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An invariance principle for non-symmetric Markov processes and reflecting diffusions in random domains

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Summary. We study an invariance principle for additive functionals of non-symmetric Markov processes with singular mean forward velocities. We generalize results of Kipnis and Varadhan [KV] and De Masi et al. [De] in two directions: Markov processes are non-symmetric, and mean forward velocities are distributions. We study continuous time Markov processes. We use our result to homogenize non-symmetric reflecting diffusions in random domains.

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1 Introduction

In [KV] Kipnis and Varadhan proved functional central limit theorems for additive functionals of stationary reversible Markov processes, and applied their results to study the asymptotic normality of tagged particles of simple exclusion processes. In [De] Masi et al. obtained a sufficient condition for the spectral condition of mean forward velocities and applied their results to various problems.

Crucial assumptions in these works are the following: (1) Markov processes are symmetric, (2) mean forward velocities are functions. The main purpose of this paper is to relax these two assumptions and generalize their results.

The results above are quite general; however assumption (1) excludes some homogenization results for non-symmetric diffusion processes studied in [PV, O1, Oe]. Our contribution is that such an invariance principle holds for certain classes of non-symmetric Markov processes. Namely, we weaken condition (1) to the so-called strong sector condition (1.1).

In [KV, De] both discrete and continuous time processes are studied. We study only the continuous time case. In this case assumption (2) seems too restrictive. Indeed when we study homogenization of reflecting diffusions or tagged particles of infinitely many particle systems with hard core interaction, the additive functionals contain local time type drifts. So their mean forward velocities are not functions. Our result admits the case that mean forward velocities are distributions.

Our result can be used to prove central limit theorems for non-symmetric reflecting diffusions in random domains, tagged particles of infinitely many particle systems with hard core interaction, and ones of non-symmetric exclusion processes. We study the first case in the present paper.

Let Θ be a Hausdorff topological space and $\mathfrak{B}(\Theta)$ denote its Borel σ-algebra. We assume $\mathfrak{B}(\Theta) = \sigma[C(\Theta)]$ where $C(\Theta)$ is the set of all continuous functions. Let μ be a probability measure on $\mathfrak{B}(\Theta)$. Let $L^2(\Theta, \mu)$ denote the real Hilbert space with the inner product $(u, v) = \int_\Theta u(\theta)v(\theta)\mu(d\theta)$, and let $\mathbf{F}$ be a dense subspace of $L^2(\Theta, \mu)$ such that $1 \in \mathbf{F}$. Let $(\mathscr{E}, \mathbf{F})$ be a (non-symmetric) Dirichlet form. We assume that $(\mathscr{E}, \mathbf{F})$ on $L^2(\Theta, \mu)$ is *quasi-regular* and that there exists a Hunt process $(\Omega, \mathfrak{F}, \mathfrak{F}_t, \xi_t, \{P_\theta\}_{\theta\in\Theta})$ *properly associated* with $(\mathscr{E}, \mathbf{F})$ on $L^2(\Theta, \mu)$. Here $\mathfrak{F}_t$ is the natural filtration of $(\Omega, \mathfrak{F}, \xi_t, \{P_\theta\}_{\theta\in\Theta})$. We refer to [MR] for the definition of quasi-regular Dirichlet form and related notions.

We assume the Dirichlet form $(\mathscr{E}, \mathbf{F})$ and the Hunt process $(\Omega, \mathfrak{F}, \mathfrak{F}_t, \xi_t, \{P_\theta\}_{\theta\in\Theta})$ satisfies the following conditions.

(1.1) Strong sector condition; there exists a constant $K_1 \geqq 1$ such that

$$|\mathscr{E}(u, v)| \leqq K_1 \mathscr{E}(u, u)^{1/2} \mathscr{E}(v, v)^{1/2} \quad \text{for all } u, v \in \mathbf{F}.$$

(1.2) Stationarity; μ is the invariant probability measure of $(\Omega, \mathfrak{F}, \mathfrak{F}_t, \xi_t, \{P_\theta\}_{\theta\in\Theta})$.

(1.3) Ergodicity; P_μ is ergodic under the time translation ϑ_t.

Let $X = X_t$ be a d-dimensional additive functional of $(\Omega, \mathfrak{F}, \mathfrak{F}_t, \xi_t, \{P_\theta\}_{\theta\in\Theta})$; that is, X_t *is* $\mathfrak{F}_t$-measurable for all t, there exist sets $\Lambda \in \mathfrak{F}$ and $\Theta_e \subset \Theta$ such that $\mathrm{Cap}(\Theta_e) = 0$, $P_\theta(\Lambda) = 1$ for all $\theta \notin \Theta_e$, $\vartheta_t \Lambda \subset \Lambda$ for all t, and for all $\omega \in \Lambda$, $X_t(\omega) \in D([0, \infty); R^d)$ and $X_{s+t}(\omega) = X_s(\omega) + X_t(\vartheta_s \omega)$ for all s, t. We note here our Hunt process is conservative; X is an additive functional in the sense of [MR].

We denote d-times products of $\mathbf{F}$ and $L^p(\Theta, \mu)$ by $\mathscr{F}$ and $\mathscr{L}^p(\Theta, \mu)$ respectively. We set $\mathscr{E}(f, g) = \sum_{i=1}^d \mathscr{E}(f^i, g^i)$ for $f = (f^i)$, $g = (g^i) \in \mathscr{F}$, and the same convention for $(\,,)$ in $\mathscr{L}^2(\Theta, \mu)$. Let $\mathbf{F}'$ be the set of the real valued linear functionals on $\mathbf{F}$, and $\mathscr{F}'$ the d-times product of $\mathbf{F}'$. We set $\varphi(f) = \sum_{i=1}^d \varphi^i(f^i)$ for $\varphi = (\varphi^i) \in \mathscr{F}'$ and $f = (f^i) \in \mathscr{F}$.

We now state our main theorem.

Theorem 1.1 *Assume that* (1.1)–(1.3) *hold. Suppose the additive functional* $X = X_t$ *satisfy the conditions in* (1.4):

$$E_\mu[|X_t|^2] < \infty \quad \textit{for all } t. \tag{1.4}$$

$$E_\theta\left[\int_0^\infty e^{-pt}|X_t|\,dt\right] < \infty \quad \textit{for all } p>0 \ \mu\text{-a.e. } \theta.$$

$$(\chi^p, \chi^p) < \infty \quad \textit{for all } p>0, \textit{ where } \chi^p(\theta) = E_\theta\left[\int_0^\infty e^{-pt}X_t\,dt\right].$$

Suppose that there exists a linear functional $\varphi \in \mathscr{F}'$ *and a constant* K_2 *satisfying* (1.5) *and* (1.6):

(1.5) $$\lim_{p\to\infty}(p^2\chi^p, f) = \varphi(f) \quad \textit{for all } f \in \mathscr{F}.$$

(1.6) $$|\varphi(f)| \leqq K_2 \mathscr{E}(f,f)^{1/2} \quad \textit{for all } f \in \mathscr{F}.$$

Let $P_\theta^\varepsilon(\cdot) = P_\theta(\varepsilon X_{t/\varepsilon^2} \in \cdot)$. *Then*

(1.7) $$\lim_{\varepsilon\to 0} P_\theta^\varepsilon = \hat{P} \quad \textit{in f.d.d. in } \mu\textit{-measure},$$

where $\hat{P}$ *is the distribution of a d-dimensional continuous martingale* $\hat{X}$ *such that*

(1.8) $$\langle \hat{X}^i, \hat{X}^j\rangle_t = 2a_{ij}t, \qquad \hat{X}_0 = 0.$$

Here a_{ij} *is a constant matrix.*

The statement (1.7) means the following: for each $t_1, \ldots, t_n$ and $F \in C_b(R^{dn})$

$$\lim_{\varepsilon\to 0}\mu(\{\theta; |E_\theta^\varepsilon[F(w_{t_1}, \ldots, w_{t_n})] - \hat{E}[F(w_{t_1}, \ldots, w_{t_n})]| \geqq \delta\}) = 0$$

for all $\delta > 0$. Here E_θ^ε and $\hat{E}$ are expectations with respect to P_θ^ε and $\hat{P}$, respectively.

Remark 1 In [KV] it was assumed that

(1.9) $$\lim_{t\to 0}\frac{1}{t}E_\theta[X_t] = \varphi(\theta) \quad \text{strongly in } \mathscr{L}^2(\Theta, \mu),$$

for some $\varphi \in \mathscr{L}^2(\Theta,\mu)$. In [De] $\mathscr{L}^2$ is replaced by $\mathscr{L}^1$. Clearly the condition (1.5) is much weaker than (1.9). We also note that X may not be a semimartingale, which is different from the case of [De]. In [De] the function $\varphi(\theta)$ in (1.9) is called *the mean forward velocity.* So we call the distribution φ in (1.5) *the mean forward velocity.*

Remark 2 In [KV, De] $(\mathscr{E}, \mathbf{F})$ is assumed to be symmetric. We assume (1.1) instead of the symmetry. We note that (1.1) is satisfied if $(\mathscr{E}, \mathbf{F})$ is symmetric, and that there are various non-symmetric models which satisfy (1.1).

