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Probability Theory and Related Fields

Asymptotic expansion of stochastic flows

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Summary. We study the asymptotic expansion in small time of the solution of a stochastic differential equation. We obtain a universal and explicit formula in terms of Lie brackets and iterated stochastic Stratonovich integrals. This formula contains the results of Doss [6], Sussmann [15], Fliess and Normand-Cyrot [7], Krener and Lobry [10], Yamato [17] and Kunita [11] in the nilpotent case, and extends to general diffusions the representation given by Ben Arous [3] for invariant diffusions on a Lie group. The main tool is an asymptotic expansion for deterministic ordinary differential equations, given by Strichartz [14].

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1. Introduction

This article is concerned with the calculation of the asymptotic expansion of the flow of a stochastic differential equation with C^∞ coefficients, which has been introduced by Gérard Ben Arous in [3]. If ξ_t is the solution of the Stratonovich stochastic differential equation:

$$\begin{cases} d\xi_t = \sum_{i=0}^{r} X_i(\xi_t)dB_t^i \\ \xi_0 = x_0 \, , \end{cases} \tag{1}$$

then our main result (stated in Theorem 2.1) asserts that $\xi_t = \exp(\zeta_t^p)(x_0) + t^{\frac{p}{2}} R_p(t)$, where $R_p(t)$ is bounded in probability when $t \to 0$. ζ_t^p is a vector field which can be expressed as:

$$\zeta_t^p = \sum_{m=1}^{p-1} \sum_{\|J\|=m} c_t^J X^J \, .$$

The coefficients c_t^J are *completely explicit* linear combinations of Stratonovich iterated integrals B_t^K, where the multi-indices J and K can be deduced one from the other by permutation. In particular, their order is the same.

This result extends to the non nilpotent case the explicit representations given by Yamato [17] and Kunita [11]. It is also more accurate than the following statement of Ben Arous [3], which says:

$$\xi_t = \exp(X_t)(x_0), \text{ with } X_t = \sum_{i=1}^{k} \left(\sum_{\|J\| \leq N} P_J^i B_t^J \right) X^{K_i} + t^{\frac{N+1}{2}} R_N(t) ,$$

where the coefficients P_J^i are given by a tedious and iterative method.

When the Lie algebra generated by the vector fields X_i is of finite dimension, Ben Arous has also obtained the following explicit representation of the stochastic flow:

$$\xi_t = \exp\left(\sum_{m=1}^{\infty} \sum_{\|J\|=m} \beta_J B_t^J \right)(x_0) ,$$

(the main result is actually the convergence of the stochastic series before a stopping time). The β_J are expressed in terms of the Campbell-Hausdorff series. In Sect. 3.2, these coefficients are algebraically computed, and this allows to connect the results of [3] and those of Theorem 2.1. The expression of β_J is now entirely independent of the Campbell-Hausdorff series. This partly explains "the miraculous interaction between purely algebraic formulas on the Campbell-Hausdorff series, and probabilistic identities between Ito and Stratonovich iterated integrals", that has been underlined by Ben Arous in [3]. This fact has independently been pointed out by Hu in [8]. Finally, we mention a work by Takanobu [16], in which a similar asymptotic expansion for ξ_t has been studied in somewhat different way.

The result of Theorem 2.1 is interesting on both practical and theoretical points of view. The explicit expression of the β_J should simplify the simulations of diffusion processes done by this method. From a more theoretical point of view, one could avoid the step which uses the Rotschild and Stein lifting in the works of Ben Arous on the asymptotic expansion of the hypoelliptic heat kernel (see [4]).

This result proves also the conviction of Léandre (see [13]), who uses the asymptotic expansion obtained in Theorem 2.1, referring to [3] without further comments.

The proof of Theorem 2.1 is based on works of Strichartz (in [14]), which give an asymptotic expansion of the solution of an ordinary differential equation for small time. The extension to the case of stochastic differential equations is performed by taking limit in probability.

In Sect. 2, the main result is stated. Its proof when the vector fields X_i generate a nilpotent Lie algebra, is done in Sect. 3. In this case, our result is not asymptotic but exact, the stochastic series being actually finite. However, all the algebraic results needed in the general case (which is proved in Sect. 4), are already present.

2. Result and notations

In this part, we introduce some notations.

Notations for the multi-indices. Let $J = (j_1, \ldots, j_m)$ be a multi-index, that is an element of $\{0, \ldots, r\}^m$. We denote:

- $|J|$ the size of J, m.
- $\|J\|$ the order of J.

$$\|J\| = |J| + \text{ Number of 0 in } J .$$

- For $X_0, \ldots, X_r$ vector fields on $\mathbb{R}^d$, X^J is the Lie bracket of the vector fields X_i,

$$X^J = [X_{j_1}[X_{j_2} \ldots [X_{j_{m-1}} X_{j_m}] \ldots] .$$

- Let $(B_t^1, \ldots, B_t^r)$ be a r-dimensional Brownian motion, and let us write for simplicity, $B_t^0 = t$. B_t^J is the Stratonovich iterated integral:

$$\int_{T_m(t)} dB_{t_1}^{j_1} \ldots dB_{t_m}^{j_m} ,$$

where $T_m(t) = \{(t_1, \ldots, t_m)/0 < t_1 < \ldots < t_m < t\}$.

Notations for the permutations. Let τ be a permutation of order m. We denote:

- $e(\tau)$ the cardinality of the set $\{j \in \{1, \ldots, m-1\}/\tau(j) > \tau(j+1)\}$. Following Strichartz, $e(\tau)$ will be called the number of errors in ordering $\tau(1), \ldots, \tau(m)$.
- If J is a multi-index of size m,

$$J o \tau = (j_{\tau(1)}, \ldots, j_{\tau(m)}) .$$

Exponential notation of a flow. When X is a vector field on $\mathbb{R}^d$, $\exp(sX)(x_0)$ denotes the solution at time s of the differential equation:

$$\begin{cases} \dfrac{du}{ds} = X(u(s)) \\ u(0) = x_0 . \end{cases}$$

With these notations, our main result is:

Theorem 2.1. *Let $X_0, \ldots, X_r$ be C^∞ bounded vector fields on $\mathbb{R}^d$, which are supposed to be Lipschitz. Let ξ be the solution on $\mathbb{R}^d$ of the Stratonovich stochastic differential equation:*

$$\begin{cases} d\xi_t = \sum_{i=0}^{r} X_i(\xi_t) dB_t^i \\ \xi_0 = x_0 \end{cases}$$

(ξ_t is well defined for all t).

