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Itô correction terms for the radial parts of semimartingales on manifolds

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Summary. We consider functions, F, of a semimartingale, X, on a complete manifold which fail to be $\mathscr{C}^2$ only on, and are sufficiently well-behaved near, a codimension 1 subset $\mathscr{S}$. We obtain an extension of the Itô formula which is valid for all time by adding a continuous predictable process given explicitly in terms of two geometric local times of X on $\mathscr{S}$ and the Gâteaux derivative of F. We then examine the cut locus of a point of the manifold in sufficient detail to show that this result applies to give a corresponding expression for the radial part of the semimartingale.

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1 Introduction

A random process X with continuous sample-paths on an m-dimensional riemannian manifold $(\boldsymbol{M}, g)$ is called a semimartingale if, for every smooth function f on $\boldsymbol{M}$, $f(X)$ is a semimartingale on $\mathbb{R}$. A semimartingale is called a Γ-martingale if, for every smooth function on $\boldsymbol{M}$,

$$f(X_t) - f(X_0) - \tfrac{1}{2} \int_0^t \mathrm{Hess}^f(\partial X_s, \partial X_s)$$

is a local martingale on $\mathbb{R}$ (cf. [4]). Geometrically, a semimartingale X on $\boldsymbol{M}$ can be constructed from a continuous semimartingale $\hat{X}$ on $\mathbb{R}^m$ via the Stratonovic stochastic differential equations

$$\partial \Xi_t = H_{\Xi_t} \partial X_t,$$
$$\partial X_t = \Xi_t \partial \hat{X}_t.$$

Here, for an orthonormal frame ξ in the tangent space $\tau_{\pi(\xi)}(M)$, H_ξ is the horizontal lifting isomorphism, supplied by the Levi–Civita connection, of $\tau_{\pi(\xi)}(M)$ onto the horizontal subspace of the tangent space $\tau_\xi(\mathcal{O}(M))$ at ξ to the orthonormal frame bundle $\mathcal{O}(M)$. Usually $\hat{X}$ is called the *stochastic development* of X and Ξ is called the *stochastic parallel transport* of X. In particular, the stochastic development of a Γ-martingale is a local martingale on $\mathbb{R}^m$ and brownian motion on M is a Γ-martingale whose stochastic development is brownian motion on $\mathbb{R}^m$.

If $\{u_i : 1 \leqq i \leqq m\}$ is a fixed orthonormal base for $\mathbb{R}^m$, with respect to the standard euclidean metric, and $\hat{X}^i$, $1 \leqq i \leqq m$, are continuous semimartingales on $\mathbb{R}$ such that $\hat{X}_t = \sum \hat{X}^i_t u_i$ is the stochastic development of a continuous semimartingale X on M, then the Itô formula for a smooth function f on M can be expressed as follows (cf. [7]):

$$df(X_t) = \sum_{i=1}^{m} \Xi_t u_i f(X_t)\, d\hat{X}^i(t) + \tfrac{1}{2} \sum_{i,j=1}^{m} \Xi_t u_i \Xi_t u_j f(X_t)\, d[\hat{X}^i, \hat{X}^j]_t,$$

which is sometimes written more simply in the form

$$df(X_t) = [\operatorname{grad} f(X_t) \cdot \Xi_t]\, d\hat{X}_t + \tfrac{1}{2} \operatorname{Hess}^f(\Xi_t, \Xi_t)\, d[\hat{X}]_t$$

(cf. [8]), where it is clear that $\operatorname{Hess}^f(\Xi_t, \Xi_t)\, d[\hat{X}]_t$ stands for $\operatorname{Hess}^f(\partial X_t, \partial X_t)$.

However, when we take our function to be the distance $\phi(x)$ from a fixed point x_0 in M, this formula for $\phi(X)$, the "radial part" of the semimartingale X, applies only up to the first time that X hits x_0 or its cut locus $C(x_0)$, since there the function $\phi(x)$ fails to be smooth. As ϕ is at least locally related to smooth convex functions, $\phi(X_t)$ will be a semimartingale when M satisfies certain conditions and X is a Γ-martingale on M (cf. [3, 8]). This makes it possible to extend the Itô formula for $\phi(X_t)$ to hold for all time including those times at which X visits $C(x_0) \cup \{x_0\}$ (cf. [8]). More precisely, when $\phi(X_t)$ is a semimartingale, if the Γ-martingale X spends almost no time in $C(x_0)$ (in the sense that $\{t : X_t \in C(x_0)\}$ is almost surely a Lebesgue null set), and if the underlying filtration is such that all continuous martingales have bracket processes which are absolutely continuous as (random) measures on the time set $\mathbb{R}_+$, then

$$d\phi(X_t) = [\operatorname{grad} \phi(X_t) \cdot \Xi_t]\, dM_t + \tfrac{1}{2} \operatorname{Hess}^\phi(\Xi_t, \Xi_t)\, d[M]_t + dL^0_t(\phi(X)) - dL_t, \quad (1)$$

where the gradient $\operatorname{grad} \phi$ and the Hessian Hess^ϕ are set to zero for $x \in C(x_0)$, $L^0(\phi(X))$ is the local time of $\phi(X)$ at zero and L is a non-decreasing process which increases only on the time set $\{t : X_t \in C(x_0)\}$.

Note that when X is brownian motion on (M, g) any fixed point is a polar set provided $m > 1$, so that the term $dL^0_t(\phi(X))$ in (1) will disappear. Details of the corresponding behaviour of L, related to a notion of local time of X on $C(x_0)$, have been given in [3]. To state that result, we need the following facts. Firstly, the one-sided Gâteaux derivative, $D_+ f$, which is defined by

$$D_+ f_x(v) = \lim_{\varepsilon \downarrow 0} \varepsilon^{-1} \{ f(\exp_x(\varepsilon v)) - f(x) \} \quad \forall v \in \tau_x(M),$$

always exists for a convex function. Secondly, a constant, c_{m-1}, times the Hausdorff measure related to the riemannian volume measure and restricted to a particular hypersurface (denoted by μ^{m-1}) gives the area measure of that hypersurface. Given any open subset $\mathscr{L}$ of a hypersurface of $\boldsymbol{M}$, write $L^{\mathscr{L}}$ for the geometric local time defined, for Borel subsets E of $\mathscr{L}$ of finite μ^{m-1}-measure, as the continuous predictable non-decreasing process $L^{\mathscr{L}}(E)$ associated with the restriction of $c_{m-1}\mu^{m-1}$ measure

$$A \to c_{m-1}\mu^{m-1}(A \cap E).$$

Then the result in [3] can be summarised as follows. If the set $\mathscr{L}$, consisting of points at which a convex function f on $(\boldsymbol{M}, \boldsymbol{g})$ fails to be smooth, is a countable union of open subsets of codimension 1 submanifolds of $\boldsymbol{M}$ then, for brownian motion X with B as its stochastic development, the stochastic differential equation of $f(X_t)$ is given by

$$df(X_t) = [\underline{\mathrm{grad}} f(X_t) \cdot \Xi_t]\, dB_t + \tfrac{1}{2}\Delta f(X_t)\, dt + \tfrac{1}{2}\{D_+ f_{X_t}(v) + D_+ f_{X_t}(-v)\, dL_t^{\mathscr{L}}, \quad (2)$$

where the gradient grad f and the Laplacian Δf are set to zero for $x \in \mathscr{L}$ and v is a (μ^{m-1}-almost everywhere defined) measurable unit normal vector-field for $\mathscr{L}$.