Remark 3 We can replace (1.5) by the weaker condition (1.5′).
(1.5′) For each $\alpha > 0$ there exists a dense subset $\mathbf{G}_\alpha$ in $L^2(\Theta, \mu)$ such that

$$\lim_{p\to\infty}(p^2\chi^p, f) = \varphi(f) \quad \text{for all } f \in \mathscr{F}_\alpha.$$

Here $\mathscr{F}_\alpha$ is the d-times product of $\mathbf{F}_\alpha$, $\mathbf{F}_\alpha = \{G^*_\alpha g; g \in \mathbf{G}_\alpha\}$ and G^*_α is the dual resolvent of $(\mathscr{E}, \mathbf{F})$ on $L^2(\Theta, \mu)$.

When X has a *mean backward velocity* φ^*, we obtain an expression of a_{ij}.

Theorem 1.2 *In addition to the assumptions of Theorem 1.1, assume that there exists a linear functional $\varphi^* \in \mathscr{F}'$ and a positive constant K^*_2 such that*

$$(1.5^*) \qquad \lim_{p\to\infty} E_\mu\left[\int_0^\infty p^2 e^{-pt} \sum_{i=1}^d X^i_t f^i(\xi_t)\,dt\right] = \varphi^*(f) \quad \text{for all } f \in \mathscr{F},$$

$$(1.6^*) \qquad |\varphi^*(f)| \leqq K^*_2 \mathscr{E}(f,f)^{1/2} \quad \text{for all } f \in \mathscr{F}.$$

Then

$$(1.10)\ a_{ij} = e(X^i, X^j) + \tilde{\mathscr{E}}^s(\psi^i, \psi^j) + \tfrac{1}{2}(\tilde{\varphi}^* - \tilde{\varphi})^i(\psi^j) + \tfrac{1}{2}(\tilde{\varphi}^* - \tilde{\varphi})^j(\psi^i),$$

where $e(X^i, X^j)$ is the mutual energy of X^i and X^j defined in Sect. 4, $\psi = (\psi^i)_{1 \leqq i \leqq d}$ is the unique solution of (2.4). $\tilde{\mathscr{E}}^s$, $\tilde{\varphi}$ and $\tilde{\varphi}^*$ are defined before Lemma 2.1. Here the superscript i indicates that the quantity is the i-th component.

Let $X^* = X^*_t$ be the additive functional such that $X^*_t = X_t \circ R_{t/2}$, where R_a is the time-reflection operator in a, $R_a(\xi)_t = \xi_{2a-t}$. Then the conditions (1.5*) and (1.6*) means that the dual process of $(\Omega, \mathfrak{F}, \mathfrak{F}_t, \xi_t, \{P_\theta\}_{\theta\in\Theta})$ on $L^2(\Theta, \mu)$ satisfies (1.5) and (1.6), where X, φ and K_2 are replaced by X^*, φ^* and K^*_2 respectively.

We give some idea of the proof. A key point of proving such a central limit theorem is to solve a Poisson-type equation such that

$$(1.11) \qquad \mathscr{E}(u,f) = \varphi(f) \quad \text{for all } f \in \mathscr{F}.$$

Unfortunately (1.11) cannot be solved in $\mathscr{F}$ except in a few cases such as periodic ones. Even in the case studied in [PV] (1.11) has no solution in $\mathscr{F}$ (but there exists a *non-stationary* solution). Hence we consider the quotient Hilbert space $\tilde{\mathscr{F}}$ of $\mathscr{F}$ and the corresponding equation (2.4) in $\tilde{\mathscr{F}}$. By (1.1) and (1.6) this equation has a unique solution $\psi \in \tilde{\mathscr{F}}$ (see Lemma 2.1). Let ψ_λ be the solution of (2.1). From the uniqueness of ψ we obtain that $\{\psi_\lambda\}_{\lambda>0}$ is $\mathscr{E}$-Cauchy and $\lim_{\lambda\to 0} \lambda(\psi_\lambda, \psi_\lambda) = 0$ (Proposition 2.2). When $(\mathscr{E}, \mathbf{F})$ is symmetric as in [KV, De], this step is clear from the spectral resolution.

We use non-symmetric Dirichlet form theory to relax the regularity of the mean forward velocity. In application we will utilize a theorem due to Ma and Röckner [MR] that the existence of Hunt processes implies the quasi-regularity of Dirichlet forms.

Homogenization of reflecting diffusions is a problem to prove a convergence to Brownian motion of diffusion process moving in random domains in R^n satisfying the reflecting boundary condition. When diffusions are reversible, this problem was studied by [Oc], [B] in the periodic case, by [O2] in the stationary case under a certain geometric condition and by [T] in Poisson blob model. In this paper we will study non-symmetric reflecting diffusions. In [O2, T] the discrete time version of the invariance principle of [De] was used.

This theorem cannot be applied to the present case because of the lack of the symmetry. Reflecting diffusions have local time type drifts. So our second generalization is useful for this problem.

The organization of this paper is as follows: In Sect. 2 we prove our main theorem (Theorem 1.1) and Theorem 1.2. In Sect. 3 we apply Theorem 1.1 to study homogenization of reflecting diffusion. In Sect. 4 we collect some results from non-symmetric Dirichlet form theory.

2 Proof of Theorems 1.1 and 1.2

In this section we prove Theorems 1.1 and 1.2. Let $\psi_\lambda = (\psi_\lambda^i)_{1 \leqq i \leqq d}$ $(\lambda > 0)$ denote the unique solution of the Eq. (2.1) in $\mathscr{F}$:

(2.1) $$\lambda(\psi_\lambda, g) + \mathscr{E}(\psi_\lambda, g) = \varphi(g) \quad \text{for all } g \in \mathscr{F},$$

Taking $g = \psi_\lambda$ in (2.1) we see from (1.6) that

(2.2) $$\mathscr{E}(\psi_\lambda, \psi_\lambda) \leqq C_1 \quad \text{for each } \lambda > 0,$$

(2.3) $$\lambda(\psi_\lambda, \psi_\lambda) \leqq C_1 \quad \text{for each } \lambda > 0, \text{ where } C_1 = K_2^2.$$

Let $\sim$ be the equivalence relation such that $f \sim g$ if and only if $\mathscr{E}(f-g, f-g) = 0$. In the following $\tilde{f}$ denotes the element of the quotient space $\mathscr{F}/\sim$ whose representative is $f \in \mathscr{F}$. Let $\tilde{\mathscr{E}}$ denote the bilinear form on $\mathscr{F}/\sim$ defined by $\tilde{\mathscr{E}}(\tilde{f}, \tilde{g}) = \mathscr{E}(f, g)$. Then by (1.1) $\tilde{\mathscr{E}}$ is well-defined. Since $\mathscr{F}/\sim$ with the inner product $\tilde{\mathscr{E}}^s(f, g) = \frac{1}{2}(\tilde{\mathscr{E}}(f, g) + \tilde{\mathscr{E}}(g, f))$ is a pre Hilbert space, we consider the completion $\tilde{\mathscr{F}}$. Let $\tilde{\varphi}: \mathscr{F}/\sim \to R$ such that $\tilde{\varphi}(\tilde{f}) \equiv \varphi(f)$. By (1.1) and (1.6) we extend $\tilde{\varphi}$ to the bounded linear functional on $\tilde{\mathscr{F}}$ denoted by the same symbol $\tilde{\varphi}$. When (1.5*) and (1.6*) are satisfied, we define $\tilde{\varphi}^*$ from φ^* similarly as $\tilde{\varphi}$. A simple consequence of this completion is the following.

Lemma 2.1 *There exists a unique $\psi \in \tilde{\mathscr{F}}$ such that*

(2.4) $$\tilde{\mathscr{E}}(\psi, g) = \tilde{\varphi}(g) \quad \textit{for all } g \in \tilde{\mathscr{F}}.$$

Proof. Let $A: \tilde{\mathscr{F}} \to \tilde{\mathscr{F}}'$ such that $Au = \tilde{\mathscr{E}}(u_1 \cdot)$. Then A is coercive, so the claim is clear (from Lemma 1.1.2 in [Ko]). □

Proposition 2.2 *Let ψ_λ, $\tilde{\psi}_\lambda$ and ψ be as above. Then*

(2.5) $$\lim_{\lambda \to 0} \tilde{\mathscr{E}}(\tilde{\psi}_\lambda - \psi, \tilde{\psi}_\lambda - \psi) = 0,$$

(2.6) $$\lim_{\lambda, \lambda' \to 0} \mathscr{E}(\psi_\lambda - \psi_{\lambda'}, \psi_\lambda - \psi_{\lambda'}) = 0,$$

(2.7) $$\lim_{\lambda \to 0} \lambda(\psi_\lambda, \psi_\lambda) = 0.$$

Proof. We first prove that

(2.8) $$\lim_{\lambda \to 0} \tilde{\psi}_\lambda = \psi \quad \text{weakly in } \tilde{\mathscr{F}}.$$

By (2.2) and $\tilde{\mathscr{E}}(\tilde{\psi}_\lambda, \tilde{\psi}_\lambda) = \mathscr{E}(\psi_\lambda, \psi_\lambda)$, $\{\tilde{\psi}_\lambda\}$ is bounded in $\tilde{\mathscr{F}}$. Let $\{\tilde{\psi}_{\lambda_h}\}$ be an arbitrary weakly convergent subsequence in $\tilde{\mathscr{F}}$ and $\psi_0 \in \tilde{\mathscr{F}}$ its limit. Then $\lim_{\lambda_h \to 0} \tilde{\mathscr{E}}(\tilde{\psi}_{\lambda_h}, \tilde{g}) = \tilde{\mathscr{E}}(\psi_0, \tilde{g})$ for each $g \in \mathscr{F}$. By (2.1) and (2.3) we see for each $g \in \mathscr{F}$

$$\lambda(\psi_\lambda, g) + \tilde{\mathscr{E}}(\tilde{\psi}_\lambda, \tilde{g}) = \tilde{\varphi}(\tilde{g}) \quad \text{and} \quad \lim_{\lambda \to 0} \lambda(\psi_\lambda, g) = 0. \tag{2.9}$$

Combining these we see ψ_0 is a solution of (2.4). Hence by Lemma 2.1 $\psi_0 = \psi$, which means (2.8).

