Probab. Theory Relat. Fields 96, 261–281 (1993)

Probability Theory and Related Fields

Stochastic flows on the boundaries of SL(n, R)

Ming Liao★
Mathematics ACA, Auburn University, Auburn, AL 36849, USA

Received August 10, 1992; in revised form January 25, 1993

Summary. We study the asymptotic stability of the stochastic flows on a class of compact spaces induced by a diffusion process in $SL(n, R)$ or $GL(n, R)$. These compact spaces are called boundaries of $SL(n, R)$, which include $SO(n)$, the flag manifold, the sphere S^{n-1} and the Grassmannians. The one point motions of these flows are Brownian motions. For almost every ω, we determine the set of stable points. This is a random open set whose complement has zero Lebesgue measure. The distance between any two points in the same component of this set tends to zero exponentially fast under the flow. The Lyapunov exponents at stable points are computed explicitly. We apply our results to a stochastic flow on S^{n-1} generated by a stochastic differential equation which exhibits some nice symmetry.

Mathematics Subject Classification (1991): 58G32, 34D08, 58F10

1 Introduction

In [7], the horizontal diffusion in a noncompact semisimple Lie group G is shown to induce stochastic flows on a class of compact spaces whose one point motions are Brownian motions. The compact spaces are the boundaries of G (The term is borrowed from Furstenberg, but our definition is slightly more general than his). Such a flow ϕ_t is asymptotically stable in the sense that for every fixed starting point x, almost surely, the distance between $\phi_t(x)$ and $\phi_t(y)$ for any near point y tends to zero exponentially fast as $t \to \infty$.

Note that the almost sure statement is stated for fixed starting point x. It does not tell us about the pathwise behavior of the flow ϕ_t, i.e., for each fixed ω in the underlying probability space Ω, the behavior of the "deterministic flow" $\phi_t(\cdot, \omega)$. Since the smooth transformation $\phi_t(\cdot, \omega)$ cannot shrink the whole space (which is

★Research supported in part by Hou Yin Dong Education Foundation of China
On leave from Nankai University, Tianjin, China

compact), it must have some unstable points, where the distance between near points is stretched under the flow $\phi_t(\cdot, \omega)$. Because of the stability of the stochastic flow ϕ_t, every point x is almost surely stable. Therefore, the unstable points should form a random set of zero Lebesgue measure.

Let K be a maximal compact subgroup of G. The homogeneous space G/K is a symmetric space. The projection of the horizontal diffusion in G into G/K is a Brownian motion. From the discussion in [7], the asymptotic behavior of the stochastic flow induced by the horizontal diffusion is essentially determined by the limiting properties of the associated Brownian motion in G/K.

In this article, we investigate the pathwise stability of stochastic flows on the boundaries of $SL(n, R)$ induced by some diffusion process g_t in $SL(n, R)$ or $GL(n, R)$. The flows are naturally expressed by $g_t^{-1}(\cdot)$. Recall that $GL(n, R)$ is the group of n by n real matrices with nonzero determinant and $SL(n, R)$ is the subgroup of $GL(n, R)$ consisting of matrices of determinant one. The boundaries of $SL(n, R)$ include orthogonal group $SO(n)$, the flag manifold (a flag is an orthonormal frame in R^n when the directions of its axes are ignored), the sphere S^{n-1} and Grassmannian manifolds. The associated symmetric space is the space V of ellipsoids of unit volume in R^n. Let y_t be the associated Brownian motion in V. It is known that almost surely the frame of the ellipsoid y_t tends to a limit as $t \to \infty$. These and some other preliminary discussions are contained in Sects. 2, 3 and 4.

In Sect. 5, we begin to discuss the asymptotic stability of stochastic flows. Although most of our discussions are centered around the flow on $SO(n)$ induced by the horizontal diffusion in $SL(n, R)$, the results for stochastic flows on the other boundaries of $SL(n, R)$ and those induced by other diffusion processes can be easily read off. We introduce an open subset Λ of $SO(n)$ whose complement has zero Lebesgue measure. Let $k_\infty \in SO(n)$ represent the limiting frame of y_t. Theorem 4 says that the stochastic flow $g_t^{-1}(\cdot)$ is stable on $k_\infty \Lambda$ in the sense that the distance between any two points contained in the same component of $k_\infty \Lambda$ tends to zero exponentially fast under the flow. An error in [7] is corrected. The proof of Theorem 4 is given in Sect. 6.

In Sect. 7, as an interesting application, we consider a stochastic flow on the sphere S^{n-1} which exhibits some nice symmetry. Consider n^2 vector fields $x_i(\partial/\partial x_j)$ on R^n and let X_{ij} be their orthogonal projections into the tangent space of S^{n-1} at every point of S^{n-1}. The latter are vector fields on S^{n-1}. Let ϕ_t be the stochastic flow on S^{n-1} generated by the Stratonovich stochastic differential equation

$$dx_t = \sum_{i,j=1}^{n} X_{ij}(x_t) \circ dw_t^{ij}$$

where $\{w_t^{ij}\}$ is an n^2-dimensional standard Brownian motion. We will show that the flow ϕ_t can be induced by a left invariant Brownian motion $\tilde{g}_t$ in $GL(n, R)$ defined by the following stochastic differential equation on $GL(n, R)$,

$$d\tilde{g}_t = \sum_{i,j=1}^{n} \tilde{g}_t E_{ij} \circ dw_t^{ij}$$

where E_{ij} is the matrix whose (i, j)-entry is one and other entries are zero. As a consequence of our results, for almost every $\omega \in \Omega$, there is a great circle C on S^{n-1} such that for any two points x and y in S^{n-1}, the distance between $\phi_t(x)$ and $\phi_t(y)$ tends to zero exponentially fast if x and y lie on the same side of C, and it tends to the diameter of the sphere if they lie on the different sides of C.

The above example can be compared with the well known gradient flow on S^{n-1}. This is generated by the stochastic differential equation

$$dx_t = \sum_{i=1}^{n} Y_i(x_t) \circ dw_t^i$$

where Y_i is the orthogonal projection of $\partial/\partial x_i$. In [1], it is shown that for the gradient flow, the set of stable points is S^{n-1} minus a single point.

In Sects. 8 and 9, we compute the Lyapunov exponents which are the exponential growth rates of the distance between near points under the flow. We show that at any point in $k_\infty A$, all exponents exist and are negative, and at any point in the complement of $k_\infty A$, there is at least one nonnegative exponent.

2 Some algebraic preliminaries

The Lie algebra of $\mathrm{GL}(n, R)$ is the space $gl(n, R)$ of all n by n real matrices. For $X, Y \in gl(n, R)$, their Lie bracket $[X, Y]$ is defined to be $XY - YX$. The Lie group $\mathrm{GL}(n, R)$ is not semisimple, but its identity component $\mathrm{GL}(n, R)^+$, the subgroup consisting of matrices with positive determinant, has the direct product decomposition

$$\mathrm{GL}(n, R)^+ = R_+ \times \mathrm{SL}(n, R) \tag{1}$$

where $R_+ = [0, \infty)$, in the sense that the map $g \mapsto (\det g, [\det g]^{-1/n} g)$ is a diffeomorphism from $\mathrm{GL}(n, R)^+$ onto $R_+ \times \mathrm{SL}(n, R)$.

Most of our discussion will be centered around the semisimple Lie group $G = \mathrm{SL}(n, R)$. Its Lie algebra $\mathscr{G}$ is $sl(n, R)$, the space of n by n real matrices of trace zero. We define an inner product $\langle \cdot, \cdot \rangle$ on $\mathscr{G}$ by

$$\langle X, Y \rangle = 2n \,\mathrm{Trace}(X^* X) \tag{2}$$

where X^* is the transpose of X. The factor $2n$ here is not important, it is chosen to make our inner product to be the one induced by the Killing form of G. Note that this inner product can be extended to $gl(n, R)$. Let $K = \mathrm{SO}(n)$ be the group of n by n orthogonal matrices of determinant one. This is a compact subgroup of G. Its Lie algebra $\mathscr{K} = o(n)$ is the space of skew-symmetric matrices. Let $\mathscr{P}$ be the space of symmetric matrices of trace zero. We have the direct sum decomposition

$$\mathscr{G} = \mathscr{K} \oplus \mathscr{P} \tag{3}$$

which is orthogonal with respect to $\langle \cdot, \cdot \rangle$. We now define the adjoint action of G on $\mathscr{G}$. For $g \in G$ and $X \in \mathscr{G}$, define $\mathrm{Ad}(g)X = gXg^{-1}$. We can show that $\mathscr{P}$ is $\mathrm{Ad}(K)$-invariant in the sense that $\mathrm{Ad}(k)\mathscr{P} \subset \mathscr{P}$, for any $k \in K$.

For $g \in G$, gg^* is a positive definite symmetric matrix of determinant one, which represents an ellipsoid of unit volume centered at origin in R^n. Let V be the space of all such ellipsoids. The map $g \mapsto gg^*$ is surjective from G onto V, whose kernel is K. Therefore, the homogeneous space G/K can be identified with V via the map $gK \mapsto gg^*$.

Let $\mathscr{A}$ be the space of n by n diagonal matrices of trace zero. This is the Lie algebra of the Lie group A of diagonal matrices with positive diagonal entries and determinant one. The space $\mathscr{A}$ is abelian in the sense that $[X, Y] = 0$ for $X, Y \in \mathscr{A}$. In fact, $\mathscr{A}$ is a maximal abelian subspace of $\mathscr{P}$. Each $k \in K$ acts on $\mathscr{P}$ via adjoint action. Let M be the subgroup of K which fixes $\mathscr{A}$ pointwise. The group M consists of diagonal matrices whose diagonal entries are 1 or -1 with an even number of -1's.

Note that $K = SO(n)$ can be identified with the set of orthonormal frames in R^n. Sometimes we may wish to identify two frames when the only differences between them are directions of their axes. For example, let $\{e_1, e_2, \ldots, e_n\}$ be the standard basis of R^n. We may wish to identify the frame $(e_1, e_2, \ldots, e_n)$ with the frame $(-e_1, e_2, \ldots, e_n)$. A frame with this identification is called a flag. It is easy to see that the set of flags can be identified with K/M.