For all integer $p \geqq 2$, we define the stochastic vector field

$$\zeta_t^p = \sum_{m=1}^{p-1} \sum_{\|J\|=m} c_t^J X^J ,$$

$$\text{where } c_t^J = \sum_{\sigma \in \sigma_{|J|}} \frac{(-1)^{e(\sigma)}}{|J|^2 \binom{|J|-1}{e(\sigma)}} B_t^{J o \sigma^{-1}} ,$$

and let $R_p(t)$ be the process defined on $\mathbb{R}^d$ by

$$\xi_t = \exp(\zeta_t^p)(x_0) + t^{\frac{p}{2}} R_p(t) . \tag{2}$$

Then, R_p is bounded in probability when t tends to 0. More precisely,

$$\exists \alpha, c > 0 \text{ such that } \forall R > c, \lim_{t \to 0} P\left[\sup_{0 \leqq s \leqq t} s^{p/2} \|R_p(s)\| \geqq R t^{p/2}\right] \leqq \exp\left(-\frac{R^\alpha}{c}\right).$$

The reader is referred to Sect. 4 for the proof of this theorem. This result gives an explicit asymptotic expansion of the stochastic flow for small time, expansion which has been introduced by Ben Arous in [3]. The coefficients of the brackets of order k are shown to be linear combinations of iterated integrals of the same order.

When the Lie algebra $\mathrm{Lie}(X_0, \ldots, X_r)$ is of finite dimension, Ben Arous has obtained a similar result which says: $\exists T$ a stopping time a.e > 0 so that for $t < T$,

$$\xi_t = \exp\left(\sum_{m=1}^{+\infty} \sum_{\|J\|=m} \beta_J B_t^J \right)(x_0) \,,$$

where β_J is a linear combination of brackets of size $|J|$. These two representations are identified in Sect. 3.2.

The result can be easily extended to the case where the vector fields are C^∞ with values in a C^∞ manifold $\mathcal{M}$, by using a system of local coordinates.

3 The nilpotent case

3.1 Proof of the result in the nilpotent case

Theorem 2.1 is proved in the case of a nilpotent Lie algebra, in order to detail the algebraic calculations which remain the same in the general case. Using the notations of Sect. 2, the result is then the following.

Proposition 3.1 *Let X_i be complete C^∞ vector fields in $\mathbb{R}^d$, such that the Lie algebra generated by the X_i is p-nilpotent. We consider the solution of the Stratonovich stochastic differential equation:*

$$\begin{cases} d\xi_t = \sum_{i=0}^{r} X_i(\xi_t) dB_t^i \\ \xi_0 = x_0 \end{cases} .$$

Then, for all t,

$$\xi_t = \exp\left(\sum_{m=1}^{p-1} \sum_{|J|=m} c_t^J X^J \right)(x_0) \quad \text{a.e.} \tag{3}$$

Proof. Let $\xi_s^{(n)}$ be the solution (defined on $[0; t]$) of the differential equation:

$$\begin{cases} \frac{d}{ds} \xi_s^{(n)} = A^n(s)(\xi_s^{(n)}) \\ \xi_0^{(n)} = x_0 \end{cases} ,$$

where

- $A^n(s) = \sum_{i=0}^{r} \frac{\delta_k^n B^i}{t_{k+1} - t_k} X_i$, if $s \in I_k = [t_k; t_{k+1}[$
- $\delta_k^n B^i = B_{t_{k+1}}^i - B_{t_k}^i$
- $t_k = \frac{k}{2^n} t$.

It is well-known that $\xi_t^{(n)} \xrightarrow{\text{Proba}} \xi_t$.

According to the results of [14], we can write $\xi_t^{(n)} = \exp(Z_t^{(n)})(x_0)$, where:

$$Z_t^{(n)} = \sum_{m=1}^{p-1} \sum_{\sigma \in \sigma_m} \frac{(-1)^{e(\sigma)}}{m^2 \binom{m-1}{e(\sigma)}} I_{m,\sigma}^{(n)}$$

and $I_{m,\sigma}^{(n)} = \int_{T_m(t)} [A^n(s_{\sigma(1)})[A^n(s_{\sigma(2)}) \dots [A^n(s_{\sigma(m-1)})\, A^n(s_{\sigma(m)})] \dots]\, ds_1 \dots ds_m$.

Remark. Actually, Strichartz does not give exactly this expression of $\zeta_t^{(n)}$. One has to transform

$$[A^n(s_{\sigma(1)})[A^n(s_{\sigma(2)}) \dots [A^n(s_{\sigma(m-1)})A^n(s_{\sigma(m)})] \dots]$$

into

$$[\dots [A^n(s_{\sigma(1)})A^n(s_{\sigma(2)})] \dots A^n(s_{\sigma(m)})]$$

to obtain the results of Strichartz. However, these two expressions are identical:

$$\sum_{\sigma \in \sigma_m} \frac{(-1)^{e(\sigma)}}{m^2 \binom{m-1}{e(\sigma)}} [A^n(s_{\sigma(1)})[A^n(s_{\sigma(2)}) \dots [A^n(s_{\sigma(m-1)})A^n(s_{\sigma(m)})] \dots]$$

$$= \sum_{\sigma \in \sigma_m} \frac{(-1)^{e(\sigma)}}{m^2 \binom{m-1}{e(\sigma)}} [\dots [A^n(s_{\sigma(1)})A^n(s_{\sigma(2)})] \dots A^n(s_{\sigma(m)})] .$$

Indeed, for $J = (j_1, \dots, j_m)$, let us denote

- $i(J) = (j_m, \dots, j_1)$;
- $X_b^J = [X_{j_1}[X_{j_2} \dots [X_{j_{m-1}} X_{j_m}] \dots]$;
- $X_s^J = [\dots [X_{j_1} X_{j_2}] \dots X_{j_{m-1}}] X_{j_m}]$.

It is easily proved by induction that $X_b^J = (-1)^{|J|-1} X_s^{i(J)}$. For $\sigma \in \sigma_m$, we define the permutation $i(\sigma)$ by $i(\sigma)(k) = \sigma(m+1-k)$. The announced identity can then be deduced from the statement: $e(i(\sigma)) = m - 1 - e(\sigma)$.