We observe

$$\begin{aligned} \tilde{\mathscr{E}}(\psi, \psi) &= \tilde{\varphi}(\psi) \quad \text{by (2.4)} \\ &= \lim_{\lambda \to 0} \tilde{\varphi}(\tilde{\psi}_\lambda) \quad \text{by (2.8)} \\ &= \lim_{\lambda \to 0} (\lambda(\psi_\lambda, \psi_\lambda) + \tilde{\mathscr{E}}(\tilde{\psi}_\lambda, \tilde{\psi}_\lambda)) \quad \text{by (2.9)} \\ &\geqq \limsup_{\lambda \to 0} \tilde{\mathscr{E}}(\tilde{\psi}_\lambda, \tilde{\psi}_\lambda). \end{aligned}$$

By (2.8) we see $\tilde{\mathscr{E}}(\psi, \psi) \leqq \liminf_{\lambda \to 0} \tilde{\mathscr{E}}(\tilde{\psi}_\lambda, \tilde{\psi}_\lambda)$. Combining these yields

$$\lim_{\lambda \to 0} \tilde{\mathscr{E}}(\tilde{\psi}_\lambda, \tilde{\psi}_\lambda) = \tilde{\mathscr{E}}(\psi, \psi) \quad \text{and} \quad \lim_{\lambda \to 0} \lambda(\psi_\lambda, \psi_\lambda) = 0.$$

Hence we obtain (2.6) and (2.7). (2.5) follows from (2.8) and $\lim_{\lambda \to 0} \tilde{\mathscr{E}}(\tilde{\psi}_\lambda, \tilde{\psi}_\lambda) = \tilde{\mathscr{E}}(\psi, \psi)$ immediately. □

Let $\mathscr{M}$ and $\mathbb{M}$ denote sets of d-dimensional martingales defined in Sect. 4. For $u \in \mathscr{F}$, we set $A_t^{[u]} = \hat{u}(\xi_t) - \hat{u}(\xi_0)$. Here $\hat{u}$ is a quasi-continuous version of u. In order to prove Theorem 1.1 we prepare two lemmas.

Lemma 2.3 *Let $M_t^\lambda = X_t + A_t^{[\psi_\lambda]} - \lambda \int_0^t \psi_\lambda(\xi_s) ds$. Then $M^\lambda = M_t^\lambda$ is an element of $\mathscr{M}$.*

Proof. Let $N_t = -X_t + \lambda \int_0^t \psi_\lambda(\xi_s) ds$. We check $N = N_t$ satisfies the conditions in Proposition 4.3. By (1.4) N satisfies (4.2 – 4.4). Let $f \in \mathscr{F}_\alpha$. Then clearly

$$\lim_{p \to \infty} \left(E_\theta \left[\int_0^\infty p^2 e^{-pt} \int_0^t \psi_\lambda(\xi_s) ds\, dt \right], f \right) = (\psi_\lambda, f).$$

Combining this with (1.5′) yields

$$\lim_{p \to \infty} \left(E_\theta \left[\int_0^\infty p^2 e^{-pt} N_t\, dt \right], f \right) = -\varphi(f) + \lambda(\psi_\lambda, f) = -\mathscr{E}(\psi_\lambda, f),$$

which means N satisfies (4.5) with $u = \psi_\lambda$. Hence by Proposition 4.3 we see the claim holds. □

Lemma 2.4 *There exists an $M = M_t \in \mathbb{M}$ such that M^λ under P_μ converges to M in $\mathbb{M}$ as $\lambda \to 0$. Moreover if (1.5*) and (1.6*) are satisfied, then*

$$e(M^i, M^j) = a_{ij}, \tag{2.10}$$

where M^i is the i-th component of M and e is the energy defined in Sect. 4, *and* a_{ij} *is defined by* (1.10).

Proof. We first prove that M^λ is e-Cauchy. By definition we see

$$M_t^\lambda - M_t^{\lambda'} = A_t^{[\psi_\lambda - \psi_{\lambda'}]} - \lambda \int_0^t \psi_\lambda(\xi_s)\,ds + \lambda' \int_0^t \psi_{\lambda'}(\xi_s)\,ds.$$

Note that $e(\int_0^{\cdot} \psi_\lambda(\xi_s)\,ds) = 0$. Hence by Lemma 4.1

$$e(M^\lambda - M^{\lambda'}) = e(A^{[\psi_\lambda - \psi_{\lambda'}]}) = \mathscr{E}(\psi_\lambda - \psi_{\lambda'}, \psi_\lambda - \psi_{\lambda'}).$$

Thus by (2.6) we see $\{M^\lambda\}$ is e-Cauchy. Combining this with (1) of Lemma 4.2 yields the first claim.
By (1.5) and (1.5*),

$$2e(X^i, A^{[\psi_\lambda], j}) = \lim_{p\to\infty} E_\mu\left[\int_0^\infty p^2 e^{-pt} X_t^i \{\psi_\lambda^j(\xi_t) - \psi_\lambda^j(\xi_0)\}\,dt\right]$$
$$= \varphi^{*,i}(\psi_\lambda^j) - \varphi^i(\psi_\lambda^j).$$

Hence by Lemma 4.1 and the definition of M^λ we obtain

$$\begin{aligned} e(M^{\lambda,i}, M^{\lambda,j}) &= e((X + A^{[\psi_\lambda]})^i, (X + A^{[\psi_\lambda]})^j) \\ &= e(X^i, X^j) + \mathscr{E}^s(\psi_\lambda^i, \psi_\lambda^j) + e(X^i, A^{[\psi_\lambda], j}) + e(A^{[\psi_\lambda], i}, X^j) \\ &= e(X^i, X^j) + \mathscr{E}^s(\psi_\lambda^i, \psi_\lambda^j) + \tfrac{1}{2}(\varphi^{*,i}(\psi_\lambda^j) - \varphi^i(\psi_\lambda^j)) \\ &\quad + \tfrac{1}{2}(\varphi^{*,j}(\psi_\lambda^i) - \varphi^j(\psi_\lambda^i)) \\ &\underset{\lambda\to 0}{\to} e(X^i, X^j) + \tilde{\mathscr{E}}^s(\psi^i, \psi^j) + \tfrac{1}{2}(\varphi^{*,i}(\psi^j) - \varphi^i(\psi^j)) \\ &\quad + \tfrac{1}{2}(\varphi^{*,j}(\psi^i) - \varphi^j(\psi^i)). \end{aligned}$$

Here we used Proposition 2.2 for the last line. Combining this with (1.10) and Lemma 4.2 (2) yields (2.10) immediately. □

Proof of Theorem 1.1 Let $N_t^\varepsilon = \varepsilon M_{t/\varepsilon^2}^{\varepsilon^2} - \varepsilon M_{t/\varepsilon^2}$, $Q_t^\lambda = A_t^{[\psi_\lambda]} - \lambda \int_0^t \psi_\lambda(\xi_s)\,ds$ and $R_t^\varepsilon = \varepsilon Q_{t/\varepsilon^2}^{\varepsilon^2}$. Then

$$\varepsilon X_{t/\varepsilon^2} = \varepsilon M_{t/\varepsilon^2} + N_t^\varepsilon + R_t^\varepsilon. \tag{2.11}$$

Since M_t^λ $(\lambda > 0)$ under P_μ have stationary increments, so does M_t. Hence by Helland theorem in [H] and the ergodic theorem we obtain for a.s.θ with respect to μ

(2.12) $\varepsilon M_{t/\varepsilon^2}$ under P_θ converge to $\hat{X}_t$ in f.d.d.