The cut locus $C(x_0)$ is contained in a countable union of open subsets of smooth hypersurfaces of $\boldsymbol{M}$ up to a Hausdorff $(m-1)$-measure zero set. It follows, by the local relation of ϕ with convex functions, that $\phi(X)$ satisfies a similar stochastic differential equation and, since $D_+ f(v) + D_+ f(-v) \geqq 0$ for any convex function f, that $D_+\phi(v) + D_+\phi(-v) \leqq 0$ (cf. [3, 8]).

In this paper we take a somewhat different approach, and also study the cut locus in more detail, to obtain a result for semimartingales generalising that in Eq. (2).

In Sect. 2 we re-express the Itô-Tanaka formula for a certain class of continuous convex functions, $f(X)$, of a continuous real-valued semimartingale, X, in terms of the local times of X at the points where f fail to be $\mathscr{C}^2$ (Proposition 1). Then, in Proposition 2, we consider the case where X is a semimartingale in $\mathbb{R}$ when it is equipped with a non-standard metric and obtain a generalisation of Proposition 1 which also generalises that case of Eq. (2). In Sect. 3 we further generalise this to allow X to be a semimartingale on a complete riemannian manifold. There we consider continuous functions of X which are $\mathscr{C}^2$ except that, in each of at most a countable number of local coordinate spaces, they restrict to different $\mathscr{C}^2$-functions on either side of a hyperplane in $\mathbb{R}^m$.

In Sect. 4 we obtain an expression for the Itô correction term for the radial part of a semimartingale in terms of the local time at zero together with what it is reasonable to regard as two geometric local times on the cut locus by showing that the distance function from a point x_0 of the manifold is a function satisfying the appropriate hypotheses of Theorem 1. To that end we first use results of Ozols to show that, except for a set whose image has Hausdorff $(m-1)$-measure zero, the cut locus of x_0 in the tangent space, outside the first conjugate locus, is a union of submanifolds for which the distance function satisfies those hypotheses. We then extend results of Warner to show that the image of the conjugate part of the cut locus, indeed of the entire conjugate locus, has Hausdorff $(m-1)$-measure zero. Note that, by [6, Theorem VII.3], it follows that this image has codimension at least two in the manifold. This results appears to be new and has analytic and geometric consequences which we shall pursue elsewhere.

Readers of [3] will be aware of our indebtedness to the authors of that paper both for inspiring our main results and for a number of the ideas within the proof. We are further indebted to W.S. Kendall and one of the referees for further helpful comments and suggestions on the first version of the paper. The latter in particular drew our attention to [1] and enabled us to simplify our proofs and to extend the results from the original context of Γ-martingales to semimartingales. We gratefully acknowledge all these debts.

In the following, for any fixed riemannian manifold $\boldsymbol{M}$, we define the operator $\underline{\text{grad}}$ from $\mathscr{C}(\boldsymbol{M})$, the space of continuous functions on $\boldsymbol{M}$, to the space of vector fields on $\boldsymbol{M}$ to be $\text{grad} f$ if the latter exists and zero otherwise. The operators $\underline{\text{Hess}}$ and $\underline{\Delta}$ are defined similarly. Note that, if the dimension of $\boldsymbol{M}$ is one, then Hess is identical with Δ and $\underline{\text{Hess}}$ with $\underline{\Delta}$. For any function f on $\mathbb{R}$, if its left-hand derivative at x exists we denote it by f'_-. We write δ_x for the Dirac measure at x. For any given semimartingale X on $\boldsymbol{M}$, we shall write $\hat{X}$ for its stochastic development and Ξ for its stochastic parallel transport except that, when X is a Γ-martingale or brownian motion, we shall write M or B respectively for its stochastic development.

2 A generalised Itô–Tanaka formula

Given a semimartingale X on $\mathbb{R}$ we denote by $L_t^{+,x}(X)$ the usual local time of X at x and by $L_t^{-,x}(X)$ the local time of $-(X-x)$ at 0. If X is continuous, the process $\{L_t^{+,x}(X): x \in \mathbb{R},\ t \in \mathbb{R}_+\}$ may be chosen such that the map $(x,t) \to L_t^{+,x}(X)$ is a.s. continuous in t and cadlag in x. Then $L_t^{-,x}(X)$ is the left limit with respect to x of $L_t^{+,x}(X)$. When X is a continuous local martingale, there is a bicontinuous modification of the family $L_t^{+,x}(X)$ of local times (cf. [11, p. 209]), and so $L_t^{+,x}(X) = L_t^{-,x}(X)$. Then it will be denoted by $L_t^x(X)$.

Proposition 1 *If the difference, f, of two continuous convex functions is $\mathscr{C}^2$ on $\mathbb{R}\backslash\{x_n^*: n \geqq 1\}$, where the set $\{x_n^*: n \geqq 1\}$ has no limit point in $\mathbb{R}$, and if X is a continuous semimartingale on $\mathbb{R}$, then*

$$df(X_t) = \underline{\text{grad}} f(X_t)\, dX_t + \tfrac{1}{2}\underline{\Delta} f(X_t)\, d[X]_t$$
$$+ \tfrac{1}{2} \sum_{n \geqq 1} \{f'_+(X_t)\, dL_t^{+,x_n^*}(X) - f'_-(X_t)\, dL_t^{-,x_n^*}(X)\}.$$

Note that in this case the stochastic parallel transport Ξ is the identity and the stochastic development of X is itself. Note also that, if the set $\{x_n^*: n \geqq 1\}$ has limit points in $\mathbb{R}$, it is sufficient to assume that the second derivative of f in the sense of distributions is absolutely continuous on $\mathbb{R}\backslash\{x_n^*: n \geqq 1\}$, to obtain a similar result.