We return now to the proof of 3.1. First of all, let us introduce some notations. When $n_1, \dots, n_k$ are k integers such that $\sum_{i=1}^{k} n_i = m$, we denote:

- $p_0 = 0$;
- for $j \in \{1, \dots, k\}$, $p_j = \sum_{i=1}^{j} n_i$;

Finally $\sigma_{n_1 \dots n_k}$ represents the subgroup of the permutation group σ_m isomorphic to the subgroup $\sigma_{n_1} \times \dots \times \sigma_{n_k}$, and given by

$$\sigma_{n_1 \dots n_k} = \{\tau \in \sigma_m / \forall i \in \{0, \dots, k-1\}, \tau(\{p_j + 1, \dots, p_{j+1}\}) = \{p_j + 1, \dots, p_{j+1}\}\} .$$

If $\sigma \in \sigma_m$, $\bar{\sigma}$ is its class in the equivalence relation defined by $\sigma_{n_1 \dots n_k}$. Then, we call (following [3])

$$\lambda(n_1 \dots n_k, \bar{\sigma}) = \{(t_1, \dots, t_m) \in \mathbb{R}^{+*m} / t_{\sigma(1)} = \dots = t_{\sigma(p_1)} < t_{\sigma(p_1+1)} = \dots$$
$$\dots = t_{\sigma(p_2)} < \dots < t_{\sigma(p_{k-1}+1)} = \dots = t_{\sigma(m)}\} .$$

It is clear that this definition does not depend on the element chosen in $\bar{\sigma}$. For simplicity, we will write $\lambda(n_1 \ldots n_k)$ for $\lambda(n_1 \ldots n_k, \overline{\mathrm{Id}})$, that is:

$$\lambda(n_1 \ldots n_k) = \{(t_1, \ldots, t_m) \in \mathbb{R}^{+*m}/t_1 = \ldots = t_{p_1} < t_{p_1+1} = \ldots = t_{p_2} < \ldots \\ \ldots < t_{p_{k-1}+1} = \ldots = t_m\} .$$

Once these notations are introduced, we have, for a given m, and a given $\sigma \in \boldsymbol{\sigma}_m$:

$$I_{m,\sigma}^{(n)} = \sum_{\substack{i_1, \ldots, i_m \\ i_1 \leqq \ldots \leqq i_m}} \int_{\substack{(I_{i_1} \times \ldots \times I_{i_m}) \\ \cap T_m(t)}} [A_{i_{\sigma(1)}}^n [A_{i_{\sigma(2)}}^n \ldots [A_{i_{\sigma(m-1)}}^n A_{i_{\sigma(m)}}^n] \ldots] ds_1 \ldots ds_m .$$

Here, A_l^n represents the value of A^n in the interval I_l.

$$A_l^n = \frac{2^n}{t} \sum_{i=0}^{r} \delta_l^n B^i X_i .$$

Let $(n_1, \ldots, n_k)$ be such that $I = (i_1, \ldots, i_m) \in \lambda(n_1 \ldots n_k)$.

$$\int_{\substack{(I_{i_1} \times \ldots \times I_{i_m}) \\ \cap T_m(t)}} ds_1 \ldots ds_m = \prod_{i=0}^{k-1} \int_{t_{k_i} < s_{p_{i+1}} < \ldots < s_{p_{i+1}} < t_{k_{i+1}}} ds_{p_i+1} \ldots ds_{p_{i+1}} \\ = \left(\frac{t}{2^n}\right)^m \frac{1}{n_1! \ldots n_k!} .$$

It follows

$$I_{m,\sigma}^{(n)} = \sum_{\substack{n_1, \ldots, n_k \\ \sum n_i = m}} \sum_{I \in \lambda(n_1 \ldots n_k)} [A_{i_{\sigma(1)}}^n [A_{i_{\sigma(2)}}^n \ldots [A_{i_{\sigma(m-1)}}^n A_{i_{\sigma(m)}}^n] \ldots] \left(\frac{t}{2^n}\right)^m \frac{1}{n_1! \ldots n_k!}$$

$$= \sum_{\substack{n_1, \ldots, n_k \\ \sum n_i = m}} \sum_{I \in \lambda(n_1 \ldots n_k)} \left(\frac{t}{2^n}\right)^m \frac{1}{n_1! \ldots n_k!}$$

$$\ldots \left(\frac{2^n}{t}\right)^m \left[\sum_{j_1=0}^{r} \delta_{i_{\sigma(1)}}^n B^{j_1} X_{j_1} \ldots \left[\sum_{j_{m-1}=0}^{r} \delta_{i_{\sigma(m-1)}}^n B^{j_{m-1}} X_{j_{m-1}} \sum_{j_m=0}^{r} \delta_{i_{\sigma(m)}}^n B^{j_m} X_{j_m} \right] \ldots \right]$$

$$= \sum_{\substack{n_1, \ldots, n_k \\ \sum n_i = m}} \sum_{I \in \lambda(n_1 \ldots n_k)} \frac{1}{n_1! \ldots n_k!} \sum_{|J|=m} (\delta_{i_{\sigma(1)}}^n B^{j_1} \ldots \delta_{i_{\sigma(m)}}^n B^{j_m}) X^J .$$

This yields the expression for $Z_t^{(n)}$:

$$Z_t^{(n)} = \sum_{m=1}^{p-1} \sum_{|J|=m} c_t^{(n),J} X^J ,$$

where

$$c_t^{(n),J} = \sum_{\sigma \in \boldsymbol{\sigma}_m} \frac{(-1)^{e(\sigma)}}{m^2 \binom{m-1}{e(\sigma)}} \sum_{\substack{n_1, \ldots, n_k \\ \sum n_i = m}} \sum_{I \in \lambda(n_1 \ldots n_k)} \frac{1}{n_1! \ldots n_k!} (\delta_{i_{\sigma(1)}}^n B^{j_1} \ldots \delta_{i_{\sigma(m)}}^n B^{j_m}) .$$

Following [3], let us call

$$\mathscr{A}_m^k = \left\{(n_1, \ldots, n_k) \in \{1, 2\}^k \Big/ \sum_{i=1}^{k} n_i = m\right\} .$$

For $v \in \mathscr{A}_m^k$, and for some martingales X^i, we define the following Ito iterated integral

$$I_t(X^1 \dots X^m, v) = \int_{T_k(t)} dY_{t_1}^1 \dots dY_{t_k}^k \quad \text{with} \quad \begin{cases} Y_s^i = X_s^{p_i} & \text{if } n_i = 1, \\ Y_s^i = \langle X^{p_i - 1}, X^{p_i} \rangle_s & \text{if } n_i = 2 . \end{cases}$$

We apply then the lemma (after Lemma 5 in [3]):

Lemma 3.1 *If* $\forall i, n_i \leqq 2$, *then* $\sum_{I \in \lambda(n_1 \dots n_k, \bar{\sigma})} \delta_{i_1}^n B^{j_1} \dots \delta_{i_m}^n B^{j_m}$ *converges (in* L^2 *sense) to the Ito iterated integral* $I_t(B^{j_{\sigma(1)}} \dots B^{j_{\sigma(m)}}, n_1 \dots n_k)$. *Otherwise, this quantity tends to* 0.