Here $\hat{X}_t$ is a d-dimensional continuous martingale such that

$$\langle \hat{X}^i, \hat{X}^j \rangle_t = t E_\mu[M_1^i \cdot M_1^j], \qquad \hat{X}_0 = 0. \tag{2.13}$$

Since $e(N^{\varepsilon}) = e(M^{\varepsilon^2} - M) \to 0$ as $\varepsilon \to 0$, we obtain by (4.1) that

(2.14) $$\lim_{\varepsilon \to 0} E_\mu[|N_t^\varepsilon|^2] = 0 \quad \text{for all } t.$$

We next prove

(2.15) $$\lim_{\varepsilon \to 0} E_\mu[|R_t^\varepsilon|^2] = 0 \quad \text{for all } t.$$

By the definition of R_t^ε we have

(2.16) $$|R_t^\varepsilon| \leqq |\varepsilon \hat{\psi}_{\varepsilon^2}(\xi_{t/\varepsilon^2})| + |\varepsilon \hat{\psi}_{\varepsilon^2}(\xi_0)| + \varepsilon \cdot \varepsilon^2 \int_0^{t/\varepsilon^2} |\psi_{\varepsilon^2}(\xi_s)|\, ds.$$

By (1.2) and (2.7) we obtain

$$E_\mu[|\varepsilon \psi_{\varepsilon^2}(\xi_{t/\varepsilon^2})|^2] = E_\mu[|\varepsilon \psi_{\varepsilon^2}(\xi_0)|^2] = \varepsilon^2(\psi_{\varepsilon^2}, \psi_{\varepsilon^2}) \to 0 \quad \text{as } \varepsilon \to 0,$$

$$E_\mu\left[\left\{\varepsilon^3 \int_0^{t/\varepsilon^2} |\psi_{\varepsilon^2}(\xi_s)|\, ds\right\}^2\right] \leqq \varepsilon^6 \frac{t}{\varepsilon^2} E_\mu\left[\int_0^{t/\varepsilon^2} |\psi_{\varepsilon^2}(\xi_s)|^2\, ds\right]$$

$$= \varepsilon^4 t \int_0^{t/\varepsilon^2} E_\mu[|\psi_{\varepsilon^2}(\xi_s)|^2]\, ds = \varepsilon^2 t^2 (\psi_{\varepsilon^2}, \psi_{\varepsilon^2}) \to 0 \quad \text{as } \varepsilon \to 0.$$

Combining these estimates with (2.16) yields (2.15).

By (2.12), (2.14) and (2.15) the one dimensional distributions of $\varepsilon X_{t/\varepsilon^2}$ converge to those of $\hat{X}_t$ weakly in μ-measure. The proof for the convergence of the k-dimensional distributions is standard. Hence we omit it. □

Proof of Theorem 1.2 By (2.10) and (4.1) we have $E_\mu[M_1^i \cdot M_1^j] = 2a_{ij}$. Combining this with (2.13) yields Theorem 1.2. □

3 Application: Homogenization of reflecting diffusions

In this section, we apply our result to the homogenization problem of non-symmetric reflecting diffusions. We quote some notations and results from [O2].

Let $\mathfrak{C}(R^d)$ denote the set of all closed sets in R^d. Let $\mathfrak{C}_\infty$ denote the subset of $\mathfrak{C}(R^d)$ such that

$$\mathfrak{C}_\infty = \{\theta \in \mathfrak{C}(R^d);\ \theta \notin \{R^d, \emptyset\},\ \theta^c \text{ is connected, } \partial\theta \text{ is smooth}\}.$$

The set $\mathfrak{C}_\infty$ is naturally regarded as a separable metric space.

Let $\tau_x (x \in R^d)$ be the translation on $\mathfrak{C}_\infty$ defined by $\tau_x \theta = \theta - x$. Let Θ be a topological subspace of $\mathfrak{C}_\infty$ such that $\tau_x \Theta = \Theta$ for all $x \in R^d$ and that Θ is a metrizable Lusin space, i.e., topologically isomorphic to a Borel subset of a Polish space. Let μ be a probability measure on $\mathfrak{B} = \mathfrak{B}(\Theta)$. We assume

(3.1) $(\Theta, \mathfrak{B}, \mu, \{\tau_x\}_{x \in R^d})$ is a d-dimensional ergodic flow.

We consider θ as obstacles in R^d; we study diffusion processes moving outside θ and reflecting when they hit the boundary of θ. By choosing Θ and μ properly, various random domains are represented by complements of sample points θ. We give here an example and refer to [O2, T] for other examples.

Example. Let $0 < L \leqq 1/2$ and $\mathbb{L} = (0, L)^{\mathbb{Z}^d}$. We set $\mathbb{A} = [0, 1)^d \times \mathbb{L}$. Let α: $\mathbb{A} \to \mathfrak{C}_\infty$ such that $\alpha(x, (l_i)_{i\in\mathbb{Z}^d}) = x + \sum_{i\in\mathbb{Z}^d} B(l_i)$, where $B(r) = \{x \in R^d;\ |x| \leqq r\}$. Let γ be a probability measure on $(0, L)$ and $\Gamma = \prod_{\mathbb{Z}^d}\gamma$. Let $\nu = dx \times \Gamma$. We set $\Theta = \alpha(\mathbb{A})$ and $\mu = \nu \circ \alpha^{-1}$. Then (Θ, μ) satisfies the conditions above. If $L < \frac{1}{2}$, then (Θ, μ) satisfies the assumption in Theorem 3.2.

Let Θ_0 be a subset of Θ defined by $\Theta_0 = \{\theta \in \Theta;\ \theta$ does not contain the origin$\}$. Then Θ_0 is an open set in Θ. Let $\bar{\Theta}_0 = \{\theta \in \Theta;\ \theta^0$ does not contain the origin$\}$, where θ^0 is the open kernel of θ in R^d.Then $\bar{\Theta}_0$ is a closed set in Θ. It is known in [O2] that $\mu(\bar{\Theta}_0) > 0$ and $\mu(\bar{\Theta}_0) = \mu(\Theta_0)$. We set

$$\mu_0(\cdot) = \mu(\cdot \cap \bar{\Theta}_0)/\mu(\bar{\Theta}_0).$$

Let $a_{ij}(\theta)$ $(1 \leqq i, j \leqq d)$ be measurable functions on Θ such that

$$\sum_{i,j=1}^{d} a_{ij}(\theta)\xi_i\xi_j \geqq K_3|\xi|^2, \quad |a_{ij}(\theta)| \leqq K_4. \tag{3.2}$$

Here K_3 and K_4 are positive constants.

Let $\mathcal{O}(\theta) = \theta^c$ and $\overline{\mathcal{O}(\theta)}$ denote the closure of $\mathcal{O}(\theta)$ in R^d. It is easy to see that $\mathcal{O}(\theta) = \{x \in R^d;\ \tau_x\theta \in \Theta_0\}$. We consider a Dirichlet form $\mathscr{E}^\theta$ with the domain $H^1(\overline{\mathcal{O}(\theta)})$ on $L^2(\overline{\mathcal{O}(\theta)}, dx)$:

$$\mathscr{E}^\theta(f, g) = \int_{\mathcal{O}(\theta)} \sum_{i,j=1}^{d} a_{ij}(\tau_x\theta)\partial_i f \partial_j g dx \quad (\partial_i = \partial/\partial x_i).$$

Let $\{P_x^\theta\}$ denote the diffusion process associated with $(\mathscr{E}^\theta, H^1(\overline{\mathcal{O}(\theta)}))$ on $L^2(\overline{\mathcal{O}(\theta)}, dx)$. If $a_{ij} = \frac{1}{2}\delta_{ij}$ (δ_{ij} is the Kronecker's delta), then $\{P_x^\theta\}$ is the reflecting barrier Brownian motion with the state space $\overline{\mathcal{O}(\theta)}$ in the sense of Fukushima [F1]. If $a_{ij} \neq a_{ji}$, then $\{P_x^\theta\}$ is a non-symmetric reflecting diffusion whose invariant measure is Lebesgue measure on $\overline{\mathcal{O}(\theta)}$. We will study the asymptotic behavior of these diffusion processes starting from the origin.

Let $\{P^\theta(t, x, dy)\}$ denote the transition probability of $\{P_x^\theta\}$. Since the boundary of $\mathcal{O}(\theta)$ is smooth, there exists a continuous density $p_t^\theta(x, y)\,dy = P^\theta(t, x, dy)$. For the sake of convenience we set $p_t^\theta(x, y) = 0$ for $(x, y) \notin \overline{\mathcal{O}(\theta)} \times \overline{\mathcal{O}(\theta)}$. Let

$$T_t^\theta f(x) = \int_{R^d} p_t^\theta(x, y) f(y)\,dy, \qquad G_p^\theta f(x) = \int_0^\infty e^{-pt} T_t^\theta f(x)\,dt,$$

$$T_t^{\theta *} f(x) = \int_{R^d} p_t^\theta(y, x) f(y)\,dy, \qquad G_p^{\theta *} f(x) = \int_0^\infty e^{-pt} T_t^{\theta *} f(x)\,dt.$$

Then T_t^θ and $T_t^{\theta *}$ can be regarded as semigroups on function spaces on $\overline{\mathcal{O}(\theta)}$. Moreover T_t^θ (resp. $T_t^{\theta *}$) is associated (resp. co-associated) with $(\mathscr{E}^\theta, L^2(\overline{\mathcal{O}(\theta)}, dx))$.

For $\theta \in \bar{\Theta}_0$ we set $P^\theta = P_0^\theta$, where the subscript 0 of P_0^θ denotes the origin. Let E^θ denote the expectation with respect to P^θ.

