Probab. Theory Relat. Fields 96, 153–176 (1993)

Probability Theory and Related Fields

Geography of the level sets of the Brownian sheet

Robert C. Dalang [1,*] **and John B. Walsh** [2]

[1] Department of Mathematics, Tufts University, Medford, MA 02155, USA
[2] Department of Mathematics, University of British Columbia, Vancouver, British Columbia, Canada V6T1Y4

Received March 31, 1992; in revised form January 4, 1993

Summary. We describe geometric properties of $\{W>\alpha\}$, where W is a standard real-valued Brownian sheet, in the neighborhood of the first hit P of the level set $\{W>\alpha\}$ along a straight line or smooth monotone curve L. In such a neighborhood we use a decomposition of the form $W(s,t)=\alpha-b(s)+B(t)+x(s,t)$, where $b(s)$ and $B(t)$ are particular diffusion processes and $x(s,t)$ is comparatively small, to show that P is not on the boundary of any connected component of $\{W>\alpha\}$. Rather, components of this set form clusters near P. An integral test for thorn-shaped neighborhoods of L with tip at P that do not meet $\{W>\alpha\}$ is given. We then analyse the position and size of clusters and individual connected components of $\{W>\alpha\}$ near such a thorn, giving upper bounds on their height, width and the space between clusters. This provides a local picture of the level set. Our calculations are based on estimates of the length of excursions of B and b and an accounting of the error term x.

Mathematics Subject Classifications (1991): 60G60, 60G15

1 Introduction

Let $\{W(s,t), s\geqq 0, t\geqq 0\}$ be a standard Brownian sheet. For any real number α, consider the level set

$$L(\alpha)=\{(s,t)\in\mathbb{R}_+^2 : W(s,t)=\alpha\}.$$

This study originated with some geometrical questions about $L(\alpha)$, not all of which are easy to make rigorous. For instance, is $L(\alpha)$ a curve – or a countable union of curves – in the classical sense, or is it like the boundary between light and dark in Escher's "Day and Night" [6]? That woodcut pictures a

* The research of this author was partially supported by NSF grant DMS-9103962, and, during the period of revision, by grant DAAL03-92-6-0323 from the Army Research Office

flock of white geese against the night sky which blends into a flock of black geese flying in the opposite direction in daylight. One side is white on a dark background, the other is black on a light background, and it is difficult to say where the one ends and the other begins.

As posed, this question is probably not answerable – the two alternatives are not mutually exclusive, for example – but Escher's image gives us a way to think of the level sets. They are quite granular in character: $L(\alpha)$ separates the open sets $L^+(\alpha)=\{W>\alpha\}$ and $L^-(\alpha)=\{W<\alpha\}$, and it is easy to see that $L^+(\alpha)$ has many small components, corresponding to local maxima of W which barely exceed α. Each of these is completely surrounded by the set $L^-(\alpha)$. Similarly, there are small components of $L^-(\alpha)$ surrounded by $L^+(\alpha)$. Let us call all these small components *bubbles*. Then we can say that on one side of $L(\alpha)$ is the set $L^+(\alpha)$ containing many bubbles of $L^-(\alpha)$ and on the other side, the set $L^-(\alpha)$ containing bubbles of $L^+(\alpha)$. Because of the irregularity of the sample paths of the Brownian sheet, we expect that each bubble will be surrounded by smaller bubbles, each in turn surrounded by even smaller bubbles, and so on. The bubbles will tend to cluster, then coalesce, at the boundary, so that one can visualize the one set of bubbles passing into the other like Escher's geese.

These considerations are implicit in an article by Kendall [9]. He showed that $L(\alpha)$ is totally disconnected at a typical point, but is not totally disconnected at every point (indeed, a set dividing the plane into two non-empty disjoint open sets cannot be totally disconnected [8]). Kendall chose his "typical" point as follows. Fix $(u, v)\in\mathbb{R}^2_+$. Let $C(u, v)$ be the component of the set $\{(s, t)\colon W(s, t) = W(u, v)\}$ which contains (u, v). He then showed that $C(u, v)$ almost surely reduces to the singleton $\{(u, v)\}$. In this case the point is fixed and the level set is chosen randomly, but making free with Fubini's theorem, this implies that for a.e. α the set $L(\alpha)$ is totally disconnected at almost every one of its points, where the "almost every" is with respect to local time on $L(\alpha)$ [3, 1]. One can see from Kendall's proof that this typical point is a.s. *not* on the boundary of any component of $L^-(\alpha)$ or $L^+(\alpha)$, but is rather a limit point of components – bubbles – of each.

We want to study points on the boundary of one of the connected components of $L^-(\alpha)$, so we will choose a different type of "typical" point as follows. Say $\alpha=1$ for simplicity. Consider the connected component C_0 of $L^-(1)\subset\mathbb{R}^2_+$ which contains the origin (and thus the coordinate axes). Fix $t_0>0$, and proceed from the t-axis along the horizontal line $t=t_0$ until first encountering the boundary of C_0 at the point (S, t_0), where $S=\inf\{s\geqq 0\colon W(s, t_0)\geqq 1\}$. Then the point (S, t_0) is in $L(1)$. It is different from the points studied by Kendall, for it is in the boundary of the component C_0.

We will study the geography of the sets $L^+(1)$ and $L^-(1)$ in a neighborhood of (S, t_0) – the "lay of the land" on a microscopic scale, so to speak. (Figure 1 shows a computer simulation of these sets on a macroscopic scale. Earlier such pictures can be found in Adler's book [2, Chap. 8, Figs. 8.0.1 and 8.0.2].) We first show that $L(1)$ is disconnected at the point (S, t_0) and that (S, t_0) is not on the boundary of any single component of $L^+(1)$, but is rather a limit point of a sequence of distinct components of that set. We then describe how these components cluster around the point (S, t_0): we show in Sect. 3.1 that there is a "thorn-shaped" neighborhood of the segment $L=]0, S[\times\{t_0\}$ which is contained in $L^-(1)$. After this, we concentrate on the distribution and size of the

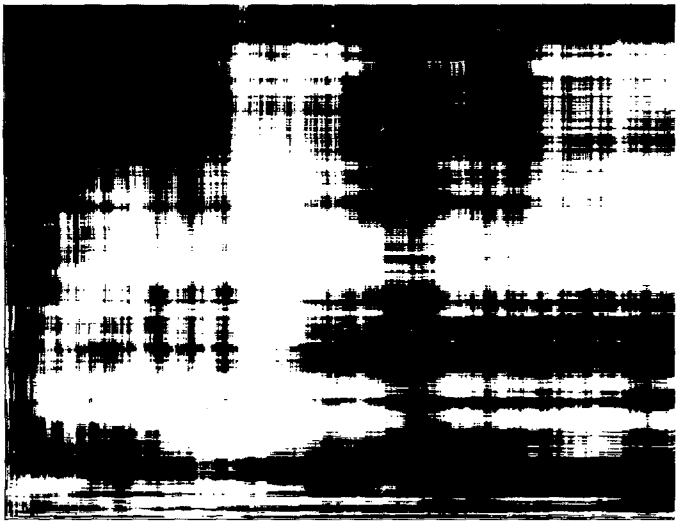

Fig. 1. A level set: the set $\{W>0\}$ is in black

bubbles near (S, t_0). As one approaches (S, t_0) along the segment L, one passes by infinitely many components of $L^+(1)$: if s_n is the s-coordinate of the position where we encounter the n-th component, then $r_n = S - s_n$ converges to 0 at a super-exponential rate (see Theorem 3.7). Bounds on the height and width of theses components are given in Theorems 3.9 and 3.11.

They key to these results is a local decomposition of the Brownian sheet in the neighborhood of (S, t_0) of the form

$$W(s, t) = 1 - b(s) + B(t) + x(s, t),$$

where b and B are particular independent diffusion processes and x is small relative to these two processes. There are four cases to distinguish, depending on the relative positions of (s, t) and (S, t_0) (see (3)–(6)). In the most interesting case, where $s<S$ and $t>t_0$, b is a Bessel process of dimension 3 and B is a standard Brownian motion. It is natural to expect that components of $L^+(1)$ are closely approximated by those of the set $\{B>b\}$. We establish this in a companion paper [5] in which we study the structure of these components.

If we interchange the coordinates s and t in the Brownian sheet, we get another Brownian sheet, so that the same results hold when we approach the level set along a vertical line as when we approach it on a horizontal line. On the other hand, as Fig. 1 makes clear, the Brownian sheet is not invariant under rotations: the horizontal and vertical are distinguished directions, so one might expect that things would be different if we approached the level set along, say, a diagonal line. However, this is *not* the case. We show in Sects. 2.3 and 3.2 that the results are unchanged for lines with positive slope, and the same methods apply to smooth monotone curves. This does not follow directly from the results about horizontal lines, and in fact, though the proofs are similar, we know of no single elegant proof which handles both cases simultaneously.

2 Points of disconnection of level sets

Let $(\Omega, \mathscr{F}, P)$ be a complete probability space and let $\{W(s,t), s \geqq 0, t \geqq 0\}$ be a Brownian sheet defined on this space. Recall [12, Chap. 3] that this is a mean zero continuous Gaussian process with covariance function

$$E\{W(s,t)\, W(u,v)\} = \min(s,u) \min(t,v).$$

Given a rectangle $R = [u_1, u_2] \times [v_1, v_2]$, we set

$$\Delta_R W = W(u_2, v_2) - W(u_2, v_1) - W(u_1, v_2) + W(u_1, v_1).$$

Recall that the map $R \mapsto \Delta_R W$ extends to a σ-additive L^2-valued measure $A \mapsto W(A)$ defined on the bounded Borel subsets of $\mathbb{R}^2_+$, such that $W(A)$ is Gaussian with mean 0 and $E\{W(A)\, W(C)\}$ equals the Lebesgue measure of $A \cap C$. In particular, if $A \cap C = \phi$, then $W(A)$ and $W(B)$ are independent.

For $\alpha > 0$, in addition to the level set $L(\alpha)$ defined in the introduction, we set

$$L^-(\alpha) = \{(s,t) \in \mathbb{R}^2_+ : W(s,t) < \alpha\}, \qquad L^+(\alpha) = \{(s,t) \in \mathbb{R}^2_+ : W(s,t) > \alpha\}.$$

Our objective is to "build a map" of these sets in the neighborhood of a particular point.