And we obtain $c_t^{(n), J} \to c_t^J$, with

$$\begin{aligned} c_t^J &= \sum_{\sigma \in \sigma_m} \frac{(-1)^{e(\sigma)}}{m^2 \binom{m-1}{e(\sigma)}} \sum_{v = (n_1, \dots, n_k) \in \mathscr{A}_m^k} \frac{1}{n_1! \dots n_k!} I_t(B^{j_{\sigma^{-1}(1)}} \dots B^{j_{\sigma^{-1}(m)}}, v) \\ &= \sum_{\sigma \in \sigma_m} \frac{(-1)^{e(\sigma)}}{m^2 \binom{m-1}{e(\sigma)}} \sum_{k = [\frac{m}{2}]}^{m} \frac{1}{2^{m-k}} \sum_{v \in \mathscr{A}_m^k} I_t(B^{j_{\sigma^{-1}(1)}} \dots B^{j_{\sigma^{-1}(m)}}, v) . \end{aligned}$$

Now, $\sum_{k=[\frac{m}{2}]}^{m} \frac{1}{2^{m-k}} \sum_{v \in \mathscr{A}_m^k} I_t (B^{j_{\sigma^{-1}(1)}} \dots B^{j_{\sigma^{-1}(m)}}, v)$ is nothing else but the expression of the Stratonovich iterated integral in terms of the Ito iterated integral (we refer the reader to Proposition 1 in [3]). This yields the expression for c_t^J:

$$c_t^J = \sum_{\sigma \in \sigma_m} \frac{(-1)^{e(\sigma)}}{m^2 \binom{m-1}{e(\sigma)}} B_t^{J \circ \sigma^{-1}} .$$

The convergence of $c_t^{(n), J}$ to c_t^J implies the L^2 convergence of $Z_t^{(n)}$ to Z_t, where:

$$Z_t = \sum_{m=1}^{p-1} \sum_{|J|=m} c_t^J X^J .$$

This ends the proof of Proposition 3.1.

3.2 Identification with already known asymptotic expansion

In the nilpotent case, Ben Arous has derived in [3] the representation

$$\xi_t = \exp\left(\sum_{m=1}^{p-1} \sum_{|J|=m} \beta_J B_t^J \right)(x_0) \tag{4}$$

where β_J corresponds to the m-homogeneous term of degree 1 in each variable in the Campbell-Hausdorff series $H(X_{j_1}, \dots, X_{j_m})$, defined by:

$$H(X_1, \dots, X_m) = \sum_{k \geqq 1} \frac{(-1)^{k-1}}{k} \sum_{P \in B_k} \frac{1}{\overline{PP!}} X^P .$$

Here, B_k, $\bar{P}$ and X^P are given by the following expressions:

$$B_k = \left\{ (p_i^j)_{\substack{i \in \{1 \dots m\} \\ j \in \{1 \dots k\}}},\ p_i^j \in \mathbb{N},\ \forall j \in \{1 \dots k\} \sum_{i=1}^{m} p_i^j > 0 \right\},$$

$$\bar{P} = \sum_{i,j} p_i^j, \quad P! = \prod_{i,j} p_i^j!\,,$$

$$X^P = \underbrace{[X_1 \dots [X_1}_{p_1^1 \text{ times}} \dots \underbrace{[X_m \dots [X_m}_{p_m^1 \text{ times}} \dots \underbrace{[X_1 \dots [X_1}_{p_1^k \text{ times}} \dots \underbrace{[X_m \dots [X_m, X_m]}_{p_m^k \text{ times}} \dots]\,.$$

That is, β_J is the coefficient of $s_1 \dots s_m$ in $H(s_1 X_{j_1}, \dots, s_m X_{j_m})$.

The identification of the expressions (3) and (4) of ξ_t, using the independence of the B^J, leads to the necessary relation:

Proposition 3.2

$$\beta_J = \sum_{\sigma \in \boldsymbol{\sigma}_m} \frac{(-1)^{e(\sigma)}}{m^2 \binom{m-1}{e(\sigma)}} [X_{j_{\sigma(1)}}[X_{j_{\sigma(2)}} \dots [X_{j_{\sigma(m-1)}} X_{j_{\sigma(m)}}] \dots]\,.$$

However, we are going to derive a purely algebraic proof of this expression for β_J. This allows us to explain a large part of the "miracle" pointed out by Ben Arous in the introduction of [3], by showing that the coefficient β_J has not much to do with the Campbell-Hausdorff series.

The extreme simplicity of that expression must also be noticed. This could be very useful in simulations of diffusion processes.

Proof. The main algebraic results used in the proof can be found in [14].

Without loss of generality, one can assume that $\forall i, j_i = i$. Thus, according to the expression of $H(X_{j_1}, \dots, X_{j_m})$,

$$\beta_J = \frac{1}{m} \sum_{k=1}^{m} \frac{(-1)^{k-1}}{k} \sum_{P \in \hat{B}_k} X^P\,,$$

with $\hat{B}_k = \left\{ (p_i^j)_{\substack{i \in \{1 \dots m\} \\ j \in \{1 \dots k\}}} \in B_k,\ \forall i \in \{1 \dots m\} \sum_{j=1}^{k} p_i^j = 1 \right\}$.

Let us define $\hat{B}_k(p^1 \dots p^k) = \left\{ P = (p_i^j) \in \hat{B}_k,\ \forall j \in \{1 \dots k\}\ p^j = \sum_{i=1}^{m} p_i^j \right\}$.