Let A_+ be the subset of A consisting of diagonal matrices with distinct and descending diagonal entries and let $\mathscr{A}_+$ be the subset of $\mathscr{A}$ consisting of trace zero diagonal matrices with distinct and descending diagonal entries. It is clear that $A_+ = \exp(\mathscr{A}_+)$. Recall that V is the space of ellipsoids of unit volume. Let V' be the open subset of V consisting of ellipsoids with distinct eigenvalues (lenthes of axes). Since any positive definite symmetric matrix can be diagonalized through an orthogonal transformation $k \in K$ such that the diagonal entries are in descending order, and the choice of k is unique up to a factor in M, we obtain the following Cartan decomposition of V'.

$$V' = (K/M) \times A_+ \tag{4}$$

in the sense that the map $(kM, a) \mapsto kaK$ is a diffeomorphism from $(K/M) \times A_+$ onto $V' \subset V = G/K$. The Cartan decomposition can be considered as an analogy of the usual polar decomposition with A_+ playing the role of radial coordinate and K/M the role of angular coordinate.

Another decomposition which is equally useful is the following Iwasawa decomposition. Recall that A is the group of diagonal matrices of positive diagonal entries and determinant one. Let N be the group of upper triangular matrices whose diagonal entries are equal to one. We have

$$G = KAN \tag{5}$$

in the sense that the map $(k, a, n) \mapsto kan$ is a diffeomorphism from $K \times A \times N$ into G. To see this, for any $g \in G$, let g_i be the i-th column vector of g. By the Gram–Schmidt orthogonalization procedure, we can find orthonormal frame $\{k_1, \ldots, k_n\}$ with positive orientation such that for $1 \leqq i \leqq n$, $k_i \in \mathrm{span}\{g_1, \ldots, g_i\}$. This implies that there is an upper triangular matrix b such that $g = kb$, where k is the matrix formed by column vectors $\{k_1, \ldots, k_n\}$. The choice of b is unique if we require that it has positive diagonal entries. This proves (5).

3 Boundaries of SL(n, R)

Let H be a closed subgroup of G. The homogeneous space G/H will be called a boundary of G if $H \supset AN$. This term is borrowed from Furstenberg [3], although our definition is slightly more general than his (see [7]). Assume G/H is a boundary of G. Let $L = H \cap K$. Then $H = LAN$ and one can identify G/H with K/L.

Any $g \in G$ induces a transformation on the boundary G/H defined by $g_1H \mapsto gg_1H$. With identification of G/H with K/L, we can describe this transformation on K/L as

$$g(kL) = hL \tag{6}$$

where h is the K-component of gk in the Iwasawa decomposition (5). It is clear that if $g \in K$, then $g(kL) = gkL$. For $g \in \mathrm{GL}(n, R)^+$, let $g' = [\det g]^{-1/n} g \in G$. We define $g(kL) = g'(kL)$. We note that for $g_1, g_2 \in \mathrm{GL}(n, R)^+$, $g_1(g_2(\cdot)) = (g_1 g_2)(\cdot)$.

If one takes $H = AN$, the boundary G/H is $K = \mathrm{SO}(n)$. If one takes $H = MAN$, the boundary G/H is K/M, the flag manifold. We will identify a matrix B in $\mathrm{SO}(n-1)$ with the one in $\mathrm{SO}(n)$ given below in block notation

$$\begin{pmatrix} 1 & 0 \\ 0 & B \end{pmatrix}.$$

One can check that $H = \mathrm{SO}(n-1)AN$ is a subgroup of G. The boundary G/H is $\mathrm{SO}(n)/\mathrm{SO}(n-1)$, which can be identified with the $(n-1)$-dimensional sphere S^{n-1}.

For $1 < i < n$, let C be an i by i matrix and let D be an $(n-i)$ by $(n-i)$ matrix. Put

$$C \times D = \begin{pmatrix} C & 0 \\ 0 & D \end{pmatrix}.$$

Let $L_1 = \mathrm{SO}(i) \times \mathrm{SO}(n-i)$ and $L_2 = [O(i) \times O(n-i)] \cap \mathrm{SO}(n)]$. One checks that $H_1 = L_1 AN$ and $H_2 = L_2 AN$ are subgroups of $G = \mathrm{SL}(n, R)$. The boundary $G/H_1 = K/L_1$ is an oriented Grassmannian manifold and the boundary $G/H_2 = K/L_2$ is an unoriented Grassmannian manifold.

Now we introduce a Riemannian metric on any boundary G/H, which is invariant under the action of K. We will identify G/H with K/L. Let $\mathscr{L}$ be the Lie algebra of $L = K \cap H$ and let $\mathscr{I}$ be the orthogonal complement of $\mathscr{L}$ in $\mathscr{K}$ with respect to $\langle \cdot, \cdot \rangle$. The space $\mathscr{I}$ can be considered as the tangent space of K/L at the coset L. Any $X \in \mathscr{I}$ is considered as the tangent vector of the curve $s \mapsto e^{sX}L$ at $s = 0$. Any $k \in K$ is a transformation on K which sends $k_1 \in K$ into kk_1. Let Dk be its differential map. $Dk(X)$ is the tangent vector of the curve $s \mapsto ke^{sX}L$ at $s = 0$. The inner product $\langle \cdot, \cdot \rangle$ restricted to $\mathscr{I}$ induces a K-invariant Riemannian metric on K/L, which is denoted by $\langle \cdot, \cdot \rangle_{K/L}$, defined by

$$\forall X, Y \in \mathscr{I}, \quad \langle Dk(X), Dk(Y) \rangle_{K/L} = \langle X, Y \rangle , \tag{7}$$

One checks that this is well defined by showing that if $k \in L$, then $Dk(X) = \mathrm{Ad}(k)X$ and $\langle \mathrm{Ad}(k)X, \mathrm{Ad}(k)Y \rangle = \langle X, Y \rangle$. The metric $\langle \cdot, \cdot \rangle_{K/L}$ is K-invariant by definition.

4 Diffusion processes in SL(n, R) and GL(n, R)

The dimension of $\mathscr{P}$ is $d = (n^2 + n - 2)/2$. Let $\{Y_i;\ 1 \leq i \leq d\}$ be an orthonormal basis of $\mathscr{P}$ with respect to $\langle \cdot, \cdot \rangle$. The horizontal diffusion g_t is a diffusion process in G obtained as the solution of the following Stratonovich stochastic differential equation in matrix form

$$dg_t = \sum_{i=1}^{d} g_t Y_i \circ dw_t^i \tag{8}$$

with $g_0 = I$ (the identity matrix), where $w_t = (w_t^1, \ldots, w_t^d)$ is a Brownian motion in R^d. If Y_i is identified with the left invariant vector field on G which is equal to Y_i at

I, the generator of g_t is $(1/2)\sum_i Y_i Y_i$. It follows that the probability law of the process g_t is independent of the choice of orthonormal basis $\{Y_i; 1 \leqq i \leqq d\}$.

Under the natural map: $G \to G/K$, g_t projects to a process $y_t = g_t K$ in G/K. Since G/K can be identified with V (the space of ellipsoids, via the map $gK \mapsto gg^*$), y_t can be regarded as an ellipsoid of unit volume centered at origin in R^n, or equivalently, as a positive definite symmetric matrix. In fact, y_t is a Brownian motion in V when it is equipped with the following metric. Identify $\mathscr{P}$ with the tangent space of $V = G/K$ at K. The inner product $\langle \cdot, \cdot \rangle$ restricted to $\mathscr{P}$ induces a G-invariant Riemannian metric on V.

The limiting behavior of y_t was first studied by Dynkin [2]. The results were later extended to general symmetric spaces, see [10, 8, 12, 6]. See also [9] for an elementary treatment for $G = \mathrm{GL}(n, R)$. The basic results are summarized in the following theorem. Let H_ρ be the diagonal matrix defined by

$$H_\rho = \frac{1}{4n}\operatorname{diag}\{n-1, n-3, \ldots, -(n-3), -(n-1)\}\,. \tag{9}$$

Recall that V' is the space of ellipsoids having distinct eigenvalues.

Theorem 1 *Almost surely, for any* $t > 0$, $y_t \in V'$. *Let* $y_t = (\bar{k}_t, a_t)$ *be the Cartan decomposition given by* (4) *and let* $a_t = \exp(A_t)$ *for* $A_t \in \mathscr{A}_+$. *Then,* $\lim_{t\to\infty} A_t/t = H_\rho$ *and* $\bar{k}_\infty$ *defined by* $\lim_{t\to\infty} \bar{k}_t$ *exists.*

An important consequence of the above theorem is that almost surely the frame of the ellipsoid y_t converges to a limit. The frame of y_t is the ordered set of axes $l_1, l_2, \ldots, l_n$ of the ellipsoid y_t with the corresponding eigenvalues in descending order. Since $y_t \in V'$, there is no ambiguity in how to define the axes of y_t. Note that the frame of an ellipsoid is a flag.

Let $G/H = K/L$ be a boundary of G as defined in Sect. 3. By (6), $g \in G$ induces a transformation $g(\cdot)$ on K/L. Let g_t be the horizontal diffusion in G. By (8) and the Ito formula applied to $g_t g_t^{-1} = I$,

$$dg_t^{-1} = -\sum_{i=1}^{n} Y_i g_t^{-1} \circ dw_t^i\,. \tag{10}$$

One checks that for fixed $s > 0$, $\psi_t = g_{s+t}^{-1}$ is a solution of (10) with g_t^{-1} and w_t^i being replaced by ψ_t and $\theta_s w_t^i = w_{s+t}^i - w_s^i$ respectively and $\psi_0 = g_s^{-1}$. It follows that $g_t^{-1}(\cdot)$ is a stochastic flow on K/L in the sense that

$$g_{s+t}^{-1}(\cdot, \omega) = g_t^{-1}(\cdot, \theta_s\omega) \circ g_s^{-1}(\cdot, \omega)$$

where θ_s is the shift operator on Ω. By [7], the one point motion of $g_t^{-1}(\cdot)$ is a Brownian motion in K/L with respect to the metric $\langle \cdot, \cdot \rangle_{K/L}$ defined in the last section. This means that for any $kL \in K/L$, $g_t^{-1}(kL)$ is a Brownian motion in K/L starting from kL.