2.1 Approaching the boundary: horizontal lines

Fix $t_0 > 0$ and define

$$S = \inf\{s > 0 : W(s, t_0) > 1\}.$$

Theorem 2.1 *Let $\varepsilon > 0$. With probability one, there exists a closed curve lying entirely in $\{(s,t) \in \mathbb{R}^2_+ : W(s,t) < 1\}$ which contains (S, t_0) in its interior and which is contained in a disc of radius ε centered at (S, t_0).*

Remark 2.2 The point (S, t_0) is of course on the boundary of the component of $L^-(1)$ which contains the origin. By Theorem 2.1, there is a sequence of curves in $L^-(1)$ decreasing to (S, t_0); these evidently disconnect the point from $L(1)$, and also from $L^+(1)$. Now the latter set, being open, is the union of a countable number of connected components. The point (S, t_0) is not in the boundary of any single one of them, for otherwise this component would intersect one of the curves. Instead, (S, t_0) must be a limit point of different components of $L^+(1)$, and we can think of $L^+(1)$ – locally at least – as a collection of bubbles clustering about the point (S, t_0). The nature of this clustering is described in Sect. 3 (see also [5]).

We will need some definitions and lemmas before we prove Theorem 2.1, but we can explain the idea right now (this idea was the key to Kendall's analysis). Near (S, t_0), $W(s,t)$ is very nearly a sum of two functions of one variable, say $W(s,t) \sim 1 + f(s) + g(t)$. We then choose $s_1 < S < s_2$ and $t_1 < t_0 < t_2$ such that $f(s_i)$ and $g(t_i)$ are negative, and then let the curve be the boundary of the rectangle $[s_1, s_2] \times [t_1, t_2]$. The sum $f(s) + g(t)$ will be negative at the corners; with some care in choosing the points, we can make it negative on the entire boundary. For ease of reference, this idea is formalized in the following elementary lemma, whose proof is left to the reader. For a real-valued function f on $\mathbb{R}_+$, we define

$$f^*(h) = \sup_{0 \leqq u \leqq h} f(u).$$

Lemma 2.3 *Let $h_1, h_2 > 0$, and consider functions f, g: $[0, 1] \to \mathbb{R}$ and γ: $[0, 1]^2 \to \mathbb{R}$ such that*

$$\max(f(h_1), g(h_2)) + \max(f^*(h_1), g^*(h_2)) + \sup_{0 \leqq u \leqq h_1, 0 \leqq v \leqq h_2} \gamma(u, v) < 0. \tag{1}$$

Then the function $(u, v) \mapsto f(u) + g(v) + \gamma(u, v)$ is strictly negative on the two segments $[0, h_1] \times \{h_2\}$ and $\{h_1\} \times [0, h_2]$.

By Brownian scaling, we can suppose that $t_0 = 1$, and we will assume this from now on. Start at the point $(S, 1)$. Move along any one of the four directions parallel to the axes. The processes we see are Brownian motions in two of the directions, a Brownian bridge in the third, and a Bessel process of "dimension 3" (or Bessel(3) process, for short) in the fourth; all four will be independent. To be more precise, note that $W(S, 1) = 1$ and define, for $u \geqq 0$ and $v \geqq 0$,

$$b(u) = 1 - W(S-u, 1); \qquad c(u) = W(S+u, 1) - 1;$$

$$B(v) = W(S, 1 + v/S) - 1; \qquad C(v) = 1 - \hat{W}(S, 1 + v/S),$$

where $\hat{W}(s, t) = t\,W(s, 1/t)$. Notice that by the time inversion property of the Brownian sheet [12, Chap. 3], $\hat{W}$ is again a Brownian sheet. Let us also define four two-parameter processes x, $\hat{x}$, X and $\hat{X}$ by

$$x(u, v) = \Delta_{R_1(u,v)} W, \qquad \hat{x}(u, v) = \Delta_{R_1(u,v)} \hat{W},$$

$$X(u, v) = \Delta_{R_2(u,v)} W, \qquad \hat{X}(u, v) = \Delta_{R_2(u,v)} \hat{W},$$

where $u, v \geqq 0$ and

$$R_1(u, v) = [S-u, S] \times [1, 1+v], \qquad R_2(u, v) = [S, S+u] \times [1, 1+v].$$

Lemma 2.4 (i) *The processes $\{B(v), v \geqq 0\}$, $\{C(v), v \geqq 0\}$, $\{c(u), u \geqq 0\}$ and $\{b(u), 0 \leqq u \leqq S\}$ are independent. The first three are Brownian motions independent of S and the fourth is equivalent in distribution to a* Bessel(3) *process $\tilde{b}$ restricted to the interval $[0, U]$, where $U = \sup\{u: \tilde{b}(u) = 1\}$.*
(ii) *X and $\hat{X}$ are independent Brownian sheets, and, conditioned on $S = s$, x and $\hat{x}$ are independent Brownian sheets restricted to $[0, s] \times \mathbb{R}_+$.*

Proof. Set $\mathscr{F}(s, t) = \sigma\{W(u, v): 0 \leqq u \leqq s, 0 \leqq v \leqq t\}$. Then S is a stopping time relative to $\{\mathscr{F}(s, \infty), s \geqq 0\}$, the process B is measurable with respect to $\mathscr{F}(S, \infty)$, and c is independent of this σ-field by the strong Markov property applied to S.

Now S is also a stopping point relative to $\{\mathscr{F}(s, 1), s \geqq 0\}$, b and C are measurable with respect to $\mathscr{F}(S, 1)$, and B and c are independent of this σ-field. The remaining independence follows by inverting time in the second variable: $\hat{W}$ equals W on the line $t = 1$, so S is also a stopping time relative to $\{\hat{\mathscr{F}}(s, 1),$

$s \geqq 0\}$. The above argument shows that c and C are independent of $\hat{\mathscr{F}}(S, 1)$, hence of b and B.

It follows that all four processes are independent. The definition of the Brownian sheet implies that the first three are Brownian motions, and the fact that b is a Bessel(3) process follows from Williams' decomposition of the Brownian excursion [11, Chap. XII (4.4)]. We get (ii) from similar considerations. $\square$

Define

$$M(h, k) = \sup\{|x(u, v)|, |X(u, v)|, |\hat{x}(u, v)|, |\hat{X}(u, v)| : u \leqq h, v \leqq k\}.$$

An upper bound on M is given by the law of the iterated logarithm for the Brownian sheet [12, Theorem 3.5.A]:

$$a < 1/2 \Rightarrow \lim_{\varepsilon \to 0} \sup_{0 \leqq h \leqq \varepsilon, 0 \leqq k \leqq \varepsilon} (h k)^{-\alpha} M(h, k) = 0. \tag{2}$$

Proof of Theorem 2.1 Recall that we have set $t_0 = 1$. For $h > 0$, consider the rectangle

$$R_h = [S - h, S + h] \times [1/(1 + h/S), 1 + h/S]$$

and the events $\Gamma_h = \{W < 1 \text{ on } \partial R_h\}$ and

$$\Lambda_h = \{\max(-b, B, -C, c)(h) < -5\sqrt{h}, \max(|B|^*, |C|^*, |c|^*)(h) < \sqrt{h}\}.$$

Note that $p \sim P(\Lambda_h)$ is strictly positive and independent of h by Brownian scaling. Let (h_n) be any sequence which decreases to 0. By Fatou's Lemma, $P(\limsup \Lambda_{h_n}) \geqq p > 0$. Then Blumenthal's zero-one law for the four dimensional diffusion (b, B, c, C) implies that $P(\limsup \Lambda_{h_n}) = 1$.

We claim that for a.e. $\omega \in \Omega$, there exists $\eta(\omega) > 0$ such that if $0 < h < \eta(\omega)$, then $\omega \in \Lambda_h$ implies $\omega \in \Gamma_h$. This will complete the proof, since then $P(\limsup \Gamma_{h_n}) \geqq P(\limsup \Lambda_{h_n}) = 1$. In order to verify the claim, for $\omega \in \Omega$, let $\eta(\omega)$ be small enough so that if $\max(h, k) \leqq \eta(\omega)$, then $M(h, k) < (h k)^{1/4}$ (such an $\eta(\omega)$ exists by (2)).

Fix $\omega \in \limsup \Lambda_{h_n}$ and choose $n \in \mathbb{N}$ such that $h_n < \eta(\omega)$ and $\omega \in \Lambda_{h_n}$. Let us show that $\omega \in \Gamma_{h_n}$. We shall omit the subscript n below. For $u, v > 0$, it is immediate to check that

$$W(S - u, 1 + v/S) = 1 - b(u) + B(v) - x(u, v/S), \tag{3}$$

$$W(S + u, 1 + v/S) = 1 + c(u) + B(v) + X(u, v/S). \tag{4}$$

It is also straightforward, but a bit lengthier, to check that

$$\begin{aligned} W(S + u, 1/(1 + v/S)) = 1 &+ c(u) - C(v) + \hat{X}(u, v/S) \\ &- (v/S)\, W(S + u, 1/(1 + v/S)), \end{aligned} \tag{5}$$

$$\begin{aligned} W(S - u, 1/(1 + v/S)) = 1 &- b(u) - C(v) - \hat{x}(u, v/S) \\ &- (v/S)\, W(S - u, 1/(1 + v/S)). \end{aligned} \tag{6}$$

Let us consider the restriction of W to the two segments

$$\partial_1 R_h = \partial R_h \cap \{(s, t) : 0 \leqq s \leqq S, t \geqq 1\}.$$

We apply Lemma 2.3 with $h_1 = h_2 = h$, $f(u) = -b(u;\omega)$, $g(u) = B(u;\omega)$ and $\gamma(u, v) = -x(u, v/S;\omega)$. Given our choice of h and since $\omega \in \Lambda_h$, condition (1) is satisfied since the left-hand side of the inequality is less than $-5\sqrt{h} + \max(0, \sqrt{h}) + \sqrt{h} < 0$. It follows from Lemma 2.3 that $W(\cdot;\omega) < 1$ on $\partial_1 R_h$.

The argument in the other three quadrants is similar. For instance, on

$$\partial_3 R_h = \partial R_h \cap \{(s, t): s \geqq S, 0 \leqq t \leqq 1\},$$

we apply Lemma 2.3 with $h_1 = h_2 = h$, $f(u) = c(u;\omega)$, $g(u) = -C(u;\omega)$ and

$$\gamma(u, v) = \hat{X}(u, v) - (v/S)\, W(S+u, 1/(1+v/S)).$$

Condition (1) is satisfied since the left-hand side of the inequality is less than $-5\sqrt{h} + \sqrt{h} + \sqrt{h} < 0$. It follows again from Lemma 2.3 that $W(\cdot;\omega) < 1$ on $\partial_3 R_h$. The proof of the remaining two cases is similar and is left to the reader. □

2.2 *Surrounding* $(S, 1)$

In [9], Kendall showed that a "typical" point in a level set is disconnected from the rest of the level set in a strong way: there exist two sets of closed curves containing the point in their interior and decreasing to this point, such that $W > 1$ on one set of curves and $W < 1$ on the other. The point $(S, 1)$ is not of this type: Theorem 2.1 shows that there are indeed curves around it on which $W < 1$, but it cannot be in the interior of a closed curve on which $W > 1$, since the segment $[0, S[\times \{1\}$ is contained in $L^-(1)$. There are two natural questions to ask about the contrast between these two types of points. First, to what extent does the set $L^+(1)$ "surround" the point $(S, 1)$? Second, how close does the set $L^+(1)$ come to the line segment $[0, S[\times \{1\}$? We will address the first question here, and leave the second to Sect. 3.1.