If $P \in \hat{B}_k(p^1 \dots p^k)$, then X^P has the form:

$$[X_{q_1}[X_{q_2} \dots [X_{q_{m-1}} X_{q_m}] \dots] \quad \text{with} \quad \begin{array}{c} q_1 < \dots < q_{p^1} \\ \vdots \\ q_{p^1 + \dots + p^{k-1} + 1} < \dots < q_{p^1 + \dots + p^k}\,. \end{array}$$

Let us call $\boldsymbol{\sigma}_m(p^1 \dots p^k) = \left\{ \sigma \in \boldsymbol{\sigma}_m,\ \begin{array}{c} \sigma(1) < \dots < \sigma(p^1), \\ \vdots \\ \sigma\left(\sum_1^{k-1} p^i + 1\right) < \dots < \sigma\left(\sum_1^{k} p^i\right) \end{array} \right\}$.

We derive:

$$\beta_J = \sum_{k=1}^{m} \sum_{\substack{p^1 \dots p^k \\ \sum p^j = m}} \sum_{\sigma \in \sigma_m (p^1 \dots p^k)} \frac{1}{m} \frac{(-1)^{k-1}}{k} [X_{\sigma(1)}[X_{\sigma(2)} \dots [X_{\sigma(m-1)} X_{\sigma(m)}] \dots]$$

$$= \sum_{k=1}^{m} \sum_{\sigma \in \sigma_m} \frac{1}{m} \frac{(-1)^{k-1}}{k} d(m, k, \sigma) [X_{\sigma(1)}[X_{\sigma(2)} \dots [X_{\sigma(m-1)} X_{\sigma(m)}] \dots]$$

$d(m, k, \sigma)$ is the number of ways of choosing k strictly positive integers $p^1 \dots p^k$

satisfying $\sum p^j = m$, and

$$\begin{cases} \sigma(1) < \dots < \sigma(p^1), \\ \vdots \\ \sigma\left(\sum_1^{k-1} p^i + 1\right) < \dots < \sigma\left(\sum_1^{k} p^i\right) \end{cases}.$$

Strichartz has stated in [14] that

$$d(m, k, \sigma) = \begin{cases} \begin{pmatrix} m - e(\sigma) - 1 \\ k - e(\sigma) - 1 \end{pmatrix} & \text{if } k \geqq e(\sigma) + 1 \\ 0 & \text{otherwise} \end{cases}.$$

To prove this assertion, he considers the sequence $1\ 2\ \dots\ m$, cut in k parts. $k - 1$ barriers have thus to be placed on $m - 1$ locations, and the integer p^k is then the number of integers in the k^{th} part.

In order to verify the condition $\sigma\left(\sum_1^{l-1} p^i + 1\right) < \dots < \sigma\left(\sum_1^{l} p^i\right)$, all the terms in each part must be well ordered. The locations of the errors must then be some barriers. If $e(\sigma) > k - 1$, the errors are more numerous than the barriers, and thus, the latter condition cannot be satisfied. There is necessarily one error between two barriers. In that case, $d(m, k, \sigma) = 0$. If $e(\sigma) \leqq k - 1$, $e(\sigma)$ of the barriers have to be located on the errors sites, and the remaining $k - 1 - e(\sigma)$ barriers can be placed on the $m - 1 - e(\sigma)$ available sites. The result follows.

$$\beta_J = \sum_{\sigma \in \sigma_m} \sum_{k=e(\sigma)+1}^{m} \frac{(-1)^{k-1}}{mk} \begin{pmatrix} m - e(\sigma) - 1 \\ k - e(\sigma) - 1 \end{pmatrix} [X_{\sigma(1)}[X_{\sigma(2)} \dots [X_{\sigma(m-1)} X_{\sigma(m)}] \dots] .$$

And the proof of the Proposition 3.2 is then ended by noticing that:

$$\sum_{k=e(\sigma)+1}^{m} \frac{(-1)^{k-1}}{mk} \begin{pmatrix} m - e(\sigma) - 1 \\ k - e(\sigma) - 1 \end{pmatrix} = \frac{(-1)^{e(\sigma)}}{m^2 \begin{pmatrix} m - 1 \\ e(\sigma) \end{pmatrix}},$$

which is also proved in [14].

4 The general case

In that section, the proof of Theorem 2.1 is given. Since we are interested in the small time behavior of ξ_t, we scale Eq. (1) by a small parameter $\varepsilon > 0$, and introduce the diffusion process ξ_t^ε on $\mathbb{R}^d$, defined by

$$\begin{cases} d\xi_t^\varepsilon = \sum_{i=1}^{r} \varepsilon X_i(\xi_t^\varepsilon) dB_t^i + \varepsilon^2 X_0(\xi_t^\varepsilon) dB_t^0 \\ \xi_0^\varepsilon = x_0 \end{cases}$$

$\xi_\bullet^\varepsilon$ has the same law as $\xi_{\varepsilon^2\bullet}$, so that we have to show:

Theorem 4.1 *Let us fix $T < 0$. Given $t < T$ and $\varepsilon \geqq 0$, we define $R_p(\varepsilon, t)$ by*

$$\xi_t^\varepsilon = \exp(\zeta_t^{\varepsilon,p})(x_0) + \varepsilon^p R_p(\varepsilon, t) \text{ a.e.} \tag{5}$$

where $\zeta_t^{\varepsilon,p} = \sum_{m=1}^{p-1} \varepsilon^m \sum_{\|J\|=m} c_t^J X^J$ a.e.

Then, $\exists \alpha, c > 0$ such that $\forall R > c$, $\lim_{\varepsilon \to 0} P\left[\sup_{0 \leqq s \leqq T} \|R_p(\varepsilon, s)\| \geqq R\right] \leqq \exp\left(-\frac{R^\alpha}{cT}\right)$.

Now, one can apply the theorem with $T = 1$. Since $\xi_\bullet^\varepsilon$ and $\zeta_\bullet^{\varepsilon,p}$ have the same laws as $\xi_{\varepsilon^2\bullet}$ $\zeta_{\varepsilon^2\bullet}$ respectively, $R_p(\varepsilon, \bullet)$ has the same law as $\bullet^{p/2} R_p(\varepsilon^2 \bullet)$. Therefore,

$$P\left[\sup_{0 \leqq s \leqq \varepsilon^2} \|R_p(s)\| s^{p/2} \geqq R\varepsilon^p\right] = P\left[\sup_{0 \leqq u \leqq 1} u^{p/2} \|R_p(\varepsilon^2 u)\| \geqq R\right]$$

$$\leqq P\left[\sup_{0 \leqq u \leqq 1} \|R_p(\varepsilon, u)\| \geqq R\right]$$

and Theorem 2.1 is deduced from Theorem 4.1.