Besides the horizontal diffusion, other diffusion processes can be equally relevant and useful. For example, let $\{X_i\}$ be an orthonormal basis of $\mathscr{G}$ and let g_t' be the solution of the stochastic differential equation

$$dg_t' = \sum_i g_t' X_i \circ dw_t^i \tag{11}$$

with $g_0' = I$. The generator of g_t' is $L = (1/2)\sum_i X_i X_i$. This is the Laplacian on G. Hence, g_t' is a Brownian motion in G with respect to the left invariant metric on

G induced by $\langle\cdot,\cdot\rangle$. We may assume $X_i = Y_i$ for $i \leqq d$, and $\{X_i;\ i > d\}$ is an orthonormal basis of $\mathscr{K}$.

Consider the expression $P = \sum_{i,j} c_{ij} X_i X_j$, which can be regarded as a left invariant differential operator on G. It is said to be Ad(K)-invariant if it is not changed when X_i are replaced by $\mathrm{Ad}(k)X_i$ for any $k \in K$. If P is Ad(K)-invariant, we can define a differential operator P' on G/K by

$$P'f(gK) = \sum_{i,j} c_{ij}(\partial^2/\partial s_i \partial s_j) f\left(g \exp\left[\sum_j s_l X_l\right] K\right)\Big|_{s=0} .$$

See [4, II.4]. One checks that if g_t is a diffusion process in G with generator P, then $g_t K$ is a diffusion process in G/K with generator P'. Since the generator of g'_t can be written

$$L = (1/2) \sum_{i=1}^{d} X_i X_i + (1/2) \sum_{i>d} X_i X_i$$

and $(\sum_{i>d} X_i X_i)' = 0$ because $X_i \in \mathscr{K}$ for $i > d$, we see that $2L'$ is the Laplacian on G/K and $y'_t = g'_t K$ is a Brownian motion in G/K. Hence, y'_t has the same probability distribution as $y_t = g_t K$, where g_t is the horizontal diffusion in G.

Our discussion will show that for any diffusion process g_t in G with a left invariant generator, the asymptotic stability of the stochastic flow $g_t^{-1}(\cdot)$ on a boundary G/H is determined by the limiting property of $y_t = g_t K$. Therefore, the stochastic flow $g_t'^{-1}(\cdot)$ has the same asymptotic stability as the flow $g_t^{-1}(\cdot)$ induced by the horizontal diffusion g_t. We now identify the one point motion of $g_t'^{-1}$. For $X \in \mathscr{G}$, let X^* be the vector field on G/H defined by $X^* f(gH) = (d/ds) f(e^{sX} gH)|_{s=0}$. One can show that if $P = \sum c_{ij} X_i X_j$ is the generator of some diffusion process g_t in G, then $P^* = \sum c_{ij} X_i^* X_j^*$ is the generator of the one point motion of the flow $g_t^{-1}(\cdot)$ on G/H (see [5]). The proof of Theorem 1 in [7] shows that

$$\sum_{i=1}^{d} X_i^* X_i^* = \sum_{i>d} X_i^* X_i^* = \text{Laplacian on } G/H .$$

It follows that the generator of the one point motion of $g_t'^{-1}$ is the Laplacian on G/H. Therefore, for any $x \in G/H$, $g_{t/2}'^{-1}(x)$ is a Brownian motion in G/H. To summarize, we have

Theorem 2 *Let g'_t be Brownian motion in $G = \mathrm{SL}(n, R)$ with respect to the left invariant metric induced by $\langle\cdot,\cdot\rangle$. Then $y'_t = g'_t K$ is a Brownian motion in G/K and for any $x \in G/H$, $g_{t/2}'^{-1}(x)$ is a Brownian motion in G/H with respect to the metric $\langle\cdot,\cdot\rangle_{K/L}$ defined in Sect.* 3.

Recall that $\mathrm{GL}(n, R)^+$ is the identity component of $\mathrm{GL}(n, R)$. Since any $g \in \mathrm{GL}(n, R)^+$ also induces a transformation on the boundary G/H as defined in Sect. 3, we may consider the stochastic flow induced by a diffusion process in $\mathrm{GL}(n, R)^+$. Let E_{ij} be the matrix whose (i, j)-entry is one and other entries are zero and let $\tilde{g}_t$ be the solution of the stochastic differential equation on $\mathrm{GL}(n, R)^+$

$$d\tilde{g}_t = (2n)^{-1/2} \sum_{i,j=1}^{n} \tilde{g}_t E_{ij} \circ dw_t^{ij} \tag{12}$$

with $\tilde{g}_0 = I$, where $\{w_t^{ij}\}$ is an n^2-dimensional standard Brownian motion. The coefficient $(2n)^{-1/2}$ is used because $(2n)^{-1/2} E_{ij}$, $1 \leqq i, j \leqq n$, are orthonormal with respect to the metric $\langle X, Y\rangle = 2n\,\mathrm{Trace}(X^*Y)$. Recall that $\mathrm{GL}(n, R)^+$ can be identified with $R_+ \times G$ via the map $g \mapsto (\det g, [\det g]^{-1/n} g)$. Let $\tilde{g}_t = (z_t, g_t')$ be the decomposition with $z_t = \det \tilde{g}_t$ and $g_t' = z_t^{-1/n}\tilde{g}_t$. Note that z_t is a process in R_+ and g_t' is a process in G. We will show that $z_t = \exp(B_t/\sqrt{2})$ for some 1-dimensional Brownian motion B_t and g_t' is a Brownian motion in G. Since $\tilde{g}_t^{-1}(\cdot) = g_t'^{-1}(\cdot)$, by Theorem 2, the stochastic flow $\tilde{g}_t^{-1}(\cdot)$ on G/H has the same asymptotic stability as the flow $g_t^{-1}(\cdot)$ induced by the horizontal diffusion g_t.

The generator of $\tilde{g}_t$ is $\tilde{L} = (1/4n)\sum E_{ij}E_{ij}$, where E_{ij} is considered as a left invariant vector field on $GL(n, R)^+$. As before, let X_i, $i = 1, 2, \ldots, n^2 - 1$, form an orthonormal basis of $\mathscr{G}$, and let $X_0 = (1/\sqrt{2n})I$. Then $\{X_i; i = 0, 1, \ldots, n^2 - 1\}$ is an orthonormal basis of $gl(n, R)$ with respect to $\langle\cdot,\cdot\rangle$. It follows that $\tilde{L} = (1/2)X_0X_0 + (1/2)\sum_{i \geqq 1} X_iX_i$. Under the decomposition $\mathrm{GL}(n, R)^+ = R_+ \times G$, the action of X_0 is tangent to R_+ and the action of X_i, for $i \geqq 1$, is tangent to G. We may regard X_0 as a vector field on R_+, then the generator of z_t is $(1/2)X_0X_0$. Similarly, the generator of g_t' is $(1/2)\sum_{i \geqq 1} X_iX_i$. Since the latter is one half of the Laplacian on G, g_t' is a Brownian motion in G. To identify z_t, note that for $g \in \mathrm{GL}(n, R)^+$,

$$\det[g \exp(sX_0)] = (\det g)\exp(s/\sqrt{2})\,.$$

It follows that for $x \in R_+$,

$$X_0X_0 f(x) = (d/ds)^2 f(x \exp(s/\sqrt{2}))|_{s=0} = (1/2)x^2 f''(x) + (1/2)x f'(x)\,.$$

Hence, the generator of z_t *is* $(1/4)(x^2 f'' + x f')$. This implies our claim. To summarize, we obtain

Theorem 3 *Let $\tilde{g}_t$ be the diffusion process in* $\mathrm{GL}(n, R)^+$ *defined by* (12) *and let $\tilde{g}_t = (z_t, g_t')$ be the decomposition with $z_t = \det\, \tilde{g}_t$ and $g_t' = z_t^{-1/n}\tilde{g}_t$. Then $z_t = \exp(B_t/\sqrt{2})$ for some 1-dimensional Brownian motion B_t and g_t' is a Brownian motion in G.*

Note that $(2n)^{-1/2}\sum X_{ij}X_{ij}$ is the Laplacian on $\mathrm{GL}(n, R)^+$ with respect to the left invariant metric induced by the inner product $\langle\cdot,\cdot\rangle$ in $gl(n, R)$. Hence, $\tilde{g}_t$ is in fact a Brownian motion in $\mathrm{GL}(n, R)^+$.

5 The global stability

We have introduced three diffusion processes g_t, g_t' and $\tilde{g}_t$ in the last section. Although we will mainly consider the asymptotic stability of the stochastic flow $g_t^{-1}(\cdot)$ on a boundary G/H of G induced by the horizontal diffusion g_t, our discussion will show that for any left invariant diffusion process g_t in G, the asymptotic stability of the stochastic flow $g_t^{-1}(\cdot)$ on G/H is determined by the limiting properties of the process g_tK in G/K. By Theorem 2 and Theorem 3, we see that the stochastic flows $g_t'^{-1}(\cdot)$ and $\tilde{g}_t^{-1}(\cdot)$ on G/H have the same asymptotic stability as $g_t^{-1}(\cdot)$.

In this section, we will study the global stability of the flow $g_t^{-1}(\cdot)$ on G/H. We will first consider the flow on the boundary $K = \mathrm{SO}(n) \cong G/H$, where $H = AN$. Because any other boundary G/H can be identified with K/L, where $L = K \cap H$, and $g(kL) = g(k)L$ for $g \in G$ and $k \in K$ by (6), the results for a general boundary of G can be essentially read off from those for $K = SO(n)$.

Let g be a matrix and $\alpha, \beta \subset \{1, 2, \ldots, n\}$. We define

$$g[\alpha | \beta]$$

to be the determinant of the submatrix of g formed by the rows indexed in α and the columns indexed in β. Let $|\alpha|$ be the cardinality of α. By convention, $g[\alpha|\beta] = 0$ if $|\alpha| \neq |\beta|$ or $|\alpha| = 0$. For $1 \leqq i \leqq n$, let

$$\alpha_i = \{1, 2, \ldots, i\} \quad \text{and} \quad \beta_i = \{n - i + 1, n - i + 2, \ldots, n\}\ .$$

Let

$$\Lambda = \{k \in K;\ k[\beta_i|\alpha_i] \neq 0 \text{ for } i = 1, 2, \ldots, n - 1\}\ . \tag{13}$$

Note that $k[\beta_n|\alpha_n] = \det(k) = 1$. The set Λ is an open subset of K whose complement has zero Lebesgue measure. Note that there are 2^{n-1} distinct sign patterns of the ordered set of real numbers

$$k[\beta_1|\alpha_1],\ k[\beta_2|\alpha_2], \ldots, k[\beta_{n-1}|\alpha_{n-1}]\ .$$

Each sign pattern corresponds to a component of Λ.