Proof. We know from the Itô–Tanaka formula that $f(X)$ satisfies the following stochastic integral equation:

$$f(X_t) = f(X_0) + \int_0^t f'_-(X_s)\,dX_s + \tfrac{1}{2}\int_{\mathbb{R}} L_t^x(X)\mu_f(dx),$$

where μ_f denotes the second derivative of f in the sense of distributions, that is μ_f is a locally finite measure such that $\int_{\mathbb{R}} f(x)\psi''(x)\,dx = \int_{\mathbb{R}}\psi(x)\mu_f(dx)$ for all smooth functions of compact support, ψ.

Without loss of generality, we may assume that f is convex, continuous on $\mathbb{R}$ and $\mathscr{C}^2$ on $\mathbb{R}\backslash\{x^*\}$. Then the measure μ_f associated with the second derivative of f in the sense of distributions is given by

$$\mu_f((x_1, x_2]) = f'_-(x_2) - f'_-(x_1) = \int_{x_1}^{x_2} \underline{\Delta} f(u)\,du + \int_{x_1}^{x_2} \zeta(u)\delta_{x^*}(du),$$

where $\zeta(x)$ is equal to $f'_+(x^*) - f'_-(x^*)$ if $x = x^*$ and zero otherwise. Thus

$$\begin{aligned}\int_{\mathbb{R}} L_t^x(X)\mu_f(dx) &= \int_{\mathbb{R}} L_t^x(X)\underline{\Delta} f(x)\,dx + \int_{\mathbb{R}} L_t^x(X)\zeta(x)\delta_{x^*}(dx)\\ &= \int_0^t \underline{\Delta} f(X_s)\,d[X]_s + \zeta(x^*)L_t^{+,x^*}(X),\end{aligned}$$

where the second equality follows from the occupation times formula (cf. [11, p. 209]). Finally, we get (cf. [11, Theorem 1.7, p. 209])

$$\begin{aligned}f(X_t) &= f(X_0) + \int_0^t \underline{\operatorname{grad}} f(X_s)\,dX_s + \int_0^t 1_{\{X_s = x^*\}} f'_-(X_s)\,dX_s\\ &\quad + \tfrac{1}{2}\int_0^t \underline{\Delta} f(X_s)\,d[X]_s + \tfrac{1}{2}\zeta(x^*)L_t^{+,x^*}(X)\\ &= f(X_0) + \int_0^t \underline{\operatorname{grad}} f(X_s)\,dX_s + \tfrac{1}{2}\int_0^t \underline{\Delta} f(X_s)\,d[X]_s\\ &\quad + \tfrac{1}{2}\{f'_+(X_t)L_t^{+,x^*}(X) - f'_-(X_t)L_t^{-,x^*}(X)\}.\end{aligned}$$

Note that the Dirac measure δ_x, which is the measure associated with $dL_t^x(B)$, differs from the Hausdorff measure related to the Lebesgue measure on $\mathbb{R}$ and restricted to x by at most a constant, and also that $v = \partial/\partial x$ is an orthonormal vector field on $\mathbb{R}$ and $f'_\pm(x) = \pm D_+ f_x(\pm v)$. Thus Proposition 1 is the generalisation to semimartingales of a special case of Eq. (2).

Now we consider the case that $\mathbb{R}$ is equipped with a general riemannian metric structure $\boldsymbol{g}$, so that the riemannian distance between any two points $x_1 \leqq x_2$ of $(\mathbb{R}, \boldsymbol{g})$ is given by

$$d(x_1, x_2) = \int_{x_1}^{x_2} \sqrt{g(x)}\,dx = s(x_2) - s(x_1),$$

where $s(x) = \operatorname{sgn}(x) d(0,x)$ is a strictly increasing function. Then a function F on $(\mathbb{R}, g)$ is called convex (cf. [11, p. 502]) if, for any $x_1 < x < x_2$,

$$(s(x_2) - s(x_1))F(x) \leqq (s(x_2) - s(x))F(x_1) + (s(x) - s(x_1))F(x_2). \tag{3}$$

Proposition 2 *If the difference, F, of two continuous convex functions on $(\mathbb{R}, g)$ is $\mathscr{C}^2$ on $\mathbb{R}\backslash\{x_n^*: n \geqq 1\}$, where the set $\{x_n^*: n \geqq 1\}$ has no limit point in $\mathbb{R}$, and if X is a continuous semimartingale on $(\mathbb{R}, g)$, then*

$$dF(X_t) = [\underline{\operatorname{grad}_g} F(X_t) \cdot \Xi_t] \, d\hat{X}_t + \tfrac{1}{2} \underline{\Delta_g} F(X_t) \, d[\hat{X}]_t$$
$$+ \tfrac{1}{2} \sqrt{g(X_t)} \sum_{n \geqq 1} \{D_+ F_{X_t}(v) \, dL_t^{+, x_n^*}(X) + D_+ F_{X_t}(-v) \, dL_t^{-, x_n^*}(X)\},$$

where $v = \{g(x)\}^{-1/2} d/dx$.

Proof. Parallel translation of cotangent vectors on $(\mathbb{R}, g)$ from 0 to x is multiplication by $\{g(x)\}^{-1/2}$ and so

$$\partial X_t = \{g(X_t)\}^{-1/2} \partial \hat{X}_t.$$

Equation (3) implies that F is convex on $(\mathbb{R}, g)$ if and only if $f = F \circ s^{-1}$ is convex on $\mathbb{R}$ equipped with the standard metric, $\imath$. Indeed the mapping s from $(\mathbb{R}, g)$ to $(\mathbb{R}, \imath)$ is an isometry and we also have that, if the second derivative of F exists at x, then

$$\left\langle \operatorname{grad} f(s(x)), \frac{d}{ds(x)} \right\rangle = \left\langle \operatorname{grad}_g F(x), \frac{1}{\sqrt{g(x)}} \frac{d}{dx} \right\rangle_g \quad \text{and} \quad \Delta f(s(x)) = \Delta_g F(x).$$

Hence, $f'_\pm(y) = \{g(x)\}^{-1/2} F'_\pm(x) = \pm D_+ F_x(\pm v)$, where $y = s(x)$.