Proof of Theorem 4.1 As in the nilpotent case, we consider the solution $\xi_s^{(n),\varepsilon}$ on $[0; t]$ of the differential equation:

$$\begin{cases} \frac{d}{ds} \xi_s^{(n),\varepsilon} = A^{\varepsilon,n}(s)(\xi_s^{(n),\varepsilon}) \\ \xi_0^{(n),\varepsilon} = x_0 \end{cases},$$

where $A^{\varepsilon,n}(s) = \frac{2^n}{t} \sum_{i=0}^{r} \varepsilon^{\|i\|} \delta_k^n B^i X_i$, when $s \in I_k = [t_k; t_{k+1}[$.

Remark. By analogy with the notation $\|J\|$, $\|i\| = \begin{cases} 1 & \text{if } i \neq 0 \\ 2 & \text{if } i = 0 \end{cases}$.

From the results of Strichartz (see Theorem 3.2 of [14]), it can be shown that for almost all ω, and for all n, there exist $\varepsilon_0(\omega, n)$ and $C_0(\omega, n)$ such that

$$\xi_t^{(n),\varepsilon}(\omega) = \exp(Z_t^{(n),\varepsilon,p}(\omega) + \varepsilon^p S_p^n(\varepsilon, t)(\omega))(x_0), \tag{6}$$

where for all $\varepsilon < \varepsilon_0(\omega, n)$ and all $x \in \mathbb{R}^d$, $\| S_p^n(\varepsilon, t)(\omega)(x) \| \leqq C_0(\omega, n)$, and

$$Z_t^{(n), \varepsilon, p} = \sum_{m=1}^{p-1} \sum_{\sigma \in \sigma_m} \frac{(-1)^{e(\sigma)}}{m^2 \binom{m-1}{e(\sigma)}} \cdots$$

$$\cdots \int_{T_m(t)} [A^{\varepsilon, n}(s_{\sigma(1)})[\ldots [A^{\varepsilon, n}(s_{\sigma(m-1)}) A^{\varepsilon, n}(s_{\sigma(m)})] \ldots] ds_1 \ldots ds_m \text{ a.e.}$$

$$= \sum_{m=1}^{p-1} \sum_{|J|=m} \varepsilon^{\|J\|} c_t^{(n), J} X^J .$$

We reorder $Z_t^{(n), \varepsilon, p}$ by increasing powers of ε, so that

$$Z_t^{(n), \varepsilon, p} = \zeta_t^{(n), \varepsilon, p} + \varepsilon^p Q_p^n(\varepsilon, t)$$

where $\zeta_t^{(n), \varepsilon, p} = \sum_{m=1}^{p-1} \varepsilon^m \sum_{\|J\|=m} c_t^{(n), J} X^J$, and where $Q_p^n(\varepsilon, t)$ is a polynomial (with random coefficients) of ε, and of the $(X^J)_{|J|<p, \|J\| \geqq p}$. Since the X_i are supposed to be bounded, there exists for almost all ω and for all n, $\varepsilon_1(\omega, n)$ and $C_1(\omega, n)$ such that for all $\varepsilon < \varepsilon_1(\omega, n)$ and all $x \in \mathbb{R}^d$, $\| S_p^n(\varepsilon, t)(\omega)(x) + Q_p^n(\varepsilon, t)(\omega)(x) \| \leqq C_1(\omega, n)$. Gronwall's Lemma gives then for almost all ω and all n, the existence of $\varepsilon_2(\omega, n)$ and $C_2(\omega, n)$ such that

$$\xi_t^{(n), \varepsilon}(\omega) = \exp(\zeta_t^{(n), \varepsilon, p}(\omega))(x_0) + \varepsilon^p R_p^n(\varepsilon, t)(\omega) \tag{7}$$

where for all $\varepsilon < \varepsilon_2(\omega, n)$, $\| R_p^n(\varepsilon, t) \| \leqq C_2(\omega, n)$.

As in Sect. 3, the convergence of $c_t^{(n), J}$ to c_t^J implies the convergence in probability of $\exp(\zeta_t^{(n), \varepsilon, p})(x_0)$ to $\exp(\zeta_t^{\varepsilon, p})(x_0)$. Since $\xi_t^{(n), \varepsilon}$ converges in probability to ξ_t^ε, $R_p(\varepsilon, t)$ can be defined as the limit in probability of $R_p^n(t, \varepsilon)$. Then, we clearly get the relation (5) by taking the limit in the expression (7). It just remains to prove that:

$$\exists \alpha, c > 0 \text{ such that } \forall R > c, \lim_{\varepsilon \to 0} P\left[\sup_{0 \leqq s \leqq T} \| R_p(\varepsilon, s) \| \geqq R \right] \leqq \exp\left(-\frac{R^\alpha}{cT} \right).$$

Now ξ_t^ε is a diffusion process with coefficients which are C^∞ in ε. It is then C^∞ with respect to ε, and its stochastic Taylor expansion (see [1]) is given by:

$$\xi_t^\varepsilon = x_0 + \sum_{i=1}^{p-1} \varepsilon^i g_i(t) + \varepsilon^p M_p(\varepsilon, t) ,$$

with $g_i(t) = \frac{1}{i!} \frac{d^i}{d\varepsilon^i} \xi_t^\varepsilon \Big|_{\varepsilon=0}$, and

$$\exists \alpha_1, c_1 > 0 \text{ such that } \forall R > c_1, \lim_{\varepsilon \to 0} P\left[\sup_{0 \leqq s \leqq T} \| M_p(\varepsilon, s) \| \geqq R \right] \leqq \exp\left(-\frac{R^{\alpha_1}}{c_1 T} \right).$$

Moreover $\exp(\zeta_t^{\varepsilon, p})(x_0)$ is the solution at time 1 of an ordinary differential equation, whose coefficients are C^∞ functions of ε. Consequently, the function $\varepsilon \to \exp(\zeta_t^{\varepsilon, p})(x_0)$ is C^∞ with respect to ε, and its Taylor expansion around 0 can be written:

$$\exp(\zeta_t^{\varepsilon, p})(x_0) = x_0 + \sum_{i=1}^{p-1} \varepsilon^i h_i(t) + \varepsilon^p P_p(\varepsilon, t) .$$

Let us assume that we have already proved that:

- $\forall i \in \{1, \ldots, p-1\}$, $h_i(t) = g_i(t)$ a.e.,
- $\exists \alpha_2, c_2 > 0$ such that $\forall R > c_2, \lim_{\varepsilon \to 0} P\left[\sup_{0 \leqq s \leqq T} \|P_p(\varepsilon, s)\| \geqq R\right] \leqq \exp\left(-\frac{R^{\alpha_2}}{c_2 T}\right)$.

