \to 0 \tag{6.8}$$

$$\int_{\mathscr{O}} d\mathbf{x}\, \left(\rho_n^- - \bar{\rho}\right)^2 \to 0 \tag{6.9}$$

as $n \to \infty$, *for almost all* ω.

Theorems 6.1, Corollary 6.1, 6.2 are also valid when (6.1), (6.2), (6.5), (6.6), (6.8) and (6.9) are averaged over ω.

For the proof we use the minimum and maximum principles to obtain upper and lower bounds, respectively, for the functionals with suitably constructed trial functions (Lemma 6.2). The gap between the upper and lower bounds is closed by proving the approximation lemma 6.3 using again the variational arguments (Theorem 6.2). The strong convergence results then follow from the convergence of functionals thanks to the variational structure (Sect. 6.1.3).

To obtain minimum or maximum principles, the partial Euler equations (4.45,4.46) and (4.48,4.49) have to be solved for selected trial functions asymptotically as $n \to \infty$. This amounts to solving the Poisson equation with rapidly oscillatory right hand side. This is the most technical part of the paper, partly because of the singular behavior near the boundary of the domain which requires careful cut-off arguments. It is presented in Sects. 8.4 and 8.5.

6.1 Convergence of functionals

As in the usual homogenization [4],[20] with the multiple scale expansion, we would like to show that the solutions of the inhomogeneous Dirichlet problems have the form

$$\rho_n^+(\mathbf{x}) \asymp \bar{\rho} + \sum_j \frac{1}{n}\chi^{+j}(n\mathbf{x})\frac{\partial\bar{\rho}}{\partial x_j}\alpha_n(\mathbf{x}) \tag{6.10}$$

$$\rho_n^-(\mathbf{x}) \asymp \bar{\rho} + \sum_j \frac{1}{n}\chi^{-j}(n\mathbf{x})\frac{\partial\bar{\rho}}{\partial x_j}\alpha_n(\mathbf{x}) \tag{6.11}$$

in the $H_0^1(\mathcal{O})$ sense or, in the symmetrized form,

$$\rho_n(\mathbf{x}) \asymp \bar{\rho} + \sum_j \frac{1}{n}\chi^{j}(n\mathbf{x})\frac{\partial\bar{\rho}}{\partial x_j}\alpha_n(\mathbf{x}) \tag{6.12}$$

$$\rho_n'(\mathbf{x}) \asymp \sum_j \frac{1}{n}\chi'^{j}(n\mathbf{x})\frac{\partial\bar{\rho}}{\partial x_j}\alpha_n(\mathbf{x}). \tag{6.13}$$

Here $\chi^{+j}, \chi^{-j}, \chi^j, \chi'^j$ are the correctors defined by (5.38, 5.39), $\bar{\rho}$ is the exact solution of the homogenized problem (6.3,6.4) and $\alpha_n(\mathbf{x})$ is a suitable cut-off function that makes (6.10)-(6.13) satisfy the Dirichlet boundary conditions. The precise way of doing the cut-off is technically important and one of the essential elements of Sect. 8.4 and 8.5. Here and below $\asymp$ denotes the asymptotic equality as $n \to \infty$.

The difficulty with (6.12)-(6.13) is that we do not know if the expansions are admissible. Is the right hand side of (6.12), (6.13) in $H_0(\Psi_n, \mathcal{O})$? The square integrability of

$$\Gamma_0\left\{\Psi_n \sum_j \nabla\chi^j(n\mathbf{x})\frac{\partial\bar{\rho}}{\partial x_j}\alpha_n(\mathbf{x})\right\} + \Gamma_0\left\{\Psi_n \sum_j \frac{1}{n}\chi^j(n\mathbf{x})\nabla\left[\frac{\partial\bar{\rho}}{\partial x_j}\alpha_n(\mathbf{x})\right]\right\} \tag{6.14}$$

is questionable because we do not know that if $\Psi_n \sum_j \nabla\chi^j$ or $\Psi_n\chi^j$ is square integrable. The estimates we obtained in Sect. 5 are not enough to ensure that we stay in the right spaces.

One of the advantages of the variational framework is that we do not have to work with the exact solutions for which we have insufficient knowledge because we can always resort to nice trial functions which approximate the exact solutions.

To make the right hand side of (6.12)-(6.13) admissible for the maximum and minimum principles, let

$$f_n(\mathbf{x}) = \rho + \sum_j \frac{1}{n} f^j(n\mathbf{x}) \frac{\partial \rho}{\partial x_j} \alpha_n(\mathbf{x}) \tag{6.15}$$

$$g_n'(\mathbf{x}) = \sum_j \frac{1}{n} g'^j(n\mathbf{x}) \frac{\partial \rho}{\partial x_j} \alpha_n(\mathbf{x}) \tag{6.16}$$

where $\rho \in C_0^\infty(\mathscr{O})$ and $f^j(\mathbf{x})$ and $g'^j(\mathbf{x})$ satisfy

$$f^j(\mathbf{0}) = 0, \quad g'^j(\mathbf{0}) = 0 \tag{6.17}$$

and have *essentially bounded* derivatives

$$\begin{aligned} \mathbf{F}^j &= \nabla f^j \in L^\infty \\ \mathbf{G}^{j'} &= \nabla g'^j \in L^\infty . \end{aligned} \tag{6.18}$$

To verify that

$$f_n, g_n' \in H_0(\Psi_m, \mathscr{O}) \tag{6.19}$$

we need the following lemma

Lemma 6.1 *If* $\nabla f = \mathbf{F}$ *is a zero mean, essentially bounded, stationary random field, then*

$$\lim_{n\to\infty} \sup_{\mathbf{x}\in\mathscr{O}} f_n^2(\mathbf{x}, \omega) = 0, \quad \textit{for almost all } \omega. \tag{6.20}$$

Where $f_n(\mathbf{x}, \omega) = \frac{1}{n} f(n\mathbf{x}, \omega)$, *with* $f(0, \omega) = 0$.

We note that the normalization $f(0, \omega) = 0$ is essential for the proof of Lemma 6.1 given in Sect. 8.2.

This lemma will eventually eliminate the difficulties that we have with the integrability of the form (6.14) when Ψ in only L^2-stationary and not in L^∞. It then follows that (6.15), (6.16) are admissible. Let

$$\mathscr{B} = \{\widetilde{\mathbf{F}} \in \mathscr{H}_g | \widetilde{\mathbf{F}} \text{ is essentially bounded}\}. \tag{6.21}$$

We now show that the functional $J_n(s + \nabla \cdot \mathbf{S})$ can be bounded from above and below with trial functions of the form (6.15), (6.16), respectively.

Lemma 6.2 *Assume that* $\langle |\widetilde{\Psi}|^2 \rangle < \infty$. *Then*

$$\begin{aligned} &\limsup_{n\to\infty} J_n(s + \nabla \cdot \mathbf{S}) \\ &\le \inf_{\rho \in C_0^\infty(\mathscr{O})} \inf_{\substack{\widetilde{\mathbf{F}}^j \in \mathscr{B} \\ \langle \widetilde{\mathbf{F}}^j \rangle = 0}} \Bigg[\frac{1}{|\mathscr{O}|} \int_{\mathscr{O}} d\mathbf{x} \sum_{i,j} \Big(\langle (\mathbf{e}^i + \widetilde{\mathbf{F}}^i) \cdot (\mathbf{e}^j + \widetilde{\mathbf{F}}^j) \rangle \\ &\quad -2\langle \widetilde{\Psi}(\mathbf{e}^i + \widetilde{\mathbf{F}}^i) \cdot \widetilde{\mathbf{F}}'^j \rangle - \langle \widetilde{\mathbf{F}}'^i \cdot \widetilde{\mathbf{F}}'^j \rangle \Big) \frac{\partial \rho}{\partial x_i} \frac{\partial \rho}{\partial x_j} + \frac{1}{|\mathscr{O}|} \int_{\mathscr{O}} d\mathbf{x}\, 2\rho(s + \nabla \cdot \mathbf{S}) \Bigg] \end{aligned} \tag{6.22}$$

and

$$\liminf_{n\to\infty} J_n(s+\nabla\cdot\mathbf{S})$$

$$\geq \inf_{\rho\in C_0^\infty(\mathcal{O})} \sup_{\substack{\widetilde{\mathbf{G}}'^j\in\mathcal{B}\\ \langle\widetilde{\mathbf{G}}'^j\rangle=0}} \Bigg[\frac{1}{|\mathcal{O}|}\int_{\mathcal{O}} d\mathbf{x}\sum_{i,j}\Big(\langle(\mathbf{e}^i+\widetilde{\mathbf{G}}^i)\cdot(\mathbf{e}^j+\widetilde{\mathbf{G}}^j)\rangle \tag{6.23}$$

$$-2\langle\widetilde{\Psi}(\mathbf{e}^i+\widetilde{\mathbf{G}}^i)\cdot\widetilde{\mathbf{G}}'^j\rangle-\langle\widetilde{\mathbf{G}}'^i\cdot\widetilde{\mathbf{G}}'^j\rangle\Big)\frac{\partial\rho}{\partial x_i}\frac{\partial\rho}{\partial x_j}+\frac{1}{|\mathcal{O}|}\int_{\mathcal{O}} d\mathbf{x}\; 2\rho(s+\nabla\cdot\mathbf{S})\Bigg]$$

for almost all ω, *where* $\widetilde{\mathbf{F}}^j, \widetilde{\mathbf{F}}'^j$ *are related through the Poisson equation (4.41) (in the form (8.53)-(8.54)), and* $\widetilde{\mathbf{G}}^j, \widetilde{\mathbf{G}}'^j$ *are similarly related through a Poisson equation (in the form(8.91)-(8.92)).*

The expressions for the infimum and the supremum in this lemma that involve $\widetilde{\mathbf{F}}^j$ and $\widetilde{\mathbf{G}}'^j$ are precisely the ones for the variational principles of Sect. 4.2.1 for the effective diffusivity. Note, however, that we cannot conclude that the upper and lower limits for J_n are the same because the infimum over $\widetilde{\mathbf{F}}^j$ and the supremum over $\widetilde{\mathbf{G}}'^j$ are restricted to bounded vector fields. To close the gap we need the following lemma.

Lemma 6.3 *We have that the effective diffusivity (4.43) is given by*

$$\begin{aligned}
&\sigma^{eff}(\mathbf{e}^i,\mathbf{e}^j)\\
&= \inf_{\substack{\widetilde{\mathbf{F}}\in\mathcal{B}\\ \langle\widetilde{\mathbf{F}}\rangle=(\mathbf{e}^i+\mathbf{e}^j)/2}} \int_\Omega P(d\omega)\left(\widetilde{\mathbf{F}}\cdot\widetilde{\mathbf{F}}-2\widetilde{\Psi}\widetilde{\mathbf{F}}\cdot\widetilde{\mathbf{F}}'-\widetilde{\mathbf{F}}'\cdot\widetilde{\mathbf{F}}'\right)\\
&= \inf_{\substack{\widetilde{\mathbf{F}}\in\mathcal{B}\\ \langle\widetilde{\mathbf{F}}\rangle=(\mathbf{e}^i+\mathbf{e}^j)/2}} \int_\Omega P(d\omega)(\widetilde{\mathbf{F}}\cdot\widetilde{\mathbf{F}}+\widetilde{\Gamma}\widetilde{\Psi}\widetilde{\mathbf{F}}\cdot\widetilde{\Gamma}\widetilde{\Psi}\widetilde{\mathbf{F}}-\widetilde{\Psi}\widetilde{\mathbf{F}}\cdot(\mathbf{e}^i-\mathbf{e}^j)\\
&\quad -|\frac{\mathbf{e}^i-\mathbf{e}^j}{2}|^2)
\end{aligned} \tag{6.24}$$

where $\widetilde{\mathbf{F}}'=-\widetilde{\Gamma}\widetilde{\Psi}\widetilde{\mathbf{F}}+(\mathbf{e}^i-\mathbf{e}^j)/2$. *Similarly, (4.44) is given by*

$$\begin{aligned}
&\sigma^{eff}(\mathbf{e}^i,\mathbf{e}^j)\\
&= \sup_{\substack{\widetilde{\mathbf{G}}'\in\mathcal{B}\\ \langle\widetilde{\mathbf{G}}'\rangle=(\mathbf{e}^i-\mathbf{e}^j)/2}} \int_\Omega P(d\omega)\left(\widetilde{\mathbf{G}}\cdot\widetilde{\mathbf{G}}-2\widetilde{\Psi}\widetilde{\mathbf{G}}\cdot\widetilde{\mathbf{G}}'-\widetilde{\mathbf{G}}'\cdot\widetilde{\mathbf{G}}'\right)\\
&= \sup_{\substack{\widetilde{\mathbf{G}}'\in\mathcal{B}\\ \langle\widetilde{\mathbf{G}}'\rangle=(\mathbf{e}^i-\mathbf{e}^j)/2}} \int_\Omega P(d\omega)(-\widetilde{\mathbf{G}}\cdot\widetilde{\mathbf{G}}-\widetilde{\Gamma}\widetilde{\Psi}\widetilde{\mathbf{G}}\cdot\widetilde{\Gamma}\widetilde{\Psi}\widetilde{\mathbf{G}}+\widetilde{\Psi}\widetilde{\mathbf{G}}\cdot(\mathbf{e}^i+\mathbf{e}^j)\\
&\quad +|\frac{\mathbf{e}^i+\mathbf{e}^j}{2}|^2)
\end{aligned} \tag{6.25}$$

where $\widetilde{\mathbf{G}}'=-\widetilde{\Gamma}\widetilde{\Psi}\widetilde{\mathbf{G}}+(\mathbf{e}^i+\mathbf{e}^j)/2$

This lemma is proved in Sect. 8.3 using a truncation argument and the variational principles again.