Recall that M is the subset of K consisting of diagonal matrices whose diagonal entries are either 1 or -1 with an even number of -1's. We see that M has 2^{n-1} elements. Let $e_1, e_2, \ldots, e_n$ be the standard basis of R^n, and let k^0 be the matrix in K defined by

$$k^0 = (\varepsilon e_n, e_{n-1}, e_{n-2}, \ldots, e_1) \tag{14}$$

where the sign $\varepsilon = \pm$ is chosen so that $\det(k^0) = 1$. It is clear that $mk^0 \in \Lambda$ for any $m \in M$. In fact, each component of Λ contains exactly one mk^0.

We will fix an $\omega \in \Omega$ throughout the rest of the paper, for which the limiting properties in Theorem 1 hold. Choose $k_\infty \in K$ such that $\bar{k}_\infty = k_\infty M$. Since M is discrete, at least for large t, there exists a continuous process $k_t \in K$ such that $\bar{k}_t = k_t M$ and $\lim_{t \to \infty} k_t = k_\infty$. By (4), we obtain the following decomposition of g_t

$$g_t = k_t a_t \tilde{k}_t \tag{15}$$

for some $\tilde{k}_t \in K$.

The following theorem says that the flow $g_t^{-1}(\cdot)$ is stable on the set $k_\infty \Lambda$. Later we will show that $k_\infty \Lambda$ can be characterized as the set of the points where all the Lyapunov exponents exist and are negative. We note that $k_\infty \Lambda$ is an open subset of K whose complement has zero Lebesgue measure. A simple geometric interpretation of $k_\infty \Lambda$ is that $k \in K$ belongs to $k_\infty \Lambda$ if and only if for any i with $1 \leqq i \leqq n - 1$, the projections of the first i column vectors of k into the space spanned by the last i column vectors of k_∞ are linearly independent.

Theorem 4 (i) *If k, k' are contained in the same component of $k_\infty \Lambda$, then the distance between $g_t^{-1}(k)$ and $g_t^{-1}(k')$ tends to zero exponentially fast in the sense that*

$$\lim_{t \to \infty} \frac{1}{t} \log d(g_t^{-1}(k), g_t^{-1}(k')) \leqq -\frac{1}{2n} \tag{16}$$

where d is the distance on K determined by the Riemannian metric.

(ii) *If* k *and* k' *belong to two different components of* $k_\infty \Lambda$ *containing* $k_\infty m k^0$ *and* $k_\infty m' k^0$ *respectively, then*

$$d(g_t^{-1}(k), g_t^{-1}(k')) \to d(mk^0, m'k^0) \quad \text{as } t \to \infty .$$

Remark 1 The statement of Theorem 4 is independent of the choice of k_∞ which represents $\bar{k}_\infty = k_\infty M$. A different choice of k_∞ corresponds to a permutation of components of Λ.

Remark 2 There is an error in the statement of Theorem 2 in [7]. It claims that for any two fixed points x and y in G/H, almost surely, the distance between $g_t^{-1}(x)$ and $g_t^{-1}(y)$ tends to zero exponentially fast. This is not true if the set of stable points is disconnected. In fact, its proof only shows that for any fixed point x, almost surely, there exists a neighborhood U of x such that the distance between x and any $y \in U$ tends to zero exponentially fast under the flow.

Corollary 1 *Let* $G/H = K/L$ *be a boundary of* G *with* $L = H \cap K$ *and let* $\Lambda' = \{kL; k \in \Lambda\}$, *the image of* Λ *under the natural map:* $K \to K/L$.
(a) *If* kL *and* $k'L$ *belong to the same component of* $k_\infty(\Lambda')$, *then the distance between* $g_t^{-1}(kL)$ *and* $g_t^{-1}(k'L)$ *tends to zero exponentially fast in the sense of* (16) *where* k *and* k' *should be replaced by* kL *and* $k'L$ *respectively.*
(b) Λ' *is an open subset of* K/L *whose complement has zero Lebesgue measure.*

Theorem 4 will be proved in the next section. (a) of Corollary 1 follows directly from (i) of Theorem 4 noting that $g_t^{-1}(kL) = g_t^{-1}(k)L$. To show (b), we note that the natural map: $K \to K/L$ is an open map, which implies that Λ' is open. We note that kL belongs to the complement of Λ' if and only if kL as a subset of K is contained in the complement of Λ. Since the complement of Λ has a positive co-dimension in K, this implies that the complement of Λ' has a positive co-dimension, hence, zero Lebesgue measure.

Example 1 (Flag manifold) If we take $H = MAN$, the boundary G/H is the flag manifold K/M. The set $k_\infty(\Lambda')$ is a connected open subset of K/M which can be characterized as the set of flags u such that the projections of the first i axes of u into the space spanned by last i axes of $\bar{k}_\infty$ are linear independent for $i = 1, 2, \ldots, n-1$. If u and u' are two flags belonging to $k_\infty(\Lambda')$, then the distance between $g_t^{-1}(u)$ and $g_t^{-1}(u')$ tends to zero exponentially fast in the sense of (16) where k and k' should be replaced by u and u' respectively.

Example 2 (Sphere) If we take $H = SO(n-1)AN$, where $SO(n-1)$ is considered a subgroup of $K = SO(n)$ as in Sect. 3, the boundary G/H is the sphere $S^{n-1} = SO(n)/SO(n-1)$. Let $e_1, e_2, \ldots, e_n$ be the standard basis of R^n. The sphere S^{n-1} can be identified with the orbit of e_1 under the action of $K = SO(n)$ on R^n. The set Λ' is the complement of the great circle on S^{n-1} which is contained in the hyperplane orthogonal to e_n. Let $\bar{k}_\infty = (l_1, l_2, \ldots, l_n)$ be the limiting flag, where $l_1, l_2, \ldots, l_n$ are the limiting axes of the ellipsoid y_t as $t \to \infty$, arranged according to the descending order of the eigenvalues. Then $k_\infty(\Lambda')$ is the complement of the great circle which is contained in the hyperplane orthogonal to l_n.

6 Proof of Theorem 4

For $k \in K$, let $|k|^2 = \sum_{i,j=1}^n |k_{ij}|^2$, where k_{ij} is the (i,j)-entry of k. We will also write $|k_j|^2$ for $\sum_{i=1}^n k_{ij}^2$. Then $|k - k'|$ can be used as a distance between k and k'. Since

K is compact, this distance is equivalent to d. With this observation and the fact that d is K-invariant, it is not hard to see that Theorem 4 follows from the following lemma.

Lemma 1 *If k belongs to the component of $k_\infty A$ containing $k_\infty mk^0$, then*

$$\lim_{t\to\infty} \frac{1}{t} \log |a_t^{-1} k_t^{-1}(k) - mk^0| \leqq -\frac{1}{2n}.$$

The rest of this section is devoted to the proof of Lemma 1.

For $g \in G$, let $g = han$ be the Iwasawa decomposition with $h \in K$, $a \in A$ and $n \in N$. Since the column vectors h_i of h are obtained from those of g through a Gram–Schmidt orthogonalization, we have

$$h_i = \sum_{p=1}^{i} c_{ip} g_p . \tag{17}$$

To determine the coefficients c_{ip}, note that $g_j \cdot h_i = 0$ for $j < i$ and $c_{ii}(g_i \cdot h_i) = 1$, where $g_j \cdot h_i$ is the usual dot product. Therefore

$$\sum_{p=1}^{i} (g^*g)_{jp} c_{ip} = 0 \quad \text{for } j < i \quad \text{and} \quad \sum_{p=1}^{i} (g^*g)_{ip} c_{ip} c_{ii} = 1 .$$

Multiplying the first $i-1$ equations by c_{ii} and then solving the system using Cramer's method, we obtain

$$c_{ip} = (-1)^{i-p} \frac{(g^*g)[\alpha_{i-1} | 1 \cdots \hat{p} \cdots i]}{\sqrt{(g^*g)[\alpha_{i-1} | \alpha_{i-1}](g^*g)[\alpha_i | \alpha_i]}} \tag{18}$$

where $\hat{p}$ means that the index p is suppressed and $(g^*g)\,[\alpha_0 | \alpha_0] = 1$.

We now introduce some notation. Let $\phi(t)$ and $\psi(t)$ be two nonnegative functions. We will denote $\phi \asymp \psi$ if they have the same exponential growth rate as $t \to \infty$, i.e., if

$$\lim_{t\to\infty} \frac{1}{t} \log \phi(t) = \lim_{t\to\infty} \frac{1}{t} \log \psi(t) .$$

We will denote $\phi \lesssim \psi$ if the exponential growth rate of ϕ is controlled by that of ψ, i.e., if

$$\limsup_{t\to\infty} \frac{1}{t} \log \phi \leqq \liminf_{t\to\infty} \frac{1}{t} \log \psi .$$

Let $g_t = k_t a_t \tilde{k}_t$ be the decomposition of the horizontal diffusion g_t given in (15) and let $a_i(t)$ be the i-th diagonal entry of a_t. By Theorem 1, $a_i(t) \asymp \exp(\lambda_i t)$, where $\lambda_i = (n - 2i + 1)/4n$. For $\alpha \subset \{1, 2, \ldots, n\}$, let $a_t[\alpha] = \prod_{i\in\alpha} a_i(t)$.

Now for $k \in K$, let $g = a_t^{-1} k_t^{-1} k$ and let $b = k_t^{-1} k$. Then $h = a_t^{-1} k_t^{-1}(k)$. We will show

$$|h_i - \varepsilon_i e_{n-i+1}| \lesssim e^{-t/2n} \tag{19}$$

where $\{e_1, e_2, \ldots, e_n\}$ is the standard basis of R^n and $\varepsilon_i = 1$ or -1. Since h_i's are column vectors of $a_t^{-1} k_t^{-1}(k)$, and $\varepsilon_i e_{n-i+1}$'s are column vectors of mk^0, this proves Lemma 1. Therefore, it suffices to show (19).