Theorem 3.1 *Suppose that (3.3) and (3.4) hold:*

(3.3)
$$\int_{\bar{\Theta}_0} E^\theta[|w_t|^2]\, d\mu_0 < \infty \quad \textit{for all } t > 0.$$

$$\int_{\bar{\Theta}_0} \int_0^\infty e^{-ps} E^\theta[|w_s|^2]\, ds\, d\mu_0 < \infty \quad \textit{for all } p > 0.$$

(3.4) $\{T_t^{\theta *}\}$ *is a semigroup of class* (C_0) *on* $L^1(\overline{\mathcal{O}(\theta)}, |x|\, dx)$, μ_0*-a.e.* θ.

$$\int_{\mathcal{O}(\theta)} \sum_{i=1}^d |\partial_i G_p^{\theta *} r|\, dx < \infty \quad \textit{for all } p > 0, r \in C_0(R^d), \mu_0\textit{-a.e. } \theta.$$

Let $P^{\theta,\varepsilon}(\cdot) = P^\theta(\varepsilon w_{t/\varepsilon^2} \in \cdot)$. *Then*

(3.5)
$$\lim_{\varepsilon \to 0} P^{\theta,\varepsilon} = \hat{P} \quad \textit{in f.d.d. in } \mu_0\textit{-measure},$$

where $\hat{P}$ *is the distribution of a d-dimensional continuous martingale* $\hat{X}$ *such that* $\langle \hat{X}^i, \hat{X}^j \rangle_t = 2\hat{a}_{ij} t$ *and* $\hat{X}_0 = 0$. *Here* $\hat{a} = \hat{a}_{ij}$ *is a constant matrix.*

Remark 1 The condition (3.4) controls the speed of the divergence of diffusion processes. We conjecture that (3.4) always holds. We give a sufficient condition for (3.3) and (3.4) in the next theorem.

If sample points θ satisfy some geometric condition, then we obtain stronger results. For a domain $\mathcal{O} \subset R^d$ we set

$$\mathscr{I}_d(\mathcal{O}) = \inf \left\{ \frac{\|(\partial q) \cap \mathcal{O}\|^d}{|q \cap \mathcal{O}|^{d-1}}; 0 < \|q \cap \mathcal{O}\| < \infty, q \text{ is open} \right\},$$

where $|\cdot|$ (respectively $\|\cdot\|$) denote the d $(d-1)$-dimensional volume. This quantity $\mathscr{I}_d(\mathcal{O})$ is called *isoperimetric constant* of $\mathcal{O}$.

Theorem 3.2 *Suppose that (3.6) holds:*

(3.6)
$$\mathscr{I}_d(\mathcal{O}(\theta)) > 0 \quad \textit{for } \mu_0\textit{-a.e. } \theta.$$

Then for each $v \in L^1(\bar{\Theta}_0, \mu_0)$ *with* $v \geqq 0$ *and* $\int v\, d\mu_0 = 1$, *it holds that*

(3.7)
$$\lim_{\varepsilon \to 0} P^{v,\varepsilon} = \hat{P} \textit{ weakly in } C([0, \infty); R^d),$$

where $P^{v,\varepsilon} = \int v P^{\theta,\varepsilon} d\mu_0$. *Moreover the matrix* $\hat{a} = \hat{a}_{ij}$ *is positive definite.*

Remark 2 By the ergodicity of μ, there exists a constant C such that $C = \mathscr{I}_d(\mathcal{O}(\theta))$ a.s. θ.

Remark 3 We have a lower bound for $\det \hat{a}$: $\det \hat{a} \geqq (2/9\pi d)^d \mathscr{I}_d(\mathcal{O}(\theta))^2$ a.s. θ.

Remark 4 A simple sufficient condition for (3.6) is known in Lemma 3 in [O2].

We apply Theorem 1.1 to Theorems 3.1 and 3.2. For this we first construct the diffusion process $\{\mathbb{P}_\theta\}$ on $\bar{\Theta}_0$ from $\{P_x^\theta\}$, and the additive functional $X = X_t$ satisfying (3.8) below. We note that $\bar{\Theta}_0$, $\{\mathbb{P}_\theta\}$ and μ_0 in this section correspond to Θ, $\{P_\theta\}$ and μ in Section 1, respectively.
Let $\{\mathbb{P}(t, \theta, \cdot)\}$ denote the family of probability measures on $\bar{\Theta}_0$ defined by

$$\mathbb{P}(t, \theta, A) = P^\theta(t, 0, A^\theta) \quad \text{where } A^\theta = \{x; \tau_z \theta \in A\}.$$

Lemma 3.3 *There exists a continuous Markov process $\{\mathbb{P}_\theta\}_{\theta \in \bar{\Theta}_0}$ on $\bar{\Theta}_0$ satisfying the following*:

(1) $\{\mathbb{P}(t, \theta, \cdot)\}$ *is the transition probability of* $\{\mathbb{P}_\theta\}$.
(2) $\{\mathbb{P}_\theta\}$ *satisfies* (1.2) *and* (1.3) *with* μ_0.
(3) $\mathbb{P}_\theta(B_\theta) = 1$ *for all* $\theta \in \bar{\Theta}_0$, *where* $B_\theta = \{\xi = \{\xi_t\} \in \Omega; \xi_t \in [\theta]$ *for all* $t\}$, $\Omega = C([0, \infty); \bar{\Theta}_0)$ *and* $[\theta] = \{\tau_x \theta; x \in \overline{\mathcal{O}(\theta)}\}$.

Proof. Lemma 3.3 follows from the proof of Theorem 2.1 in [O2]. □

Let $B = \bigcup_{\theta \in \bar{\Theta}_0} B_\theta$. Let X: $B \to C([0, \infty); R^d)$ be the function defined by $X(\xi) = X_t(\xi)$:

$$X_t(\xi) = w_t - w_0 \quad \text{for all } t,$$

where $w = w_t \in C([0, \infty); R^d)$ such that $\tau_{w_t - w_0} \xi_0 = \xi_t$ for all t. Note that $\mathbb{P}_\theta(B) = 1$ for all θ and that the domain of X can be extended to Ω in such a way that X can be regarded as a continuous additive functional. Moreover we see

$$P^\theta = \mathbb{P}_\theta X^{-1} \quad \text{for each } \theta \in \bar{\Theta}_0. \tag{3.8}$$

Let $\{U_x\}$ be the family of unitary operators on $L^2(\Theta, \mu)$ defined by $U_x f(\theta) = f(\tau_x \theta)$. Let $\tilde{D}_i$ be its infinitesimal generators and D_i the restriction of $\tilde{D}_i$ on Θ_0 (see [O2] for the precise definition.) We set $\tilde{\mathbf{F}} = \bigcap_{i=1}^d \mathcal{D}(\tilde{D}_i)$ and $\mathbf{F} = \bigcap_{i=1}^d \mathcal{D}(D_i)$. Let $(\mathcal{E}, \mathbf{F})$ be the Dirichlet form defined by

$$\mathcal{E}(f, g) = \int_{\Theta_0} \sum_{i,j=1}^d a_{ij}(\theta) D_i f D_j g \, d\mu_0 \quad \text{with the domain } \mathbf{F}.$$

Lemma 3.4 (1) $\{\mathbb{P}_\theta\}$ *is a diffusion.*
(2) $\{\mathbb{P}_\theta\}$ *is properly associated with* $(\mathcal{E}, \mathbf{F})$ *on* $L^2(\bar{\Theta}_0, \mu_0)$. *Moreover* $(\mathcal{E}, \mathbf{F})$ *on* $L^2(\bar{\Theta}_0, \mu_0)$ *is quasi-regular.*
(3) $\mathcal{E}$ *satisfies* (1.1).

Proof. For (1) it remains only to prove the strong Markov property of $\{\mathbb{P}_\theta\}$. This follows from that of $\{P_x^\theta\}$, combined with (3.8) and (3) of Lemma 3.3.

We can adapt the proof of Lemma 2.4 in [O2] to prove $\{\mathbb{P}_\theta\}$ is associated with $(\mathcal{E}, \mathbf{F})$ on $L^2(\bar{\Theta}_0, \mu_0)$. Recall that $\bar{\Theta}_0$ is a closed set in Θ and that Θ is a metrizable Lusin space. Then $\bar{\Theta}_0$ is also a metrizable Lusin space. Hence applying Theorem 5.1 and Remark 1.14 in Ch. 4 in [MR], we obtain (2). (3) is clear from (3.2). □

Let $D = (D_i)_{1 \leqq i \leqq d}$ and $aD = (\sum_{j=1}^d a_{ij} D_j)_{1 \leqq i \leqq d}$. We set $\mathscr{F}$ and $\mathscr{F}'$ from **F** in the same fashion in Sect. 1. Let $\varphi \in \mathscr{F}'$ such that

(3.9) $$\varphi(\cdot) = -\int_{\bar{\Theta}_0} aD \, d\mu_0 .$$

Proposition 3.5 $(\{\mathbb{P}_\theta\}, X, \varphi)$ *satisfies the condition* (1.5′).

Remark 5 If there exists an $h \in \mathscr{F}$ such that $X_t = \hat{h}(\xi_t) - \hat{h}(\xi_0)$, then we obtain $\varphi(\cdot) = -\mathscr{E}(h, \cdot)$. (See Theorem 2.13 (iii) on p. 22 in [MR].) However there exists no such an h. Only a non-measurable function h satisfying $X_t = h(\xi_t) - h(\xi_0)$ exists.

The proof of Proposition 3.5 is somewhat complicated. So we postpone it. We now prove Theorems 3.1 and 3.2.

Proof of Theorem 3.1 By (3.8) Theorem 3.1 is reduced to Theorem 1.1 if X in (3.8) satisfies the assumptions in Theorem 1.1. (1.1)–(1.3) follow from Lemmas 3.3 and 3.4.