With this lemma we can close the gap in Lemma 6.2.

Theorem 6.2 *We have*

$$\lim_{n\to\infty} J_n(s+\nabla\cdot\mathbf{S}) = \inf_{\rho\in C_0^\infty(\mathscr{O})} \frac{1}{|\mathscr{O}|}\int_{\mathscr{O}} d\mathbf{x} \times \left(\sum_{i,j} 1/2\left(\sigma_{ij}^{eff}+\sigma_{ij}^{eff\dagger}\right)\frac{\partial\rho}{\partial x_i}\frac{\partial\rho}{\partial x_j}+2\rho(s+\nabla\cdot\mathbf{S})\right), \tag{6.26}$$

for almost all ω.

We remark that the theorem is valid also when all the expressions are averaged over ω with respect to P.

Before getting into the proof, let us explain the notation we will use. We denote by $f^j(\mathbf{x},\omega), f'^j(\mathbf{x},\omega), g^j(\mathbf{x},\omega), g'^j(\mathbf{x},\omega)$, $j=1,\ldots,d$, non-stationary random functions whose gradients

$$\nabla f^j(\mathbf{x},\omega) = \mathbf{F}^j(\mathbf{x},\omega) = T_{\mathbf{x}}\widetilde{\mathbf{F}}^j(\omega) \tag{6.27}$$

$$\nabla f'^j(\mathbf{x},\omega) = \mathbf{F}'^j(\mathbf{x},\omega) = T_{\mathbf{x}}\widetilde{\mathbf{F}}'^j(\omega) \tag{6.28}$$

$$\nabla g^j(\mathbf{x},\omega) = \mathbf{G}^j(\mathbf{x},\omega) = T_{\mathbf{x}}\widetilde{\mathbf{G}}^j(\omega) \tag{6.29}$$

$$\nabla g'^j(\mathbf{x},\omega) = \mathbf{G}'^j(\mathbf{x},\omega) = T_{\mathbf{x}}\widetilde{\mathbf{G}}'^j(\omega) \tag{6.30}$$

are in $\mathscr{H}_g(\Omega)$, the space of L^2, stationary, gradient fields with zero mean.

The scaled functions $f_n^j(\mathbf{x},\omega), f_n'^j(\mathbf{x},\omega), g_n^j(\mathbf{x},\omega), g_n'^j(\mathbf{x},\omega)$, are defined by

$$f_n^j(\mathbf{x},\omega) = \frac{1}{n}f^j(n\mathbf{x},\omega), \quad f_n'^j(\mathbf{x},\omega) = \frac{1}{n}f'^j(n\mathbf{x},\omega) \tag{6.31}$$

$$g_n^j(\mathbf{x},\omega) = \frac{1}{n}g^j(n\mathbf{x},\omega), \quad g_n'^j(\mathbf{x},\omega) = \frac{1}{n}g'^j(n\mathbf{x},\omega). \tag{6.32}$$

and are uniquely determined up to constant. The normalization constant is essential in determining the right trial functions.

6.1.1 Upper bound. For the minimum principle of Sect. 4.2.2, consider the trial function

$$f_n(\mathbf{x},\omega) = \rho(\mathbf{x}) + \sum_j f_n^j(\mathbf{x},\omega)\frac{\partial\rho(\mathbf{x})}{\partial x_j}\alpha_n(\mathbf{x}) \tag{6.33}$$

where f^j satisfies (6.18) and $\alpha_n(\mathbf{x})$ is the cut-off function defined in the Sect. 8.4, 8.5. We show in Lemma 8.4 that

$$f_n'(\mathbf{x},\omega) = \sum_j f_n'^j(\mathbf{x},\omega)\frac{\partial\rho(\mathbf{x})}{\partial x_j}\alpha_n(\mathbf{x}). \tag{6.34}$$

solves the Poisson problem for the minimum principle asymptotically in $H_0^1(\mathcal{O})$, under the assumptions of Lemma 8.4 and if $\widetilde{\mathbf{F}}^j(\omega), \widetilde{\mathbf{F}}'^j(\omega), j = 1, ..., d$, satisfy (8.53)-(8.54) with $f_n'^j(\mathbf{x}, \omega)$ satisfying the normalization (8.56).

Thus $f_n(\mathbf{x}, \omega), f_n'(\mathbf{x}, \omega)$ are a legitimate pair of trial functions for the minimum principle of Sect. 4.2.2 in the asymptotic sense as $n \to \infty$. Substituting (6.33), (6.34) into the minimum principle and collecting similar terms we get

$$\begin{aligned} J_n(s + \nabla \cdot \mathbf{S}) &\leq \frac{1}{|\mathcal{O}|} \int_{\mathcal{O}} d\mathbf{x} \left(\nabla f_n \cdot \nabla f_n - 2\Psi_n \nabla f_n \cdot \nabla f_n' \right. \\ &\left. - \nabla f_n' \cdot \nabla f_n' + 2 f_n (s + \nabla \cdot \mathbf{S}) \right) + o(1) \\ &\leq \frac{1}{|\mathcal{O}|} \int_{\mathcal{O}} d\mathbf{x} \Big(\sum_{i,j} \big((\mathbf{e}^i + \nabla f_n^i) \cdot (\mathbf{e}^j + \nabla f_n^j) \\ &- 2\Psi_n \left(\mathbf{e}^i + \nabla f_n^i\right) \cdot \nabla f_n'^j \\ &- \nabla f_n'^i \cdot \nabla f_n'^j \big) \frac{\partial \rho}{\partial x_i} \frac{\partial \rho}{\partial x_j} + 2\rho(s + \nabla \cdot \mathbf{S}) \Big) + o(1) \end{aligned} \tag{6.35}$$

in view of (8.58), (8.57) and (8.82). Passing to the limit in (6.35) using the individual ergodic theorem gives

$$\begin{aligned} \limsup_{n \to \infty} J_n(s + \nabla \cdot \mathbf{S}) &\leq \frac{1}{|\mathcal{O}|} \int_{\mathcal{O}} d\mathbf{x} \sum_{i,j} \Big(\langle (\mathbf{e}^i + \widetilde{\mathbf{F}}^i) \cdot (\mathbf{e}^j + \widetilde{\mathbf{F}}^j) \rangle \\ &- 2 \langle \widetilde{\Psi} (\mathbf{e}^i + \widetilde{\mathbf{F}}^i) \cdot \widetilde{\mathbf{F}}'^j \rangle - \langle \widetilde{\mathbf{F}}'^i \cdot \widetilde{\mathbf{F}}'^j \rangle \Big) \frac{\partial \rho}{\partial x_i} \frac{\partial \rho}{\partial x_j} + 2\rho(s + \nabla \cdot \mathbf{S}) \end{aligned} \tag{6.36}$$

Minimizing the right side of (6.36) over $\widetilde{\mathbf{F}}^i, i = 1, ..., d,$, bearing in mind Lemma (6.3) and the identity (4.23), we obtain

$$\begin{aligned} \limsup_{n \to \infty} J_n(s + \nabla \cdot \mathbf{S}) &\leq \frac{1}{|\mathcal{O}|} \int_{\mathcal{O}} d\mathbf{x} \\ &\times \left(\sum_{i,j} 1/2 \left(\sigma_{ij}^{eff} + \sigma_{ij}^{eff\dagger} \right) \frac{\partial \rho}{\partial x_i} \frac{\partial \rho}{\partial x_j} + 2\rho(s + \nabla \cdot \mathbf{S}) \right) \end{aligned} \tag{6.37}$$

where we use the variational definition of the effective diffusivity (Sect. 4.2.1). Note that only its symmetric part appears on the right side.

6.1.2 Lower bound. To get lower bounds, we use the maximum principle of Sect. 4.2.2. The analysis for the lower bound is different from that for the upper bound in the following aspect: the quantity ρ the macroscopic scale is no longer independent from the quantities on the microscopic scale.

Consider the trial function g_n' defined by (6.16). As stated in Lemma 8.6, g_n defined by

$$g_n(\mathbf{x}) = \rho + \sum_j g_n^j(\mathbf{x}) \frac{\partial \rho}{\partial x_j} \alpha_n \tag{6.38}$$

solves the Poisson problem for the maximum principle asymptotically, as $n \to \infty$, in $H_0^1(\mathcal{O})$ if $\nabla g^j = \widetilde{\mathbf{G}}^j$ and $\nabla g'^j = \widetilde{\mathbf{G}}'^j$ satisfy (8.91)-(8.92), g_n^j the normalization (8.95) and ρ the equations

$$\sum_{i,j} \frac{\partial}{\partial x_i}(\delta_{ij} + < \tilde{\Psi}\widetilde{\mathbf{G}}'^i > \cdot \mathbf{e}_j)\frac{\partial}{\partial x_j}\rho = s + \nabla \cdot \mathbf{S} \tag{6.39}$$

$$\rho = 0, \quad \text{on } \partial\mathcal{O}. \tag{6.40}$$

Thus g_n', g_n are a legitimate pair of trial functions for the maximum principle of Sect. 4.2.2 in the asymptotic sense as $n \to \infty$.

Substituting g_n', g_n and passing to the limit, using the individual ergodic theorem, gives

$$\begin{aligned}
&\liminf_{n\to\infty} J_n(s + \nabla \cdot \mathbf{S}) \\
&\geq \frac{1}{|\mathcal{O}|}\int_{\mathcal{O}} d\mathbf{x} \sum_{i,j}\Big(\langle(\mathbf{e}^i + \widetilde{\mathbf{G}}^i)\cdot(\mathbf{e}^j + \widetilde{\mathbf{G}}^j)\rangle - 2\langle\widetilde{\Psi}(\mathbf{e}^i + \widetilde{\mathbf{G}}^i)\cdot\widetilde{\mathbf{G}}'^j\rangle \\
&\quad -\langle\widetilde{\mathbf{G}}'^i\cdot\widetilde{\mathbf{G}}'^j\rangle\Big)\frac{\partial\rho}{\partial x_i}\frac{\partial\rho}{\partial x_j} + 2\rho(s + \nabla\cdot\mathbf{S})
\end{aligned} \tag{6.41}$$

where ρ solves (6.39) and (6.40).

We may assume the right side of (6.41) is minimized over $\rho(\mathbf{x})$ so that it decouples from $\widetilde{\mathbf{G}}'^j$. Maximizing the right side of (6.41) over $\widetilde{\mathbf{G}}'^j$, $j = 1, \ldots, d$, and using Lemma 6.3 and the identity (4.23), we obtain

$$\begin{aligned}
&\liminf_{n\to\infty} J_n(s + \nabla \cdot \mathbf{S}) \\
&\geq \frac{1}{|\mathcal{O}|}\int_{\mathcal{O}} d\mathbf{x}\left(\sum_{i,j} 1/2\left(\sigma_{ij}^{eff} + \sigma_{ij}^{eff\dagger}\right)\frac{\partial\rho}{\partial x_i}\frac{\partial\rho}{\partial x_j} + 2\rho(s + \nabla\cdot\mathbf{S})\right)
\end{aligned} \tag{6.42}$$

Since $\widetilde{\mathbf{E}}_{jj}'$ are the maximizer of the quadratic terms involving $\widetilde{\mathbf{G}}^i$ and $\widetilde{\mathbf{G}}'^j$, the $\widetilde{\mathbf{G}}'^j$-depending ρ solves the Dirichlet problem for the Poisson equation (6.39) with the elliptic coefficients

$$\delta_{ij} + < \widetilde{\Psi}\widetilde{\mathbf{E}}_{ii}' > \cdot \mathbf{e}_j \tag{6.43}$$

which is exactly the effective diffusivity given by (4.25). Therefore, (6.42) is equivalent to

$$\begin{aligned}
&\liminf_{n\to\infty} J_n((s + \nabla \cdot \mathbf{S})) \geq \min_{\rho} \frac{1}{|\mathcal{O}|}\int_{\mathcal{O}} d\mathbf{x} \\
&\times\left(\sum_{i,j} 1/2\left(\sigma_{ij}^{eff} + \sigma_{ij}^{eff\dagger}\right)\frac{\partial\rho}{\partial x_i}\frac{\partial\rho}{\partial x_j} + 2\rho(s + \nabla\cdot\mathbf{S})\right).
\end{aligned} \tag{6.44}$$

In view of (6.37), we then conclude that

$$\lim_{n\to\infty} J_n((s+\nabla\cdot\mathbf{S})) = \min_{\rho} \frac{1}{|\mathscr{O}|}\int_{\mathscr{O}} d\mathbf{x}$$
$$\times\left(\sum_{i,j} 1/2\left(\sigma_{ij}^{eff}+\sigma_{ij}^{eff\dagger}\right)\frac{\partial\rho}{\partial x_i}\frac{\partial\rho}{\partial x_j}+2\rho(s+\nabla\cdot\mathbf{S})\right). \tag{6.45}$$

This completes the proof of Theorem 6.2.

6.2 Convergence of solutions: proof of Theorem 6.1

In this section we complete the proof of Theorem 6.1 and Corollary 6.1 stated in the beginning of Sect. 6, using Theorem 6.2. Because of the variational structure, much of the proof of Theorem 6.1 is contained in the preceeding analysis.