We can rephrase the first question as follows. What kind of curves in $L^+(1)$, connected but not necessarily closed, lie near $(S, 1)$? We will show that the point can be "surrounded" on its right hand side by such curves.

More exactly, for each $h > 0$ let Γ_h be the square of side $2h$ centered at the point $(S, 1)$ and let γ_h be the curve consisting of the top, right hand, and bottom sides of Γ_h (so γ_h consists of three sides of the square, open on the left). Then we have the following result.

Theorem 2.5 *With probability one, there exists a sequence $a_n \downarrow 0$ such that W is strictly greater than* 1 *on each γ_{a_n}.*

Proof. For a process $Y(t)$, let $Y_*(t) = \inf\{Y(v): 0 \leqq v \leqq t\}$. Set $\Gamma_h = \{W > 1 \text{ on } \gamma_h\}$ and

$$\Lambda_h = \{\min(B, c, -C)(h) > 5\sqrt{h},\ \min(-b^*, B_*, c_*, (-C)_*)(h) > -\sqrt{h}\}.$$

As in the proof of Theorem 2.1, note that $P(\Lambda_h)$ is strictly positive and independent of h, and thus $P(\limsup \Lambda_{h_n}) = 1$ if $h_n \downarrow 0$. And as in that proof, we only need to show that for a.e. $\omega \in \Omega$, there is $\eta(\omega) > 0$ such that if $0 < h < \eta(\omega)$, then

$\omega \in \Lambda_h$ implies $\omega \in \Gamma_h$. Using (2), we choose $\eta(\omega)$ so that $\max(h, k) < \eta(\omega)$ implies $M(h, k) < (hk)^{1/4}$.

Fix $\omega \in \limsup \Gamma_{h_n}$ and let $n \in \mathbb{N}$ be such that $h_n < \eta(\omega)$ and $\omega \in \Lambda_{h_n}$. We show that $\omega \in \Gamma_{h_n}$. We will omit the subscript n below. Looking at the decompositions (4) and (5) of W near $(S, 1)$, it is clear that a slight variation on Lemma 2.3 implies that $W(\cdot\,; \omega) > 1$ on $\Gamma_h \cap \{(s, t)\colon s \geqq S,\ t \geqq 0\}$. In order to handle the segment $[S-h, S] \times \{1+h\}$, we use (3). The lower bound on $B(h)$, together with the lower bound on $-b^*(h)$ and $-x(u, v/S)$ and another slight variation on Lemma 2.3 show that $W(\cdot\,; \omega) < 1$ on this segment. The last segment is handled in a similar fashion using (6). □

2.3 *Approaching the boundary: other lines*

By symmetry, the result of Theorem 2.1 will hold if we approach the boundary of C_0 along a vertical rather than a horizontal line. Since most properties of the Brownian sheet depend strongly upon orientation, it is not obvious that the conclusion of Theorem 2.1 remains valid if we approach the boundary of C_0 along lines with positive slope. We will show here how to modify the arguments of Sect. 2.1 to handle this situation.

We can reduce the problem to one special case. If $a > 0$, and if $\bar{W}$ is defined by $\bar{W}(s, t) = W(s\sqrt{a}, t/\sqrt{a})$, then $\bar{W}$ is a Brownian sheet. Thus to understand the behavior of $\bar{W}$ along the line $t = as$, it is sufficient to study the behavior of W along the diagonal $t = s$. We will limit ourselves to this case. Let $S = \inf\{s\colon W(s, s) \geqq 1\}$.

Theorem 2.6 *Let $\varepsilon > 0$. With probability one, there exists a closed curve lying entirely in $\{(s, t) \in \mathbb{R}_+^2\colon W(s, t) < 1\}$ which contains (S, S) in its interior and which is contained in a disc of radius ε centered at (S, S).*

The key to Theorem 2.1 was the approximation of W by a sum of two one-parameter processes. We will use the same idea but need different processes here. Set $\varphi(u) = u^{\frac{1}{2}}$ and

$$R_u^1 = \{(s, t)\colon 0 \leqq t \leqq s \leqq u\}, \qquad R_u^2 = \{(s, t)\colon 0 \leqq s \leqq t \leqq u\}.$$

Consider the stochastic processes

$$B^1(u) = W(R^1_{\varphi(u)}), \qquad B^2(u) = W(R^2_{\varphi(u)}).$$

Lemma 2.7 *Set $X(u) = B^1(u) + B^2(u)$ and $Y(u) = B^1(u) - B^2(u)$. Then $X = \{X(u),\ u \geqq 0\}$ and $Y = \{Y(u),\ u \geqq 0\}$ are independent Brownian motions.*

Proof. Clearly both X and Y are continuous Gaussian processes of independent increments and therefore are continuous martingales. Standard properties of the Brownian sheet show that $E\{X(u)^2\} = E\{Y(u)^2\} = u$, and a simple calculation shows that $E\{X(u)\,Y(v)\} = 0$, so these two Gaussian processes are independent Brownian motions. □

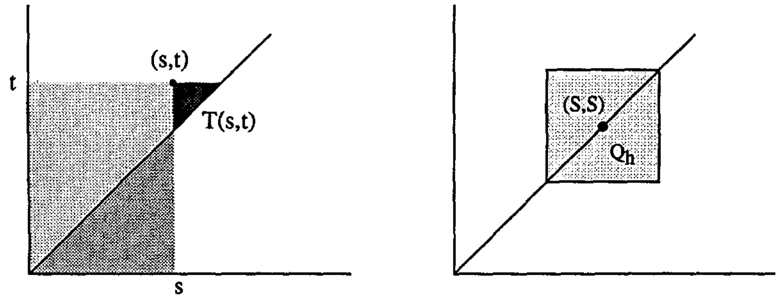

Fig. 2. Decomposition of W and the square Q_h

Note that along the diagonal, $W(t,t)=X(t^{\frac{1}{2}})$. Set $\sigma=\inf\{u\geqq 0: X(u)=1\}$ and for $u\geqq 0$ define

$$b(u)=X(\sigma)-X(\sigma-u), \qquad c(u)=X(\sigma+u)-X(\sigma),$$
$$B(u)=\sigma^{-\frac{1}{2}}(Y(\sigma)-Y(\sigma-\sigma u)), \quad C(u)=\sigma^{-\frac{1}{2}}(Y(\sigma+\sigma u)-Y(\sigma)).$$

Lemma 2.8 *The processes* $\{b(u), 0\leqq u\leqq\sigma\}$, $\{c(u), u\geqq 0\}$, $\{B(u), 0\leqq u\leqq 1\}$ *and* $\{C(u), 0\leqq u\leqq 1\}$ *are independent. The first is a* Bessel(3) *process run until it last hits one, and the other three are Brownian motions.*

Proof. The fact that b is a Bessel(3) process follows directly from [11, Chap. XII (4.4)], and the fact that c and C are Brownian motions follows from Lemma 2.7 and the strong Markov property. Independence of these two processes is checked straightforwardly by using the definition of independence and conditioning on σ. Observe that the conditional law of B given $\sigma=s$ is the same as that of $s^{-\frac{1}{2}}(Y(s)-Y(s-su))$, which is that of Brownian motion. Since this conditional distribution does not depend on s, B itself is a Brownian motion.

To see the independence, notice that c and C are independent of the pre-σ sigma-field of X and Y, and hence of b and B, which are measurable with respect to this sigma-field. On the other hand, for all choices of n, times $u_0<\dots<u_n$ and Borel sets $A_0,\dots,A_n$, we have

$$\begin{aligned}P\{&B(u_0)\in A_0, B(u_k)-B(u_{k-1})\in A_k, k=1,\dots,n\,|\,\sigma(X)\}\\ &=P\{\sigma^{-\frac{1}{2}}(Y(\sigma)-Y(\sigma-\sigma u_0))\in A_0, \sigma^{-\frac{1}{2}}(Y(\sigma-\sigma u_{k-1})\\ &\quad -Y(\sigma-\sigma u_k))\in A_k, k=1,\dots,n\,|\,X(u), u\geqq 0\}\\ &=g(\sigma),\end{aligned}$$

where

$$\begin{aligned}g(s)&=P\{s^{-\frac{1}{2}}(Y(s)-Y(s-su_0))\in A_0, s^{-\frac{1}{2}}(Y(s-su_{k-1})\\ &\qquad -Y(s-su_k))\in A_k, k=1,\dots,n\}\\ &=P\{Y(u_0)\in A_0, Y(u_k)-Y(u_{k-1})\in A_k, k=1,\dots,n\}.\end{aligned}$$

Therefore $g(s)$ does not depend on s. It follows that the conditional probability above is constant, which proves that B and $\sigma(X)$ are independent. Thus all four processes are independent. □

For $(s,t)\in\mathbb{R}^2_+$, let $T(s,t)$ be the right triangle whose apex is at (s,t) and whose hypotenuse is on the main diagonal (see Fig. 2). Let $z(s,t)=W(T(s,t))$ and note that

$$\begin{aligned}(7)\qquad W(s,t)&=B^1(s^2)+B^2(t^2)-z(s,t)\\ &=\tfrac{1}{2}(X(s^2)+Y(s^2)+X(t^2)-Y(t^2))-z(s,t).\end{aligned}$$

The processes X and Y can be expressed in terms of b, B, c, and C, but the expression depends on the values of s and t. If $s<S$, then

(8) $$X(s^2)=X(\sigma)-b(\sigma-s^2) \quad \text{and} \quad Y(s^2)=Y(\sigma)-\sigma^{\frac{1}{2}}B(1-s^2/\sigma),$$

while if $s>S$, then

(9) $$X(s^2)=X(\sigma)+c(s^2-\sigma) \quad \text{and} \quad Y(s^2)=Y(\sigma)+\sigma^{\frac{1}{2}}C(s^2/\sigma-1).$$

Finally, the following supplies a global bound on the error term $z(s,t)$, which will play the role of (2) in Sect. 2.1. For each $1>\delta>0$ and $K>0$,

(10) $$\lim_{\varepsilon\to 0}\varepsilon^{-(1-\delta)}\sup\{|z(s,t)|: s+t\leqq 2K, |s-t|<\varepsilon\}=0 \quad \text{a.s.}$$

This is a consequence of [7, Theorem 2], which implies that in the region where $|s-t|<\eta$ and $s+t\leqq 2K$, the modulus of continuity $\omega(\varepsilon)$ of z is bounded by $\sqrt{\eta}\,\varepsilon^{\frac{1}{2}-\delta}$, for any $\delta>0$. Thus, taking $\eta=\varepsilon$, (10) follows.