By (3.8) and $w_0 = 0$ a.s. P^θ we see $\mathbb{E}_{\mu_0}[|X_t|^2] = \int_{\bar{\Theta}_0} E^\theta[|w_t|^2] \, d\mu_0$, where $\mathbb{E}_{\mu_0}$ is the expectation with respect to $\mathbb{P}_{\mu_0}$. Then by (3.3) X satisfies the first condition in (1.4). And other conditions in (1.4) also follow from (3.3) similarly. (1.5′) follows from Proposition 3.5. (1.6) is an immediate consequence of (3.2) and (3.9). □

Proof of Theorem 3.2 By estimates in [D], (3.6) implies

(3.10) $$p^\theta(t, x, y) \leqq K t^{-d/2} \exp\left(-\frac{|x-y|^2}{Kt} \right) \quad \text{for } 0 < t < \infty, \quad x, y \in \overline{\mathcal{O}(\theta)}.$$

Here K is a constant depending only on d and $\mathscr{I}_d(\mathcal{O}(\theta))$. (3.3) and the first condition in (3.4) follow from (3.10) immediately.

We next prove the second condition in (3.4). Without loss of generality we can assume $r \geqq 0$ and $\int_{\mathcal{O}(\theta)} r(x) \, dx = 1$. Let $r_t(x) = T_t^{\theta *} r(x)$. Then for all $t > 0$, $r_t > 0$ on $\mathcal{O}(\theta)$ and $\int_{\mathcal{O}(\theta)} r_t \, dx = 1$.

We introduce *entropy* $E(t)$ and *moment* $M(t)$ of r_t; $E(t) = -\int_{\mathcal{O}(\theta)} r_t \log r_t \, dx$ and $M(t) = \int_{\mathcal{O}(\theta)} r_t \sum_{i=1}^d |x^i| \, dx$. By (3.10) and $r \in C_0(R^d)$, there exists a constant C_1 satisfying the following:

(3.11) $$\sup_{x \in R^d} r_t(x) \leqq C_1 \rho(t)^d, \quad M(t) \leqq C_1 \rho(t) \qquad \text{for all } t \geqq 0.$$

where $\rho(t) = \min\{1, t^{-1/2}\}$. By (3.11) we see $\log r_t \leqq \log C_1 + d \log \rho(t)$. Then

(3.12) $$E(t) \geqq -\log C_1 - d \log \rho(t) \geqq -\log C_1 \quad \text{for all } t \geqq 0.$$

It is easy to check that $-s \log s \leqq \alpha s + e^{-\alpha - 1}$ for all $s > 0$. Let $s = r_t$ and $\alpha = \sum_{i=1}^d |x^i|$; integrate over $\mathcal{O}(\theta)$. Then we have $E(t) \leqq M(t) + C_2$ for all $t \geqq 0$, where $C_2 = \int_{R^d} \exp(-\sum_{i=1}^d |x^i| - 1) \, dx$. Hence by (3.11)

(3.13) $$\int_0^\infty e^{-pt} E(t) \, dt < \infty, \quad \lim_{t \to \infty} e^{-pt} E(t) = 0.$$

Let $C_3 = d^{1/2}K_3^{-1/2}$. We now see

$$\int_{\mathcal{O}(\theta)} \sum_{i=1}^{d} |\partial_i r_t|\, dx = \int_{\mathcal{O}(\theta)} \sum_{i=1}^{d} |\partial_i \log r_t| r_t\, dx$$
$$\leqq d^{1/2}\left\{ \int_{\mathcal{O}(\theta)} \sum_{i=1}^{d} |\partial_i \log r_t|^2 r_t\, dx \right\}^{1/2} \left(\text{by} \int_{\mathcal{O}(\theta)} r_t\, dx = 1\right)$$
$$\leqq C_3 \left\{ \int_{\mathcal{O}(\theta)} \sum_{i,j=1}^{d} a_{ij}\partial_i \log r_t \cdot \partial_j \log r_t \cdot r_t\, dx \right\}^{1/2} \quad \text{(by (3.2))}$$
$$= C_3 \left\{ \int_{\mathcal{O}(\theta)} \sum_{i,j=1}^{d} a_{ij}\partial_i (1 + \log r_t) \cdot \partial_j r_t\, dx \right\}^{1/2}$$
$$= C_3 \left\{ - \int_{\mathcal{O}(\theta)} (1 + \log r_t)\partial_t r_t\, dx \right\}^{1/2} \quad (\text{by } r_t = T_t^{\theta *} r)$$
$$= C_3\{\partial_t E(t)\}^{1/2} \quad (\text{by } \partial_t(r_t \log r_t) = (1 + \log r_t)\partial_t r_t).$$

Hence

$$\text{(3.14)} \qquad \int_0^\infty e^{-pt} \int_{\mathcal{O}(\theta)} \sum_{i=1}^{d} |\partial_i r_t|\, dx\, dt \leqq C_3 \int_0^\infty e^{-pt}\{\partial_t E(t)\}^{1/2}\, dt$$
$$\leqq C_3 p^{-1/2} \left\{ \int_0^\infty e^{-pt}\partial_t E(t)\, dt \right\}^{1/2} < \infty .$$

Here we used (3.12), (3.13) and integration by parts for the last line. Since

$$\int_{\mathcal{O}(\theta)} \sum_{i=1}^{d} |\partial_i G_p^{\theta *} r|\, dx \leqq \int_0^\infty e^{-pt} \int_{\mathcal{O}(\theta)} \sum_{i=1}^{d} |\partial_i r_t|\, dx\, dt,$$

(3.14) implies the second condition in (3.4).

We thus obtain (3.5), which implies $\lim_{\varepsilon\to 0} P^{\nu,\varepsilon} = \hat{P}$ in f.d.d.. By (3.10) and Remark 2, $\{P^{\nu,\varepsilon}\}_{\varepsilon>0}$ is tight in $C([0, \infty); R^d)$. Hence we complete the proof of Theorem 3.2. □

To prove Proposition 3.5 we prepare four lemmas.

For $f: \Theta \to R$, we set $f_\theta(x) = f(\tau_x\theta)$. Let $C^\infty(\Theta) = \{f \in \tilde{\mathbf{F}};\, f, \tilde{D}_i f \in L^\infty(\Theta), f_\theta \in C_b^\infty(R^d) \text{ a.e. } \theta\}$, where $\tilde{\mathbf{F}}$ is defined before Lemma 3.4. Then $C^\infty(\Theta)$ is dense in $\tilde{\mathbf{F}}$ (see Lemma 2.2 in [O2]).

In the rest of this section we set $v(x) \equiv (v^i(x))_{1\leqq i\leqq d} = x \in R^d$ and $g^\theta(x) = G_\alpha^{\theta *} r(x)$, where $r \in C_0(R^d)$ and $\alpha > 0$. Let $J_p^\theta = p(G_p^\theta - 1_{\bar{\Theta}_0}(\tau_x\theta)\cdot)$ and $J_p^{\theta *} = p(G_p^{\theta *} - 1_{\bar{\Theta}_0}(\tau_x\theta)\cdot)$. Set $J_p^\theta v = (J_p^\theta v^i)_{1\leqq i\leqq d}$. Then

$$\text{(3.15)} \qquad \int_{R^d} J_p^\theta v \cdot g^\theta\, dx = \int_{R^d} v \cdot J_p^{\theta *} g^\theta\, dx.$$

Here we used $J_p^\theta f(x) = J_p^{\theta *} f(x) = 0$ for $x \notin \overline{\mathcal{O}(\theta)}$, which follows from the definition of $p_t^\theta(x, y)$, and $1_{\overline{\mathcal{O}(\theta)}}(x) = 1_{\bar{\Theta}_0}(\tau_x\theta)$.

We denote by $(,)$ the inner product of $L^2(\bar{\Theta}_0, \mu_0)$. For $a = (a^i)_{1\leqq i\leqq d} \in \mathscr{L}^2(\bar{\Theta}_0, \mu_0)$ and $b \in L^2(\bar{\Theta}_0, \mu_0)$ we set $(a, b) = ((a^i, b))_{1\leqq i\leqq d}$. Recall that $\chi^p = \mathbb{E}_\theta[\int_0^\infty e^{-pt} X_t\, dt]$.

Lemma 3.6 *For $f \in C^\infty(\Theta)$ and $x \in R^d$,*

$$(p^2\chi^p, g^{\tau_{-x}\theta}(x) f(\tau_{-x}\theta)) = \left\{ \int_\Theta J_p^\theta v(x) \cdot g^\theta(x) f(\theta)\, d\mu \right\} \mu(\Theta_0)^{-1}.$$

Proof. For $h(\theta, x)$ we see

$$(\mathbb{E}_\theta[X_t], h(\tau_{-x}\theta, x)) \cdot \mu(\bar{\Theta}_0) = \int_{R^d} \int_\Theta 1_{\bar{\Theta}_0}(\theta) p_t^\theta(0, y) v(y) h(\tau_{-x}\theta, x)\, d\mu\, dy$$

$$= \int_{R^d} \int_\Theta 1_{\bar{\Theta}_0}(\tau_x\theta) p_t^{\tau_x\theta}(0, y) v(y) h(\theta, x)\, d\mu\, dy \quad (\text{by } \mu = \mu\tau_x^{-1})$$

$$= \int_\Theta \int_{R^d} 1_{\bar{\Theta}_0}(\tau_x\theta) p_t^{\tau_x\theta}(0, z - x) v(z - x) h(\theta, x)\, dz\, d\mu$$

$$= \int_\Theta \int_{R^d} 1_{\bar{\Theta}_0}(\tau_x\theta) p_t^\theta(x, z) v(z - x) h(\theta, x)\, dz\, d\mu$$

$$= \int_\Theta (T_t^\theta - 1_{\bar{\Theta}_0}(\tau_x\theta) \cdot) v(x) \cdot h(\theta, x)\, d\mu \quad (\text{by } v(z - x) = v(z) - v(x)).$$

Here we used the fact that $p_t^{\tau_a\theta}(x, y) = p_t^\theta(x + a, y + a)$ for the fourth line.