Define the differences

$$\mathbf{r}_n = \nabla\rho_n - \nabla f_n \tag{6.46}$$
$$\mathbf{r}_n' = \nabla\rho_n' - \nabla f_n' \tag{6.47}$$

where ρ_n, ρ_n' are the solutions of the system of symmetrized inhomogeneous boundary value problems and f_n, f_n' are given by (6.33), (6.34) with ρ replaced by the exact solution $\bar\rho$ of the homogenized problem (6.3), (6.4). Then

$$\begin{aligned}&\int_{\mathscr{O}} d\mathbf{x}\ \left(\mathbf{r}_n\cdot\mathbf{r}_n+\mathbf{r}_n'\cdot\mathbf{r}_n'\right)\\ &=\int_{\mathscr{O}} d\mathbf{x}\ \left(\nabla\rho_n\cdot\nabla\rho_n+\nabla\rho_n'\cdot\nabla\rho_n'\right)\\ &\quad+\int_{\mathscr{O}} d\mathbf{x}\ \left(\nabla f_n\cdot\nabla f_n+\nabla f_n'\cdot\nabla f_n'\right)\\ &\quad-2\int_{\mathscr{O}} d\mathbf{x}\ \left(\nabla\rho_n\cdot\nabla f_n+\nabla\rho_n'\cdot\nabla f_n'\right)\end{aligned} \tag{6.48}$$

Since ρ_n is the solution of the symmetrized boundary value problem

$$\nabla\cdot(I-\Psi_n\Gamma_0\Psi_n)\nabla\rho_n = s+\nabla\cdot\mathbf{S}, \quad \text{in } \mathscr{O} \tag{6.49}$$
$$\rho_n = 0, \quad \text{on } \mathscr{O} \tag{6.50}$$

whose weak formulation is given by

$$\int_{\mathscr{O}} d\mathbf{x}\ (\nabla\phi\cdot\nabla\rho_n+\Gamma_0\Psi_n\nabla\phi\cdot\Gamma_0\Psi_n\nabla\rho_n) = -\int_{\mathscr{O}} d\mathbf{x}\ \phi(s+\nabla\cdot\mathbf{S}) \tag{6.51}$$

for all $\phi\in H_0^1(\mathscr{O})$ (cf. Theorem 5.2). When f_n is substituted in (6.51), we have

$$\int_{\mathscr{O}} d\mathbf{x}\ \left(\nabla f_n\cdot\nabla\rho_n+\nabla f_n'\cdot\nabla\rho_n'\right)\asymp-\int_{\mathscr{O}} d\mathbf{x}\, f_n(s+\nabla\cdot\mathbf{S}), \quad \text{as } n\to\infty \tag{6.52}$$

since $\nabla\rho_n' = \Gamma_0\Psi_n\nabla\rho_n$ and $\nabla f_n' \asymp -\Gamma_0\Psi_n\nabla f_n$ according to Lemma 8.4 (cf. Sect. 8.4). Here $\asymp$ denotes the asymptotic equality as $n \to \infty$. It is also clear that

$$\int_{\mathcal{O}} d\mathbf{x} \; \left(\nabla\rho_n \cdot \nabla\rho_n + \nabla\rho_n' \cdot \nabla\rho_n'\right) = -\int_{\mathcal{O}} d\mathbf{x} \; \rho_n(s + \nabla\cdot\mathbf{S}) = -J_n(s + \nabla\cdot\mathbf{S}) \tag{6.53}$$

when the exact solution ρ_n is inserted in (4.47). Thus, using (6.52) and (6.53) we have that

$$\begin{aligned} &\int_{\mathcal{O}} d\mathbf{x} \; \left(\mathbf{r}_n \cdot \mathbf{r}_n + \mathbf{r}_n' \cdot \mathbf{r}_n'\right) \\ &\asymp -J_n(s + \nabla\cdot\mathbf{S}) + 2\int_{\mathcal{O}} d\mathbf{x}\, f_n(s + \nabla\cdot\mathbf{S}) \\ &\quad + \int_{\mathcal{O}} d\mathbf{x} \; \left(\nabla f_n \cdot \nabla f_n + \nabla f_n' \cdot \nabla f_n'\right) \end{aligned} \tag{6.54}$$

which in the limit can be made arbitrarily close to zero in view of the proof of Theorem 6.2.

To prove Theorem 6.1, we need to show now that for any given $\delta > 0$ there exist f_n and f_n' in the form (6.33) and (6.34) respectively such that

$$\int_{\mathcal{O}} d\mathbf{x} \left(\nabla\left(f_n - \bar{\rho} - \sum_j \chi_n^j(\mathbf{x},\omega)\frac{\partial\bar{\rho}(\mathbf{x})}{\partial x_j}\right)\right)^2 \le c\delta \tag{6.55}$$

$$\int_{\mathcal{O}} d\mathbf{x} \left(\nabla\left(f_n' - \sum_j \chi_n'^j(\mathbf{x},\omega)\frac{\partial\bar{\rho}(\mathbf{x})}{\partial x_j}\right)\right)^2 \le c\delta \tag{6.56}$$

in the limit $n \to \infty$ for some constant c independent of δ and $\widetilde{\mathbf{F}}^j, \widetilde{\mathbf{F}}'^j \in \mathscr{B}$ (in the definitions (6.33), (6.34) respectively.

For this, we need the following lemma:

Lemma 6.4 *Given $\delta > 0$, there exists $\widetilde{\mathbf{F}}^j \in \mathscr{B}$ with zero mean such that*

$$||(\mathbf{e}_j + \widetilde{\mathbf{F}}^j) - \widetilde{\mathbf{E}}_{jj}||_{\widetilde{\Psi}} \le \delta. \tag{6.57}$$

Proof: Consider the following identities

$$\begin{aligned} &||(\mathbf{e}_j + \widetilde{\mathbf{F}}^j) - \widetilde{\mathbf{E}}_{jj}||_{\widetilde{\Psi}} \\ &= <\widetilde{\mathbf{E}}_{jj} \cdot \widetilde{\mathbf{E}}_{jj}> + <\widetilde{\Gamma}\widetilde{\Psi}\widetilde{\mathbf{E}}_{jj} \cdot \widetilde{\Gamma}\widetilde{\Psi}\widetilde{\mathbf{E}}_{jj}> \\ &\quad + <(\mathbf{e}_j + \widetilde{\mathbf{F}}^j)\cdot(\mathbf{e}_j + \widetilde{\mathbf{F}}^j)> + <\widetilde{\Gamma}\widetilde{\Psi}(\mathbf{e}_j + \widetilde{\mathbf{F}}^j)\cdot\widetilde{\Gamma}\widetilde{\Psi}(\mathbf{e}_j + \widetilde{\mathbf{F}}^j)> \\ &\quad -2<(\mathbf{e}_j + \widetilde{\mathbf{F}}^j)\cdot\widetilde{\mathbf{E}}_{jj}> -2<\widetilde{\Gamma}\widetilde{\Psi}\widetilde{\mathbf{E}}_{jj}\cdot\widetilde{\Gamma}\widetilde{\Psi}(\mathbf{e}_j + \widetilde{\mathbf{F}}^j)> \end{aligned} \tag{6.58}$$

where $\widetilde{\mathbf{E}}_{jj}, j = 1, ..., d$, are the solutions of the symmetrized cell problems (4.9)-(4.12) and $\widetilde{\mathbf{F}}^j$ zero mean bounded gradient fields. Recall the weak formulation for $\widetilde{\mathbf{E}}_{jj}$:

$$< \widetilde{\mathbf{E}}_{jj} \cdot \widetilde{\mathbf{F}} > + < \widetilde{\Gamma}\widetilde{\Psi}\widetilde{\mathbf{E}}_{jj} \cdot \widetilde{\Gamma}\widetilde{\Psi}\widetilde{\mathbf{F}} >= 0 \tag{6.59}$$

for any zero mean gradient field $\widetilde{\mathbf{F}} \in \mathscr{H}_g(\widetilde{\Psi})$ which contains $\mathscr{B}$. Thus the last two terms in (6.58) equal to

$$\begin{aligned} & 2 < (\mathbf{e}_j + \widetilde{\mathbf{F}}^j) \cdot \widetilde{\mathbf{E}}_{jj} > +2 < \widetilde{\Gamma}\widetilde{\Psi}\widetilde{\mathbf{E}}_{jj} \cdot \widetilde{\Gamma}\widetilde{\Psi}(\mathbf{e}_j + \widetilde{\mathbf{F}}^j) > \\ & = 2 < \widetilde{\mathbf{E}}_{jj} \cdot \mathbf{e}_j > +2 < \widetilde{\Gamma}\widetilde{\Psi}\widetilde{\mathbf{E}}_{jj} \cdot \widetilde{\Gamma}\widetilde{\Psi}\mathbf{e}_j >= 2\sigma_{jj}^{eff} \end{aligned} \tag{6.60}$$

in view of (4.17). It is also clear that the first two terms in (6.58) equal to σ_{jj}^{eff}. By the approximation Lemma 6.3 the middle two terms in (6.58) can be made arbitrarily close to σ_{jj}^{eff} and therefore $||(\mathbf{e}_j + \widetilde{\mathbf{F}}^j) - \widetilde{\mathbf{E}}_{jj}||_{\widetilde{\Psi}}$ can be made arbitrarily small as stated in Lemma 6.4.

(6.55) and thus Theorem 6.1 and Corollary 6.1 follow from Lemma 6.4 and

$$\int_{\mathscr{O}} d\mathbf{x}\, (\chi_n^j)^2 \to 0 \tag{6.61}$$

almost surely as $n \to \infty$. (see Lemma 8.2).

7 Probabilistic convergence theorem: compactness of the processes

In this section we prove that the rescaled processes $\mathbf{x}_n(\cdot)$

$$\mathbf{x}_n(\cdot) = \frac{1}{n}\mathbf{x}(n^2 t) \tag{7.1}$$

with $\mathbf{x}(t)$ defined by (1.2) satisfy the tightness condition

$$\lim_{h\downarrow 0} \overline{\lim_{n\uparrow\infty}} \text{ Prob}\left\{\sup_{\substack{0<s<t<T \\ |s-t|\leq h}} |\mathbf{x}_n(s) - \mathbf{x}_n(t)| > \delta\right\} = 0 \tag{7.2}$$

for each $\delta > 0$ and $T < \infty$. We will assume now the the velocity field is smooth (differentiable) almost surely and that it is bounded by a linear function of the coordinates almost surely. This will be the case for stationary Gaussian velocity fields with smooth covariance. Under these conditions the process $\mathbf{x}_n$ is well defined as a process with continuous paths.

The unboundedness of Ψ makes the use of the Nash estimates, or generalizations [19], impossible. Thus, the compactness of the processes $\mathbf{x}_n(t)$ is no longer straightforward. We reduce the estimate of the probability in (7.2) to a resolvent estimate which we can study using variational methods except that in this case we need an L^∞, rather than L^2, estimate. This comes when we average over the ensemble of flows $\mathbf{u}$. So we no longer have convergence with probability one, as in Theorem 1, but convergence in measure with respect to the flows (cf. Theorem 7.1).

It is enough for (7.2) to obtain the estimate

$$\lim_{h\downarrow 0} \frac{1}{h} \overline{\lim_{n\uparrow\infty}} \text{ Prob}\left\{\sup_{0<s<h} |\mathbf{x}_n(s) - \mathbf{x}_n(0)| > \delta\right\} = 0 \tag{7.3}$$

for each $\delta > 0$. To prove (7.3) it suffices to obtain for each component $x_i(s)$ an estimate of the form

$$\lim_{h\downarrow 0} \frac{1}{h} \overline{\lim_{n\uparrow\infty}} \text{ Prob}\left\{\sup_{0<s<hn^2} |x_i(s) - x_i(0)| \geq \delta n\right\} = 0 \tag{7.4}$$

for each $\delta > 0$. Let τ_L be the time it takes for $x_i(t)$ to reach level L assuming $x_i(0) = 0$. Then (7.4) reduces to

$$\lim_{h\downarrow 0} \frac{1}{h} \overline{\lim_{n\uparrow\infty}} \text{ Prob}\left\{\tau_{\delta n} \leq hn^2\right\} = 0 \tag{7.5}$$

for each $\delta > 0$. From the Tchebyshev inequality

$$\text{Prob}\left\{\tau_{\delta n} \leq hn^2\right\} \leq e^{\alpha h} E\left\{e^{-n^{-2}\alpha\tau_{\delta n}}\right\}. \tag{7.6}$$

Therefore, (7.5) can be deduced from

$$\overline{\lim_{n\uparrow\infty}} E\left\{e^{-n^{-2}\alpha\tau_{\delta n}}\right\} \leq M(\alpha, \delta), \tag{7.7}$$

for each $\alpha > 0$, and

$$\inf_{\alpha>0} e^{\alpha h} M(\alpha, \delta) = o(h) \tag{7.8}$$

as $h \to 0$ for each $\delta > 0$.

Let $\mathscr{L}$ be the generator of the processes

$$\mathscr{L} = \nabla \cdot [(\sigma I + \Psi_n(\mathbf{x}))\nabla\ \cdot] \tag{7.9}$$

and consider the solution of

$$\alpha\rho_\alpha - \mathscr{L}\rho_\alpha = 0 \tag{7.10}$$

for $x_1 < L$ with

$$\rho_\alpha = 0, \text{for} \quad x_1 = -\infty, \quad \rho_\alpha = 1 \quad \text{for} \quad x_1 = L \tag{7.11}$$

Write $\mathbf{x} = (x_1, \mathbf{x}^1)$ with $\mathbf{x}^1 = (x_2, \ldots, x_d)$. and let

$$\int P(d\omega)\rho_\alpha((0, \mathbf{x}^1), L, \omega) = \theta_\alpha(L), \tag{7.12}$$

which does not depend on $\mathbf{x}^1$, by stationarity. To get (7.8) we need an estimate for $\theta_{n^{-2}\alpha}(\delta n)$ as $n \to \infty$.

Since averaging with respect to ω allows us to use stationarity, starting from zero and going to L is equivalent to starting from $-L$ and going to level 0. Therefore, we may consider

$$\alpha\rho_\alpha - \mathscr{L}\rho_\alpha = 0, \qquad x_1 < 0 \tag{7.13}$$

$$\rho_\alpha = 0, \quad \text{for} \quad x_1 = -\infty, \quad \rho_\alpha = 1, \quad \text{for} \quad x_1 = 0 \tag{7.14}$$

and let

$$\int P(d\omega)\rho_\alpha(\mathbf{x}, L, \omega) = \theta_\alpha(x_1) \tag{7.15}$$

We will study the asymptotic behavior of $\theta_{n^{-2}\alpha}(-\delta n)$ as $n \to \infty$.