Note that if $k \in k_\infty A$, then $b \in A$ for large t. For $\alpha, \alpha' \subset \{1, 2, \ldots, n\}$ with $|\alpha| = |\alpha'| = i$, we have by the Binet–Cauchy formula,

$$(g^*g)[\alpha|\alpha'] = \sum_\beta g^*[\alpha|\beta]g[\beta|\alpha'] = \sum_\beta g[\beta|\alpha]g[\beta|\alpha']$$

$$= \sum_\beta a_t[\beta]^{-2} b[\beta|\alpha]b[\beta|\alpha'] .$$

One checks that for any $\beta \subset \{1, 2, \ldots, n\}$ with $|\beta| = i$ and $\beta \neq \beta_i$, $a_t[\beta]^{-1} \leqq a_t[\beta_i]^{-1} e^{-t/2n}$. It follows that for $|\alpha| = |\alpha'| = i$,

$$(g^*g)[\alpha|\alpha'] = a_t[\beta_i]^{-2} b[\beta_i|\alpha]b[\beta_i|\alpha'] + \xi_i = g[\beta_i|\alpha]g[\beta_i|\alpha'] + \xi_i \quad (20)$$

where $\xi_i \leqq a_t[\beta_i]^{-2} e^{-t/n}$. In particular, we have

$$(g^*g)[\alpha_i|\alpha_i] = a_t[\beta_i]^{-2} b[\beta_i|\alpha_i]^2 + \xi_i = g[\beta_i|\alpha_i]^2 + \xi_i . \quad (21)$$

For $1 \leqq r, i \leqq n$, we have

$$e_r \cdot h_i = \sum_{p=1}^{i} c_{ip} g_{rp}$$

$$= \frac{1}{\sqrt{(g^*g)[\alpha_{i-1}|\alpha_{i-1}](g^*g)[\alpha_i|\alpha_i]}} \sum_{p=1}^{i} (-1)^{i-p}(g^*g)[\alpha_{i-1}|1 \cdots \hat{p} \cdots i]g_{rp} \quad (22)$$

$$\sum_{p=1}^{i} (-1)^{i-p}(g^*g)[\alpha_{i-1}|1 \cdots \hat{p} \cdots i]g_{rp} = \sum_{p=1}^{i} (-1)^{i-p} \sum_\alpha g[\alpha|\alpha_{i-1}]g[\alpha|1 \cdots \hat{p} \cdots i]g_{rp}$$

$$= \sum_\alpha g[\alpha|\alpha_{i-1}]\varepsilon(\alpha, r)g[\alpha \cup \{r\}|\alpha_i]$$

where $\varepsilon(\alpha, r) = 0$ if $r \in \alpha$ and $\varepsilon(\alpha, r) = \pm 1$ otherwise. Since $\beta_{i-1} \cup \{n - i + 1\} = \beta_i$, $\varepsilon(\beta_{i-1}, n - i + 1) = (-1)^{i-1}$.

When $r = n - i + 1$, the above is equal to

$$(-1)^{i-1} g[\beta_{i-1}|\alpha_{i-1}]g[\beta_i|\alpha_i] + \zeta_i$$

where

$$|\zeta_i| = \left| \sum_{\alpha \neq \beta_{i-1}} \varepsilon(\alpha, r)g[\alpha|\alpha_{i-1}]g[\alpha \cup \{r\}|\alpha_i] \right| \leqq a_t[\beta_{i-1}]^{-1} a_t[\beta_i]^{-1} e^{-t/n} .$$

By (21) and (22),

$$e_{n-i+1} \cdot h_i = (-1)^{i-1} sgn(g[\beta_{i-1}|\alpha_{i-1}]g[\beta_i|\alpha_i]) + \zeta_i' \quad (23)$$

where $|\zeta_i'| \leqq e^{-t/n}$. Let $\varepsilon_i = \pm 1$ be the first term on the right, we have

$$|h_i - \varepsilon_i e_{n-i+1}|^2 = 2 - 2\varepsilon_i e_{n-i+1} \cdot h_i \leqq e^{-t/n} .$$

This proves Lemma 1.

7 Stochastic flow on the sphere

As an application of our results, we will consider the stochastic flow generated by a stochastic differential equation on the sphere S^{n-1} which exhibits some nice symmetry.

Let S^{n-1} be embedded in R^n as the unit sphere $\sum_{i=1}^n x_i^2 = 1$. At any point $x \in S^{n-1}$, let $X_{ij}(x)$ be the vector tangent to S^{n-1} at x obtained as the orthogonal projection of the vector field $x_i(\partial/\partial x_j)$ on R^n. A simple computation yields

$$X_{ij} = x_i(\partial/\partial x_j) - x_i x_j D_r \tag{24}$$

where $D_r = \sum_{i=1}^n x_i(\partial/\partial x_i)$. Consider the following stochastic differential equation on S^{n-1}

$$dx_t = \sum_{i,j=1}^{n} X_{ij}(x_t) \circ dw_t^{ij} \tag{25}$$

where $\{w_t^{ij}\}$ is an n^2-dimensional standard Brownian motion.

Recall that $\tilde{g}_t$ is the left invariant Brownian motion in $\mathrm{GL}(n, R)^+$ defined in Sect. 4. We have seen that it induces a stochastic flow $\tilde{g}_t^{-1}(\cdot)$ on any boundary of $G = \mathrm{SL}(n, R)$, in particular, on S^{n-1}. We have also noted that it has the same asymptotic stability as the flow $g_t^{-1}(\cdot)$ induced by the horizontal diffusion g_t in $G = \mathrm{SL}(n, R)$.

In Sect. 3, we introduced a K-invariant metric $\langle \cdot, \cdot \rangle_{K/L}$ on S^{n-1} which corresponds to the Killing form $2n$ Trace$(X^* Y)$. But the standard metric on S^{n-1}, the one induced by the Euclidean metric on R^n, corresponds to (1/2)Trace$(X^* Y)$. Therefore, the standard metric on S^{n-1} is equal to $(1/4n)\langle \cdot, \cdot \rangle_{K/L}$.

Theorem 5 *Let ϕ_t be the stochastic flow on S^{n-1} generated by* (25). *Then ϕ_t is identical in law with the stochastic flow $\tilde{g}_{2nt}^{-1}(\cdot)$ on S^{n-1}. Consequently, the one point motion of ϕ_t is a Brownian motion on S^{n-1} with respect to the standard metric and for almost all ω, there is a great circle C on S^{n-1} such that*
(a) *if x and y lie on the same side of C, then the distance between $\phi_t(x)$ and $\phi_t(y)$ tends to zero exponentially fast in the sense that*

$$\lim_{t\to\infty} \frac{1}{t} \log d(\phi_t(x), \phi_t(y)) \leqq -1$$

where $d(x, y)$ is the distance between x and y;
(b) *if x and y lie on different sides of C, then*

$$d(\phi_t(x), \phi_t(y)) \to 2, \textit{ the diameter of } S^{n-1}.$$

First assume that ϕ_t is identical in law with the stochastic flow $\tilde{g}_{2nt}^{-1}(\cdot)$ on S^{n-1}. By Theorem 2 and Theorem 3, the one point motion of $\tilde{g}_{t/2}^{-1}(\cdot)$ is a Brownian motion on S^{n-1} with respect to the metric $\langle \cdot, \cdot \rangle_{K/L}$. It follows that the one point motion of $\tilde{g}_{2nt}^{-1}$, hence, of ϕ_t, is a Brownian motion on S^{n-1} with respect to the standard metric. Let $\bar{k}_\infty = (l_1, l_2, \ldots, l_n)$ be the limiting flag of Example 2 and let C be the great circle on S^{n-1} which is orthogonal to l_n. We see that (a) above follows directly from (a) of Corollary 1 with a rescaling factor $2n$.

The axe l_n cuts the sphere S^{n-1} at two antipotal points x_0 and x_0'. If we identify S^{n-1} with $K(e_1)$ as in Example 2, we may write $x_0 = k_\infty k^0(e_1)$ and $x_0' = k_\infty m k^0(e_1)$ for some $m \in M$. See Sect. 5 for the definition of k^0. Let $x \in S^{n-1}$ lie on the same side of C as x_0 and let $x' \in S^{n-1}$ lie on the same side of C as x'_0. We can choose $k, k' \in K$ such that $x = k(e_1)$ and $x' = k'(e_1)$. Moreover, we may assume that k and $k_\infty k^0$ lie in the same component of $k_\infty \Lambda$, and k' and $k_\infty m k^0$ lie in the same component of $k_\infty \Lambda$. Let $\tilde{g}_t = (z_t, g'_t)$ be the decomposition with $z_t = \det \tilde{g}_t$ and $g'_t = z_t^{-1/n} \tilde{g}_t$, and let $g'_t = k_t a_t \tilde{k}_t$ be the decomposition (15). By Lemma 1, the distance between $a_t^{-1} k_t^{-1}(k)$ and k^0 tends to zero. Since $\tilde{k}_t^{-1}$ is an isometric transformation,

$$d(\tilde{g}_t^{-1}(x), \tilde{k}_t^{-1} k_\infty^{-1}(x_0)) = d(\tilde{k}_t^{-1} a_t^{-1} k_t^{-1}(k(e_1)), \tilde{k}_t^{-1}(k^0(e_1)))$$
$$= d(a_t^{-1} k_t^{-1}(k)(e_1), k^0(e_1)) \to 0 .$$

Similarly, we have

$$d(\tilde{g}_t^{-1}(x'), \tilde{k}_t^{-1} k_\infty^{-1}(x'_0)) = d(a_t^{-1} k_t^{-1}(k')(e_1), mk^0(e_1)) \to 0 .$$

Since $d(\tilde{k}_t^{-1} k_\infty^{-1}(x_0), \tilde{k}_t^{-1} k_\infty^{-1}(x'_0)) = d(x_0, x'_0)$, we see that $d(\tilde{g}_t^{-1}(x), \tilde{g}_t^{-1}(x'))$ tends to $d(x_0, x'_0) = 2$. This proves (b).