Proof of Theorem 2.6 The curves around (S,S) on which $W<1$ will again be rectangles, with sides parallel to the axes rather than at 45° angles. Set $Q_h=[\sqrt{\sigma-h},\sqrt{\sigma+h}]^2$. Let $\Gamma_h=\{W<1 \text{ on } \partial Q_h\}$ and

$$\Lambda_h=\{\max(-b,c)(h)<-5\sqrt{h}, \max(|c|^*(h), \sigma^{\frac{1}{2}}|B|^*(h/\sigma), \sigma^{\frac{1}{2}}|C|^*(h/\sigma))<\sqrt{h}\}.$$

As in the proof of Theorem 2.1, note that $p\equiv P(\Lambda_h)$ is strictly positive and independent of h, and this implies $P(\limsup\Lambda_{h_n})=1$ if $h_n\downarrow 0$.

The remainder of the proof is also similar to that of Theorem 2.1. This will be clear once we write out the analogues of (3), (4), (5) and (6). By (7), (8) and (9), the first two become

(11) $$W^{-,+}(u,v)=1-(b(u)+\sigma^{\frac{1}{2}}B(u/\sigma))/2+(c(v)-\sigma^{\frac{1}{2}}C(v/\sigma))/2-z^{-,+}(u,v)$$

(12) $$W^{+,+}(u,v)=1+(c(u)+\sigma^{\frac{1}{2}}C(u/\sigma))/2+(c(v)-\sigma^{\frac{1}{2}}C(v/\sigma))/2-z^{+,+}(u,v)$$

and the last two are

(13) $$W^{+,-}(u,v)=1+(c(u)+\sigma^{\frac{1}{2}}C(u/\sigma))/2-(b(v)-\sigma^{\frac{1}{2}}B(v/\sigma))/2-z^{+,-}(u,v)$$

(14) $$W^{-,-}(u,v)=1-(b(u)+\sigma^{\frac{1}{2}}B(u/\sigma))/2-(b(v)-\sigma^{\frac{1}{2}}B(v/\sigma))/2-z^{-,-}(u,v),$$

where

$$W^{\pm,\pm}(u,v)=W(\sqrt{\sigma\pm u},\sqrt{\sigma\pm v}), \qquad z^{\pm,\pm}(u,v)=z(\sqrt{\sigma\pm u},\sqrt{\sigma\pm v}).$$

For example, in case (11), we would apply Lemma 2.3 with $h_1=h_2=h$,

$$f(u)=-(b(u)+\sigma^{\frac{1}{2}}B(u/\sigma))/2, \qquad g(u)=(c(v)-\sigma^{\frac{1}{2}}C(v/\sigma))/2,$$

and $\gamma(u,v)=z(\sqrt{\sigma-u},\sqrt{\sigma+v})$. Condition (1) is satisfied since by (10), for small h, the left hand-side of the inequality is $<-4\sqrt{h}/2+2\sqrt{h}/2+\sqrt{h}=0$. The other cases are similar and are left to the reader. $\square$

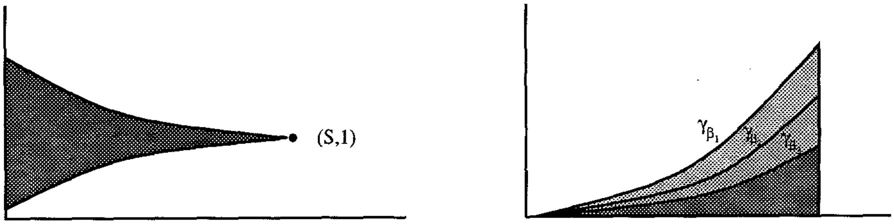

Fig. 3. A thorn, and the sets $T_{\psi_{\beta_i}}$ and curves $\gamma_{\beta_i}=\gamma_{\psi_{\beta_i}}$, $\beta_1<\beta_2<\beta_3$

Remark 2.9 The result of Theorem 2.6 remains valid if one approaches the boundary of C_0 along smooth monotone curves, say. This can be checked using the same methods as above.

3 Local geography of the level set

In order to understand the geography of $L(1)$ in the neighborhood of $(S, 1)$, where

$$S=\inf\{s>0\colon W(s, 1)=1\},$$

we consider $W(S-s, 1+t)$ for small s and t. It is clearly smaller than 1 if $t=0$ and $0<s<S$, and by continuity it must be smaller than 1 in some neighborhood of the segment $[0, S[\times\{1\}$. What does this neighborhood look like? It can be rather complicated – it will not be simply connected, for instance – but we can ask whether or not there is a neighborhood of a simpler character. The class of convex neighborhoods is not appropriate, for any convex neighborhood of the segment intersects $\{W>1\}$ (see Theorem 3.1(b) below). The natural neighborhoods to consider are shaped like a thorn, with the tip at $(S, 1)$ (see the first picture in Fig. 3). The purpose of this section is to determine the exact shape of these particular neighborhoods and to describe the size and position of the connected components of $\{W>1\}$ just beyond the thorn. The shape of individual clusters and components is described in the companion paper [5].

If A is an event in $\sigma\{b\}$ (resp. $\sigma\{B\}$), then $P^c(A)$ denotes the conditional probability of A given that $b(0)=c$ (resp. $B(0)=c$).

3.1 *The thorn: horizontal lines*

It is easiest to visualize neighborhoods of $[0, S[\times\{1\}$ if we translate the origin to the point $(S, 1)$ and reverse the s-axis: let $\tilde{W}(s, t)=W(S-s, 1+t/S)-1$. Clearly, $\{W>1\}$ corresponds to $\{\tilde{W}>0\}$. By (3),

$$\tilde{W}(s, t)=B(t)-b(s)-x(s, t/S), \qquad s>0,\ t>0, \tag{15}$$

where B is a standard Brownian motion starting from 0, b is a Bessel(3) process starting at 0 and independent of B and x is comparatively small by (2). By (6), a similar decomposition holds for $s>0$ and $t<0$. We only study the first case, since it is then not difficult to check that the results are not affected by the additional error term in (6), and so it is sufficient to consider thorns which are symmetric with respect to the segment $[0, S[\times\{1\}$.

Let $\tau(s)$ be a continuous non-decreasing function on $\mathbb{R}_+$ with $\tau(0)=0$ and $\tau(s)>0$ if $s>0$. Let T_τ be the thorn-shaped neighborhood $T_\tau=\{(s, t)\colon s\geqq 0, 0\leqq t \leqq\tau(s)\}$ and let γ_τ be the upper boundary curve of T_τ, i.e. the graph of $t=\tau(s)$. We say that T_τ is *initially in* $\{\tilde{W}<0\}$ if with probability one, there exists $\eta>0$ such that $T_\tau\cap\{(s, t)\colon 0\leqq s\leqq\eta, t\geqq 0\}\subset\{\tilde{W}<0\}$.

Theorem 3.1 *Suppose that* $s\mapsto\tau(s)/s$ *is increasing for* $s>0$. *Then*
(a) T_τ *is initially in* $\{\tilde{W}<0\}$ *if and only if*

$$I(\tau)=\int_{0+}\left(\frac{\tau(s)}{s}\right)^{\frac{1}{2}}\frac{ds}{s}<\infty.$$

(b) *If this integral is infinite, then* γ_τ *is* not *initially in* $\{\tilde{W}<0\}$. *Moreover, for all* $\kappa>0$, *there is a sequence* $(s_n)\downarrow 0$ *such that* $\tilde{W}(s_n), \tau(s_n)>\kappa\,\tau(s_n)^{\frac{1}{2}}$, *for all n.*

Remark 3.2 (a) For $\beta\geqq 0$, let

$$\psi_\beta(s)=s\left(\log\frac{1}{s}\right)^{-2}\left(\log\log\frac{1}{s}\right)^{-\beta}.$$

Then $s\mapsto\psi_\beta(s)/s$ is increasing and $I(\psi_\beta)<+\infty$ if and only if $\beta>2$.
(b) If we had set $\tilde{W}(s, t)=W(1-s, 1+t)-1$, Theorem 3.1 would remain valid. Indeed, (15) would become $\tilde{W}(s, t)=B(St)-b(s)-x(s, t)$. But the distribution of $B(S\,\cdot)$ is the same as that of $S^{\frac{1}{2}}B(\cdot)$, and the factor $S^{\frac{1}{2}}$ clearly does not affect the calculation below.
(c) For later reference, we point out that a proof similar to that of Theorem 3.1(b) shows in fact that for all $\kappa>0$, the events

$$\{\tilde{W}(v_n, \psi_\beta(e^{-n}))>\kappa\,\psi_\beta(e^{-n})^{\frac{1}{2}}, b(v_n)<\psi_\beta(e^{-n})^{\frac{1}{2}}\}$$

occur infinitely often, where v_n is the unique time in $[e^{-n}, e^{1-n}]$ such that $b(v_n) = \min_{u\in[e^{-n}, e^{1-n}]} b(u)$.

The proof of Theorem 3.1 uses the following property of the Bessel(3) process b.