The idea is to show that the averaged moment-generating functions $\theta_{n^{-2}\alpha}(-\delta n)$ of the exit time for the processes $\mathbf{x}_n$ is very close to that of Brownian motion for which we have the estimate (7.8). With slight modifications, it is routine to check that strong convergence holds. In particular,

$$\lim_{Q\to R^{d-1}} \frac{1}{Q}\int_Q d\mathbf{x}^1 \int_{-\infty}^0 dx_1 \int_\Omega P(d\omega)\,(\rho_{n^{-2}\alpha}(n\mathbf{x},\omega) - \bar\rho_\alpha(\mathbf{x}))^2 \to 0 \tag{7.16}$$

as $n \to \infty$. Here $\bar\rho_\alpha(\mathbf{x})$ is the moment generating function of the exit time for Brownian motion with variance coefficient $\frac{1}{2}(\sigma^{\mathit{eff}} + \sigma^{\mathit{eff}\dagger})$. The techniques developed in the previous sections apply equally well here, with some modifications needed to account for the α dependence and the semi-infinite domain $\{\mathbf{x}|x_1 \leq 0\}$. Therefore,

$$\int_{-\infty}^0 dx_1\,(\theta_{n^{-2}\alpha}(nx_1) - M(\alpha, x_1))^2 \to 0, \tag{7.17}$$

as $n \to \infty$, where $M(\alpha, x_1)$ is the moment-generating function of the exit time for the one-dimensional Brownian motion starting at x_1. Here we use the ω-average version of homogenization theorems as noted in the remark after the statements of Theorem 6.1and 6.1. However, from (7.7) and (7.8) we see that what is needed here is not the $L^2(dx_1)$ convergence but convergence pointwise in x_1. But both $\theta_{n^{-2}\alpha}(nx_1)$ and $M(\alpha, x_1)$ are monotone so $L^2(dx_1)$ convergence actually implies uniform convergence

$$\sup_{x_1\leq 0}\,(\theta_{n^{-2}\alpha}(nx_1) - M(\alpha, x_1))^2 \to 0 \tag{7.18}$$

as $n \to \infty$. From the Laplace transform of the heat equation on the semi-infinite line, we know that as $\alpha \to \infty$

$$M(\alpha, \delta) = O(e^{-c\delta\sqrt{\alpha}}). \tag{7.19}$$

for some positive constant c. The tightness condition (7.2) then follows from

$$\inf_{\alpha\geq 0} e^{\alpha h} M(\alpha, \delta) = O(e^{-c_1/h}), \tag{7.20}$$

as $h \to 0$, for some positive constant $c_1 = c^2\delta^2/4$, by taking $\alpha = \frac{c^2\delta^2}{4h^2}$. We have thus proven Theorem 2 of the Introduction, which we restate here.

Theorem 7.1 *The family $\mathbf{x}_n(t)$ of stochastic processes defined by (1.2) is uniformly tight in measure with respect to the ensemble of media $P(d\omega)$ and therefore we have weak convergence to Brownian motion in the space of continuous functions in R^d, in measure with respect to $P(d\omega)$.*

8 The proofs of some technical lemmas

8.1 L^2-sublinear growth of random functions with L^2-derivatives

The main result of this section is the proof of the almost sure L^2-sublinear growth estimate of Lemma 8.2 which is the strengthened version of the standard L^2-sublinear growth estimate stated in the following Lemma 8.1. Lemma 8.2 is needed for the proof of Lemma 6.1, 8.4 and 8.6.

Lemma 8.1 *Let* $\tilde{\mathbf{F}} \in \mathscr{H}_g$. *There exists a uniquely defined process* $f(\mathbf{x}, \omega) \in H^1_{\mathrm{loc}}(R^d; L^2(\Omega))$, *it is not stationary,* $f(0, \omega) = 0$ *and*

$$\nabla f(\mathbf{x}, \omega) = \mathbf{F}(\mathbf{x}, \omega) = \tilde{\mathbf{F}}(\tau_{-\mathbf{x}}\omega). \tag{8.1}$$

For any compact subset $K \subset R^d$, *we have*

$$\lim_{n\to\infty} \sup_K \left\langle \left[\frac{1}{n} f(n\mathbf{x}, \omega)\right]^2 \right\rangle = 0. \tag{8.2}$$

Proof. This proof follows Papanicolaou and Varadhan[20]. Define $f(\mathbf{x}, \omega)$ by

$$f(\mathbf{x}, \omega) = \int_{R^d} \frac{e^{i\mathbf{x}\cdot\mathbf{k}} - 1}{|\mathbf{k}|^2} (-i\mathbf{k}) \cdot U(d\mathbf{k})\tilde{\mathbf{F}}(\omega) \tag{8.3}$$

where $U(d\mathbf{k})$ is the spectral resolution of the unitary group $\{T_{\mathbf{x}}\}$, i.e.,

$$T_{\mathbf{x}} = \int_{\mathbf{k}\in R^d} e^{i\mathbf{k}\cdot\mathbf{x}} U(d\mathbf{k}). \tag{8.4}$$

The process $f(\mathbf{x}, \omega)$ is not stationary because it is not of the form $f(\mathbf{x}, \omega) = T_{\mathbf{x}}\tilde{f}(\omega)$. It is easy to see that $f(0, \omega) = 0$ and $\nabla f(\mathbf{x}, \omega) = \mathbf{F}(\mathbf{x}, \omega)$ and, as a consequence, it is in $H^1_{\mathrm{loc}}(R^d; L^2(\Omega))$. It remains to show (8.2). We have the identity

$$\int_\Omega P(d\omega) \left(\frac{1}{n} f(n\mathbf{x}, \omega)\right)^2 \tag{8.5}$$

$$= \int_{R^d} \left| \frac{e^{in\mathbf{x}\cdot\mathbf{k}} - 1}{n\mathbf{k}} \right|^2 \sum_{i,j=1}^d \frac{k_i k_j}{|\mathbf{k}|^2} \hat{R}_{ij}(d\mathbf{k}) \tag{8.6}$$

where

$$\hat{R}_{ij}(d\mathbf{k}) = \int_\Omega P(d\omega) U(d\mathbf{k}) \tilde{\mathbf{F}}_i(\omega) \tilde{\mathbf{F}}_j(\omega) \tag{8.7}$$

is the power spectral measure of $\mathbf{F}_i(\mathbf{x}, \omega) = \tilde{\mathbf{F}}_j(\tau_{-\mathbf{x}}\omega)$. From the estimate

$$\frac{1}{|\mathbf{k}|^2} \sum_{i,j=1}^d k_i k_j \hat{R}_{ij}(d\mathbf{k}) \le \sum_{i=1}^d \hat{R}_{ii}(d\mathbf{k}), \tag{8.8}$$

we obtain

$$\int_{\Omega} P(d\omega) \left(\frac{1}{n} f(n\mathbf{x}, \omega)\right)^2 \leq \int_{R^d} \left|\frac{e^{in\mathbf{x}\cdot\mathbf{k}} - 1}{n\mathbf{k}}\right|^2 \sum_{i=1}^{d} \hat{R}_{ii}(d\mathbf{k}). \tag{8.9}$$

By ergodicity and $\langle\tilde{\mathbf{F}}\rangle = 0$, it follows that $\hat{R}_{ii}(\{0\}) = 0$. The Lebesque convergence theorem then yields the result.

Let

$$f_n(\mathbf{x}, \omega) = \frac{1}{n} f(n\mathbf{x}, \omega). \tag{8.10}$$

Then (8.2) implies that

$$\langle \int_K d\mathbf{x}\, f_n^2 \rangle \to 0 \tag{8.11}$$

as $n \to \infty$. Consider also

$$f_n'(\mathbf{x}, \omega) \equiv f_n(\mathbf{x}, \omega) - a_n(\omega) \tag{8.12}$$

where $a_n(\omega) = \frac{1}{|K|} \int_K d\mathbf{x}\, f_n(\mathbf{x}, \omega)$. It is easy to see that (8.11) implies that

$$\langle \int_K d\mathbf{x}\, (f_n')^2 \rangle \to 0 \tag{8.13}$$

as $n \to \infty$, since

$$\langle a_n^2 \rangle \leq \langle \frac{1}{|K|} \int_K d\mathbf{x}\, f_n^2 \rangle \to 0 \tag{8.14}$$

as $n \to \infty$.

The constant $a_n(\omega)$ in (8.12) is essential for the proof of the following strengthened version of (8.13) that the convergence holds without the average $\langle\cdot\rangle$.

Lemma 8.2 *For P almost all $\omega \in \Omega$*

$$\int_K d\mathbf{x}\, (f_n')^2 \to 0, \quad \text{as } \mathrm{n} \to \infty. \tag{8.15}$$

Proof. Without loss of generality, we may assume $K = \{|\mathbf{x}| < a\}$, for some $a > 0$. From the definition (8.12), we have

$$\nabla f_n'(\mathbf{x}, \omega) = \mathbf{F}(n\mathbf{x}, \omega) \in L^2_{\mathrm{loc}}(R^d) \tag{8.16}$$

for almost all $\omega \in \Omega$. Furthermore, given $\delta > 0$, there exists $n_0(\omega, \delta)$ such that, for $n > n_0(\omega, \delta)$

$$\frac{1}{|K|} \int_K d\mathbf{x}\, (\nabla f_n')^2 \leq |\tilde{\mathbf{F}}|_{L^2(\Omega)} + \delta \tag{8.17}$$

for almost all $\omega \in \Omega$, by ergodicity. The uniform estimate (8.17) and the mean zero property $\int_K d\mathbf{x}\, f_n' = 0$ imply that $\{f_n'\}$ is precompact in the strong L^2 sense. Consider any convergent subsequence, still denoted by $\{f_n'\}$. There exists a function $g(\mathbf{x}, \omega) \in L^2(K)$ such that

$$\int_K d\mathbf{x}\, (f_n' - g)^2 \to 0 \tag{8.18}$$

as $n \to \infty$, for almost all ω.

On the other hand, (8.13) implies that the sequence of positive random variables $\{\int_K d\mathbf{x}\, (f_n')^2\}$ converges to zero in probability with respect to P and in particular there exists a subsequence $\{\int_K d\mathbf{x}\, (f_{nj}')^2\}$ converging to zero for P almost all $\omega \in \Omega$. Thus $g(\mathbf{x}, \omega) = 0$, for almost all ω. This proves the lemma.

8.2 L^∞-sublinear growth of random functions with L^∞-derivatives

In this section we prove Lemma 6.1 which is essential in our estimates for the Poisson problems in Sect. 8.4 and 8.5. Note that, contrary to the constant $a_n(\omega)$, a different normalization has been taken in Lemma 6.1.

Proof of Lemma 6.1. In view of boundedness of the domain $\mathcal{O}$ and $\nabla f_n(\mathbf{x}, \omega) = \mathbf{F}(n\mathbf{x}, \omega)$, the pointwise convergence to zero

$$\lim_{n \to \infty} f_n^2(\mathbf{x}, \omega) = 0, \quad \forall \mathbf{x} \in \mathcal{O} \tag{8.19}$$

implies the uniform convergence (6.20). It remains to prove (8.19) and this is done by contradiction.

Suppose (8.19) fails at a point $\mathbf{x}_o \in \mathcal{O}$. We select a convergent subsequence, still denoted by $f_n(\mathbf{x}_o, \omega)$ such that

$$f_n(\mathbf{x}_o, \omega) \to \alpha_\omega \neq 0. \tag{8.20}$$

By the boundedness of $\widetilde{\mathbf{F}}$ and the normalization $f_n(0, \omega) = 0$ it follows that there exists a $\delta > 0$ and $n_0 > 0$ such that for all $n > n_0$, we have

$$|f_n(\mathbf{x}, \omega) - \alpha_\omega| \leq \alpha_\omega / 3, \quad \text{for} \quad |\mathbf{x} - \mathbf{x}_o| < \delta \tag{8.21}$$

$$|f_n(\mathbf{x}, \omega)| \leq \alpha_\omega / 3, \quad \text{for} \quad |\mathbf{x}| < \delta \tag{8.22}$$

Now consider the cylinder set $\mathcal{O}'$ of radius δ, with $\overline{\mathbf{ox}_o}$ as its axis. By ergodicity and the zero mean property $\langle \widetilde{\mathbf{F}} \rangle = 0$, we have

$$\frac{1}{|\mathcal{O}'|} \int_{\mathcal{O}'} d\mathbf{x}\, \mathbf{F}(n\mathbf{x}, \omega) \to 0, \quad \text{for almost all } \omega \tag{8.23}$$

But, from (8.21)-(8.22), it follows that $\frac{1}{|\mathcal{O}'|} \int_{\mathcal{O}'} d\mathbf{x}\, \nabla f_n(\mathbf{x}, \omega)$ has a nonzero component in the direction of $\overline{\mathbf{ox}_o}$, which is larger than

$$\frac{\alpha_\omega |B_\delta^{d-1}|}{3|\mathcal{O}'|} > 0. \tag{8.24}$$

Here B_δ^{d-1} is the $d-1$ dimensional ball of radius δ. Thus, $\alpha_\omega = 0$ and the proof is complete.

8.3 Approximation lemmas

First we introduce the level M truncation $\widetilde{\Psi}^{(M)}$ of the stream matrix:

$$\widetilde{\Psi}_{ij}^{(M)} = \begin{cases} \widetilde{\Psi}_{i,j}, & \text{for } |\widetilde{\Psi}_{ij}| < M \\ \text{sign}\,(\widetilde{\Psi}_{ij})M, & \text{for } |\widetilde{\Psi}_{ij}| \geq M \end{cases} \tag{8.25}$$

for all i,j. Thus $|\widetilde{\Psi}_{ij}^{(M)}| \leq M, \forall i,j$.