It remains to prove that ϕ_t is identical in law with the stochastic flow $\tilde{g}_{2nt}^{-1}(\cdot)$ on S^{n-1}. Applying Ito's formula to $\tilde{g}_t \tilde{g}_t^{-1} = I$ and using (12), we obtain

$$d\tilde{g}_t^{-1} = -(2n)^{-1/2} \sum_{i,j=1}^{n} E_{ij} \tilde{g}_t^{-1} \circ dw_t^{ij} . \tag{26}$$

The stochastic differential equation satisfied by $\tilde{g}_{2nt}^{-1}$ is

$$d\tilde{g}_{2nt}^{-1} = -\sum_{i,j=1}^{n} E_{ij} \tilde{g}_{2nt}^{-1} \circ dw_t^{ij} . \tag{27}$$

We may assume that $\{w_t^{ij}\}$ above is the n^2-dimensional Brownian motion appearing in (25).

For $x \in S^{n-1}$, let $\tilde{X}_{ij}$ be the tangent vector of the curve $s \mapsto \exp(sE_{ji})(x)$ in S^{n-1} at $s = 0$. Let $x_t = \tilde{g}_{2nt}^{-1}(x)$. Since $\exp(sE_{ji})(x_t) = (\exp(sE_{ji})\tilde{g}_{2nt}^{-1})(x)$, it follows from Ito's formula and (27) that

$$dx_t = -\sum_{i,j=1}^{n} \tilde{X}_{ji}(x_t) \circ dw_t^{ij} . \tag{28}$$

Note that $\{-w_t^{ji}\}$ is also an n^2-dimensional standard Brownian motion. Comparing (28) with (25), we see that in order to prove Theorem 5, it suffices to show $\tilde{X}_{ij} = X_{ij}$.

Recall that $\{e_1, e_2, \ldots, e_n\}$ is the standard basis of R^n. Any $x \in S^{n-1}$ can be written as $x = h(e_1)$ for some $h \in K$. Let

$$b(s) = \exp(-s\delta_{ij}/n) \exp(sE_{ji})h . \tag{29}$$

Since $\det[\exp(sE_{ji})h] = \exp(s\delta_{ij})$, $b(s) \in G = \mathrm{SL}(n, R)$. We have

$$b(s) = [1 - s\delta_{ij}/n]h + sE_{ji}h + O(s^2) . \tag{30}$$

Let $h(s)$ be the K-component of $b(s)$ in the Iwasawa decomposition (5). We have $b(0) = h(0) = h$. For $g \in G$ and $x \in S^{n-1}$, $g(x) = k(x)$, where k is the K-component of g in the Iwasawa decomposition. It follows that $\exp(sE_{ji})\, h(e_1) = h(s)(e_1)$. There exists $Y_{ij} \in \mathscr{K}$ such that

$$h(s) = h \exp(sY_{ij}) + O(s^2) . \tag{31}$$

We see that $\tilde{X}_{ij}$ is the tangent vector of the curve $s \mapsto h \exp(sY_{ij})(e_1)$ at $s = 0$.

Let b_p and b_{pq} be respectively the p-th column vector and the (p, q)-entry of b. By (30),

$$b_p(s) = (1 - s\delta_{ij}/n)h_p + sh_{ip}e_j + O(s^2)$$

$$\dot{b}_p = (d/ds)b_p(0) = -(1/n)\delta_{ij}h_p + h_{ip}e_j \,.$$

Since the column vectors of $h(s)$ are obtained from those of $b(s)$ through a Gram–Schmidt orthogonalization, we see that

$$h_p(s) = \sum_{r=1}^{p} H_{pr}(s)b_r(s)$$

for some $H_{pr}(s)$ satisfying $H_{pr}(0) = \delta_{pr}$. Let $\dot{h}_p = (d/ds)h_p(0)$. It follows that

$$\dot{h}_p = \sum_{r=1}^{p} \dot{H}_{pr}(0)h_r + \dot{b}_p = \sum_{r=1}^{p} \dot{H}_{pr}(0)h_r + h_{ip}e_j - (1/n)\delta_{ij}h_p \,. \tag{32}$$

By (31), $h + s\dot{h} + O(s^2) = h + shY_{ij} + O(s^2)$. It follows that $\dot{h} = hY_{ij}$ and $Y_{ij} = h^*\dot{h}$. Since $Y_{ij} \in \mathscr{K}$, for $p < q$, let $(Y_{ij})_{pq}$ be the (p, q)-entry of the matrix Y_{ij}, we have

$$(Y_{ij})_{pq} = h_p \cdot \dot{h}_q = -\dot{h}_p \cdot h_q = -h_{ip}h_{jq} \,.$$

The above last equality follows from (32). Note that $(Y_{ij})_{pp} = 0$ and for $p > q$,

$$(Y_{ij})_{pq} = -(Y_{ij})_{qp} = h_{iq}h_{jp} \,. \tag{33}$$

Recall that $\tilde{X}_{ij}$ is the tangent vector of the curve $s \mapsto h\exp(sY_{ij})(e_1)$ at $s = 0$. This is a tangent vector of S^{n-1} at $x = h(e_1)$. Since

$$h\exp(sY_{ij})(e_1) = h(e_1) + shY_{ij}(e_1) + O(s^2)$$

the tangent vector viewed as a vector in R^n is $hY_{ij}(e_1)$. Its r-th component is

$$\sum_{p=1}^{n} h_{rp}(Y_{ij})_{p1} = \sum_{p=2}^{n} h_{rp}h_{i1}h_{jp} = \sum_{p=1}^{n} h_{rp}h_{i1}h_{jp} - h_{r1}h_{i1}h_{j1}$$

$$= \delta_{rj}h_{i1} - h_{r1}h_{i1}h_{j1} \,.$$

Since h_{r1} is the r-th Euclidean coordinate x_r of $x = h(e_1)$, we see that the r-th component of the vector $\tilde{X}_{ij}$ is $\delta_{rj}x_i - x_ix_jx_r$. By (24), $\tilde{X}_{ij} = X_{ij}$. This proves Theorem 5.

8 Lyapunov exponents

The local stability of a stochastic flow ϕ_t on a Riemannian manifold S can be described by its Lyapunov exponents. For a tangent vector X at $x \in S$, the limit

$$\lambda = \lim_{t\to\infty} \frac{1}{t} \log \| D\phi_t(X) \|_S$$

when it exists, is called a Lyapunov exponent of the flow ϕ_t at x, where $\| X \|_S$ is the length of the vector X determined by the Riemannian metric (The Lyapunov exponent is in fact independent of the metric). Let $\exp_x$ be the exponential map at x.

Roughly speaking, the distance between $\phi_t(x)$ and $\phi_t(\exp_x(sX))$, for small s, grows like $e^{\lambda t}$.

If all the Lyapunov exponents exist at some point x and

$$\lambda_1 < \lambda_2 < \cdots < \lambda_k$$

are the distinct exponents, then the tangent space T_xS at x has a filtration of subspaces

$$\{0\} = V_0 \subset V_1 \subset V_2 \subset \cdots \subset V_k = T_xS$$

such that if $X \in V_i - V_{i-1}$, $1 \leqq i \leqq k$, then

$$\lim_{t\to\infty} \frac{1}{t} \log \| D\phi_t(X) \|_S = \lambda_i .$$

A point $x \in S$ will be called stable for the flow ϕ_t if all the Lyapunov exponents at x exist and are negative.

Now let $S = G/H$, a boundary of $G = \mathrm{SL}(n, R)$. We identify G/H with K/L, where $L = K \cap H$. Recall that Λ is an open subset of K defined by (13) and Λ' is its image under the natural map: $K \to K/L$. We have seen that the distance between any two points in the same component of $k_\infty(\Lambda')$ tends to zero exponentially fast under the stochastic flow $g_t^{-1}(\cdot)$ on K/L. We will show that any $kL \in k_\infty(\Lambda')$ is a stable point.

Let $g_t = n_t' a_t' k_t'$ be the Iwasawa decomposition with $n_t' \in N$, $a_t' \in A$ and $k_t' \in K$, and let A_t' be the process in $\mathscr{A}$ such that $a_t' = \exp(A_t')$. By [8], almost surely,

(A) $\lim_{t\to\infty} A_t'/t = -H_\rho$
(B) $n_\infty' = \lim_{t\to\infty} n_t'$ exists.

Recall that $y_t = g_t K$ is a Brownian motion in $V = G/K$. Let ρ be the Riemannian distance on V. By [11], almost surely, there exists a positive integer L and a real number η with $0 < \eta < 1$ such that

(C) $\forall$ integer $k \geqq L$, $\sup\{\rho(y_t, y_k);\ k \leqq t \leqq k+1\} \leqq k^\eta$.

Let E_{ij} be the matrix whose (i, j)-entry is one and other entries are zero. Define $h_{ij}^{pq}(t)$ by

$$\mathrm{Ad}(n_t'^{-1} n_\infty') E_{ij} = \sum_{p,q} h_{ij}^{pq}(t) E_{pq} . \tag{34}$$

By Lemma 2 in [7], if $\omega \in \Omega$ is a path which satisfies (A), (B) and (C), then

$$| h_{ij}^{pq}(t) | \lesssim \exp[- |(\lambda_i - \lambda_j) - (\lambda_p - \lambda_q)| t] \tag{35}$$

where $\lambda_i = (n - 2i + 1)/4n$ is the i-th diagonal entry of H_ρ. Note that the proof of (35) depends only on the assumptions (A), (B) and (C).

Now let $k \in K$ and let

$$k^{-1} g_t = n_t' a_t' k_t' \tag{36}$$

be the Iwasawa decomposition with $n_t' \in N$, $a_t' \in A$ and $k_t' \in K$. Note that n_t', a_t' and k_t' have been redefined. As before, let A_t' be the process in $\mathscr{A}$ such that $a_t' = \exp(A_t')$.

We will show that if $k \in k_\infty \Lambda$, then (A) and (B) hold. Since k^{-1} is an isometry on V, we see that (C) holds with $y_t = g_t K$ being replaced by $k^{-1} g_t K$. It follows that (35) holds with our new n'_t.