We deduce then that almost everywhere, $R_p(\varepsilon, t) = M_p(\varepsilon, t) - P_p(\varepsilon, t)$, and therefore that,

$$\exists \alpha, c > 0 \text{ such that } \forall R > c, \lim_{\varepsilon \to 0} P\left[\sup_{0 \leqq s \leqq T} \|R_p(\varepsilon, s)\| \geqq R\right] \leqq \exp\left(-\frac{R^{\alpha}}{cT}\right),$$

that is the desired result.

So, it only remains to show the two preceding assertions.

Step 1 We first prove that $\forall i \in \{1, \ldots, p-1\}$, $h_i(t) = g_i(t)$ a.e.

We define $g_i^{(n)}(s) = \frac{1}{i!} \frac{\partial}{\partial \varepsilon^i} \xi_s^{(n),\varepsilon} \Big|_{\varepsilon=0}$.

For $g_1, \ldots, g_l$ in $\mathbb{R}^d$, and x in $\mathbb{R}$, $C_i^l(g_1, \ldots, g_l)$ denotes the coefficient of x^l in the expansion of $X_i(x_0 + xg_1 + \ldots + x^l g_l)$ (composition of formal power series). From the differential equation satisfied by $\xi^{(n),\varepsilon}$, we have:

$$\begin{cases} \frac{d}{ds} g_1^{(n)}(s) = \sum_{i=1}^{r} \frac{2^n}{t} \delta_k^n B^i X_i(x_0) \\ \frac{d}{ds} g_2^{(n)}(s) = \sum_{i=1}^{r} \frac{2^n}{t} \delta_k^n B^i C_i^1(g_1^{(n)}) + X_0(x_0) \\ \vdots \\ \frac{d}{ds} g_l^{(n)}(s) = \sum_{i=1}^{r} \frac{2^n}{t} \delta_k^n B^i C_i^{l-1}(g_1^{(n)}, \ldots, g_{l-1}^{(n)} + C_0^{l-2}(g_1^{(n)}, \ldots, g_{l-2}^{(n)}) \end{cases} \quad \text{for } s \in I_k .$$

This can be written (using the appropriate functions F_i^{l-1})

$$\frac{d}{ds}\begin{pmatrix} g_1^{(n)}(s) \\ \vdots \\ g_l^{(n)}(s) \end{pmatrix} = \sum_{i=1}^{r} F_i^{l-1}\begin{pmatrix} g_1^{(n)} \\ \vdots \\ g_{l-1}^{(n)} \end{pmatrix} \frac{2^n}{t} \delta_k^n B^i + F_0^{l-2}\begin{pmatrix} g_1^{(n)} \\ \vdots \\ g_{l-2}^{(n)} \end{pmatrix} \text{ for } s \in I_k .$$

It is known from [1] that the coefficients of the stochastic Taylor expansion of the diffusion ξ_t^ε, satisfy:

$$d\begin{pmatrix} g_1(s) \\ \vdots \\ g_l(s) \end{pmatrix} = \sum_{i=1}^{r} F_i^{l-1}\begin{pmatrix} g_1(s) \\ \vdots \\ g_{l-1}(s) \end{pmatrix} dB_s^i + F_0^{l-2}\begin{pmatrix} g_1(s) \\ \vdots \\ g_{l-2}(s) \end{pmatrix} ds .$$

Therefore, $\forall t, \forall l, (g_1^{(n)}(t), \ldots, g_l^{(n)}(t)) \xrightarrow{\text{Proba}} (g_1(t), \ldots, g_l(t))$.

Let $h_i(t) = \frac{1}{i!} \frac{\partial^i}{\partial \varepsilon^i} \exp(\zeta_t^{\varepsilon, p})(x_0) \Big|_{\varepsilon=0}$

and $h_i^{(n)}(t) = \frac{1}{i!} \frac{\partial^i}{\partial \varepsilon^i} \exp(\zeta_t^{(n), \varepsilon, p})(x_0) \Big|_{\varepsilon=0}$.

We want to show that $h_i^{(n)}(t) \xrightarrow{\text{Proba}} h_i(t), \forall i$. For that purpose, we introduce the map ϕ defined for the appropriate l, by:

$$\begin{array}{cccc} \phi\colon & \mathbb{R}^l & \to & \mathbb{R}^d \\ & (v_J)_{\|J\|<p} & \mapsto & \exp\left(\sum_{J, \|J\|<p} v_J X^J\right)(x_0) \end{array}$$

ϕ is C^∞ (this is a well-known result about the dependence of the solution of a differential equation on its parameters), and it is clear that

$$\exp(\zeta_t^{\varepsilon, p})(x_0) = \phi((\varepsilon^{\|J\|} c_t^J)_{\|J\|<p})$$

$$\exp(\zeta_t^{(n), \varepsilon, p})(x_0) = \phi((\varepsilon^{\|J\|} c_t^{(n), J})_{\|J\|<p}) .$$

The composition of the Taylor expansions of ϕ on one part, and of the mapping $\varepsilon \mapsto (\varepsilon^{\|J\|} c_t^{(n), J})_{\|J\|<p}$ on the other part, allows us to express $h_i^{(n)}(t)$ as a universal polynomial of the derivatives of ϕ at zero up to the order i, and of the coefficients $(c_t^{(n), J})_{\|J\| \leqq i}$. The expression of $h_i(t)$ can be obtained in a very similar way, by substituting the $c_t^{(n), J}$ by the c_t^J. The convergence of $c_t^{(n), J}$ to c_t^J, already seen in Sect. 3, leads to the convergence of $h_i^{(n)}(t)$ to $h_i(t)$.