Associated with this bounded stream matrix $\widetilde{\Psi}^{(M)}$, there is the effective diffusivity matrix $\sigma^{(M)}$ which admits the same variational principles as (4.43) and (4.44):

$$\sigma_{ij}^{(M)} = \inf_{\substack{\tilde{\nabla}\times\tilde{\mathbf{F}}=0 \\ \langle\tilde{\mathbf{F}}\rangle=(\mathbf{e}^i+\mathbf{e}^j)/2}} \int_{\Omega} P(d\omega)$$

$$\times\Big(\widetilde{\mathbf{F}}\cdot\widetilde{\mathbf{F}} + \tilde{\Gamma}\widetilde{\Psi}^{(M)}\widetilde{\mathbf{F}}\cdot\tilde{\Gamma}\widetilde{\Psi}^{(M)}\widetilde{\mathbf{F}} - \widetilde{\Psi}^{(M)}\widetilde{\mathbf{F}}\cdot(\mathbf{e}^i-\mathbf{e}^j) - |\frac{\mathbf{e}^i-\mathbf{e}^j}{2}|^2\Big) \tag{8.26}$$

$$= \sup_{\substack{\tilde{\nabla}\times\tilde{\mathbf{F}}'=0 \\ \langle\tilde{\mathbf{F}}'\rangle=(\mathbf{e}^i-\mathbf{e}^j)/2}} \int_{\Omega} P(d\omega)$$

$$\times\Big(-\widetilde{\mathbf{F}}'\cdot\widetilde{\mathbf{F}}' - \tilde{\Gamma}\widetilde{\Psi}^{(M)}\widetilde{\mathbf{F}}'\cdot\tilde{\Gamma}\widetilde{\Psi}^{(M)}\widetilde{\mathbf{F}}' + \widetilde{\Psi}^{(M)}\widetilde{\mathbf{F}}'\cdot(\mathbf{e}^i+\mathbf{e}^j) + |\frac{\mathbf{e}^i+\mathbf{e}^j}{2}|^2\Big). \tag{8.27}$$

In the variational principles (4.43) and (4.44) for σ^{eff} the operator $\widetilde{\Gamma}\widetilde{\Psi}$ has domain defined by

$$\mathscr{D} = \{\widetilde{\mathbf{F}} \in L^2 | \widetilde{\Psi}\widetilde{\mathbf{F}} \in L^2\}. \tag{8.28}$$

To approximate σ^{eff} with arbitrary accuracy it is thus enough to consider the trial fields $\widetilde{\mathbf{F}} \in \mathscr{D}$. That is, $\mathscr{D}$ is dense in $\mathscr{H}_g(\widetilde{\Psi})$ in the $\|\cdot\|_{\widetilde{\Psi}}$ norm. On the other hand, since $\widetilde{\Psi}^{(M)}$ is bounded (and hence $\widetilde{\Gamma}\widetilde{\Psi}^{(M)}$ is bounded in L^2), a suitable domain for (8.26) and (8.27) is the space $\mathscr{B}$ of bounded gradient fields which is dense in $\mathscr{H}_g$ in the usual L^2 sense. The extremal values of (8.26) and (8.27) can be achieved within arbitrary accuracy in $\mathscr{B}$. Clearly, we have $\mathscr{B} \subset \mathscr{D}$.

Before proving Lemma 8.3 and 6.3, we note that for any $\widetilde{\mathbf{F}} \in \mathscr{D}$, we have

$$< [(\widetilde{\Psi} - \widetilde{\Psi}^{(M)})\widetilde{\mathbf{F}}]^2 > \rightarrow 0. \tag{8.29}$$

as $M \rightarrow \infty$. Consequently,

$$< \widetilde{\Gamma}(\widetilde{\Psi} - \widetilde{\Psi}^{(M)})\widetilde{\mathbf{F}} \cdot \widetilde{\Gamma}(\widetilde{\Psi} - \widetilde{\Psi}^{(M)})\widetilde{\mathbf{F}} > \rightarrow 0 \tag{8.30}$$

as $M \rightarrow \infty$.

A crucial step toward establishing the approximation Lemma 6.3 is the convergence of $\sigma^{(M)}$ as $M \rightarrow \infty$:

Lemma 8.3

$$\lim_{M\rightarrow\infty} \sigma^{(M)} = \sigma^{eff}. \tag{8.31}$$

Proof. We first show the upper bound: $\limsup_{M\to\infty} \sigma_{ij}^{(M)} \leq \sigma_{ij}^{eff}, \forall i, j$, using the minimum principles (4.43) and (8.26).

By the previous remark and (4.43), for given $\epsilon > 0$ there exists a $\widetilde{\mathbf{F}} \in \mathscr{D}$ such that

$$< \widetilde{\mathbf{F}}\cdot\widetilde{\mathbf{F}} > + < \tilde{\Gamma}\tilde{\Psi}\widetilde{\mathbf{F}}\cdot\tilde{\Gamma}\tilde{\Psi}\widetilde{\mathbf{F}} > - < \tilde{\Psi}\widetilde{\mathbf{F}}\cdot(\mathbf{e}^i - \mathbf{e}^j) > - |\frac{\mathbf{e}^i - \mathbf{e}^j}{2}|^2 \leq \sigma^{eff} + \epsilon. \quad (8.32)$$

By (8.29) and (8.30), we also know that the left-side of (8.32) is bigger than

$$< \widetilde{\mathbf{F}}\cdot\widetilde{\mathbf{F}} > + < \tilde{\Gamma}\tilde{\Psi}^{(M)}\widetilde{\mathbf{F}}\cdot\tilde{\Gamma}\tilde{\Psi}^{(M)}\widetilde{\mathbf{F}} > - < \tilde{\Psi}^{(M)}\widetilde{\mathbf{F}}\cdot(\mathbf{e}^i - \mathbf{e}^j) > - |\frac{\mathbf{e}^i - \mathbf{e}^j}{2}|^2 - \epsilon \quad (8.33)$$

which in turn is bigger than

$$\sigma^{(M)} - \epsilon \quad (8.34)$$

in view of (8.26). Thus we have that

$$\sigma_{ij}^{eff} \geq \sigma_{ij}^{(M)} + 2\epsilon \quad (8.35)$$

for sufficiently large M. This proves the upper bound.

We turn to the lower bound: $\liminf_{M\to\infty} \sigma_{ij}^{(M)} \geq \sigma_{ij}^{eff}, \forall i, j$.

By the maximum principle (4.44), there exists $\widetilde{\mathbf{F}}' \in \mathscr{D}$ for given $\epsilon > 0$, such that

$$\sigma_{ij}^{eff} - \epsilon \leq < -\widetilde{\mathbf{F}}'\cdot\widetilde{\mathbf{F}}' > - < \tilde{\Gamma}\tilde{\Psi}\widetilde{\mathbf{F}}'\cdot\tilde{\Gamma}\tilde{\Psi}\widetilde{\mathbf{F}}' >$$
$$+ < \tilde{\Psi}^{(M)}\widetilde{\mathbf{F}}'\cdot(\mathbf{e}^i + \mathbf{e}^j) > + |\frac{\mathbf{e}^i + \mathbf{e}^j}{2}|^2. \quad (8.36)$$

In view of (8.29) and (8.30), the right side of (8.36) in turn is bounded by

$$- < \widetilde{\mathbf{F}}'\cdot\widetilde{\mathbf{F}}' > - < \tilde{\Gamma}\tilde{\Psi}^{(M)}\widetilde{\mathbf{F}}'\cdot\tilde{\Gamma}\tilde{\Psi}^{(M)}\widetilde{\mathbf{F}}' >$$
$$+ < \tilde{\Psi}^{(M)}\widetilde{\mathbf{F}}'\cdot(\mathbf{e}^i + \mathbf{e}^j) > + |\frac{\mathbf{e}^i + \mathbf{e}^j}{2}|^2 + \epsilon \quad (8.37)$$

for sufficiently large M. Combining (8.36), (8.37) and (8.27), we have

$$\sigma_{ij}^{eff} - 2\epsilon \leq \liminf_{M\to\infty} \sigma_{ij}^{(M)} \quad (8.38)$$

for any $\epsilon > 0$. This completes the proof.

Using Lemma 8.3, we now prove Lemma 6.3.

Proof of Lemma 6.3. For the minimum principle (4.43), suffice it to show that given $\epsilon > 0$, there exists bounded gradient field $\widetilde{\mathbf{F}}$ with $< \widetilde{\mathbf{F}} > = \frac{\mathbf{e}_i + \mathbf{e}_j}{2}$ such that

$$\sigma_{ij}^{eff} + \epsilon \geq < \widetilde{\mathbf{F}}\cdot\widetilde{\mathbf{F}} > + < \tilde{\Gamma}\tilde{\Psi}\widetilde{\mathbf{F}}\cdot\tilde{\Gamma}\tilde{\Psi}\widetilde{\mathbf{F}} > - < \tilde{\Psi}\widetilde{\mathbf{F}}\cdot(\mathbf{e}^i - \mathbf{e}^j) > - |\frac{\mathbf{e}^i - \mathbf{e}^j}{2}|^2. \quad (8.39)$$

By Lemma 8.3, we have that

$$\sigma_{ij}^{eff} + \frac{\epsilon}{2} \geq \sigma_{ij}^{(M)} \quad (8.40)$$

for sufficiently large M. By the remark in the begining of the section, there exists $\widetilde{\mathbf{F}} \in \mathscr{B}$ such that

$$\sigma_{ij}^{(M)} + \frac{\epsilon}{2} \geq < \widetilde{\mathbf{F}} \cdot \widetilde{\mathbf{F}} > + < \tilde{\Gamma}\tilde{\Psi}^{(M)}\widetilde{\mathbf{F}} \cdot \tilde{\Gamma}\tilde{\Psi}^{(M)}\widetilde{\mathbf{F}} >$$
$$- < \tilde{\Psi}^{(M)}\widetilde{\mathbf{F}} \cdot (\mathbf{e}^i - \mathbf{e}^j) > - |\frac{\mathbf{e}^i - \mathbf{e}^j}{2}|^2 \tag{8.41}$$

Moreover, (8.41) is valid uniformly in M by (8.29) and (8.30). Equations (8.40), (8.41) together with (8.29), (8.30) imply (8.39).

We turn to the maximum principle (4.44). Suffice it to show that given $\epsilon > 0$, there exists $\widetilde{\mathbf{F}}' \in \mathscr{B}$ with $< \widetilde{\mathbf{F}}' > = \frac{\mathbf{e}_i - \mathbf{e}_j}{2}$ such that such that

$$\sigma_{ij}^{eff} - \epsilon \leq - < \widetilde{\mathbf{F}}' \cdot \widetilde{\mathbf{F}}' > - < \tilde{\Gamma}\tilde{\Psi}^{(M)}\widetilde{\mathbf{F}}' \cdot \tilde{\Gamma}\tilde{\Psi}^{(M)}\widetilde{\mathbf{F}}' >$$
$$+ < \tilde{\Psi}^{(M)}\widetilde{\mathbf{F}}' \cdot (\mathbf{e}^i + \mathbf{e}^j) > + |\frac{\mathbf{e}^i + \mathbf{e}^j}{2}|^2. \tag{8.42}$$

By Lemma 8.3, we have that

$$\sigma_{ij}^{eff} - \frac{\epsilon}{2} \leq \sigma_{ij}^{(M)} \tag{8.43}$$

for sufficiently large M. Thus it follows from (8.27) and the density of $\mathscr{B}$ for bounded $\widetilde{\Psi}^{(M)}$ that there exists $\mathbf{F}' \in \mathscr{B}$ such that

$$\sigma_{ij}^{eff} - \epsilon \leq - < \widetilde{\mathbf{F}}' \cdot \widetilde{\mathbf{F}}' > - < \tilde{\Gamma}\tilde{\Psi}^{(M)}\widetilde{\mathbf{F}}' \cdot \tilde{\Gamma}\tilde{\Psi}^{(M)}\widetilde{\mathbf{F}}' >$$
$$+ < \tilde{\Psi}^{(M)}\widetilde{\mathbf{F}}' \cdot (\mathbf{e}^i + \mathbf{e}^j) > + |\frac{\mathbf{e}^i + \mathbf{e}^j}{2}|^2. \tag{8.44}$$

Note that (8.44) is valid for all sufficiently large M due to (8.29) and (8.30). Thus, (8.42) follows by passing to the limit $M \to \infty$.

Before leaving this section let us remark that Lemma 6.3 is essentially equivalent to Lemma 6.4 (the latter appears stronger). Lemma 6.4 states that the closure of $\mathscr{B}$, with respect to the norm $\| \cdot \|_{\widetilde{\Psi}}$, contains the exact solutions $\widetilde{\mathbf{E}}_{jj}, j = 1, 2, 3..., d$ in $\mathscr{H}_g(\widetilde{\Psi})$, but it is not clear if $\mathscr{B}$ is dense everywhere in $\mathscr{H}_g(\widetilde{\Psi})$. In other words, we do not know if $\mathscr{B}$ is a core for the unbounded operator $\widetilde{\Gamma}\widetilde{\Psi}$.