Recall that we have fixed an $\omega \in \Omega$ which satisfies the limiting properties in Theorem 1. Now we assume that it also satisfies (C). By excluding a null set in Ω, every ω satisfies these hypotheses.

Theorem 6 *If* $k \in k_\infty \Lambda$, *then* (A) *and* (B) *hold.*

Let $g_t = k_t a_t \tilde{k}_t$ be the decomposition given in (15). We have

$$k_t'^{-1} a_t'^{-1} n_t'^{-1} = g_t^{-1} k = \tilde{k}_t^{-1} a_t^{-1} k_t^{-1} k \,.$$

As in Sect. 6, let $g = a_t^{-1} k_t^{-1} k$. Then $g n'_t a'_t = \tilde{k}_t k_t'^{-1}$. We see that $\tilde{k}_t k'_t$ plays the role of h in Sect. 6. By (17), for $p \leqq i$, $(n'_t a'_t)_{pi} = c_{ip}$. Since a'_t is a diagonal matrix and n'_t is an upper triangular matrix whose diagonal entries are equal to one, by (18), we obtain

$$(a'_t)_{ii} = c_{ii} = \sqrt{\frac{(g^* g)[\alpha_{i-1} | \alpha_{i-1}]}{(g^* g)[\alpha_i | \alpha_i]}}$$

and

$$(n'_t)_{pi} = (-1)^{i-p} \frac{(g^* g)[\alpha_{i-1} | 1 \ldots \hat{p} \ldots i]}{(g^* g)[\alpha_{i-1} | \alpha_{i-1}]} \,.$$

Since $k \in k_\infty \Lambda$, $(k_\infty^{-1} k)[\beta_i | \alpha_i] \neq 0$ for any i. By (20),

$$(g^* g)[\alpha_i | \alpha_i] = a_t[\beta_i]^{-2} (k_t^{-1} k)[\beta_i | \alpha_i]^2 + \xi_i$$

where $\xi_i \leqq a_t[\beta_i]^{-2} e^{-t/n}$. Because

$$a_t[\beta_i] / a_t[\beta_{i-1}] \asymp \exp(\lambda_{n-i+1}\, t) = \exp(-\lambda_i t)$$

it follows that $A'_t/t \to -H_p$. This proves (A). By the expression for $(n'_t)_{pi}$ and (20), we see that for $p \leqq i$,

$$(n'_\infty)_{pi} = \lim_{t \to \infty} (n'_t)_{pi} = (-1)^{i-p} \frac{(k_\infty^{-1} k)[\beta_{i-1} | 1 \ldots \hat{p} \ldots i]}{(k_\infty^{-1} k)[\beta_{i-1} | \alpha_{i-1}]} \,. \tag{37}$$

This proves (B) hence Theorem 6.

Any $X \in \mathscr{G}$ can be considered as a tangent vector on K/L at the coset L, i.e., the tangent vector of the curve $s \to e^{sX} H$ in $G/H = K/L$ at $s = 0$. In Sect. 3, we introduced a Riemannian metric $\langle \cdot , \cdot \rangle_{K/L}$ on the boundary K/L. The length of X, determined by the Riemannian metric, is $\| X \|_{K/L} = \sqrt{\langle X, X \rangle_{K/L}}$. Note that if $X \in \mathscr{H}$, the Lie algebra of H, then $\| X \|_{K/L} = 0$.

For $X \in \mathscr{G}$, let $\mathrm{ad}(X)$ be the linear transformation on $\mathscr{G}$ defined by $\mathrm{ad}(X) Y = [X, Y]$. One checks that $\{\mathrm{ad}(W);\ W \in \mathscr{A}\}$ is a commutative family of symmetric transformations on $\mathscr{H}$ with respect to the inner product $\langle \cdot , \cdot \rangle$. Therefore, their common eigenvectors form a basis of $\mathscr{H}$. It is easy to show that $X \in \mathscr{H}$ is a common eigenvector if and only if $X = c E_{ij}$ for some constant c. It follows that $\mathscr{H}$ is spanned by a subset of $\{E_{ij};\ 1 \leqq i, j \leqq n\}$. Let

$$J = \{(i, j);\ E_{ij} \notin \mathscr{H}\} \,. \tag{38}$$

We note that $E_{ij} \in \mathscr{H}$ if $i \leqq j$, so, $(i, j) \in J$ implies $i > j$. We also note that the Lie algebra $\mathscr{L}$ of L is spanned by $\{E_{ij} - E_{ji};\ (i, j) \notin J, i > j\}$. Let $\mathscr{J}$ be the orthogonal complement of $\mathscr{L}$ in $\mathscr{K}$ with respect to the inner product $\langle \cdot, \cdot \rangle$. We see that $\mathscr{J}$ is spanned by $\{E_{ij} - E_{ji};\ (i, j) \in J\}$. It follows that E_{ij}, $(i, j) \in J$, considered as tangent vectors on K/L, are mutually orthogonal and form a basis of the tangent space at the coset L.

Now assume $kL \in k_\infty(A')$. By the definition of A', we may assume $k \in k_\infty A$. By the remark following (36), the estimate (35) holds. For $Z \in \mathscr{G}$, $Dk(Z)$ is the tangent vector of the curve $s \mapsto ke^{sZ}H$ in $G/H = K/L$ at $s = 0$. For $(i, j) \in J$, let $Z = \mathrm{Ad}(n'_\infty)E_{ij}$. We see that $Dg_t^{-1}(Dk(Z))$ is the tangent vector of the curve $s \mapsto g_t^{-1} k n'_\infty \exp(sE_{ij})H$ at $s = 0$, which is the same as the tangent vector of the curve

$$s \mapsto k_t'^{-1} a_t'^{-1} n_t'^{-1} n'_\infty \exp(sE_{ij})H = k_t'^{-1} \exp[s\,\mathrm{Ad}(a_t'^{-1} n_t'^{-1} n'_\infty)E_{ij}]\,H\,.$$

Since the Riemannian metric on K/L is K-invariant, $k_t'^{-1}$ is an isometry on K, we have

$$\| Dg_t^{-1}(Dk(Z)) \|_{K/L} = \| \mathrm{Ad}(a_t'^{-1} n_t'^{-1} n'_\infty) E_{ij} \|_{K/L}\,. \tag{39}$$

Since $\mathrm{Ad}(e^X) = e^{\mathrm{ad}(X)}$, $\mathrm{Ad}(a_t'^{-1}) = \exp[-\mathrm{ad}(A'_t)]$.

$$\mathrm{ad}(A'_t)E_{pq} = [A'_t, E_{pq}] = [A'_p(t) - A'_q(t)]\,E_{pq}\,.$$

We have

$$\mathrm{Ad}(a_t'^{-1} n_t'^{-1} n'_\infty)E_{ij} = \exp(-\mathrm{ad}(A'_t))\mathrm{Ad}(n_t'^{-1} n'_\infty)E_{ij}$$

$$= \exp(-\mathrm{ad}(A'_t)) \sum_{p,q} h_{ij}^{pq}(t)E_{pq} = \sum_{p,q} h_{ij}^{pq}(t) \exp[-(A'_p(t) - A'_q(t))]\,E_{pq}\,.$$

Since $(A'_i(t)) - (A'_j(t)/t \to -(\lambda_i - \lambda_j)$ and $h_{ij}^{ij}(t) \to 1$, the term on the right hand side with $p = i$ and $q = j$ grows like $\exp[(\lambda_i - \lambda_j)t]$. Note that the terms with $(p, q) \notin J$ correspond to zero vector on K/L and the other terms correspond to mutually orthogonal vectors on K/L. By (35), the exponential growth rates of these terms are controlled by $\exp[(\lambda_i - \lambda_j)t]$. It follows that for $(i, j) \in J$,

$$\lim_{t \to \infty} \frac{1}{t} \log \| \mathrm{Ad}(a_t'^{-1} n_t'^{-1} n'_\infty) E_{ij} \|_{K/L} = \lambda_i - \lambda_j = -\frac{i - j}{2n}\,. \tag{40}$$

Note that the tangent space of K/L at kL is spanned by $\{Dk(\mathrm{Ad}(n'_\infty)E_{ij});\ (i, j) \in J\}$. We have proved the following results.

Theorem 7 *At $kL \in k_\infty(A')$, all the Lyapunov exponents of the stochastic flow $g_t^{-1}(\cdot)$ exist and are given by $-(i - j)/2n$ for $(i, j) \in J$. Note that $i > j$ for $(i, j) \in J$.*

Consider the sphere S^{n-1} as a boundary of $G = \mathrm{SL}(n, R)$. If we identify S^{n-1} with $K/\mathrm{SO}(n-1)$, where $\mathrm{SO}(n-1)$ is identified with a subgroup of $K = \mathrm{SO}(n)$ as in Sect. 3, we see that

$$J = \{(2, 1), (3, 1), \ldots, (n, 1)\}\,.$$

As a direct consequence of Theorem 7, we obtain

Corollary 2 *Let ϕ_t be the stochastic flow on S^{n-1} generated by* (25) *and for almost every ω, let C be the great circle in Theorem 5. At any point $x \in S^{n-1} - C$, the Lyapunov exponents of ϕ_t are $-1, -2, \ldots, -(n-1)$.*

9 Instable points

By Theorem 7, any $k \in k_\infty \Lambda$ is stable for the stochastic flow $g_t^{-1}(\cdot)$ on $K = \mathrm{SO}(n)$ in the sense that all the Lyapunov exponents at k are negative. In this section, we will show that at $k \in K - k_\infty \Lambda = k_\infty(K - \Lambda)$, there is at least one nonnegative exponent.

Recall that $g_t = k_t a_t \tilde{k}_t$ is the decomposition (15). Define $f_{ij}^{pq}(t)$ by

$$\mathrm{Ad}(k_t^{-1} k_\infty) E_{ij} = \sum_{p,q} f_{ij}^{pq}(t) E_{pq} . \tag{41}$$

By Lemma 10 in [6],

$$|f_{ij}^{pq}(t)| \leqq \exp[-|(\lambda_i - \lambda_j) - (\lambda_p - \lambda_q)| t] . \tag{42}$$

We note that the proof of this lemma shows that (42) holds for any path ω which satisfies the limiting properties in Theorem 1 and (C).