Furthermore, the identification of the Taylor expansions of $\exp(\zeta_t^{(n), \varepsilon, p})(x_0)$ and of $\xi_t^{(n), \varepsilon}$ in expression (7) leads to the equality $g_i^{(n)}(t) = h_i^{(n)}(t)$ a.e. By taking the limit, we obtain:

$$\forall i \leqq p-1, h_i(t) = g_i(t) \text{ a.e.}$$

Step 2 It remains now to prove that $\lim_{\varepsilon \to 0} P\left[\sup_{0 \leqq s \leqq T} \|P_p(\varepsilon, s)\| \geqq R\right] \leqq \exp\left(-\frac{R^\alpha}{cT}\right)$.

Let K be a compact set of $\mathbb{R}^l$ containing 0, and let T_K^ε be the infimum of the set $\{t, (\varepsilon^{\|J\|} c_t^J)_{\|J\|<p} \notin K\}$. We are going to show that:

Lemma 4.1 *Let T be strictly positive. Then, $\exists \alpha$, c strictly positive such that*

$$\forall R \geqq c, t \leqq T, \varepsilon \leqq 1, P\left[\sup_{0 \leqq s \leqq t} \|P_p(\varepsilon, s)\| \geqq R; t < T_K^\varepsilon\right] \leqq \exp\left(-\frac{R^\alpha}{cT}\right).$$

Before proving the lemma, we recall the following definitions, first introduced by Azencott in [1]. Let ζ be a stopping time, and let X_t be a stochastic process with values in $\mathbb{R}^l$, which is continuous on $[0; \zeta[$. X is said to be in $\mathscr{W}(\alpha, c, \zeta)$ if and only if

$$\forall R \geqq c, P\left[\sup_{0 \leqq u \leqq t} \|X_t\| \geqq R; t < \zeta\right] \leqq \exp\left(-\frac{R^\alpha}{cT}\right).$$

The following properties are then obvious:

(P1) Let ϕ_t be a continuous process on $[0; \zeta[$ with values in the space of the polynomials of degree less than q, in p Euclidean variables, with coefficients bounded by some constant A on $[0; \zeta[$. The image of $\mathscr{W}(\alpha_1, c_1, \zeta) \times \ldots \times \mathscr{W}(\alpha_p, c_p, \zeta)$ by the mapping

$$(X^1, \ldots, X^p) \mapsto Y, \quad \text{where} \quad Y_t = \phi_t(X_t^1, \ldots, X_t^p),$$

is in some $\mathscr{W}(\alpha, c, \zeta)$, with α, c determined by A, p, q, $\alpha_1, c_1, \ldots, \alpha_p, c_p$.

(P2) If ζ is bounded by some fixed T, the image of $\mathscr{W}(\alpha_1, c_1, \zeta)$ by the mapping $X \mapsto Y$, where $Y_t = \int_0^t X_u dB_u$, is in some $\mathscr{W}(\alpha, c, \zeta)$. This is also the case for the mapping $X \mapsto Y$, $Y_t = \int_0^t X_u du$.

We consider again the map ϕ introduced in step 1, and the identity:

$$\exp(\zeta_t^{\varepsilon, p})(x_0) = \phi((\varepsilon^{\|J\|} c_t^J)_{\|J\| < p}).$$

By composition of the Taylor expansion of ϕ around 0, and of the one of the mapping $\varepsilon \mapsto (\varepsilon^{\|J\|} c_t^J)_{\|J\| < p}$, we obtain, for all $t < T_K^\varepsilon$, and all compact neighborhood L of K in $\mathbb{R}^l$,

$$\exp(\zeta_t^{\varepsilon, p})(x_0) = x_0 + \sum_{i=1}^{p-1} \varepsilon^i h_i(t) + \varepsilon^p P_p(\varepsilon, t),$$

where $\|P_p(\varepsilon, t)\| \leqq C \left\| \frac{\partial^p \phi}{\partial V^p} \right\|_L \|(\varepsilon^{\|J\|-1} c_t^J)_{\|J\| < p})\|^p$.

$X_t = t$, and $X_t = B_t$ (where B_t is a Brownian), are in some $\mathscr{W}(\alpha, c, T \wedge T_K^\varepsilon)$. Property (P2) allows then to say that the iterated integrals B^J are also in some $\mathscr{W}(\alpha, c, T \wedge T_K^\varepsilon)$. Now, $c_t^J = \sum_{\sigma \in \sigma_m} \frac{(-1)^{e(\sigma)}}{m^2 \binom{m-1}{e(\sigma)}} B_t^{J \circ \sigma^{-1}}$ is a linear combination of the B^J. Property (P1) shows then that for $\varepsilon \leqq 1$, $P_p(\varepsilon, t)$ is in some $\mathscr{W}(\alpha, c, T \wedge T_K^\varepsilon)$ (with α et c independent of ε). And the proof of Lemma 4.1 is complete.

Let us fix $T > 0$. Let K be a compact set of $\mathbb{R}^l$ of the form $\prod_{J, \|J\| < p} [-k_J; k_J]$.

$$P\left[\sup_{0 \leqq t \leqq T} \|P_p(\varepsilon, t)\| \geqq R\right] \leqq P\left[\sup_{0 \leqq t \leqq T} \|P_p(\varepsilon, t)\| \geqq R; T < T_K^\varepsilon\right] + P[T \geqq T_K^\varepsilon].$$

According to Lemma 4.1, the first term is bounded up by $\exp\left(-\frac{R^\alpha}{cT}\right)$ for sufficiently large R, and for $\varepsilon \leqq 1$. With regards to the second term, we have, for sufficiently small ε,

$$\begin{aligned} P[T \geqq T_K^\varepsilon] &\leqq \sum_{J, \|J\| < p} P\left[\sup_{0 \leqq t \leqq T} |\varepsilon^{\|J\|} c_t^J| \geqq k_J\right] \\ &\leqq \sum_{J, \|J\| < p} P\left[\sup_{0 \leqq t \leqq T} |c_t^J| \geqq \frac{k_J}{\varepsilon^{\|J\|}}\right] \\ &\leqq \sum_{J, \|J\| < p} \exp\left(-\frac{k_J^{\alpha_J}}{c_J T \varepsilon^{\alpha_J \|J\|}}\right). \end{aligned}$$

This ends the proof of step 2.

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