8.4 Poisson problem for upper bound

Without loss of generality, let the domain $\mathscr{O}$ be the square $|x_i| \leq 1, \ i = 1, ..., d$. Consider the inhomogeneous boundary value problem

$$\Delta f_n(\mathbf{x}, \omega) + \nabla \cdot \Psi_n \mathbf{F}_n = 0, \quad \text{in } \mathscr{O} \tag{8.45}$$
$$f_n = 0, \quad \text{on } \partial\mathscr{O}. \tag{8.46}$$

where the inhomogeneous term $\mathbf{F}_n(\mathbf{x}, \omega)$ has the form

$$\mathbf{F}_n(\mathbf{x},\omega) = \sum_j \nabla(\alpha_n(\mathbf{x}) f_n^j(\mathbf{x},\omega) \frac{\partial \rho}{\partial x_j}(\mathbf{x})) + \nabla \rho(\mathbf{x}), \quad \text{in } \mathscr{O}. \tag{8.47}$$

Here $\rho(\mathbf{x}) \in C_0^\infty(\mathscr{O})$ and $f_n^j(\mathbf{x},\omega), j = 1,...,d$ are non-stationary random functions whose gradients are

$$(\nabla f_n^j)(\mathbf{x},\omega) = \mathbf{F}^j(n\mathbf{x},\omega) = T_{n\mathbf{x}} \widetilde{\mathbf{F}}^j(\omega) \in \mathscr{H}_g(\Omega). \tag{8.48}$$

We take the gradients $\widetilde{\mathbf{F}}^i, i = 1,...,d$, to be *essentially bounded.* The cut-off function is

$$\alpha_n(\mathbf{x}) = \prod_{i=1}^d \gamma(\frac{1+x_i}{\tau_n})\gamma(\frac{1-x_i}{\tau_n}) \tag{8.49}$$

with $\gamma(s) \in C^\infty(R)$ such that

$$0 \le \gamma(s) \le 1 \tag{8.50}$$

$$\gamma(s) = \begin{cases} 1, & |s| \ge 2, \\ 0, & |s| \le 1. \end{cases} \tag{8.51}$$

and τ_n is a decreasing sequence of positive numbers with a rate that will be determined later and depends on $f^j(\mathbf{x},\omega), j = 1,...,d$. We denote the set $\{\mathbf{x} \mid \alpha_n(\mathbf{x}) = 1\}$ by $\mathscr{O}'$.

We shall show how to solve (8.45), (8.46) in terms of $f_n^{'i}(\mathbf{x},\omega), i = 1,...,d$, whose gradients

$$(\nabla f_n^{'i})(\mathbf{x},\omega) = \mathbf{F}^{'i}(n\mathbf{x},\omega) = T_{n\mathbf{x}} \widetilde{\mathbf{F}}^{'i}(\omega) \in \mathscr{H}_g(\Omega) \tag{8.52}$$

satisfy

$$\widetilde{\nabla} \cdot \widetilde{\mathbf{F}}^{'i} + \widetilde{\nabla} \cdot \widetilde{\Psi}(\widetilde{\mathbf{F}}^i + \mathbf{e}^i) = 0, \tag{8.53}$$

$$\langle \widetilde{\mathbf{F}}^{'i} \rangle = 0 \tag{8.54}$$

We impose the normalization conditions

$$f_n^i(0,\omega) = 0 \tag{8.55}$$

$$\int_{\mathscr{O}} d\mathbf{x}\, f_n^{'i}(\mathbf{x},\omega) = 0, \tag{8.56}$$

so that, by Lemmas 8.2 and 6.1,

$$\int_{\mathscr{O}} d\mathbf{x}\, (f_n^{'i})^2(\mathbf{x},\omega) \to 0, \tag{8.57}$$

$$\sup_{\mathbf{x}\in\mathscr{O}} (f_n^i)^2(\mathbf{x},\omega) \to 0, \tag{8.58}$$

in the limit $n \to \infty$, with probability one.

We prove

Lemma 8.4 *Let z_n be defined by*

$$z_n(\mathbf{x},\omega) = f_n(\mathbf{x},\omega) - \sum_j f_n'^{j}(\mathbf{x},\omega)\frac{\partial \rho(\mathbf{x})}{\partial x_j}\alpha_n(\mathbf{x}), \tag{8.59}$$

assume that $\langle|\widetilde{\Psi}|^2\rangle < \infty$, $\widetilde{\mathbf{F}}^j \in \mathscr{H}_g(\Omega)$(defined in (3.3)), $j = 1, ..., d$, is essentially bounded and the normalization conditions (8.55), (8.56) hold. Then

$$\int_{\mathscr{O}} d\mathbf{x}\ (\nabla z_n)^2 \to 0, \quad as \quad n \to \infty, \tag{8.60}$$

for almost all ω.

Proof of Lemma 8.4. Under the same assumptions of Lemma 8.4, we first prove a lemma which implies that z_n tends to zero in the L^2-norm in the limit $n \to \infty$.

Lemma 8.5 *For almost all ω, ∇f_n converges weakly to zero and*

$$\int_{\mathscr{O}} d\mathbf{x}\ g_n^2 \to 0 \tag{8.61}$$

as $n \to \infty$.

Proof of Lemma 8.5. Multiplying (8.45) by any test function $\phi \in C_0^\infty(\mathscr{O})$ and integrating by parts give

$$\int_{\mathscr{O}} d\mathbf{x}\ \nabla\phi \cdot \nabla f_n + \int_{\mathscr{O}} d\mathbf{x}\ \Psi_n \mathbf{F}_n \cdot \nabla\phi = 0 \tag{8.62}$$

The second integral vanishes in the limit. It follows that $\{\nabla f_n\}$ converges weakly to zero.

The energy estimate for (8.45), (8.46) gives

$$\begin{aligned}\int_{\mathscr{O}} d\mathbf{x}\ (\nabla f_n)^2 &= -\int_{\mathscr{O}} d\mathbf{x}\ \Psi_n \mathbf{F}_n \cdot \nabla f_n \\ &\leq |\Psi_n \mathbf{F}_n|_{L^2(\mathscr{O})} |\nabla f_n|_{L^2(\mathscr{O})} \\ &\leq (|\widetilde{\Psi}\widetilde{\mathbf{F}}|_{L^2(\Omega)} + \delta)|\nabla f_n|_{L^2(\mathscr{O})}\end{aligned} \tag{8.63}$$

for any given $\delta > 0$ and $n > n_o(\delta, \omega)$, by ergodicity. Hence

$$|\nabla f_n|_{L^2(\mathscr{O})} \leq c \tag{8.64}$$

where c is a constant independent of n. This implies that $\{f_n\}$ is precompact in the strong L^2 sense. Thus, by the strong compactness, $\{f_n\}$ converges strongly to zero.

We return now to the proof of Lemma 8.4 and note that Lemma 8.5 and (8.57) imply that

$$\nabla z_n \to 0, \text{ weakly} \tag{8.65}$$

and

$$\int_{\mathscr{O}} d\mathbf{x}\ z_n^2 \to 0 \tag{8.66}$$

as $n \to \infty$. From equations (8.45), (8.53) it follows that

$$\begin{aligned}
\Delta z_n &= -\nabla \cdot \Psi_n \left(\nabla \left(\sum_j f_n^j \frac{\partial \rho}{\partial x_j} \alpha_n \right) + \nabla \rho \right) - \sum_j \Delta f_n^{\prime j} \frac{\partial \rho}{\partial x_j} \alpha_n \\
&\quad - \sum_j \nabla f_n^{\prime j} \cdot \nabla \left(\frac{\partial \rho}{\partial x_j} \alpha_n \right) - \sum_j \nabla \cdot \left(f_n^{\prime j} \nabla \left(\frac{\partial \rho}{\partial x_j} \alpha_n \right) \right) \\
&= -\nabla \cdot \Psi_n \sum_j \nabla f_n^j \frac{\partial \rho}{\partial x_j} \alpha_n - \nabla \cdot \Psi_n \sum_j f_n^j \nabla \left(\frac{\partial \rho}{\partial x_j} \alpha_n \right) - \nabla \cdot \Psi_n \nabla \rho \\
&\quad + \nabla \cdot \Psi_n \sum_j \left(\nabla f_n^j + \mathbf{e}^j \right) \frac{\partial \rho}{\partial x_j} \alpha_n - \sum_j \nabla f_n^{\prime j} \cdot \nabla \left(\frac{\partial \rho}{\partial x_j} \alpha_n \right) \\
&\quad - \sum_j \nabla \cdot \left(f_n^{\prime j} \nabla \left(\frac{\partial \rho}{\partial x_j} \alpha_n \right) \right) \\
&= -\nabla \cdot \Psi_n \nabla \rho (1 - \alpha_n) - \nabla \cdot \Psi_n \sum_j f_n^j \nabla \left(\frac{\partial \rho}{\partial x_j} \alpha_n \right) \\
&\quad - \sum_j \nabla f_n^{\prime j} \cdot \nabla \left(\frac{\partial \rho}{\partial x_j} \alpha_n \right) - \sum_j \nabla \cdot \left(f_n^{\prime j} \nabla \left(\frac{\partial \rho}{\partial x_j} \alpha_n \right) \right) \qquad (8.67)
\end{aligned}$$

The major terms $\nabla \cdot \Psi_n \sum_j \nabla f_n^j \frac{\partial \rho}{\partial x_j} \alpha_n$, $\nabla \cdot \Psi_n \nabla \rho$ and $\sum_j \Delta f_n^{\prime j} \frac{\partial \rho}{\partial x_j} \alpha_n$ nearly cancel because of (8.53) and the residual is $\nabla \cdot \Psi_n \nabla \rho (1 - \alpha_n)$. Multiplying (8.67) by z_n, integrating by parts and using the Schwartz inequality gives

$$\begin{aligned}
&\int_{\mathcal{O}} d\mathbf{x} \, (\nabla z_n)^2 \\
&= -\int_{\mathcal{O}} d\mathbf{x} \, \Psi_n \nabla \rho (1 - \alpha_n) \cdot \nabla z_n - \int_{\mathcal{O}} d\mathbf{x} \, \Psi_n \nabla \rho \cdot \nabla (1 - \alpha_n) z_n \\
&\quad - \int_{\mathcal{O}} d\mathbf{x} \, \Psi_n \sum_j \nabla f_n^j \cdot \nabla \left(\frac{\partial \rho}{\partial x_j} \alpha_n \right) z_n - \int_{\mathcal{O}} d\mathbf{x} \, \Psi_n \sum_j f_n^j \nabla \left(\frac{\partial \rho}{\partial x_j} \alpha_n \right) \cdot \nabla z_n \\
&\quad + \int_{\mathcal{O}} d\mathbf{x} \, \sum_j \nabla f_n^{\prime j} \cdot \nabla \left(\frac{\partial \rho}{\partial x_j} \alpha_n \right) z_n - \int_{\mathcal{O}} d\mathbf{x} \, \sum_j f_n^{\prime j} \nabla \left(\frac{\partial \rho}{\partial x_j} \alpha_n \right) \cdot \nabla z_n \\
&\leq \int_{\mathcal{O}} d\mathbf{x} \, |\Psi_n \nabla \rho (1 - \alpha_n) \cdot \nabla z_n| + \int_{\mathcal{O}} d\mathbf{x} \, |\Psi_n \nabla \rho \cdot \nabla (1 - \alpha_n) z_n| \\
&\quad + \int_{\mathcal{O}} d\mathbf{x} \, |\Psi_n \sum_j \nabla f_n^j \cdot \nabla \frac{\partial \rho}{\partial x_j} \alpha_n z_n| + \int_{\mathcal{O}} d\mathbf{x} \, |\Psi_n \sum_j \nabla f_n^j \frac{\partial \rho}{\partial x_j} \cdot \nabla \alpha_n z_n| \\
&\quad + \int_{\mathcal{O}} d\mathbf{x} \, |\Psi_n \sum_j f_n^j \nabla \frac{\partial \rho}{\partial x_j} \alpha_n \cdot \nabla z_n| + \int_{\mathcal{O}} d\mathbf{x} \, |\Psi_n \sum_j f_n^j \frac{\partial \rho}{\partial x_j} \nabla \alpha_n \cdot \nabla z_n| \\
&\quad + \int_{\mathcal{O}} d\mathbf{x} \, |\sum_j \nabla f_n^{\prime j} \nabla \frac{\partial \rho}{\partial x_j} \alpha_n z_n| + \int_{\mathcal{O}} d\mathbf{x} \, |\sum_j \nabla f_n^{\prime j} \frac{\partial \rho}{\partial x_j} \cdot \nabla \alpha_n z_n|
\end{aligned}$$