When X is regarded as a continuous semimartingale on $(\mathbb{R}, \imath)$, the relation between the local times of X and its stochastic development, $\hat{X}$, is given as follows (cf. [11, p. 212]):

$$L_t^{+, s(x)}(\hat{X}) = \lim_{\varepsilon \downarrow 0} \left\{ \frac{1}{s(x+\varepsilon) - s(x)} \int_0^t 1_{[s(x), s(x+\varepsilon))}(\hat{X}_s) \, d[\hat{X}]_s \right\}$$
$$= \lim_{\varepsilon \downarrow 0} \left\{ \frac{\varepsilon}{s(x+\varepsilon) - s(x)} \frac{1}{\varepsilon} \int_0^t 1_{[x, x+\varepsilon)}(X_s) g(X_s) \, d[X]_s \right\}$$
$$= \sqrt{g(x)} L_t^{+, x}(X). \tag{4}$$

Thus, by Proposition 1, we have

$$F(X_t) = f(\hat{X}_t) = f(\hat{X}_0) + \int_0^t \underline{\operatorname{grad}} f(\hat{X}_s) \, d\hat{X}_s + \tfrac{1}{2} \int_0^t \underline{\Delta} f(\hat{X}_s) \, d[\hat{X}]_s$$
$$+ \tfrac{1}{2} \sum_{n \geqq 1} \{f'_+(\hat{X}_t) L_t^{+, s(x_n^*)}(\hat{X}) - f'_-(\hat{X}_t) L_t^{-, s(x_n^*)}(\hat{X})\}$$
$$= F(X_0) + \int_0^t [\underline{\operatorname{grad}_g} F(X_s) \cdot \Xi_s] \, d\hat{X}_s + \tfrac{1}{2} \int_0^t \underline{\Delta_g} F(X_s) \, d[\hat{X}]_s$$
$$+ \tfrac{1}{2} \sum_{n \geqq 1} \{D_+ F_{X_t}(v) L_t^{+, x_n^*}(X) + D_+ F_{X_t}(-v) L_t^{-, x_n^*}(X)\} \sqrt{g(X_t)}.$$

Note that the random measures $\sqrt{g(X_t)}\, dL_t^{\pm, x_n^*}(X)$ are invariant with respect to the choice of the coordinates and take the geometric structure of the space into account.

When X is a Γ-martingale on $(\mathbb{R}, g)$, its stochastic development is a local martingale M on $(\mathbb{R}, \imath)$ and so $L^{\pm, s(x)}(M) = L^{s(x)}(M)$. Thus, the above proof implies the following result.

Corollary *If X in Proposition 2 is a Γ-martingale on $(\mathbb{R}, g)$, then*

$$dF(X_t) = [\underline{\mathrm{grad}_g} F(X_t)\cdot \Xi_t]\, dM_t + \tfrac{1}{2} \underline{\Delta_g} F(X_t)\, d[M]_t$$
$$+ \tfrac{1}{2} \sum_{n \geqq 1} \{D_+ F_{X_t}(v) + D_+ F_{X_t}(-v)\} \sqrt{g(X_t)}\, dL_t^{x_n^*}(X).$$

The following result and proof are extentions of those for the case when X is brownian motion on $(\mathbb{R}, \imath)$ (cf. [11, p. 385]).

Proposition 3 *If X is brownian motion on $(\mathbb{R}, g)$ and if A is a continuous additive functional of X with associated measure μ_A, then*

$$A_t = \int_{\mathbb{R}} L_t^x(X) \sqrt{g(x)} \mu_A(dx).$$

Proof. It follows from Eq. (4) that the measure associated with $L^x(X)$ is $\mu_{L^x} = \{g(x)\}^{-1/2} \delta_x$. Write

$$\tilde{A}_t = \int_{\mathbb{R}} L_t^x(X) \sqrt{g(x)} \mu_A(dx).$$

Then $\tilde{A}$ is a continuous additive functional of X. Write $\mu_{\tilde{A}}$ for the measure associated with $\tilde{A}$ and m for the volume measure of $(\mathbb{R}, g)$. Then m is the invariant measure of X and so, for any bounded non-negative measurable function ψ,

$$\begin{aligned}
\mu_{\tilde{A}}(\psi) &= E_m\left[\int_0^1 \psi(X_s)\, d\tilde{A}_s\right] \\
&= E_m\left[\int_{\mathbb{R}} \left(\int_0^1 \psi(X_s)\, dL_s^x(X)\right)\sqrt{g(x)}\, \mu_A(dx)\right] \\
&= \int_{\mathbb{R}} \sqrt{g(x)}\, \mu_A(dx) E_m\left[\int_0^1 \psi(X_s)\, dL_s^x(X)\right] \\
&= \int_{\mathbb{R}} \sqrt{g(x)}\, \mu_{L^x}(\psi) \mu_A(dx) \\
&= \int_{\mathbb{R}} \psi(x) \mu_A(dx) = \mu_A(\psi).
\end{aligned}$$

Thus, $\mu_{\tilde{A}} = \mu_A$ and so (up to equivalence) $A = \tilde{A}$.

An immediate consequence of Proposition 3 is that, if X is brownian motion on $(\mathbb{R}, g)$, A is a continuous additive functional of X and ψ is a bounded non-negative measurable function, then

$$\int_{\mathbb{R}} \psi(X) L_t^x(x) \sqrt{g(x)}\, \mu_A(dx) = \int_0^t \psi(X_s)\, dA_s.$$

The measure $\delta_{x^*}(dx)$ differs from the Hausdorff measure related with the corresponding riemannian measure of $(\mathbb{R}, g)$ and restricted to x^* by at most a constant and is the associated measure for the geometric local time, $\tilde{L}^{x^*}(X)$, of brownian motion on $(\mathbb{R}, g)$ at x^*. The above equation implies that $\tilde{L}^x(X) = \sqrt{g(x)} L^x(X)$. Hence Eq. (2), for the case $(\mathbb{R}, g)$, is a corollary of Proposition 2.

3 The Itô correction terms for certain functions of semimartingales

We turn now to the general case of semimartingales on a complete riemannian manifold $(\boldsymbol{M}, \boldsymbol{g})$. We shall consider continuous functions on $\boldsymbol{M}$ which fail to be differentiable on at most the countable disjoint union $\mathscr{L}$ of open subsets O_i of codimension one two-sided submanifolds of $\boldsymbol{M}$, where for each i there is an open subset U_i of $\boldsymbol{M}$ such that $O_i = \mathscr{L} \cap U_i$ and $U_i \backslash O_i$ has two components. We choose a never-zero unit normal field v on $\mathscr{L}$. For each i let U_i^+ be the component of $U_i - O_i$ into which v points, U_i^- be the other component and $\mathscr{H}_i^{\pm} = O_i \cup U_i^{\pm}$. Then we require our functions, F, to be continuous functions satisfying the following hypothesis:

(*H*) *For each i there are* $\mathscr{C}^2$-*functions* $G_{i,\pm}$ *on* U_i *such that*

$$F \mid \mathscr{H}_i^{\pm} = G_{i,\pm} \mid \mathscr{H}_i^{\pm}.$$

We denote by $F \circ \pi$ the composition of F with the orthogonal projection π onto $\mathscr{L}$, uniquely defined on a sufficiently small neighbourhood of $\mathscr{L}$ in $\boldsymbol{M}$. Then our basic result is the following.