Taking $h(\theta, x) = f(\theta) g^\theta(x)$, multiplying the both sides by $p^2 e^{-pt} dt$ and integrating over $[0, \infty)$, we complete the proof. □

We set $\mathscr{E}^\theta(v, g^\theta) = (\mathscr{E}^\theta(v^i, g^\theta))_{1 \leqq i \leqq d}$.

Lemma 3.7 *For $f \in C^\infty(\Theta)$*

$$\lim_{p \to \infty} \int_\Theta \left\{ \int_{R^d} J_p^\theta v \cdot g^\theta\, dx \right\} f(\theta)\, d\mu = -\int_\Theta \mathscr{E}^\theta(v, g^\theta) f(\theta)\, d\mu.$$

Proof. Recall that $\{T_t^{\theta *}\}$ is a semigroup of class (C_0) on $L^1(\mathcal{O}(\theta), |x|\, dx)$. Let $L^{\theta *}$ be the generator of $\{T_t^{\theta *}\}$ on $L^1(\mathcal{O}(\theta), |x|\, dx)$. Since $g^\theta = G_\alpha^{\theta *} r$ is an element of the domain of $L^{\theta *}$, we see

$$\lim_{p \to \infty} \int_{R^d} v \cdot J_p^{\theta *} g^\theta\, dx = \int_{R^d} v \cdot L^{\theta *} g^\theta\, dx,$$

$$\left| \int_{R^d} v \cdot J_p^{\theta *} g^\theta\, dx \right| \leqq d^{1/2} \| J_p^{\theta *} g^\theta \|_{L^1(\mathcal{O}(\theta), |x| dx)}$$

$$\leqq C_4 \| L^{\theta *} g^\theta \|_{L^1(\mathcal{O}(\theta), |x| dx)}$$

$$= C_4 \| \alpha G_\alpha^{\theta *} r - r \|_{L^1(\mathcal{O}(\theta), |x| dx)}.$$

Here C_4 is a constant independent of p. Let $h(x) = \sup_y |r(x)||y|/(1 + |(x - y|^2)$ and $G_\alpha^\theta(x, y) = \int_0^\infty e^{-\alpha t} p_t^\theta(x, y)\, dt$. Then we have

$$\| \alpha G_\alpha^{\theta *} r \|_{L^1(\mathcal{O}(\theta), |x| dx)} \leqq \int_{R^d} \int_{R^d} h(x)(1 + |x - y|^2) \alpha G_\alpha^\theta(x, y)\, dx\, dy$$

$$= \int_{R^d} \int_{R^d} h(x)(1 + |x - y|^2) \alpha G_\alpha^{\tau_x\theta}(0, y - x)\, dx\, dy$$

$$= \int_{R^d} h(x)\left(1 + \int_0^\infty \alpha e^{-\alpha t} 1_{\Theta_0}(\tau_x\theta) E^{\tau_x\theta}[|w_t|^2]\,dt\right)dx$$

$$\in L^1(\Theta, \mu) \quad (\text{by (3.3) and } \mu = \mu\tau_x^{-1}).$$

Collecting these we apply Lebesgue's convergence theorem to obtain

$$\lim_{p\to\infty} \int_\Theta \left\{ \int_{R^d} v \cdot J_p^{\theta *} g^\theta\, dx \right\} f(\theta)\, d\mu = \int_{R^d} v \cdot L^{\theta *} g^\theta\, dx. \tag{3.16}$$

Let $v_n \in C_0^\infty(R^d; R^d)$ satisfy $v_n(x) = v(x)$ for $|x| \leqq n$, $|v_n(x)| \leqq |v(x)|$ for all x, and $\sum_{i=1}^d |\partial_i v(x)| \leqq d$ for all x. Then by (3.4) we have

$$\int_{R^d} v \cdot L^{\theta *} g^\theta\, dx = \lim_{n\to\infty} \int_{R^d} v_n \cdot L^{\theta *} g^\theta\, dx = -\lim_{n\to\infty} \mathscr{E}^\theta(v_n, g^\theta) = -\mathscr{E}^\theta(v, g^\theta).$$

Combining this with (3.15) and (3.16) completes the proof. □

Lemma 3.8 *Let $f \in C^\infty(\Theta)$, set $f^x(\theta) = f(\tau_x\theta)$. Then*

$$\int_\Theta \mathscr{E}^\theta(v, g^\theta) f(\theta)\, d\mu = \left\{ \int_\Theta aD\left(\int_{R^d} g^{\tau_{-x}\theta}(x) f^{-x}(\theta)\, dx \right) d\mu_0 \right\} \mu(\bar\Theta_0). \tag{3.17}$$

Proof. Let $\partial = (\partial_i)_{1 \leqq i \leqq d}$ and $a\partial = (\sum_{j=1}^d a_{ij}\partial_j)_{1 \leqq i \leqq d}$. Note that

$$\partial(f(\tau_{-x}\theta)) = -Df^{-x}(\theta), \qquad \partial(g^{\tau_{-x}\theta}(x)) = -Dg^{\tau_{-x}\theta}(x) + (\partial g)^{\tau_{-x}\theta}(x).$$

Then

$$\begin{aligned}
\int_\Theta \mathscr{E}^\theta(v, g^\theta) f(\theta)\, d\mu &= \int_\Theta \int_{R^d} 1_{\Theta_0}(\tau_x\theta) a(\tau_x\theta) \partial g^\theta(x) f(\theta)\, dx\, d\mu \\
&= \int_{R^d} \int_\Theta 1_{\Theta_0}(\theta) a(\theta) (\partial g)^{\tau_{-x}\theta}(x) f(\tau_{-x}\theta)\, d\mu\, dx \\
&= \int_{R^d} \int_\Theta 1_{\Theta_0}(\theta) a(\theta) \{ Dg^{\tau_{-x}\theta}(x) \cdot f^{-x}(\theta) \\
&\qquad + g^{\tau_{-x}\theta}(x) Df^{-x}(\theta) \}\, d\mu\, dx \\
&= \left\{ \int_{R^d} \int_\Theta aD(g^{\tau_{-x}\theta}(x) f^{-x}(\theta))\, d\mu_0\, dx \right\} \mu(\bar\Theta_0).
\end{aligned}$$

Since the operations $\int_{R^d} dx$ and D are commutable, we obtain Lemma 3.8. □

Let $\mathbb{G}_\alpha^*$ denote the dual resolvent of $\mathscr{E}$ on $L^2(\bar\Theta_0, \mu_0)$. Then it is easy to see that

$$\mathbb{G}_\alpha^* f(\theta) = \int_{R^d} G_\alpha^\theta(y, 0) f(\tau_y\theta)\, dy. \tag{3.18}$$

Here $G_\alpha^\theta(x, y) = \int_0^\infty e^{-\alpha t} p_t^\theta(x, y)\, dt$. Let $\tilde f(\theta) = \int_{R^d} r(z) f(\tau_{-z}\theta)\, dz$.

Lemma 3.9

$$\int_{R^d} g^{\tau_{-x}\theta}(x) f^{-x}(\theta)\, dx = \mathbb{G}_\alpha^* \tilde f(\theta).$$

Proof.

$$\int_{R^d} g^{\tau_{-x}\theta}(x) f^{-x}(\theta)\,dx = \int_{R^d}\int_{R^d} G_\alpha^{\tau_{-x}\theta}(y,x) r(y) f(\tau_{-x}\theta)\,dy\,dx$$
$$= \int_{R^d}\int_{R^d} G_\alpha^{\theta}(y-x,0) r(y) f(\tau_{-x}\theta)\,dy\,dx$$
$$= \int_{R^d}\int_{R^d} G_\alpha^{\theta}(z,0) r(z+x) f(\tau_{-x}\theta)\,dz\,dx$$
$$= \int_{R^d} G_\alpha^{\theta}(z,0) \bar{f}(\tau_z\theta)\,dz.$$

Combining this with (3.18) completes the proof of Lemma 3.9. □

Proof of Proposition 3.5 By Lemmas 3.6–3.9, we obtain

$$\lim_{p\to\infty} (p^2\chi^p, \mathbb{G}_\alpha^* \bar{f}) = -\int_{\bar{\Theta}_0} aD\mathbb{G}_\alpha^* \bar{f}\,d\mu_0. \tag{3.19}$$

Since $\{\bar{f}(\theta) = \int_{R^d} f(\tau_{-z}\theta) r(z)\,dz; f\in C^\infty(\Theta),\ r\in C_0(R^d)\}$ is dense in $L^2(\bar{\Theta}_0, \mu_0)$, we obtain (1.5′) with φ defined by (3.9). □

4 Appendix: Preparation from non-symmetric Dirichlet form theory

In this section we prepare some results from non-symmetric Dirichlet form theory in [Ki, MR, Os]. We use these results in previous sections. We assume the Dirichlet space $((\mathscr{E}, \mathbf{F}), L^2(\Theta, \mu))$ and the Hunt process $(\Omega, \mathfrak{F}, \mathfrak{F}_t, \xi_t, \{P_\theta\}_{\theta\in\Theta})$ satisfy the same conditions as in Sect. 1. In [Ki, Os] locally compactness of state spaces and the regularity of Dirichlet forms were assumed. These assumptions can be replaced by the quasi-regularity, $1\in\mathbf{F}$, and the existence of properly associated Hunt processes.