Let $\eta_t = k_t^{-1} k_\infty$. We have

$$f_{ij}^{pq}(t) = [\mathrm{Ad}(\eta_t) E_{ij}]_{pq} = [\eta_t E_{ij} \eta_t^{-1}]_{pq} = \eta_{pi}(t) \eta_{qj}(t) .$$

Setting $i = p$ and noting $\eta_{ii}(t) \to 1$, we obtain

$$|\eta_{qj}(t)| \lesssim \exp(-|\lambda_q - \lambda_j| t) = \exp\left(-\frac{|q-j|}{2n} t\right) . \tag{43}$$

Let $\alpha, \beta \subset \{1, 2, \ldots, n\}$ with $\alpha = \{i_1, \ldots, i_k\}$ and $\beta = \{j_1, \ldots, j_k\}$. Assume $i_1 < i_2 < \cdots < i_k$ and $j_1 < j_2 < \cdots < j_k$. Define $|\alpha - \beta|$ to be $\sum_{s=1}^k |i_s - j_s|$. Note that this is only defined when α and β have the same cardinality. By (43), we see that

$$\eta_t[\alpha | \beta] \lesssim \exp\left(-\frac{|\alpha - \beta|}{2n} t\right) . \tag{44}$$

We note that $\eta_t[\alpha | \alpha] \to 1$ as $t \to \infty$.

As in Sect. 6, let $g = a_t^{-1} k_t^{-1} k$ for some $k \in K$ and let $a_t[\alpha] = \prod_{i \in \alpha} a_i(t)$, where $a_i(t)$ is the i-th diagonal entry of a_t. Recall $a_i(t) \asymp \exp[(n - 2i + 1)t/4n]$. As in Sect. 6, let $\beta_i = (n - i + 1, n - i + 2, \ldots, n)$. We have $a[\beta_i]^{-2} \asymp \exp[i(n-i)t/2n]$. It follows that for $|\beta| = i$,

$$a_t[\beta]^{-2} \asymp \exp\left[\frac{i(n-i) - 2|\beta - \beta_i|}{2n} t\right] . \tag{45}$$

By the Binet–Cauchy formula,

$$(g^* g)[\alpha | \alpha] = \sum_\beta g[\beta | \alpha]^2 = \sum_\beta a[\beta]^{-2} (k_t^{-1} k)[\beta | \alpha]^2 . \tag{46}$$

Since

$$(k_t^{-1} k)[\beta | \alpha] = (\eta_t k_\infty^{-1} k)[\beta | \alpha] = \sum_\gamma \eta_t[\beta | \gamma] (k_\infty^{-1} k)[\gamma | \alpha]$$

by (44), (45) and (46), we see that for any $\alpha \subset \{1, 2, \ldots, n\}$, there is an integer μ such that

$$(g^* g)[\alpha | \alpha] \asymp \exp\left(\frac{\mu}{2n} t\right) . \tag{47}$$

Now let h be the K-component of g in the Iwasawa decomposition (5). We have $g_t^{-1}(k) = \tilde{k}_t^{-1} h$. For any $X \in \mathscr{K}$, let $h(s)$ be the K-component of ge^{sX} in the Iwasawa decomposition. Then $h(0) = h$. There exists $Y \in \mathscr{K}$ such that the tangent vector of the curve $s \mapsto he^{sY}$ at $s = 0$ is the same as the tangent vector of the curve $s \mapsto h(s)$. We have

$$Dg_t^{-1}(Dk(X)) = D(\tilde{k}_t^{-1} h)(Y) . \tag{48}$$

Let h_p be the p-th column vector of h and let $\dot{h}$ be $(d/dt)h(0)$. We have

$$h + shY + O(s^2) = he^{sY} = h(s) + O(s^2) = h + s\dot{h} + O(s^2) .$$

Hence, $Y = h^{-1}\dot{h}$. Let y_{pq} be the (p, q)-entry of Y. We have

$$y_{pq} = h_p \cdot \dot{h}_q = -\dot{h}_p \cdot h_q .$$

For $1 \leqq r \leqq n-1$, define $X_r = E_{r,r+1} - E_{r+1,r}$. Let $X = X_r$ and let Y_r be the corresponding Y. Then the i-th column vector of ge^{sX} is g_i for $i \neq r, r+1$, it is $g_r - sg_{r+1} + O(s^2)$ for $i = r$ and $g_{r+1} + sg_r + O(s^2)$ for $i = r+1$. Since the column vectors of $h(s)$ are obtained from those of ge^{sX} through a Gram–Schmidt diagonalization, we see that $h_i(s) = h_i$ for $i \neq r, r+1$ and

$$h_r(s) = \sum_{p=1}^{r} c_{rp}(s) g_p - sc_{rr}(s) g_{r+1} + O(s^2)$$

where $c_{rp}(s)$ satisfies $c_{rp}(0) = c_{rp}$ in (18). It follows that $\dot{h}_i = 0$ for $i \neq r, r+1$ and

$$\dot{h}_r = \sum_{p=1}^{r} \dot{c}_{rp}(0) g_p - c_{rr} g_{r+1} .$$

Since $g_p \cdot h_{r+1} = 0$ for $p \leqq r$,

$$y_{r,r+1} = -\dot{h}_r \cdot h_{r+1} = c_{rr} g_{r+1} \cdot h_{r+1} = c_{rr} \sum_{p=1}^{r+1} c_{r+1,p} g_{r+1} \cdot g_p .$$

By (18), the above is equal to

$$\begin{aligned} &c_{rr}((g^*g)[\alpha_r|\alpha_r](g^*g)[\alpha_{r+1}|\alpha_{r+1}])^{-1/2} \\ &\quad \times \sum_{p=1}^{r+1} (-1)^{r+1-p} (g^*g)[\alpha_r|1, \ldots, \hat{p}, \ldots, r+1](g^*g)_{r+1,p} \\ &= c_{rr}((g^*g)[\alpha_r|\alpha_r](g^*g)[\alpha_{r+1}|\alpha_{r+1}])^{-1/2}(g^*g)[\alpha_{r+1}|\alpha_{r+1}] \\ &= \sqrt{(g^*g)[\alpha_{r-1}|\alpha_{r-1}](g^*g)[\alpha_{r+1}|\alpha_{r+1}]}/(g^*g)[\alpha_r|\alpha_r] . \end{aligned}$$

Hence

$$Y_r = \frac{\sqrt{(g^*g)[\alpha_{r-1}|\alpha_{r-1}](g^*g)[\alpha_{r+1}|\alpha_{r+1}]}}{(g^*g)[\alpha_r|\alpha_r]} X_r . \tag{49}$$

Note that $(g^*g)[\alpha_0|\alpha_0] = (g^*g)[\alpha_n|\alpha_n] = 1$. It follows that

$$\prod_{r=1}^{n-1} \| Y_r \| = 1/\sqrt{(g^*g)[\alpha_1|\alpha_1](g^*g)[\alpha_{n-1}|\alpha_{n-1}]} .$$

If $k \in k_\infty A$, by (20), $\prod_{r=1}^{n-1} \| Y_r \| \asymp \exp[-(n-1)t/2n]$. If $k \notin k_\infty A$, then the exponential growth rate of $(g^*g)[\alpha_{n-1} | \alpha_{n-1}]$ becomes smaller and that of $(g^*g)[\alpha_1 | \alpha_1]$ does not become greater. Therefore, the exponential growth rate of $\prod \| Y_r \|$ is greater than that of $\exp[-(n-1)t/2n]$. On the other hand, by (47) and (49), $\| Y_r \| \asymp \exp(vt/2n)$ for some integer v. It follows that if $k \notin k_\infty A$, then for some r, v is a nonnegative integer. We have proved the following result.

Theorem 8 *If $k \in K - k_\infty A$, then there exists a tangent vector Z on K at k such that*

$$\lim_{t \to \infty} \frac{1}{t} \log \| Dg_t^{-1}(Z) \| = \frac{v}{2n} \tag{50}$$

for some nonnegative integer v.

References

1. Baxendale, P.H.: Asymptotic behaviour of stochastic flows of diffeomorphisms: two case studies. Probab. Theory Relat. Fields **73**, 51–85 (1986)
2. Dynkin, E.B.: Non-negative eigenfunctions of the Laplace-Betrami operators and Brownian motion in certain symmetric spaces. Dokl. Akad. Nauk. SSSR **141**, 1433–1436 (1961)
3. Furstenberg, H.: A Poisson formula for semisimple Lie groups. Ann Math **77** (no 2), 335–386 (1962)
4. Helgason, S.: Group and geometric analysis. New York London: Academic Press 1984
5. Liao, M.: The existence of isometric stochastic flows for Riemannian Brownian motions. In: Pinsky, M., Wihstutz (eds.) Diffusion processes and the related problems in analysis, vol. II. Boston Basel Stuttgart: Birkhäuser 1992
6. Liao, M.: The Brownian motion and the canonical stochastic flow on a symmetric space. Trans. Am. Math. Soc. (to appear)
7. Liao, M.: Stochastic flows on the boundaries of Lie groups. Stochastics Stochastic Rep. **39**, 213–237 (1992)
8. Malliavin, M.P., Malliavin, P.: Factorisations et lois limites de la diffusion horizontale au-dessus d'un espace Riemannien symmetrique. In: Théorie du Potentiel et Analyse Harmonique. (Lect. Notes Math., vol. 404, pp. 164–217) Berlin Heidelberg New York: Springer 1974
9. Norris, J.R., Rogers L.C.G., Williams, D.: Brownian motion of ellipsoids. Trans. Am. Math. Soc. **294**, 757–765 (1986)
10. Orihara, A.: On random ellipsoid. J. Fac. Sci. Univ. Tyoko, Sect. IA Math. **17**, 73–85 (1970)
11. Prat, M.J.: Étude asymptotique et convergence angulaire du mouvement brownien sur une variété à courbure négative. C.R. Acad. Sci., Paris, Sér. A **280**, 1539–1524 (1975)
12 Taylor, J.C.: Brownian motion on a symmetric space of non-compact type: asymptotic behaviour in polar coordinates. Can. J. Math. **43**, 1065–1085 (1991)