Theorem 1 *Let X be a semimartingale on a complete riemannian manifold* $(\boldsymbol{M}, \boldsymbol{g})$ *and let* $\mathscr{L}$, *its normal field v and associated sets* O_i, U_i *be as described above. Then there are two non-decreasing continuous predictable processes* $L^{\pm v, \mathscr{L}}(X)$, *which are functionals of X and whose associated random measures* $dL^{\pm v, \mathscr{L}}(X)$ *are a.s. carried by* $\mathscr{L}$ *such that, for any continuous function F on* $\boldsymbol{M}$ *satisfying the hypothesis (H),*

$$\begin{aligned} dF(X_t) &= [\underline{\operatorname{grad}}\, F(X_t) \cdot \Xi_t]\, d\hat{X}_t + \tfrac{1}{2} \underline{\operatorname{Hess}}^F(\Xi_t, \Xi_t)\, d[\hat{X}]_t \\ &\quad + 1_{\{X_t \in \mathscr{L}\}} \{[\operatorname{grad}(F \circ \pi)(X_t) \cdot \Xi_t]\, d\hat{X}_t + \tfrac{1}{2} \operatorname{Hess}^{F \circ \pi}(\Xi_t, \Xi_t)\, d[\hat{X}]_t\} \\ &\quad + \tfrac{1}{2} \{D_+ F_{X_t}(v)\, dL_t^{+v, \mathscr{L}}(X) + D_+ F_{X_t}(-v)\, dL_t^{-v, \mathscr{L}}(X)\}. \end{aligned} \tag{5}$$

Proof. By localisation we may assume that $\boldsymbol{M} = \mathbb{R}^m$, with the induced metric $\boldsymbol{g}$, and that $\mathscr{L}$ is given by $x_1 = 0$ and $\mathscr{H}^+$ is the set $\{(x_1, \ldots, x_m) \colon x_1 \geqq 0\}$. Let $F \colon \mathbb{R}^m \to \mathbb{R}$ be a continuous function such that there are $\mathscr{C}^2$-functions $G_{\pm}$ on $\mathbb{R}^m$ with $F \mid \mathscr{H}^{\pm} = G_{\pm} \mid \mathscr{H}^{\pm}$. Write $x^{\pm} = (x_1^{\pm}, x_2, \ldots, x_m)$, where $x_1^+ = \max\{0, x_1\}$

and $x_1^- = x_1 - x_1^+$, and $F^0(x) = F(0, x_2, \dots, x_m)$. Then

$$F(x) = G_+(x^+) + G_-(x^-) - F^0(x)$$

and so we have (cf. [1, Proposition 5]) that

$$\begin{aligned} dF(X_t) = {} & [\underline{\operatorname{grad}} F(X_t)\cdot \Xi_t]\, d\hat{X}_t + \tfrac{1}{2}\underline{\operatorname{Hess}}^F(\Xi_t, \Xi_t)\, d[\hat{X}]_t \\ & + 1_{\{X_t \in \mathscr{L}\}}\, dF^0(X_t) + \tfrac{1}{2}\left\{\frac{\partial G_+(X_t)}{\partial x_1}\, dL_t^{+,0}(X^1) - \frac{\partial G_-(X_t)}{\partial x_1}\, dL_t^{-,0}(X^1)\right\}. \end{aligned} \tag{6}$$

In order to re-express the second line of (6) in the form given in the theorem we first note that

$$\begin{aligned} & [\operatorname{grad}(F\circ\pi)(X_t)\cdot \Xi_t]\, d\hat{X}_t + \tfrac{1}{2}\operatorname{Hess}^{F\circ\pi}(\Xi_t, \Xi_t)\, d[\hat{X}]_t \\ & = 1_{\{X_t \in \mathscr{L}\}}\left\{dF^0(X_t) - \sum_{i=2}^m \frac{g^{1i}(X_t)}{g^{11}(X_t)}\frac{\partial F(X_t)}{\partial x_i}\, dX_t^1\right\}. \end{aligned} \tag{7}$$

This follows from the following three observations. Firstly, since ν is the normal vector field on $\mathscr{L}$ pointing into $\mathscr{H}^+$, it is orthogonal to $\partial/\partial x_i$, $2 \leqq i \leqq m$, and we have

$$\nu = \sqrt{g^{11}}\frac{\partial}{\partial x_1} + \sum_{i=2}^m \frac{g^{1i}}{\sqrt{g^{11}}}\frac{\partial}{\partial x_i}.$$

Thus

$$[\operatorname{grad}(F\circ\pi)(X_t)\cdot \Xi_t]\, d\hat{X}_t = \sum_{i=2}^m \frac{\partial F^0}{\partial x_i}\left\{-\frac{g^{1i}}{\sqrt{g^{11}}}\, dX_t^1 + dX_t^i\right\}.$$

Secondly, if we denote by N the local martingale in the Doob–Meyer decomposition of X^1, we have $\int_{\mathbb{R}_+} 1_{\{X_t \in \mathscr{L}\}}\, dN_t = 0$. This implies that

$$\begin{aligned} \sum_{i,j=1}^m \frac{\partial^2 (F\circ\pi)(X_t)}{\partial x_i \partial x_j}\, d[X^i, X^j]_t = 1_{\{X_t \in \mathscr{L}\}} & \left\{\sum_{i,j=2}^m \frac{\partial^2 F(X_t)}{\partial y_i \partial y_j}\, d[Y^i, Y^j]_t \right. \\ & \left. + \sum_{i,j,k=2}^m \frac{\partial F(X_t)}{\partial y_k}\frac{\partial^2 y_k(X_t)}{\partial x_i \partial x_j}\, d[X^i, X^j]_t\right\}, \end{aligned}$$

where y_i is the ith coordinate of $\pi(x_1, \dots, x_m)$. Finally, $(x_1, y_2, \dots, y_m)$ is also a coordinate system on $\boldsymbol{M}$ and so $(x_2, \dots, x_m) \to (y_2, \dots, y_m)$, when it is restricted to $\mathscr{L}$, represents a change of coordinates.

Equation (7), together with the fact that

$$D_+F_x(\pm\nu) = \pm\sqrt{g^{11}}\frac{\partial G_\pm}{\partial x_1} \pm \sum_{i=2}^m \frac{g^{1i}}{\sqrt{g^{11}}}\frac{\partial F^0}{\partial x_i},$$

implies that, if we define

$$L_t^{\pm\nu,\mathscr{L}}(X) = \int_0^t \{g^{11}(X_s)\}^{-1/2}\, dL_s^{\pm,0}(X^1), \tag{8}$$

then Eqs. (5) and (6) are identical. Clearly, the processes $L^{\pm \nu, \mathscr{L}}(X)$ defined in this way are non-decreasing and continuous and the random measures $dL_t^{\pm \nu, \mathscr{L}}(X)$ are a.s. carried by the set $\mathscr{L}$.