Lemma 3.3 *There is a positive constant k such that*

$$P\{\min_{1\leqq s\leqq e} b(s)<a\}\leqq k\,a, \textit{for all } a>0,$$

and a positive constant k' *such that* $P\{\min_{1\leqq s\leqq e} b(s)<a\}\geqq k' a$ *for all sufficiently small a.*

Proof. The first statement follows immediately from [11, Chap. 6, Corollary 3.4], since using the explicit density of $b(1)$ (see (22)), the probability in question is

$$\leqq P\{\min_{s\geqq 1} b(s)<a\}=a\,E\{1/b(1)\}<\infty.$$

Concerning the second statement, let α be a random variable with a uniform distribution on $[0, c]$, $c>0$, and set $T(a)=\inf\{t\geqq 1: \bar{B}(t)=a\}$, where $\bar{B}$ is a Brownian motion independent of α. We apply a path decomposition theorem [11, Chap. 6, Theorem 3.11] to see that

$$\begin{aligned} P^c\{\min_{0\leqq s\leqq e-1} b(s)\geqq a\} &= P^c(\{\alpha\geqq a\}\cup\{\alpha\leqq a, T(a)>e-1\}) \\ &= 1-a/c+(a/c)\,P^c\{T(a)>e-1\} \\ &\leqq 1-a/c+(a/c)\,P^c\{T(0)>e-1\}. \end{aligned}$$

Thus

$$P^c\{\min_{0\leqq s\leqq e-1} b(s)\leqq a\}\geqq (a/c)\,P^c\{T(0)<e-1\}.$$

Integrating with respect to the density of $b(1)$ and using the Markov property of b gives the conclusion. □

Proof of Theorem 3.1 First suppose that the integral is finite. We claim that T_τ is initially in $\{\tilde{W}<0\}$. Set $\phi(s)=\tau(s)/s$, put $\varepsilon_n=e^{(1-n)(1-\delta)}$, where $0<\delta<1/4$, and let

$$m_n=\min\{b(s): e^{-n}\leqq s\leqq e^{1-n}\}, \qquad A_n=\{m_n\leqq B^*(\tau(e^{1-n}))+\varepsilon_n\}.$$

Now $t<s$ in a neighborhood T_τ, so by (2), for all large enough n and $s\in[e^{-n}, e^{1-n}]$, we have $|x(s,t)|\leqq (s\vee t)^{1-\delta}\leqq\varepsilon_n$. If A_n does not occur, then

$$s\in[e^{-n}, e^{1-n}],\ t\leqq\tau(e^{1-n}) \Rightarrow B(t)-b(s)+\varepsilon_n<0$$

since τ is non-decreasing. Thus $\tilde{W}(s,t)\leqq B(t)-b(s)+\varepsilon_n<0$. It follows that

$$T_\tau\cap([e^{-n}, e^{1-n}]\times[0, \tau(e^{1-n})])\subset\{\tilde{W}<0\} \qquad \text{on } \Omega\setminus A_n.$$

So it will be sufficient to prove that A_n occurs only finitely often. Since b and B are independent by Lemma 2.4, Brownian scaling yields

$$\begin{aligned} P(A_n) &= P\{e^{-n/2}\, m_0\leqq \tau(e^{1-n})^{\frac{1}{2}}\, B^*(1)+\varepsilon_n\} \\ &= P\{m_0\leqq K\,\phi(e^{1-n})^{\frac{1}{2}}\, B^*(1)+e^{n/2}\,\varepsilon_n\} \end{aligned}$$

where K denotes a constant whose value may change from line to line, and we have used the fact that $e^n\,\tau(e^{1-n})=e\,\phi(e^{1-n})$. Since $\delta<1/4$, $e^{n/2}\,\varepsilon_n\ll e^{-n/4}$ for large n, so that by Lemma 3.3, the last expression is

$$\begin{aligned} &\leqq K\int_0^\infty \phi(e^{1-n})^{\frac{1}{2}}\, a\, P\{B^*(1)\in d a\}+K\,e^{-n/4} \\ &= K\,\phi(e^{1-n})^{\frac{1}{2}}\, E\{B^*(1)\}+K\,e^{-n/4}. \end{aligned}$$

Now sum this over n. The sum of the exponentials is finite and ϕ is monotone, so we can estimate the sum as an integral to see that

$$\sum_{n=1}^\infty P(A_n)<\infty \Leftrightarrow \int_1^\infty \phi(e^{1-x})^{\frac{1}{2}}\, d x<\infty \Leftrightarrow \int_0^1 \phi(s)^{\frac{1}{2}}\, s^{-1}\, d s<\infty,$$

which, thanks to the Borel-Cantelli Lemma, proves our claim. This implies the "if" part of (a).

Now consider the process along the top boundary γ_τ of T_τ. Suppose the integral in (a) is infinite, and fix $\kappa>0$. We claim there exist $s_n \to 0$ such that $\tilde{W}(s_n, \tau(s_n)) > \kappa \tau(s_n)^{\frac{1}{2}}$. This will prove (b) and the "only if" part of (a) at the same time. Define $\tilde{b}(u)=e^{u/2}\, b(e^{-u})$. Then $b(s)=s^{\frac{1}{2}}\,\tilde{b}(\log(1/s))$ and

$$\tilde{W}(s, \tau(s)) > \kappa \tau(s_n)^{\frac{1}{2}} \Leftrightarrow B(\tau(s)) > \kappa \tau(s_n)^{\frac{1}{2}} + s^{\frac{1}{2}}\, \tilde{b}(\log(1/s)) + x(s, \tau(s)).$$

So it will be sufficient to show that there is a sequence $u_n \uparrow \infty$ such that

$$B(\tau(e^{-u_n})) > \kappa \tau(e^{-u_n})^{\frac{1}{2}} + e^{-u_n/2}\, \tilde{b}(u_n) + x(e^{-u_n}, \tau(e^{-u_n})). \tag{16}$$

Fix $0<C<D$ and let $\rho_0 = \inf\{u>0: \tilde{b}(u)=D\}$ and, having defined $\rho_0, \sigma_1, \ldots, \sigma_n, \rho_n$, put

$$\sigma_{n+1} = \inf\{u>\rho_n: \tilde{b}(u)=C\}, \qquad \rho_{n+1} = \inf\{u>\sigma_{n+1}: \tilde{b}(u)=D\}.$$

Since $\tilde{b}$ is a stationary Markov process, $(\sigma_{n+1}-\sigma_n, n\in\mathbb{N})$ and $(\rho_{n+1}-\rho_n, n\in\mathbb{N})$ are both iid sequences with finite mean [10, Sect. 5.4, p. 145], and, in particular, $\sigma_n < \rho_n < \infty$, for all $n\in\mathbb{N}$. Let v_n be the (unique) time at which $\tilde{b}$ takes on its minimum in $[\sigma_n, \rho_n]$ and define

$$F_n = \{\min_{\sigma_n \leq u \leq \rho_n} \tilde{b}(u) \leqq \phi(e^{-v_n})^{\frac{1}{2}} - e^{-v_n(\frac{1}{2}-\delta)}\},$$

$$G_n = \{B(\tau(e^{-v_n})) \geqq (\kappa+1)\, \tau(e^{-v_n})^{\frac{1}{2}}\}.$$

On $F_n \cap G_n$, there is $u_n \in [\sigma_n, \rho_n]$ such that (16) holds, namely the time $u_n = v_n$. So we only need to show that $F_n \cap G_n$ occurs infinitely often. This is just a property of the law of the independent processes $\tilde{b}$ and B, which we now prove, without using the fact that these two processes come from a Brownian sheet.

We can assume that $\tilde{b}$ and B are canonically defined on the product of two canonical probability spaces $(\Omega_1, \mathscr{F}_1, P_1)$ and $(\Omega_2, \mathscr{F}_2, P_2)$, i.e. $\tilde{b}(\omega_1, \omega_2)$ (resp. $B(\omega_1, \omega_2)$) does not depend on ω_2 (resp. ω_1). We can also assume that $F_n \in \mathscr{F}_1$, and replace $F_n \cap G_n$ by $(F_n \times \Omega_2) \cap G_n$.

Observe that the events F_n depend only on $\tilde{b}$ and are independent of each other, and, by the strong law of large numbers, that $\rho_n/n \to \alpha > 0$. In addition, setting $\tilde{m}_0 = \min\{\tilde{b}(u): 0 \leqq u \leqq \rho_1\}$, letting P_1^C denote conditional probability given $\tilde{b}_0 = C$ and using Lemma 3.3, we see that

$$\begin{aligned} P_1(F_n) &= P_1^C\{\tilde{m}_0 \leqq \phi(e^{-v_n})^{\frac{1}{2}} - e^{-v_n(\frac{1}{2}-\delta)}\} \\ &\geqq P_1^C\{\tilde{m}_0 \leqq \phi(e^{-(\alpha+1)n})^{\frac{1}{2}} - e^{\alpha(\frac{1}{2}-\delta)n}\} \\ &\geqq K(\phi(e^{-(\alpha+1)n})^{\frac{1}{2}} - e^{\alpha(\frac{1}{2}-\delta)n}). \end{aligned}$$

Since the sum of the exponentials is finite, it follows that

$$\sum_{n\in\mathbb{N}} P_1(F_n) \geqq \int_1^\infty K\, \phi(e^{-(\alpha+1)u})^{\frac{1}{2}}\, du + K' = \tilde{K} \int_{0+} \phi(s)^{\frac{1}{2}}\, s^{-1}\, ds + K' = +\infty.$$

By the converse of the Borel-Cantelli Lemma, $P_1(\limsup_{n\in\mathbb{N}} F_n) = 1$.

Now fix $\omega_1 \in \limsup F_n$. There is a sequence $(n_k(\omega_1), k \in \mathbb{N})$ increasing to $+\infty$ such that $\omega_1 \in F_{n_k(\omega_1)}$, for all $k \in \mathbb{N}$. The ω_1-section of $G_{n_k(\omega_1)}$ is

$$G_{n_k}(\omega_1) = \{\omega_2 \in \Omega_2 : B(\tau(e^{-\nu_{n_k}(\omega_1)})) \geqq (\kappa+1)\, \tau(e^{-\nu_{n_k}(\omega_1)})^{\frac{1}{2}}\}.$$

Now, $P_2(G_{n_k}(\omega_1)) = P_2\{B(1) \geqq \kappa + 1\} > 0$ and $\tau(e^{-\nu_{n_k}(\omega_1)}) \downarrow 0$. Since

$$P_2(\limsup_{k \in \mathbb{N}} G_{n_k}(\omega_1)) \geqq \limsup_{k \in \mathbb{N}} P_2(G_{n_k}(\omega_1)) > 0,$$

it follows by the 0–1 Law that $P_2(\limsup G_{n_k}(\omega_1)) = 1$. Thus P_1-almost all sections of $\limsup(F_n \times \Omega_2) \cap G_n$ have full P_2-measure, hence the $P_1 \times P_2$-measure of this set is 1. The proof is complete. □

3.2 The thorn: other lines

We would like to know whether the results of Sect. 2.3 hold when we approach the set $\{W > 1\}$ along lines which are not horizontal or vertical. Specifically, we want to know about thorns along lines with positive slope. We will use the setting and notation of Sect. 2.3. As mentioned there, we need only consider the main diagonal.

Let us rotate coordinates by setting $\xi = 2^{-\frac{1}{2}}(t+s)$ and $\eta = 2^{-\frac{1}{2}}(t-s)$, so that the ξ-axis lies along the main diagonal. Define $\hat{W}$ by $\hat{W}(\xi, \eta) = W(s, t)$, let $\hat{S} = \inf\{\xi \geqq 0 : \hat{W}(\xi, 0) = 1\}$ and let $\bar{W}(\xi, \eta) = \hat{W}(\hat{S} - \xi, \eta) - 1$.