$$
\begin{aligned}
&+\int_{\mathcal{O}} d\mathbf{x}\ \Big|\sum_j f_n'^j \nabla \frac{\partial \rho}{\partial x_j}\alpha_n \cdot \nabla z_n\Big| + \int_{\mathcal{O}} d\mathbf{x}\ \Big|\sum_j f_n'^j \frac{\partial \rho}{\partial x_j}\nabla\alpha_n \cdot \nabla z_n\Big| \\
&\leq \left(\int_{\mathcal{O}} d\mathbf{x}\ (\Psi_n \nabla\rho(1-\alpha_n))^2\right)^{1/2}\left(\int_{\mathcal{O}\setminus\mathcal{O}'} d\mathbf{x}\ (\nabla z_n)^2\right)^{1/2} \\
&+\left(\int_{\mathcal{O}} d\mathbf{x}\ (\Psi_n \nabla\rho\cdot\nabla(1-\alpha_n))^2\right)^{1/2}\left(\int_{\mathcal{O}\setminus\mathcal{O}'} d\mathbf{x}\ z_n^2\right)^{1/2} \\
&+\left(\int_{\mathcal{O}} d\mathbf{x}\ \left(\Psi_n \sum_j \nabla f_n^j \cdot \nabla \frac{\partial \rho}{\partial x_j}\alpha_n\right)^2\right)^{1/2}\left(\int_{\mathcal{O}} d\mathbf{x}\ z_n^2\right)^{1/2} \\
&+\left(\int_{\mathcal{O}} d\mathbf{x}\ \left(\Psi_n \sum_j \nabla f_n^j \frac{\partial \rho}{\partial x_j}\cdot\nabla\alpha_n\right)^2\right)^{1/2}\left(\int_{\mathcal{O}\setminus\mathcal{O}'} d\mathbf{x}\ z_n^2\right)^{1/2} \\
&+\left(\int_{\mathcal{O}} d\mathbf{x}\ \left(\Psi_n \sum_j f_n^j \nabla \frac{\partial \rho}{\partial x_j}\alpha_n\right)^2\right)^{1/2}\left(\int_{\mathcal{O}} d\mathbf{x}\ (\nabla z_n)^2\right)^{1/2} \\
&+\left(\int_{\mathcal{O}} d\mathbf{x}\ \left(\Psi_n \sum_j f_n^j \frac{\partial \rho}{\partial x_j}\nabla\alpha_n\right)^2\right)^{1/2}\left(\int_{\mathcal{O}\setminus\mathcal{O}'} d\mathbf{x}\ (\nabla z_n)^2\right)^{1/2} \\
&+\left(\int_{\mathcal{O}} d\mathbf{x}\ \left(\sum_j \nabla f_n'^j \nabla \frac{\partial \rho}{\partial x_j}\alpha_n\right)^2\right)^{1/2}\left(\int_{\mathcal{O}} d\mathbf{x}\ z_n^2\right)^{1/2} \\
&+\left(\int_{\mathcal{O}} d\mathbf{x}\ \left(\sum_j \nabla f_n'^j \frac{\partial \rho}{\partial x_j}\cdot\nabla\alpha_n\right)^2\right)^{1/2}\left(\int_{\mathcal{O}\setminus\mathcal{O}'} d\mathbf{x}\ z_n^2\right)^{1/2} \\
&+\left(\int_{\mathcal{O}} d\mathbf{x}\ \left(\sum_j f_n'^j \nabla \frac{\partial \rho}{\partial x_j}\alpha_n\right)^2\right)^{1/2}\left(\int_{\mathcal{O}} d\mathbf{x}\ (\nabla z_n)^2\right)^{1/2} \\
&+\left(\int_{\mathcal{O}} d\mathbf{x}\ \left(\sum_j f_n'^j \frac{\partial \rho}{\partial x_j}\nabla\alpha_n\right)^2\right)^{1/2}\left(\int_{\mathcal{O}\setminus\mathcal{O}'} d\mathbf{x}\ (\nabla z_n)^2\right)^{1/2} \qquad (8.68)
\end{aligned}
$$

There are no boundary contributions because of the boundary condition for z_n and the cut-off function α_n.

We note that the Poincare inequality for z_n in $\mathcal{O} \setminus \mathcal{O}'$ gives

$$\int_{\mathscr{O}\backslash\mathscr{O}'} d\mathbf{x}\, z_n^2 \leq c_1\tau_n^2 \int_{\mathscr{O}\backslash\mathscr{O}'} d\mathbf{x}\ (\nabla z_n)^2$$
$$\leq c_1\tau_n^2 \int_{\mathscr{O}} d\mathbf{x}\ (\nabla z_n)^2 , \tag{8.69}$$

since $\mathscr{O}\backslash\mathscr{O}'$ is a strip of width τ_n near $\partial\mathscr{O}$ and z_n vanishes on $\partial\mathscr{O}$. The estimate (8.69) holds for all $H^1(\mathscr{O})$-functions with zero Dirichlet data and the constant c_1 depends on the domain $\mathscr{O}$. For the cutoff function we have the estimates

$$|\nabla\alpha_n|^2 \leq \frac{c_2}{\tau_n^2}, \tag{8.70}$$

$$\int_{\mathscr{O}} d\mathbf{x}\ (\nabla\alpha_n)^2 \leq \frac{c_3}{\tau_n}, \tag{8.71}$$

We now use (8.69), (8.70) and (8.71) whenever the integral $\int_{\mathscr{O}\backslash\mathscr{O}'} d\mathbf{x}\, z_n^2$ occurs. Lemma 8.4 then follows from (8.66) and

$$\int_{\mathscr{O}\backslash\mathscr{O}'} d\mathbf{x}\, \Psi_n^2 \rightarrow 0, \tag{8.72}$$

$$\int_{\mathscr{O}\backslash\mathscr{O}'} d\mathbf{x}\ \left(\nabla f_n^j\right)^2 \rightarrow 0, \quad j = 1, ..., d \tag{8.73}$$

$$\int_{\mathscr{O}\backslash\mathscr{O}'} d\mathbf{x}\ \left(\nabla f_n^{'j}\right)^2 \rightarrow 0, \quad j = 1, ..., d \tag{8.74}$$

$$\int_{\mathscr{O}\backslash\mathscr{O}'} d\mathbf{x}\ \left(\Psi_n \nabla f_n^j\right)^2 \rightarrow 0, \quad j = 1, ..., d \tag{8.75}$$

$$\frac{1}{\tau_n^2}\int_{\mathscr{O}} d\mathbf{x}\ (f_n^{'j})^2 \rightarrow 0, \quad j = 1, ..., d, \tag{8.76}$$

$$\frac{1}{\tau_n^2}\left(\sup_{\mathbf{x}\in\mathscr{O}}(f_n^j)^2\right)\left(\int_{\mathscr{O}} d\mathbf{x}\, \Psi_n^2\right) \rightarrow 0, \quad j = 1, ..., d. \tag{8.77}$$

for almost all ω. The estimate (8.77) is used for the integrals whose integrands involve $\Psi_n f_n^j$.

The estimates (8.72)-(8.75) are immediate consequences of the ergodicity, since $\tau_n \rightarrow 0$. To get (8.76) and (8.77), we first note that

$$\int_{\mathscr{O}} d\mathbf{x}\, \Psi_n^2 \leq \langle|\widetilde{\Psi}|^2\rangle + \delta \tag{8.78}$$

for any $\delta > 0$, for n sufficiently large. So we must now choose a proper cut-off rate τ_n. Let

$$\eta_n = \max\left\{\int_{\mathscr{O}} d\mathbf{x}\ (f_n^{'i})^2,\ \sup_{\mathbf{x}\in\mathscr{O}}(f_n^i)^2,\ i = 1, ..., d\right\}. \tag{8.79}$$

We know from (8.58), (8.57) that

$$\eta_n \rightarrow 0, \quad \text{as} \quad n \rightarrow \infty, \quad \text{with probability one.} \tag{8.80}$$

The desired result (8.60) and Lemma 8.4 follow from

$$\tau_n \to 0, \quad \frac{\eta_n}{\tau_n^2} \to 0, \quad \text{as } n \to \infty. \tag{8.81}$$

when we let $\tau_n = \eta_n^{1/4}$.

Once Lemma 8.4 is proved, the cut-off function α_n can be omitted in the limit as $n \to \infty$. We have

$$\int_{\mathcal{O}} d\mathbf{x} \, \left(\nabla z_n'\right)^2 \to 0, \tag{8.82}$$

where

$$z_n'(\mathbf{x}, \omega) = f_n(\mathbf{x}, \omega) - \sum_j f_n'^j(\mathbf{x}, \omega) \frac{\partial \rho(\mathbf{x})}{\partial x_j}. \tag{8.83}$$

because

$$\int_{\mathcal{O}} d\mathbf{x} \, \left(\sum_j \nabla f_n'^j \frac{\partial \rho}{\partial x_j} (1 - \alpha_n) \right)^2 \to 0, \tag{8.84}$$

$$\int_{\mathcal{O}} d\mathbf{x} \, \left(\sum_j f_n'^j \nabla \frac{\partial \rho}{\partial x_j} (1 - \alpha_n) \right)^2 \to 0, \tag{8.85}$$

$$\int_{\mathcal{O}} d\mathbf{x} \, \left(\sum_j f_n'^j \frac{\partial \rho}{\partial x_j} \nabla \alpha_n \right)^2 \to 0, \tag{8.86}$$

as $n \to \infty$, for almost all ω, for $j = 1, 2, 3, ..., d$.

We note that Lemma 8.4 is also valid if the left side of (8.60) is averaged over ω. This can be seen by applying the Lebesgue dominated convergence theorem.

8.5 Poisson problem for lower bound

We again assume, without loss of generality, that the domain $\mathcal{O}$ is the square $|x_i| \leq 1, \ i = 1, ..., d$.

The Poisson problem in this case is

$$\Delta g_n'(\mathbf{x}, \omega) + \nabla \cdot \Psi_n \mathbf{G}_n' = s + \nabla \cdot \mathbf{S}, \quad \text{in } \mathcal{O} \tag{8.87}$$

$$g_n' = 0, \quad \text{on } \partial\mathcal{O}. \tag{8.88}$$

where the inhomogeneous term $\mathbf{G}_n'(\mathbf{x}, \omega)$ has the form

$$\mathbf{G}_n'(\mathbf{x}, \omega) = \sum_j \nabla(\alpha_n(\mathbf{x}) g_n'^j(\mathbf{x}, \omega) \frac{\partial \rho}{\partial x_j}(\mathbf{x})) \tag{8.89}$$

Here $\rho(\mathbf{x})$ is some function to be determined later, α_n is a cutoff function as defined in (8.49)-(8.50) with the cut-off rate τ_n to be determined later and $g_n'^j(\mathbf{x}, \omega)$ are related to $\widetilde{\mathbf{G}}'^i$ via (8.52).

Euqation (8.89) differs from equation (8.45) in the additional terms $s+\nabla\cdot\mathbf{S}$. To balance the inhomogenous terms $s+\nabla\cdot\mathbf{S}$, we introduce an additional term ρ in the ansatz:

$$\rho+\sum_j g_n^j(\mathbf{x},\omega)\frac{\partial\rho(\mathbf{x})}{\partial x_j}\alpha_n(\mathbf{x}). \tag{8.90}$$

Here $g_n^j(\mathbf{x},\omega)$ are related to $g_n^{\prime j}(\mathbf{x},\omega)$ in the following way: Let $\tilde{\mathbf{G}}^i$ solve the equations

$$\widetilde{\nabla}\cdot\widetilde{\mathbf{G}}^i+\widetilde{\nabla}\cdot\widetilde{\Psi}\widetilde{\mathbf{G}}'^i=0,\quad i=1,\ldots,d \tag{8.91}$$

$$\langle\widetilde{\mathbf{G}}^i\rangle=0,\quad i=1,\ldots,d \tag{8.92}$$

with given stationary fields $\widetilde{\mathbf{G}}'^i$. The solutions $\mathbf{G}^i(\mathbf{x},\omega)$ can be written as gradients of nonstationary random functions $g^i(\mathbf{x},\omega)$. Let the rescaled random functions be, as before,

$$g_n^i(\mathbf{x},\omega)=\frac{1}{n}g^i(n\mathbf{x},\omega). \tag{8.93}$$

Both $g_n^i(\mathbf{x},\omega)$ and $g_n^{\prime j}(\mathbf{x},\omega)$ are determined uniquely by the following normalization conditions

$$g_n^{\prime i}(0,\omega)=0 \tag{8.94}$$

$$\int_{\mathcal{O}}d\mathbf{x}\ g_n^i(\mathbf{x},\omega)=0. \tag{8.95}$$

By Lemma 8.2 and 6.1,

$$\int_{\mathcal{O}}d\mathbf{x}\ (g_n^i)^2(\mathbf{x},\omega)\to 0, \tag{8.96}$$

$$\sup_{\mathbf{x}\in\mathcal{O}}(g_n^{\prime i})^2(\mathbf{x},\omega)\to 0, \tag{8.97}$$

in the limit $n\to\infty$.

Due to the inhomogeneous terms $s+\nabla\cdot\mathbf{S}$, we do not expect Lemma 8.5 to hold for g_n'. However, if ρ is properly chosen, the difference

$$z_n=g_n'-\rho-\sum_j g_n^j(\mathbf{x},\omega)\frac{\partial\rho(\mathbf{x})}{\partial x_j}\alpha_n(\mathbf{x}) \tag{8.98}$$

can be made to satisfy

$$\int_{\mathcal{O}}(z_n)^2d\mathbf{x}\to 0 \tag{8.99}$$

as $n\to\infty$. For this, we must assume ρ to solve

$$\Delta\rho+\sum_i<\tilde{\Psi}\tilde{\mathbf{G}}'^i>\cdot\frac{\partial}{\partial x_i}\nabla\rho=s+\nabla\cdot\mathbf{S} \tag{8.100}$$

$$\rho=0,\quad \text{on }\partial\mathcal{O} \tag{8.101}$$

or

$$\sum_{i,j} \frac{\partial}{\partial x_i}(\delta_{ij} + < \tilde{\Psi}\tilde{\mathbf{G}}'^{i} > \cdot \mathbf{e}_j)\frac{\partial}{\partial x_j}\rho = s + \nabla \cdot \mathbf{S} \tag{8.102}$$

$$\rho = 0, \quad \text{on } \partial\mathcal{O}. \tag{8.103}$$

Note that, in view of (4.17) and (4.18),

$$\delta_{ij} + < \tilde{\Psi}\tilde{\mathbf{G}}'^{i} > \cdot \mathbf{e}_j \tag{8.104}$$

is precisely the symmetric part of the effective diffusivity $\frac{1}{2}(\sigma^{eff}_{ij} + \sigma^{eff}_{ji})$ if $\tilde{\mathbf{G}}'^{i} = \frac{1}{2}(\tilde{\mathbf{E}}^{+}_{\mathbf{e}_i} - \tilde{\mathbf{E}}^{-}_{\mathbf{e}_i})$.