To complete the proof of the theorem we need to check that the expression (8) is invariant with respect to the choice of coordinates. Suppose that $\tilde{x} = (\tilde{x}_1, \ldots, \tilde{x}_m)$ gives another coordinate system on $(\mathbb{R}^m, g)$ with the properties that $x_1 = 0$ if and only if $\tilde{x}_1 = 0$ and that $x_1 > 0$ if and only if $\tilde{x}_1 > 0$. Write $(\tilde{g}^{i,j})$ for the inverse of the riemannian metric matrix with respect to the new coordinates $\tilde{x}$. Then, on the one hand, we have by the Itô–Tanaka formula that

$$d|\tilde{X}_t^1| = \operatorname{sgn}(\tilde{X}_t^1)\, d\tilde{X}_t^1 + \tfrac{1}{2}\{dL_t^{+,0}(\tilde{X}^1) + dL_t^{-,0}(\tilde{X}^1)\},$$

where the function sgn is defined by $\operatorname{sgn}(x)$ is equal to 1, if $x > 0$, -1, if $x < 0$, and 0, if $x = 0$. On the other hand, we have by the theorem proved using the coordinates $(x_1, \ldots, x_m)$ that

$$d|\tilde{X}_t^1| = \operatorname{sgn}(\tilde{X}_t^1)\, d\tilde{X}_t^1 + \tfrac{1}{2}\frac{\partial \tilde{x}_1(X_t)}{\partial x_1}\{dL_t^{+,0}(X^1) + dL_t^{-,0}(X^1)\},$$

since $\tilde{x}_1(0, x_2, \ldots, x_m) = 0$ and thus $\partial \tilde{x}_1(0, x_2, \ldots, x_m)/\partial x_i = \partial^2 \tilde{x}_1(0, x_2, \ldots, x_m)/\partial x_i \partial x_j = 0$, $2 \leqq i, j \leqq m$. It follows that

$$dL_t^{+,0}(\tilde{X}^1) + dL_t^{-,0}(\tilde{X}^1) = \frac{\partial \tilde{x}_1(X_t)}{\partial x_1}\{dL_t^{+,0}(X^1) + dL_t^{-,0}(X^1)\}.$$

If we repeat the above argument for $\tilde{x}_1(X)$ instead of $|\tilde{x}_1(X)|$, we shall get

$$dL_t^{+,0}(\tilde{X}^1) - dL_t^{-,0}(\tilde{X}^1) = \frac{\partial \tilde{x}_1(X_t)}{\partial x_1}\{dL_t^{+,0}(X^1) - dL_t^{-,0}(X^1)\}.$$

Therefore,

$$dL_t^{\pm,0}(\tilde{X}^1) = \frac{\partial \tilde{x}_1(X_t)}{\partial x_1}\, dL_t^{\pm,0}(X^1).$$

Thus the invariance of expression (8) follows from $\tilde{g}^{11} = (\partial \tilde{x}_1/\partial x_1)^2 g^{11}$ on $\mathscr{L}$ which, again, is a consequence of $\partial \tilde{x}_1(0, x_2, \ldots, x_m)/\partial x_i = 0$, $2 \leqq i \leqq m$.

Note that, as in the 1-dimensional case, the processes $L^{\pm \nu, \mathscr{L}}(X)$ are not only invariant with respect to the choice of coordinates, but also take the geometric structure of M into account. In fact the above proof shows that, if we define Y to be $\operatorname{dist}(X, \mathscr{L})$ on the side of $\mathscr{L}$ into which ν points and $-\operatorname{dist}(X, \mathscr{L})$ on the other, then $L^{\pm \nu, \mathscr{L}}(X) = L^{\pm,0}(Y)$.

When X is a Γ-martingale with $\int_{\mathbb{R}_+} 1_{\{X_t \in \mathscr{L}\}}\, g(dX_t, dX_t) = 0$, then $L^{+\nu, \mathscr{L}}(X) = L^{-\nu, \mathscr{L}}(X) = L^{\mathscr{L}}(X)$ and the terms in the second line of (5) vanish and we get the following corollary.

Corollary *Let X be a Γ-martingale on a complete riemannian manifold (M, g) and let $\mathscr{L}$ and its normal field ν be as described above. Assume that $\int_{\mathbb{R}^+} 1_{\{X_t \in \mathscr{L}\}}\, g(dX_t, dX_t) = 0$. Then there is a non-decreasing continuous predictable process $L^{\mathscr{L}}(X)$, which is a functional of X and whose associated random measure*

$dL^{\mathscr{L}}(X)$ is a.s. carried by $\mathscr{L}$ such that, for any continuous function F on M satisfying the hypothesis (H),

$$dF(X_t) = [\underline{\operatorname{grad}} F(X_t) \cdot \Xi_t]\, d\hat{X}_t + \tfrac{1}{2} \underline{\operatorname{Hess}}^F(\Xi_t, \Xi_t)\, d[\hat{X}]_t$$
$$+ \tfrac{1}{2}\{D_+ F_{X_t}(v) + D_+ F_{X_t}(-v)\}\, dL_t^{\mathscr{L}}(X).$$

When X is brownian motion on M, the above corollary is analogous to the Eq. (2). To see this, it is sufficient to show that the associated measure of $L^{\mathscr{L}}(X)$ is the hyperplane measure $\sqrt{g_{22}(0, x_2)}\, dx_2$ of $\mathscr{L}$.

Since $d[X^1]_t = g^{11}(X_t)\, dt$, we have

$$dL_t^{x_1}(X^1) = \lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon} 1_{[x_1, x_1 + \varepsilon)}(X_t^1) g^{11}(X_t)\, dt.$$