Let τ be a continuous non-decreasing function on $\mathbb{R}_+$ with $\tau(0) = 0$ and $\tau(\xi) > 0$ if $\xi > 0$. Let T_τ be the thorn-shaped neighborhood $T_\tau = \{(\xi, \eta) : 0 < \xi < \hat{S}, 0 \leqq \eta \leqq \tau(\xi)\}$ and let γ_τ be the upper boundary curve of T_τ. Then we have the following analogue of Theorem 3.1.

Theorem 3.4 *Suppose that $\xi \mapsto \tau(\xi)/\xi$ is increasing for $\xi > 0$. Then*
(a) *T_τ is initially in $\{\bar{W} < 0\}$ if and only if*

$$I(\tau) \equiv \int_{0+} \left(\frac{\tau(u)}{u}\right)^{\frac{1}{2}} \frac{du}{u} < \infty.$$

(b) *If $I(\tau)$ is infinite, then γ_τ is* not *initially in $\{\hat{W} < 0\}$.*

Remark 3.5 This states that the thorn has exactly the same shape as in the horizontal and vertical cases. This might seem surprising in light of the strong horizontal-vertical orientation of the set $\{W > 0\}$ which is so visible in Fig. 1. However, Theorem 3.4 is only an order-of-magnitude result: it tells us that the tip of the thorn has the same order of sharpness in the two cases, but it does not distinguish between one thorn and another which is, for example, twice as sharp. Finer results might well show a difference.

Proof of Theorem 3.4 We can reduce the proof of this theorem to the calculations we did in Theorem 3.1. Let B^1, B^2, X and Y be defined as in Sect. 2.3 and note that along the first diagonal $\hat{W}(\xi, 0) = X(\xi^2/2) = B^1(\xi^2/2) + B^2(\xi^2/2)$, and that Eq. (7) becomes

$$\begin{aligned}\hat{W}(\xi, \eta) &= B^1((\xi-\eta)^2/2) + B^2((\xi+\eta)^2/2) - \hat{z}(\xi, \eta) \\ &= \tfrac{1}{2}((X+Y)((\xi-\eta)^2/2) + (X-Y)((\xi+\eta)^2/2)) - \hat{z}(\xi, \eta),\end{aligned}$$

where, by (10), $\hat{z}(\xi, \eta)$ is small compared to X and Y if η is small. We will express this in terms of Bessel processes and Brownian motions. Let $\sigma = \hat{S}^2/2$ and recall that

$$b(u) = X(\sigma) - X(\sigma - u), \qquad B(u) = \sigma^{-\frac{1}{2}}(Y(\sigma) - Y(\sigma - \sigma u))$$

are, respectively, a Bessel and an independent Brownian motion (Lemma 2.8). Thus $\bar{W}(\xi, \eta)$ is equal to

$$\begin{aligned}&\tfrac{1}{2}[\sigma^{\frac{1}{2}}(B(1-(\xi+\eta)^2/(2\sigma)) - B(1-(\xi-\eta)^2/(2\sigma))) \\ &\quad - b(\sigma-(\xi+\eta)^2/2) - b(\sigma-(\xi-\eta)^2/2)] - z(\xi, \eta).\end{aligned}$$

Let $u = \hat{S} - \xi$, and $v = \eta$. Then $1-(\xi \pm \eta)^2/(2\sigma) = (2\hat{S}(u \mp v) - (u \mp v)^2)/(2\sigma)$ and $\sigma - (\xi \pm \eta)^2/2 = \hat{S}(u \mp v) - (u \mp v)^2/2$.

When u and v are small, we can neglect $\hat{z}(\xi, \eta)$ and the second order terms. Using Brownian scaling, we see that

$$\bar{W}(\xi, \eta) \approx \frac{\hat{S}^{\frac{1}{2}}}{2}(B(u-v) - B(u+v) - b(u-v) - b(u+v)).$$

But when u is small and $0 < v \ll u$, as we have near the tip of the thorn, the distribution of $(b(u+v) + b(u-v))/2$ is essentially the same as that of $b(u)$, whereas the distribution of $(B(u+v) - B(u-v))/2$ is exactly that of $B(v)/\sqrt{2}$. This means that up to negligeable terms, $\hat{W}$ has essentially the same decomposition as $\tilde{W}$ in (15). Therefore, the remainder of the proof of this theorem will be essentially identical to the proof of Theorem 3.1 concerning the case of horizontal lines. The details are left to the reader. □

3.3 Supergeometric spacing of bubbles near the thorn

As we move away from the thorn T_β, we encounter components of $\{\tilde{W} > 0\}$. We are going to describe the position and shape of these components. Intuitively, the first components we encounter will be small, and then we will encounter larger and larger components. We are going to make these statements precise. Set

$$b^+(s) = b(s) + s^{3/4}, \qquad b^-(s) = b(s) - s^{3/4}.$$

Notice that the thorn T_β is contained in $\{(s, t): s \geqq t\}$. By (15) and (2), the following implications are true for all small s and t such that $s \geqq t$:

$$b^+(s) < B(t) \Rightarrow \tilde{W}(s, t) > 0 \Rightarrow b^-(s) < B(t) \tag{17}$$

$$b^-(s) > B(t) \Rightarrow \tilde{W}(s, t) < 0 \Rightarrow b^+(s) > B(t). \tag{18}$$

Observe that $b^{+}(s)>0$ for all s, and, by [10, Example 5.4.7], $s \mapsto s^{3/4}$ is a lower escape function for b, so there is a random variable $\eta^{-}>0$ such that $b^{-}(s)>0$ when $0<s<\eta^{-}$. In particular, by (18), $\tilde{W}(s, t)<0$ on the horizontal segments where $s \geqq t$ and $B(t)=0$. Set

$$Q=\{(s, t): s>t \geqq 0, \tilde{W}(s, t)>0\}. \tag{19}$$

Remark 3.6 For small s and t, any point $(s, t) \in Q$ is contained in a horizontal strip of the form $\mathbb{R}_{+} \times [t_1, t_2]$, where $[t_1, t_2]$ is an interval where B is accomplishing an excursion above 0. Points in Q which are sufficiently near the origin and are in different horizontal strips are in distinct connected components of Q. Indeed, assume $A_1=(u_1, v_1) \in Q$, $A_2=(u_2, v_2) \in Q$, with $u_1<\eta^{-}$, $u_2<\eta^{-}$ and

$$t_1<v_1<t_2<t_3<v_2<t_4<\tau(\eta^{-}),$$

where $\tau=\psi_a$ (defined in Remark 3.2), $a>2$, and both $[t_1, t_2]$ and $[t_3, t_4]$ are intervals where B is accomplishing an excursion above 0. Then by Theorem 2.1, any curve Γ in Q with extremities A_1 and A_2 must be contained in $Q \cap (T_\tau)^c$. But if $i=3$, 4 and $0<s<\eta^{-}$, then $b^{-}(s)>B(t_i)=0$, so by (18), $\tilde{W}(s, t)<0$ on the horizontal segment $]0, \eta^{-}[\times\{t_i\}$, $i=3$, 4. Now Γ must pass through one of these two segments, a contradiction.

When $\beta \leqq 2$ and we follow the curve γ_{ψ_β} towards the origin, we will encounter components of $\tilde{W}>0$ infinitely many times. Due to the irregularity of the sample paths of $\tilde{W}$, these components will never be isolated. But they will tend to occur in clusters, and the space between clusters will get larger and larger relative to the remaining distance to $(S, 1)$. The following theorem makes this statement precise.

Let $\mathscr{T}$ be the class of continuous increasing functions τ defined in some interval $[0, \eta[$ – where η may depend on τ – such that $\tau(0)=0$, $\tau(s)>0$ if $s>0$, and $s \mapsto \tau(s)/s$ is increasing on $]0, \eta[$. Let Q be defined as in (19). If a given point $(s, \tau(s))$ is in Q, let Q_s denote the connected component of Q which contains $(s, \tau(s))$.

Theorem 3.7 *Consider $\tau_1 \in \mathscr{T}$ such that $I(\tau_1)=+\infty$. Fix $0<\tilde{c}<1$ and let τ_2, $\tau_3 \in \mathscr{T}$ be such that $s^{3/2} \leqq \tau_1(s) \leqq \tau_2(s) \leqq \tau_3(s) \leqq \tilde{c}\, s$ and $s \mapsto \tau_2(s)/\tau_3(s)$ is increasing and $\tau_3(s)/\tau_3(\tilde{c}\, s)$ is bounded in a neighborhood of the origin. Assume*

$$\tilde{I}(\tau_1, \tau_2, \tau_3)=\int_{0+}\left(\frac{\tau_1(s)}{s} \frac{\tau_2(s)}{\tau_3(s)}\right)^{\frac{1}{2}} \frac{d s}{s}<+\infty. \tag{20}$$

Then with probability 1, *there exists $\eta>0$ such that for $0<s<\eta$,*

$$Q_s \neq \phi \Rightarrow ([\tau_3(s), \tilde{c}\, s] \times [0, \tau_2(s)]) \subset \{\tilde{W}<0\}.$$

Remark 3.8 (a) Consider the function ψ_β defined in Remark 3.2. If $\tau_i(s)=\psi_{\beta_i}(s)$, $i=1, 2, 3$, with $\beta_3<\beta_2<\beta_1 \leqq 2$, then the conditions of Theorem 3.7 are satisfied and $\tilde{I}(\tau_1, \tau_2, \tau_3)<+\infty$ if and only if $\beta_1+\beta_2-\beta_3>2$.
(b) If $s_1>s_2>\ldots$ denote the s-coordinates of one point from each cluster encountered along γ_{ψ_β}, $\beta \leqq 2$, as we approach the origin, then $s_n/s_{n+1} \geqq s_n/\psi_\beta(s_n) \uparrow \infty$ as $n \rightarrow \infty$. For a geometric sequence, this ratio is constant: this justifies the title of this subsection.

(c) Theorem 3.7 shows in particular that there are no long bubbles parallel to γ_{ψ_2}. Rather, the shape of the thorn is due to isolated short bubbles.
(d) In view of Theorem 3.1, the assumption $\tau_1(s) > s^{3/2}$ is not a serious restriction.