Equations (8.100) and (8.101) follow naturally from the following consideration: From the equations (8.87), (8.91) it follows that

$$\begin{aligned}
\Delta z_n &= s + \nabla \cdot \mathbf{S} - \left((\nabla \cdot \Psi_n) \sum_j \nabla g'^{j}_n \frac{\partial \rho}{\partial x_j}\alpha_n + (\nabla \cdot \Psi_n) \sum_j g'^{j}_n \nabla \left(\frac{\partial \rho}{\partial x_j}\alpha_n\right)\right) \\
&\quad - \left(\sum_j \Delta g^{j}_n \frac{\partial \rho}{\partial x_j}\alpha_n + \Delta\rho + \sum_j \nabla g^{j}_n \cdot \nabla \left(\frac{\partial \rho}{\partial x_j}\alpha_n\right)\right. \\
&\quad \left.\left. + \sum_j \nabla \cdot \left(g^{j}_n \nabla \left(\frac{\partial \rho}{\partial x_j}\alpha_n\right)\right)\right)\right) \\
&= s + \nabla \cdot \mathbf{S} - \Delta\rho - (\nabla \cdot \Psi_n)\sum_j g'^{j}_n \nabla \left(\frac{\partial \rho}{\partial x_j}\alpha_n\right) - \sum_j \nabla g^{j}_n \cdot \frac{\partial \rho}{\partial x_j}\nabla\alpha_n \\
&\quad - \sum_j \nabla g^{j}_n \cdot \frac{\partial \nabla\rho}{\partial x_j}\alpha_n - \sum_j \nabla \cdot \left(g^{j}_n \nabla \left(\frac{\partial \rho}{\partial x_j}\alpha_n\right)\right)
\end{aligned} \tag{8.105}$$

The leading order $O(n)$-terms $\nabla \cdot \Psi_n \sum_j \nabla g'^{j}_n \frac{\partial \rho}{\partial x_j}\alpha_n$ and $\sum_j \Delta g^{j}_n \frac{\partial \rho}{\partial x_j}\alpha_n$ cancel because of (8.91).

Multiplying (8.105) by any test function ϕ, integrating by parts and using (8.100) give that

$$\begin{aligned}
&- \int_{\mathcal{O}} d\mathbf{x} \nabla z_n \cdot \nabla\phi \\
&= \int_{\mathcal{O}} d\mathbf{x}\Psi_n \sum_j (\Psi_n \nabla g'^{j}_n - < \tilde{\Psi}\tilde{\mathbf{G}}'^{j} >)\nabla(\frac{\partial \rho}{\partial x_j}\alpha_n)\phi \\
&+ \int_{\mathcal{O}} d\mathbf{x}\Psi_n \sum_j g'^{j}_n \nabla \left(\frac{\partial \rho}{\partial x_j}\alpha_n\right) \cdot \nabla\phi \\
&+ \int_{\mathcal{O}} d\mathbf{x}\Psi_n \sum_j g'^{j}_n \Delta \left(\frac{\partial \rho}{\partial x_j}\alpha_n\right)\phi - \int_{\mathcal{O}} d\mathbf{x} \sum_j \nabla g^{j}_n \cdot \frac{\partial \rho}{\partial x_j}\nabla\alpha_n\phi \\
&- \int_{\mathcal{O}} d\mathbf{x} \sum_j \nabla g^{j}_n \cdot \frac{\partial \nabla\rho}{\partial x_j}\alpha_n\phi + \int_{\mathcal{O}} d\mathbf{x} \sum_j g^{j}_n \nabla \left(\frac{\partial \rho}{\partial x_j}\alpha_n\right) \cdot \nabla\phi.
\end{aligned} \tag{8.106}$$

The right side of (8.106) vanishes in the limit $n \to \infty$ due to (8.96), (8.97) and

$$\int_{calO} d\mathbf{x}(\sum_j \Psi_n \nabla g_n^{'j} - < \tilde{\Psi}\mathbf{G}^{'j} >) \to 0. \tag{8.107}$$

This implies that ∇z_n converges weakly to zero. It is also clear that

$$\int_{\mathscr{O}} d\mathbf{x}|\nabla z_n|^2 < c < \infty. \tag{8.108}$$

Thus z_n is strongly pre-compact in L^2 and has zero as limit, namely, we have (8.99) as a consequence of (8.100).

We now prove

Lemma 8.6 *Assume that* $\langle|\widetilde{\Psi}|^2\rangle < \infty$. *Let* $\widetilde{\mathbf{G}}^{'j} \in \mathscr{H}_g(\Omega), j = 1, ..., d$, *be essentially bounded and let the normalization conditions (8.94), (8.95) hold. Assume also that* ρ *satifies (8.100) and (8.101). Then*

$$\int_{\mathscr{O}} d\mathbf{x}\ (\nabla z_n)^2 \to 0, \quad as \quad n \to \infty, \tag{8.109}$$

for almost all ω, *where*

$$z_n(\mathbf{x}, \omega) = g_n'(\mathbf{x}, \omega) - \rho - \sum_j g_n^j(\mathbf{x}, \omega)\frac{\partial \rho(\mathbf{x})}{\partial x_j}\alpha_n(\mathbf{x}). \tag{8.110}$$

Proof. Multiplying (8.105) by z_n, integrating by parts and using Schwartz inequality give that

$$\begin{aligned}
&\int_{\mathscr{O}} d\mathbf{x}\ (\nabla z_n)^2 \\
&\le \left(\int_{\mathscr{O}} d\mathbf{x}\ \left(\Psi_n \sum_j g_n^{'j} \nabla \frac{\partial \rho}{\partial x_j}\alpha_n\right)^2\right)^{1/2} \left(\int_{\mathscr{O}} d\mathbf{x}\ (\nabla z_n)^2\right)^{1/2} \\
&+ \left(\int_{\mathscr{O}} d\mathbf{x}\ \left(\Psi_n \sum_j g_n^{'j} \frac{\partial \rho}{\partial x_j}\nabla\alpha_n\right)^2\right)^{1/2} \left(\int_{\mathscr{O}\backslash\mathscr{O}'} d\mathbf{x}\ (\nabla z_n)^2\right)^{1/2} \\
&+ \left(\int_{\mathscr{O}} d\mathbf{x}\left(\sum_j (\Psi_n \nabla g_n^{'j} - < \tilde{\Psi}\tilde{\mathbf{G}}^{'j} >)\nabla \frac{\partial \rho}{\partial x_j}\alpha_n\right)^2\right)^{1/2} \left(\int_{\mathscr{O}} d\mathbf{x}\, (z_n)^2\right)^{1/2} \\
&+ \left(\int_{\mathscr{O}\backslash\mathscr{O}'} d\mathbf{x}\ \left(\Psi_n \sum_j \nabla g_n^{'j} \frac{\partial \rho}{\partial x_j}\nabla\alpha_n\right)^2\right)^{1/2} \left(\int_{\mathscr{O}\backslash\mathscr{O}'} d\mathbf{x}\ (z_n)^2\right)^{1/2} \\
&+ \left(\int_{\mathscr{O}} d\mathbf{x}\ \left(\sum_j \nabla g_n^j \cdot \nabla \frac{\partial \rho}{\partial x_j}\alpha_n\right)^2\right)^{1/2} \left(\int_{\mathscr{O}} d\mathbf{x}\ (z_n)^2\right)^{1/2}
\end{aligned}$$

$$+\left(\int_{\mathcal{O}\setminus\mathcal{O}'} d\mathbf{x}\,\left(\sum_j \nabla g_n^j \frac{\partial\rho}{\partial x_j}\nabla\alpha_n\right)^2\right)^{1/2}\left(\int_{\mathcal{O}\setminus\mathcal{O}'} d\mathbf{x}\,(z_n)^2\right)^{1/2}$$

$$+\left(\int_{\mathcal{O}} d\mathbf{x}\,\left(\sum_j g_n^j \nabla\frac{\partial\rho}{\partial x_j}\alpha_n\right)^2\right)^{1/2}\left(\int_{\mathcal{O}} d\mathbf{x}\,(\nabla z_n)^2\right)^{1/2}$$

$$+\left(\int_{\mathcal{O}} d\mathbf{x}\,\left(\sum_j g_n^j \frac{\partial\rho}{\partial x_j}\nabla\alpha_n\right)^2\right)^{1/2}\left(\int_{\mathcal{O}\setminus\mathcal{O}'} d\mathbf{x}\,(\nabla z_n)^2\right)^{1/2} \tag{8.111}$$

We note that similar to the proof of Lemma 8.5, (8.65) and (8.66) also hold here.

Lemma 8.6 then follows from (8.69), (8.99),

$$\int_{\mathcal{O}\setminus\mathcal{O}'} d\mathbf{x}\,\left(\nabla g_n^j\right)^2 \to 0 \tag{8.112}$$

$$\int_{\mathcal{O}\setminus\mathcal{O}'} d\mathbf{x}\,\left(\Psi_n \nabla g_n'^j\right)^2 \to 0 \tag{8.113}$$

$$\frac{1}{\tau_n^2}\int_{\mathcal{O}} d\mathbf{x}\,(g_n^j)^2 \to 0, \tag{8.114}$$

$$\frac{1}{\tau_n^2}\left(\sup_{\mathbf{x}\in\mathcal{O}}(g_n'^j)^2(\mathbf{x},\cdot)\right)\left(\int_{\mathcal{O}} d\mathbf{x}\,\Psi_n^2\right) \to 0, \quad j = 1,\ldots,d. \tag{8.115}$$

for almost all ω. As in the proof of Lemma 8.4, we have to choose the cutoff rate τ_n to satisfy (8.114) and (8.115). This completes the proof.

Similar to (8.82), we have

$$\int_{\mathcal{O}} d\mathbf{x}\,\left(\nabla z_n'\right)^2 \to 0, \tag{8.116}$$

where

$$z_n'(\mathbf{x},\omega) = g_n(\mathbf{x},\omega) - \rho - \sum_j g_n^j(\mathbf{x},\omega)\frac{\partial\rho(\mathbf{x})}{\partial x_j}. \tag{8.117}$$

Namely the effect of the cut-off functions is negligible.

A Summary of notation used

Notations	Definitions	
$\mathcal{H}_g$	Stationary gradient fields	(3.3)
$\mathcal{H}_c$	Stationary curl fields	(3.4)
$H_0(\Psi_n, \mathcal{O})$	The solution spaces for the Dirichlet problems	(3.12)
$\mathcal{H}_g(\tilde{\Psi})$	The solution space for the cell problems	(3.41)
$\|\| \cdot \|\|_{\Psi_n}$	The norms associated with the spaces $H_0(\Psi_n, \mathcal{O})$	(3.13)
$\|\| \cdot \|\|_{\tilde{\Psi}}$	The norm associated with the space $\mathcal{H}_g(\tilde{\Psi})$	(3.42)
Γ_o	The projection into the gradient of $H_0^1(\mathcal{O})$	(3.11)
$\tilde{\Gamma}$	The projection into the space $\mathcal{H}_g$	(3.40)
$\tilde{\mathbf{E}}^+_{\mathbf{e}^i}$	The intensity fields	(4.1), (4.3), (4.4)
$\tilde{\mathbf{E}}^-_{\mathbf{e}^i}$	The adjoint intensity fields	(4.2), (4.3), (4.4)
$\tilde{\mathbf{E}}_{ij}$	The average of $\tilde{\mathbf{E}}^+_{\mathbf{e}^i}$ and $\tilde{\mathbf{E}}^+_{\mathbf{e}^j}$	(4.5)
$\tilde{\mathbf{E}}'_{ij}$	The half difference of $\tilde{\mathbf{E}}^+_{\mathbf{e}^i}$ and $\tilde{\mathbf{E}}^+_{\mathbf{e}^j}$	(4.6)
$\tilde{\mathbf{E}}_i$	The fluctuating intensity fields $\tilde{\mathbf{E}}^+_{\mathbf{e}^i} - \mathbf{e}^i$	(3.33)-(3.36)
$\mathbf{E}_i(\mathbf{x}, \omega)$	The $\mathbf{x}$-translate of $\tilde{\mathbf{E}}_i$	(3.25)-(3.28)
$\tilde{\mathbf{D}}^+_{\mathbf{e}^i}$	The flux fields	(4.1), (4.3), (4.4)
$\tilde{\mathbf{D}}^-_{\mathbf{e}^i}$	The adjoint flux fields	(4.2), (4.3), (4.4)
$\tilde{\mathbf{D}}_{ij}$	The average of $\tilde{\mathbf{D}}^+_{\mathbf{e}^i}$ and $\tilde{\mathbf{D}}^+_{\mathbf{e}^j}$	(4.7)
$\tilde{\mathbf{D}}'_{ij}$	The half difference of $\tilde{\mathbf{D}}^+_{\mathbf{e}^i}$ and $\tilde{\mathbf{D}}^+_{\mathbf{e}^j}$	(4.8)
$\tilde{\mathbf{D}}_i$	The fluctuating flux fields $\tilde{\mathbf{D}}^+_{\mathbf{e}^i}$	(3.33)-(3.36)
$\mathbf{D}_i(\mathbf{x}, \omega)$	The $\mathbf{x}$-translate of $\tilde{\mathbf{D}}_i$	(3.25)-(3.28)
ρ_n^+	The solutions of the Dirichlet problems.	(4.26)-(4.27)
ρ_n^-	The solutions of the adjoint BV problems	(4.28)-(4.29)
ρ_n	The average of ρ_n^+, ρ_n^-	(4.30)
ρ_n'	The half difference of ρ_n^+, ρ_n^-	(4.31)
$\chi_i^+(\mathbf{x}, \omega)$	The correctors	(5.38)
$\chi_i^-(\mathbf{x}, \omega)$	The adjoint correctors	(5.38)
$\chi(\mathbf{x}, \omega)$	The average of $\chi_i^+(\mathbf{x}, \omega), \chi_i^-(\mathbf{x}, \omega)$	(5.39)
$\chi'(\mathbf{x}, \omega)$	The half difference of $\chi_i^+(\mathbf{x}, \omega), \chi_i^-(\mathbf{x}, \omega)$	(5.39)
$f_n^i(\mathbf{x}, \omega), g_n^i(\mathbf{x}, \omega)$... etc.	Rescaled functions	(6.31), (6.32)

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