Thus, by the definition and using the fact that the volume measure is an invariant measure for brownian motion, we see that the associated measure of $L^{\mathscr{L}}(X)$ is given by

$$\begin{aligned}
\mu_{L^{\mathscr{L}}}(dx_2) &= \lim_{t \downarrow 0} \frac{1}{t} \int_{\mathbb{R}^2} E_{(u,v)}\left[\int_0^t 1_{[x_2, x_2 + dx_2)}(X_s^2)\, dL_s^{\mathscr{L}}(X) \right] \sqrt{\det(g)}\, du\, dv \\
&= \lim_{t \downarrow 0} \frac{1}{t} \int_{\mathbb{R}^2} E_{(u,v)}\left[\int_0^t 1_{[x_2, x_2 + dx_2)}(X_s^2) \{g^{11}(X_s)\}^{-1/2}\, dL_s^0(X^1) \right] \\
&\quad \times \sqrt{\det(g(u, x_2))}\, du\, dv \\
&= \lim_{t, \varepsilon \downarrow 0} \frac{1}{\varepsilon t} \int_{\mathbb{R}^2} E_{(u,v)}\left[\int_0^t 1_{[x_2, x_2 + dx_2)}(X_s^2) 1_{[0, \varepsilon)}(X_s^1) \{g^{11}(0, x_2)\}^{1/2}\, ds \right] \\
&\quad \times \sqrt{\det(g(u, x_2))}\, du\, dv \\
&= \lim_{t, \varepsilon \downarrow 0} \frac{1}{\varepsilon t} \int_{\mathbb{R}^2} E_{(u,v)}\left[\int_0^t 1_{[0, \varepsilon)}(X_s^1)\, 1_{[x_2, x_2 + dx_2)}(X_s^2)\, ds \right] \\
&\quad \times \sqrt{g_{22}(u, v)}\, du\, dv \\
&= \sqrt{g_{22}(0, x_2)}\, dx_2.
\end{aligned}$$

In the case of brownian motion, the authors of [3] refer to $L^{\mathscr{L}}(X)$ as the geometric local time of X on $\mathscr{L}$ and we shall use similar terminology in the general case.

4 The cut locus and the radial part of a semimartingale

We now apply Theorem 1 to the radial part of a semimartingale. To do that we shall first need to study the cut locus since the result we shall use to establish the condition, (H), which we require for Theorem 1 is not explicitly mentioned, so far as we can ascertain, in the geometric literature.

We consider a point x_0 of a complete riemannian manifold $(\boldsymbol{M}, \boldsymbol{g})$ of dimension m. For each v in $\tau_{x_0}(\boldsymbol{M})$, the tangent space to $\boldsymbol{M}$ at x_0, the ray, r_v, determined by v is

$$r_v : \mathbb{R}_+ \to \tau_{x_0}(\boldsymbol{M}); \quad t \mapsto tv.$$

Then the exponential map at x_0 is the smooth map

$$\exp_{x_0} : \tau_{x_0}(\boldsymbol{M}) \to \boldsymbol{M}$$

of the tangent space at x_0 onto $\boldsymbol{M}$ which embeds each ray r_v smoothly into the geodesic which starts at x_0 and has initial tangent vector v. Thus, in particular, for $v \in \tau_{x_0}(\boldsymbol{M})$ the curve

$$\gamma_v : [0, 1] \to \boldsymbol{M}; \quad t \to \exp_{x_0}(tv)$$

is a geodesic segment from x_0 to $\exp_{x_0}(v)$ of length $\|v\| = g_{x_0}(v, v)^{1/2}$. The minimum length of geodesic segments between x and y determines a metric $d(x, y)$ on $\boldsymbol{M}$. We define $\mathscr{C}(x_0)$, the *cut locus of* x_0 *in* $\tau_{x_0}(\boldsymbol{M})$, to be the set

$$\{v \in \tau_{x_0}(\boldsymbol{M}) \colon d(x_0, \exp_{x_0}(v)) = \|v\| \colon d(x_0, \exp_{x_0}(\rho v)) < \rho \|v\|, \ \forall \rho > 1\};$$

the *interior set*, $\mathscr{I}(x_0)$, *in* $\tau_{x_0}(\boldsymbol{M})$ to be

$$\{v \in \tau_{x_0}(\boldsymbol{M}) \colon \exists \rho > 1 \text{ s.t. } d(x_0, \exp_{x_0}(\rho v)) = \rho \|v\|\};$$

and the *first conjugate locus*, $\mathscr{Q}(x_0)$, *of* x_0 *in* $\tau_{x_0}(\boldsymbol{M})$ to be the set

$$\{v \in \tau_{x_0}(\boldsymbol{M}) \colon \text{rank of } d(\exp_{x_0})(v) < m \text{ and rank of } d(\exp_{x_0})(\rho v) = m \ \forall \rho \in [0, 1)\}.$$

Then $C(x_0) = \exp_{x_0}(\mathscr{C}(x_0))$ and $Q(x_0) = \exp_{x_0}(\mathscr{Q}(x_0))$ are the *cut and first conjugate loci*, respectively, *of* x_0 (in $\boldsymbol{M}$) and $\exp_{x_0}$ restricts to a diffeomorphism of $\mathscr{I}(x_0)$ onto $\boldsymbol{M} - C(x_0)$, the complement of the cut locus in $\boldsymbol{M}$, so that $\exp_{x_0}$ map $\mathscr{C}(x_0) \cup \mathscr{I}(x_0)$ onto the whole of $\boldsymbol{M}$. In general $Q(x_0)$ or both of $C(x_0)$ and $Q(x_0)$ could be empty and neither need include the other.

All the above may be found in standard references, such as [2, Sect. 11.6].

Theorem 2 *Except for a set, E, of Hausdorff $(m - 1)$-measure zero the cut locus $C(x_0)$ is the countable disjoint union, $\mathscr{L}$, of open subsets of two-sided $(m - 1)$-dimensional submanifolds such that, locally, the distance from x_0 restricted to one side of such a subset extends as a smooth function to the other side.*

Proof. What we show in fact is that, except for a set of Hausdorff $(m - 1)$-measure zero, the cut locus is a disjoint union of open subsets N_{ij} of two-sided $(m - 1)$-dimensional submanifolds, where each N_{ij} is the diffeomorphic image under $\exp_{x_0}$ of precisely two submanifolds $\mathscr{N}_i$, $\mathscr{N}_j$ of $\mathscr{C}(x_0)$ such that on one side of N_{ij} distances from x_0 are realised by geodesics which are the images of rays in $\tau_{x_0}(\boldsymbol{M})$ leading to points of $\mathscr{N}_i$, and on the other side of N_{ij} distances are similarly realised using $\mathscr{N}_j$. The theorem follows since, on the image of the interior set, the distance of x from x_0 is just the norm of $\exp_{x_0}^{-1}(x)$, which obviously extends smoothly across $\mathscr{N}_i$ or $\mathscr{N}_j$.