Proof of Theorem 3.7 We shall show that there are only finitely many s for which

$$b^-(s) < B^*(\tau_1(s)) \quad \text{and} \quad \min_{\tau_3(s) < u < \tilde{c}s} b^-(u) < B^*(\tau_2(s)).$$

By (18), this will complete the proof. Set $\delta = 1/4$, $c = \sqrt{\tilde{c}}$, $\varepsilon'_n = c^{(n-2)(1-\delta)}$ and $A_n = A'_n \cap A''_n$, where

$$A'_n = \{\min_{c^{n-1} < u < c^{n-2}} b(u) \leqq B^*(\tau_1(c^{n-2})) + \varepsilon'_n\},$$

$$A''_n = \{\min_{\tau_3(c^n) < u < c^n} b(u) \leqq B^*(\tau_2(c^{n-2})) + \varepsilon'_n\}.$$

Since $\tilde{c}s < c^n$ for $s \in]c^{n-1}, c^{n-2}[$, it will be sufficient to show that $P(\limsup A_n) = 0$, and, by the Borel-Cantelli Lemma, we only need to show that $\sum P(A_n) < \infty$. By Brownian scaling, $P(A_n) = P(C'_n \cap C''_n)$, where

$$C'_n = \{\min_{c^{-1} < u < c^{-2}} b(u) \leqq K B^*(1)\, \phi_1(c^{n-2})^{\frac{1}{2}} + \varepsilon'_n c^{-n/2}\},$$

$$C''_n = \{\min_{\phi_3(c^n) < u < 1} b(u) < K B^*(1)\, \phi_2(c^{n-2})^{\frac{1}{2}} + \varepsilon'_n c^{-n/2}\},$$

and $\phi_i(s) = \tau_i(s)/s$, $i = 1, 2, 3$, and K denotes a constant. Thus

$$P(C'_n \cap C''_n) = \int_0^\infty P(D'_n(a) \cap D''_n(a))\, \nu(d\,a),$$

where $\nu(d\,a)$ is the density of $B^*(1)$ and $D'_n(a)$ and $D''_n(a)$ are defined by replacing $B^*(1)$ by a in the definition of C'_n and C''_n, respectively. Let

$$\alpha_i(n, a) = K\, a\, \phi_i(c^{n-2})^{\frac{1}{2}} + \varepsilon'_n c^{-n/2}, \qquad i = 1, 2.$$

Using the Markov property of b, we obtain

$$\begin{aligned} P(D'_n(a) \cap D''_n(a)) &= E\{P(D''_n(a) \mid b(1))\, P^{b(1)}(\tilde{D}'_n(a))\} \\ &\leqq P(D''_n(a))\, P^0(\tilde{D}'_n(a)), \end{aligned}$$

where

$$\tilde{D}'_n(a) = \{\min_{d_1 < u < d_2} b(u) \leqq \alpha_1(n, a)\}$$

and $d_i = c^{-i} - 1 > 0$, $i = 1, 2$. By Lemma 3.3, $P^0(\tilde{D}_n'(a)) \leqq K\alpha_1(n, a)$. In order to estimate $P(D_n''(a))$, we again use the scaling property of b and Lemma 3.3 to observe that

$$P(D_n''(a)) = P\{\min_{1<u<1/\phi_3(c^n)} b(u) < \alpha_2(n, a)\,\phi_3(c^n)^{-\frac{1}{2}}\}$$
$$\leqq K(a+1)\left(\frac{\phi_2}{\phi_3}(c^{n-2})\right)^{\frac{1}{2}}$$

(the last inequality is valid for large n and uses the fact that $\varepsilon_n' c^{-n/2} < \phi_2(c^{n-2})^{\frac{1}{2}}$ since $s^{3/2} < \tau_2(s)$ by assumption, and that $K\phi_3(c^n) \geqq \phi_3(c^{n-2})$, also by assumption). Thus, for large n,

$$P(A_n) \leqq \int_0^\infty \left[K(a+1)\,\varepsilon_n' c^{-n/2}\left(\frac{\phi_2}{\phi_3}(c^{n-2})\right)^{\frac{1}{2}} + K a(a+1)\left(\frac{\phi_1\phi_2}{\phi_3}(c^{n-2})\right)^{\frac{1}{2}}\right] \nu(da)$$
$$\leqq K E\{B^*(1)+1\}\,\varepsilon_n' c^{-n/2} + K E\{B^*(1)^2\}\left(\frac{\phi_1\phi_2}{\phi_3}(c^{n-2})\right)^{\frac{1}{2}},$$

since ϕ_2/ϕ_3 is bounded for large n by assumption. Summing over n, using the monotonicity of $\phi_1\phi_2/\phi_3$ to replace the sum by an integral and applying (20), we conclude that $\sum P(A_n) < \infty$. □

3.4 Height of bubbles near the thorn

The following theorem shows that bubbles that meet a particular curve near the thorn will not be very tall. Let Q be defined as in (19). As before, K denotes a constant whose value may change from line to line.

Theorem 3.9 *Let $\tau_1, \tau_2 \in \mathscr{T}$ be such that $s^{5/4} \leqq \tau_1(s) \leqq \tau_2(s)$, $s \mapsto \tau_1(s)/\tau_2(s)$ is increasing with limit 0 as $s \downarrow 0$ and, for each $c > 1$, $\tau_2(cs)/\tau_2(s)$ is bounded in a neighborhood of the origin. Suppose*

$$J(\tau_1, \tau_2) = \int_{0+} \frac{\tau_1(s)}{(s\,\tau_2(s))^{\frac{1}{2}}}\frac{ds}{s} < \infty.$$

Then with probability one, there is $\eta > 0$ such that if $0 < s < \eta$ and $(s, \tau_1(s)) \in Q$, then Q_s is contained in $\mathbb{R}_+ \times [0, \tau_2(s)]$.

Remark 3.10 (a) If $\tau_1 = \psi_\beta$, $\beta \leqq 2$, as defined in Remark 3.2, and $\beta' \leqq \beta$, then $J(\psi_\beta, \psi_{\beta'}) < +\infty$ if and only if $\beta' < 2\beta - 2$.
(b) If

$$I_1 \equiv \int_{0+} (\tau_1(s)/s)^{\frac{1}{2}} s^{-1}\, ds < \infty,$$

the integral in (i) is always finite, but the theorem is empty since Q does not intersect γ_1. Thus the theorem is only of interest when I_1 diverges.

Proof of Theorem 3.9 Suppose the integral is finite. Let γ_i be the graph of τ_i. Both τ_1 and τ_2 are defined near 0, so for all large n we can define

$$A_n = \{\exists\, s \in [e^{-n}, e^{1-n}] : (s, \tau_1(s)) \in Q,\ Q_s \cap (\mathbb{R}_+ \times [\tau_2(s), \infty[\,) \neq \phi\}.$$

We must show A_n occurs only finitely often. Let $\delta = 1/4$, $\varepsilon_n = e^{(1-n)(1-\delta)}$ and

$$m_n = \inf_{e^{-n} \leqq s \leqq e^{1-n}} b(s), \qquad M_n = \sup_{\tau_1(e^{-n}) \leqq t \leqq \tau_1(e^{1-n})} B(t).$$

By (18), if $m_n \geqq M_n + \varepsilon_n$, then Q does not intersect γ_1 between $s = e^{-n}$ and $s = e^{1-n}$. On the other hand, since $Q_s \subset (T_{\psi_3})^c$ by Theorem 3.1 and Remark 3.2(a), if there is some $t \in [\tau_1(e^{1-n}), \tau_2(e^{-n})]$ such that $B(t) \leqq -\varepsilon_n$, then $\tilde{W}(s, t) < 0$ for all small s, so $Q_s \subset \mathbb{R}_+ \times [0, t]$. Since $M_n \leqq B^*(\tau_1(e^{1-n}))$, it follows that

$$P(A_n) \leqq P\{m_n < B^*(\tau_1(e^{1-n})) + \varepsilon_n, \inf_{\tau_1(e^{1-n}) \leqq t \leqq \tau_2(e^{-n})} B(t) > -\varepsilon_n\}.$$

Let $\lambda_n = \tau_2(\mathrm{e}^{-n})/\tau_1(e^{1-n})$ and observe that $e^{n/2}\varepsilon_n < e^{-n/4}$. Set $\phi_i(s) = \tau_i(s)$, $i = 1, 2$. Using Brownian scaling and the independence of b and B, the last probability above is

$$= P\{m_0 \leqq K\phi_1(e^{1-n})^{\frac{1}{2}} B^*(1) + e^{n/2}\varepsilon_n,\ \tau_1(e^{1-n})^{\frac{1}{2}} \inf_{1 \leqq t \leqq \lambda_n} B(t) > -\varepsilon_n\}$$

$$\leqq \int_0^\infty P\{m_0 \leqq K\phi_1(e^{1-n})^{\frac{1}{2}} a + e^{-n/4}\}\, P\{B^*(1) \in da, \inf_{1 \leqq t \leqq \lambda_n} B(t) > -e^{-n/8}\},$$

since $\varepsilon_n \tau_1(e^{1-n})^{-\frac{1}{2}} < e^{-n/8}$ because $s^{5/4} < \tau_1(s)$ by assumption. By Lemma 3.3, the first probability is dominated by $K(\phi_1(e^{1-n})^{\frac{1}{2}} a + e^{-n/4})$. To bound the second factor, apply the reflection principle at $T(a)$, the first hit of a by B:

$$P\{\inf_{1 \leqq u \leqq \lambda_n} B(u) > -e^{-n/8} \mid T(a) \leqq 1\} = P\{\sup_{1 \leqq u \leqq \lambda_n} B(u) < 2a + e^{-n/8} \mid T(a) \leqq 1\}$$

$$\leqq P^0\{(\lambda_n - 1)^{\frac{1}{2}} B^*(1) < a + e^{-n/8}\}$$

by the strong Markov property at $T(a)$ and scaling. Applying the reflection principle again and since $B(1)$ is standard Normal, this is

$$= P^0\{|B(1)| < (a + e^{-n/8})(\lambda_n - 1)^{-\frac{1}{2}}\} \leqq K(a + e^{-n/8})(\lambda_n - 1)^{-\frac{1}{2}}.$$

Thus, the above integral is

$$\leqq K \int_0^\infty (\phi_1(e^{1-n})^{\frac{1}{2}} a + e^{-n/4})(a + e^{-n/8})(\lambda_n - 1)^{-\frac{1}{2}} P\{B^*(1) \in da\}.$$

Since ϕ_1 is increasing, this is

$$\leqq K(e^{-3n/8} + e^{-n/8} E\{B^*(1)\} + \phi_1(e^{1-n})^{\frac{1}{2}} E\{B^*(1)^2\})(\lambda_n - 1)^{-\frac{1}{2}}.$$

It follows from our assumptions that $(\lambda_n-1)^{-\frac{1}{2}}$ is $\leqq K(\phi_1/\phi_2(e^{1-n}))^{\frac{1}{2}}$ for large n, which is decreasing in n by hypothesis, hence the last expression is bounded by

$$K(e^{-3n/8}+e^{-n/8}+\phi_1(e^{1-n})\,\phi_2(e^{1-n})^{-\frac{1}{2}}).$$

Summing over n and dominating the sum by an integral, we conclude that $\sum_n P(A_n)<\infty$ since

$$\int_0^\infty \phi_1(e^{-x})\,\phi_2(e^{-x})^{-\frac{1}{2}}\,dx=\int_0^1 \phi_1(y)\,\phi_2(y)^{-\frac{1}{2}}\,y^{-1}\,dy<\infty$$

by hypothesis. Thus A_n only happens finitely often by the Borel-Cantelli Lemma, proving the theorem. □

3.5 *Width of bubbles*

Recall that $Q=\{(s,t)\colon s>t\geqq 0,\ \tilde{W}(s,t)>0\}$. K and c are constants whose values may change from line to line. We work with the functions ψ_β defined in Remark 3.2.