For 1-dimensional processes X and Y on $(\Omega, \mathfrak{F}, P_\mu)$, we define the energy $e(X, Y)$ by

$$e(X,Y) = \lim_{p\to\infty} \frac{1}{2} E_\mu\left[\int_0^\infty p^2 e^{-pt} X_t Y_t\,dt\right].$$

if expectations and the limit exist. For d-dimensional processes $X = (X^i)$ and $Y = (Y^i)$ we set $e(X,Y) = \sum_{i=1}^d e(X^i, Y^i)$ and $e(X) = e(X,X)$. X is called of finite energy if $e(X) < \infty$.

For $u\in\mathscr{F}$ we set $A_t^{[u]} = \hat{u}(\xi_t) - \hat{u}(\xi_0)$, where $\hat{u}$ is a quasi-continuous version of $u\in\mathscr{F}$. It is known [MR] that $A^{[u]}$ is an additive functional.

Lemma 4.1 (1) *Let* $u, v\in\mathscr{F}$. *Then*

$$e(A^{[u]}, A^{[v]}) = \mathscr{E}^s(u,v) \quad \text{where } \mathscr{E}^s(u,v) = (\mathscr{E}(u,v) + \mathscr{E}(v,u))/2.$$

(2) *Let A and B be additive functionals of finite energy. Suppose $e(B) = 0$. Then $A + B$ is an additive functional of finite energy such that $e(A+B) = e(A)$.*

Proof. See Lemma 4.5 in [Ki] for (1). We note here our Dirichlet form is conservative. (2) is clear from the definition of e. □

Let $\mathscr{M}$ denote the collection of d-dimensional additive functionals satisfying, for μ-a.e. θ, $E_\theta[M_t] = 0$ for all t and $E_\mu[|M_t|^2] < \infty$ for all t. It is easy to see that $M \in \mathscr{M}$ is a martingale under P_μ; indeed

$$\begin{aligned} E_\mu[M_{s+t} | \mathfrak{F}_s] &= E_\mu[M_s | \mathfrak{F}_s] + E_\mu[M_t(\vartheta_s \cdot) | \mathfrak{F}_s] \\ &= M_s + E_{\xi_s}[M_t] \quad P_\mu\text{-a.s.} \end{aligned}$$

Since $P_\mu \circ \xi_s^{-1} = \mu$ by (1.2), $E_{\xi_s}[M_t] = 0$ P_μ-a.s., which means M under P_μ is a martingale. We note that M under P_μ has stationary increments by (1.2). Note that $M \in \mathscr{M}$ is not necessarily a martingale additive functional in the sense of [MR]; however, this is sufficient for our purpose.

Let $\mathbb{M}$ be the collection of d-dimensional L^2-martingales on $(\Omega, \mathfrak{F}, \mathfrak{F}_t, P_\mu)$ such that $M_t \in D([0, \infty); R^d)$ and $M_0 = 0$ a.s.. Then $\mathbb{M}$ is a complete metric space with the metric induced by $\|\cdot\|$, where $\|M\| = \sum_{n=1}^{\infty} 2^{-n} \min\{1, E_\mu[|M_n|^2]^{1/2}\}$ (see for example [IW, p. 47]). We easily see $\mathscr{M} \subset \mathbb{M}$, that is, $M \in \mathscr{M}$ under P_μ are elements of $\mathbb{M}$.

Suppose that $M = (M^i)_{1 \leq i \leq d} \in \mathbb{M}$ has stationary increments. Then we see

$$E_\mu[M_t^i M_t^j] = t \cdot E_\mu[M_1^i M_1^j] = 2t \cdot e(M^i, M^j). \tag{4.1}$$

Lemma 4.2 *Let $\{M^k\} = \{(M^{k,i})_{1 \leq i \leq d}\}$ be an e-Cauchy sequence in $\mathscr{M}$. Then there exists an $M = (M^{(i)})_{1 \leq i \leq d} \in \mathbb{M}$ satisfying the following:*

(1) $\quad$ *M^k under P_μ converge to M in $\mathbb{M}$.*

(2) $$e(M^{(i)}, M^{(j)}) = \lim_{k \to \infty} e(M^{k,i}, M^{k,j}).$$

Proof. By (4.1) $E[|M_t^k - M_t^l|^2] = 2t \cdot e(M^k - M^l)$. Hence $\{M^k\}$ under P_μ is a $\|\cdot\|$-Cauchy sequence in $\mathbb{M}$, which implies (1). Since M^k under P_μ are martingales with stationary increments, so is M. Hence (2) follows from (4.1) and (1). □

The next proposition is a modification of Theorem 5.3.1 in [F2] and Theorem 5.2.5 in [Os].

Proposition 4.3 *Let $u \in \mathscr{F}$. Let N be a d-dimensional additive functional satisfying (4.2)–(4.5) below. Then $A^{[u]} - N \in \mathscr{M}$.*

$$E_\mu[|N_t|^2] < \infty \quad \textit{for all } t. \tag{4.2}$$

$$E_\theta\left[\int_0^\infty e^{-pt} |N_t| \, dt\right] < \infty \quad \textit{for all } p > 0 \quad \textit{for } \mu\textit{-a.e. } \theta. \tag{4.3}$$

$$(\mathscr{N}^p, \mathscr{N}^p) < \infty \quad \textit{for all } p, \textit{ where } \mathscr{N}^p(\theta) = E_\theta\left[\int_0^\infty e^{-pt} N_t \, dt\right]. \tag{4.4}$$

$$\lim_{p \to \infty} (p^2 \mathscr{N}^p, f) = -\mathscr{E}(u, f) \quad \textit{for all } f \in \bigcup_{\alpha > 0} \mathscr{F}_\alpha. \tag{4.5}$$

Here $\mathscr{F}_\alpha$ is defined in Remark 3 in Sect. 1.

To prove Proposition 4.3 we prepare two lemmas. Let $R_p f(\theta) = E_\theta[\int_0^\infty e^{-pt} f(\xi_t)\,dt]$. Then $R_p f$ is a μ-version of $G_p f$ for all $p > 0$, $f \in \mathscr{B}_b(\Theta) \cap L^2(\Theta, \mu)$. Here $(G_p)_{p>0}$ are resolvents of $\mathscr{E}$ on $L^2(\Theta, \mu)$. We prepare a resolvent-type equation of additive functionals.

Lemma 4.4 *For μ-a.e. θ*

(4.6) $$R_q(p\mathscr{N}^p) = R_p(q\mathscr{N}^q) \quad \textit{for all } p, q > 0.$$

Proof. We can prove Lemma 4.4 similarly as Proposition 1.6 in Ch. 10 in [RY]. □

Lemma 4.5 *For μ-a.e. θ, $p\mathscr{N}^p = pR_p\hat{u} - \hat{u}$ for all $p > 0$, where $\hat{u}$ is a quasi-continuous version of $u \in \mathscr{F}$.*

Proof. Let G_p^* denote the dual resolvent of $\mathscr{E}$ on $L^2(\Theta, \mu)$. For $f = (f^i)$, $f^i \in G_p$ (G_p is defined in Remark 3 in Section 1),

$$\begin{aligned}(p\mathscr{N}^p, f) &= \lim_{q\to\infty} (p\mathscr{N}^p, qG_q^* f)\\ &= \lim_{q\to\infty} (q^2\mathscr{N}^q, G_p^* f) \quad \text{by (4.6)}\\ &= -\mathscr{E}(u, G_p^* f) \quad \text{by (4.5)}\\ &= (pR_p\hat{u} - \hat{u}, f).\end{aligned}$$

Hence for all $p > 0$, $p\mathscr{N}^p = pR_p\hat{u} - \hat{u}$ a.e.. Since for a.e. θ both sides are continuous in p, we have Lemma 4.5. □

Proof of Proposition 4.3 By Lemma 4.5 we see for a.e. θ

$$\int_0^\infty e^{-pt} E_\theta[A_t^{[u]} - N_t]\,dt = \frac{1}{p}(pR_p\hat{u} - \hat{u} - p\mathscr{N}^p) = 0 \quad \text{for all } p > 0.$$

Hence $E_\theta[A_t^{[u]} - N_t] = 0$ for all t for a.e. θ. By $E_\mu[|A_t^{[u]}|^2] < \infty$ and (4.2), $E_\mu[|A_t^{[u]} - N_t|^2] < \infty$ for all t. We thus complete the proof. □

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