We consider first that part, $C(x_0) \backslash Q(x_0)$, of the cut locus which lies outside the first conjugate locus and which gives rise to the submanifolds N_{ij} above. If

$\mathscr{U}_i$ is any open subset of the tangent space $\tau_{x_0}(M)$ which is mapped diffeomorphically onto an open subset U of M, then, we may define a smooth function d_i on U by $d_i(x) = \|(\exp_{x_0}|\mathscr{U}_i)^{-1}(x)\|$. Thus d_i is the distance to points of U measured along geodesics which are the images of rays from the origin in $\tau_{x_0}(M)$ to points of $\mathscr{U}_i$. The function $d_i(x)$ only agrees with the metric distance $d(x_0, x)$ on $\exp_{x_0}\{\overline{\mathscr{I}(x_0)\cap\mathscr{U}_i}\}$. In [10] Ozols uses the differences $d_{ij} = d_i - d_j$ between pairs of such functions to characterise $C(x_0)\backslash Q(x_0)$. He shows [10, Proposition 2.3] that each point, x, of $C(x_0)\backslash Q(x_0)$ has a neighbourhood U, which is the image of finitely many $\mathscr{U}_i$ such that the subsets $V_{ij} = d_{ij}^{-1}(0)\cap U$ are codimension one submanifolds. Moreover, writing C_{ij} for the subset $\{x\colon d_i(x) = d_j(x) \leqq d_k(x)$ for all $k\}$ of V_{ij}, we have $C(x_0)\cap U = \bigcup_{i<j} C_{ij}$. Now if $x \in C_{ij}\cap C_{kl}$ we must have $d_i(x) = d_j(x) = d_k(x) = d_l(x)$ so that, if all of i, j, k and l are distinct, x would already lie in $C_{ij}\cap C_{ik}$. However Ozols [10, Proposition 2.6] shows that then the intersection between V_{ij} and V_{ik} is transverse and so is an $(m-2)$-dimensional submanifold. It follows that, except for a subset of Hausdorff $(m-1)$-measure zero contained in the union of such intersections, $C(x_0)\cap U$ is a disjoint union of the relatively open subsets,

$$N_{ij} = \{x\colon d_i(x) = d_j(x) < d_k(x),\ \forall k \neq i, j\}$$

of V_{ij}. The subset N_{ij} is the intersection with V_{ij} of the open subset

$$U_{ij} = \{x\colon d_i(x) < d_k(x),\ d_j(x) < d_k(x),\ \forall k \neq i, j\}$$

of M. Then on one side, $\{x\colon d_i(x) \leqq d_j(x)\}$, of N_{ij} in U_{ij} the distance $d(x, x_0)$ from x_0 is given by d_i, and on the other side it is given by d_j and both of these functions are smooth and defined on the whole of U_{ij}.

Turning to the conjugate part, $C(x_0)\cap Q(x_0)$, of the cut locus, we shall require Warner's results of [12, 13]. In [13, Lemma 1.1] he shows that the image, under the exponential map, of the conjugate locus has Hausdorff $(m-1)$-measure zero except, possibly, for the image of an open $(m-1)$-dimensional submanifold, $\mathscr{W}$, of $\tau_{x_0}(M)$ which has the following properties. The manifold $\mathscr{W}$, which would be $C_1 - T$ in the notation of [13], is part of the regular conjugate locus. That is, every point p of $\mathscr{W}$ has a neighbourhood $\mathscr{U}$ in $\tau_{x_0}(M)$ such that each ray in $\tau_{x_0}(M)$ meets $\mathscr{U}\cap\mathscr{Q}(x_0)$ in at most one point. In fact $\mathscr{W}$ is comprised of those points, p, of the first conjugate locus of order one where the (one-dimensional) kernel of $d(\exp_{x_0})(p)$ does not lie in $\tau_p(\mathscr{W})$. Now in [12] Warner had already given a local representation of the exponential map at such points. Namely there are coordinates $y_1, \ldots, y_m$ in $\tau_{x_0}(M)$ about p and coordinates $x_1, \ldots, x_m$ about $\exp_{x_0}(p)$ in M with respect to which $\exp_{x_0}$ is given by

$$x_1 = y_1, \ldots, x_{m-1} = y_{m-1}, x_m = y_m^2$$

so that, in particular, $\mathscr{W}$ is given locally by $y_m = 0$.

Consider however the effect that this map must have on the rays which meet $\mathscr{W}$. Since the rays are embedded smoothly in M as geodesics and since both sides of $\mathscr{W}$ in $\tau_{x_0}(M)$ are mapped to the same side of $\exp_{x_0}(\mathscr{W})$, given locally by $x_m \geqq 0$, all the rays which meet $\mathscr{W}$ must do so tangentially. Through each point of $\mathscr{W}$ there is such a ray, so that $\mathscr{W}$ has an everywhere radial never zero tangential vector field. Let $c(t)$ be an integral curve in $\mathscr{W}$ of this field. Then, if $\mathscr{S}$ is the unit sphere in $\tau_{x_0}(M)$, the radial projection of $c(t)$ on $\mathscr{S}$ will be an integral curve of the projection onto the tangent bundle of $\mathscr{S}$ of the tangent field $c'(t)$ to c. Since $c'(t)$ is everywhere radial this projected field will be zero and so the projection of $c(t)$ on $\mathscr{S}$ will be constant. In other words $c(t)$ is a, possibly reparametrised, segment of a ray lying in $\mathscr{W}$. However that contradicts the regularity of $\mathscr{W}$. Hence that part of the conjugate locus in the tangent space cannot exist and so the only part of the cut locus in M where we cannot obtain the distance functions we require has Hausdorff $(m-1)$-measure zero.

Finally, from Theorems 1 and 2, we obtain the stochastic differential equation for the radial part of a semimartingale X, which is valid for all time.

Theorem 3 *Suppose that E and $\mathscr{L}$ are as given in Theorem 2 and that E is a polar set of X. Then the radial part $\phi(X) = d(X, x_0)$ of X is a semimartingale on $\mathbb{R}$ and its stochastic differential equation is given by the following full Itô formula:*

$$\begin{aligned} d\phi(X_t) = {} & [\underline{\operatorname{grad}}\,\phi(X_t)\cdot \Xi_t]\, d\hat{X}_t + \tfrac{1}{2}\underline{\operatorname{Hess}}^{\phi}(\Xi_t, \Xi_t)\, d[\hat{X}]_t \\ & + 1_{\{X_t \in \mathscr{L}\}}\{[\operatorname{grad}\phi\circ\pi(X_t)\cdot \Xi_t]\, d\hat{X}_t + \tfrac{1}{2}\operatorname{Hess}^{\phi\circ\pi}(\Xi_t, \Xi_t)\, d[\hat{X}]_t\} \\ & + dL_t^0(\phi(X)) + \tfrac{1}{2}\{D_+\phi_{X_t}(v)\, dL_t^{+v,\mathscr{L}}(X) + D_+\phi_{X_t}(-v)\, dL_t^{\mathscr{L}}(X)\}, \end{aligned}$$

where $L^{\pm v, \mathscr{L}}(X)$ are the geometric local times of X on $\mathscr{L}$.

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