Theorem 3.11 *Let $\beta\leqq 2$ and fix $\beta'<2\beta-2$ and $\varepsilon>0$. For $s>0$, let $V_s^\varepsilon=]s-\varepsilon\psi_{\beta'}(s),\ s+\varepsilon\psi_{\beta'}(s)[\times[0,\infty[$. Then with probability one, there exists an $\eta>0$ such that if $0<s<\eta$ and if Q_s is the connected component of Q which contains the point $(s,\psi_\beta(s))$, then Q_s is contained in the vertical strip V_s^ε.*

There is one calculation to be made before we prove this. We separate it out as a lemma.

Lemma 3.12 *Let $a>0$, $\lambda>0$, and put $L=a\lambda^{-\frac{1}{2}}$. Set $q(a,\lambda)=P^a\{b$ ever has an excursion below a of duration $\geqq 2\lambda\}$. Then there exists a constant C, independent of a and λ, such that*

$$q(a,\lambda)\leqq C(L+1)^4\,e^{-1/(CL^2)}. \tag{21}$$

Proof. By Brownian scaling, $q(a,\lambda)=\hat{q}(L)$, where $\hat{q}(L)=P^L\{b$ has an excursion below L of duration $\geqq 2\}$. Let $T(L)$ be the first time b hits L and let $S(L)$ be the first time after $t=1$ that b hits L. Consider time 1. If $b(1)<L$, then b must stay below L at least one unit longer to complete its excursion, and if $b(1)>L$, b must first return to L, then make its excursion. Thus $\hat{q}(L)\leqq p_1+p_2$, where $p_1=P^L\{b(1)<L,\ S(L)\geqq 2\}$ and

$$p_2=P^L\{S(L)<\infty, \text{ there is an excursion below } L \text{ of duration } \geqq 2 \text{ after } S(L)\}.$$

Applying the Markov property of b at time 1, we have

$$p_1\leqq P^L\{b(1)<L\}\ P^0\{T(L)\geqq 1\}.$$

Clearly

$$P^L\{b(1)\leqq L\}\leqq P^0\{b(1)\leqq L\}=K\int_0^L x^2\,e^{-x^2/2}\,dx, \tag{22}$$

where we have used the density of $b(1)$ (see e.g. [11, Chap. 11, Sect. 1]). By [4, Theorem 2], the second probability on the right hand side is bounded by $c\exp(-1/(cL^2))$. With this, we see that $p_1 \leqq CL^3 \exp(-1/(cL^2))$. Turning to p_2, we note that if $S(L)<\infty$, then $b(S(L))=L$, and the probability of having the excursion after $S(L)$ is just $\hat{q}(L)$. Thus $p_2 = P^L\{S(L)<\infty\}\,\hat{q}(L)$, hence

$$\hat{q}(L) \leqq cL^3 e^{-1/(cL^2)} + P^L\{S(L)<\infty\}\,\hat{q}(L),$$

or equivalently,

$$\hat{q}(L) \leqq cL^3 e^{-1/(cL^2)} (1 - P^L\{S(L)<\infty\})^{-1}. \tag{23}$$

Now by [11, Corollary 3.4]

$$P^x\{T(L)<\infty\} = P^x\{\min_{s\geqq 0} b(s) \leqq L\} = L/x. \tag{24}$$

It follows that

$$P^L\{S(L)=\infty\} = \int_L^\infty (1-L/x)\, P^L\{b(1)\in dx\}.$$

Notice that for $x>L$, $P^L\{b(1)\leqq x\} \leqq P^L\{B(1)\leqq x\}$, where B is a standard Brownian motion (since b has the same distribution as the modulus of a three-dimensional Brownian motion) so that this is

$$\geqq \int_L^\infty (1-L/x)\, c\, e^{-(x-L)^2/2}\, dx = c \int_0^\infty (\sqrt{2u}+L)^{-1} e^{-u}\, du$$

after the change of variable $u=(x-L)^2/2$. It is easy to see that this is a continuous decreasing function of L which is asymptotic to $1/L$ for large L. Thus there is some constant K such that $P^L\{S(L)=\infty\} \geqq K/(L+1)$ for all L, which, together with (23), proves the lemma. □

Proof of Theorem 3.11 Fix β'' such that $\beta'<\beta''<2\beta-2$ (which is $\leqq\beta$ since $\beta\leqq 2$). Let $I_n=[e^{-n}, e^{1-n}]$ and let A_n be the event on which there exists a component Q_s of $\tilde{Q}$ which contains a point $(s, \psi_\beta(s))$ for some $s\in I_n$ and which is not contained in the corresponding strip V_s^ε. Then there exist u and v such that the points $(s, \psi_\beta(s))$ and (u, v) are in Q_s while $|u-s|>\varepsilon\psi_{\beta'}(s)\geqq\varepsilon\psi_{\beta'}(e^{-n})$.

Let J be the interval with endpoints u and s. Since Q_s is connected, there is a continuous curve $\{(g_1(x), g_2(x)), 0\leqq x\leqq 1\}$ in Q_s with $(g_1(0), g_2(0))=(s, \psi_\beta(s))$ and $(g_1(1), g_2(1))=(u, v)$. By (17), it follows that on the set A_n, we have $B(g_2(x)) > b^-(g_1(x))$ for all $x\in[0,1]$. Now $\psi_{\beta''}(s)\leqq\psi_{\beta''}(e^{1-n})$ so by Theorem 3.9 and Remark 3.10(a), if n is large enough, then $Q_s\subset\mathbb{R}_+\times[0, \psi_{\beta''}(e^{1-n})]$. This implies that $B^*(\psi_{\beta''}(e^{1-n}))\geqq b^-(g_1(x))$ for each x. Since g_1 runs through the interval J,

$$B^*(\psi_{\beta''}(e^{1-n})) \geqq b^-(u), \qquad \forall u\in J.$$

Thus, J is contained in an excursion interval of b^- below the level $B^*(\psi_{\beta''}(e^{1-n}))$. Let us simplify notation by setting $\delta=1/4$, $\varepsilon_n'=e^{(2-n)(1-\delta)}$ and

$$\tau'=\psi_{\beta'}(e^{-n}), \qquad \tau''=\psi_{\beta''}(e^{1-n}).$$

For large n, if there is $s \in I_n$ and a component Q_s of $\tilde{Q}$ which contains $(s, \psi_\beta(s))$ and intersects $(V_s^\varepsilon)^c$, then

(25) there exists an interval J of length at least $\varepsilon\tau'$ which intersects I_n and is an excursion interval of b below the level $B^*(\tau'') + \varepsilon_n'$.

Let D_n be the event on which (25) happens and let us estimate $P(D_n)$. Now $P(D_n) \leqq P(D_{n,1}) + P(D_{n,2})$, where

$$D_{n,1} = D_n \cap \{b(e^{-n}) \leqq (\log n)^{\beta''/2} B^*(\tau'')\},$$

$$D_{n,2} = D_n \cap \{b(e^{-n}) > (\log n)^{\beta''/2} B^*(\tau'')\}.$$

Using scaling, the independence of b and B, the density of $b(1)$ (see (22)) and the definition of $\psi_{\beta''}$ in Remark 3.2(a), we obtain

$$P(D_{n,1}) = P\{b(1) \leqq K B^*(1)/n\} \leqq K E\{B^*(1)^3\}/n^3.$$

Thus $\sum \mathscr{P}(D_{n,1}) < \infty$. Note that $(\log n)^{\beta''/2} \geqq 1$ for $n \geqq 3$, so $e^{-n} \notin J$ on $D_{n,2}$. It follows that the excursion interval J starts after time $s = e^{-n}$. Use again independence and scaling to see that $P(D_{n,2})$ is the probability that $b(1) > B^*(1)/n$ and b has an excursion below $e^{n/2}\sqrt{\tau''}\, B^*(1) + \varepsilon_n' e^{n/2}$ after time $s = 1$ of duration $\geqq \varepsilon e^n \tau'$. Setting $\alpha(a, n) = a e^{n/2}\sqrt{\tau''} + \varepsilon_n' e^{n/2}$, conditioning on $B^*(1)$ and using the strong Markov property of b at time $S(\alpha(a, n))$, this probability is

$$\leqq \int_0^\infty P^{a/n}\{\min_{u \geqq 1} b(u) \leqq \alpha(a, n)\}\, q(\alpha(a, n), \varepsilon e^n \tau')\, \nu(d a),$$

where $\nu(d a)$ is the density of $B^*(1)$ and $q(\cdot, \cdot)$ is defined in Lemma 3.12. Let $\beta'' - \beta' = 2d > 0$ and $L = \alpha(a, n)(\varepsilon e^n \tau')^{-\frac{1}{2}} = \varepsilon^{-\frac{1}{2}}(a(\log n)^{-d} + \varepsilon_n'(\tau')^{-\frac{1}{2}})$. For large n, since

$$\hat{q}(L) \leqq K(a+2)^4 \exp(-(a+1)^{-2}(\log n)^{2d}/C)$$

by Lemma 3.12, Lemma 3.3 implies that

$$P(D_{n,2}) \leqq K \int_0^\infty \alpha(a, n)(a+2)^4 \exp(-(a+1)^{-2}(\log n)^{2d}/C)\, \nu(d a).$$

Sum this over $n \geqq 2$, estimate the sum by an integral and interchange order to see that $\sum_{n=2}^\infty P(D_{n,2})$ is

$$\leqq c + \int_0^\infty \nu(d a)(a+2)^4 a \int_2^\infty 2x^{-1}(\log x)^{-\beta''/2} \exp(-(a+1)^{-2}(\log x)^{2d}/C)\, d x.$$

Let $y = (\log x)/(a+1)^{1/d}$ to see that the inner integral is

$$\leqq (a+1)^{(2-\beta'')/(2d)} \int_0^\infty y^{-\beta''/2} \exp(-y^{2d}/C)\, d y.$$

This integral converges at 0 since $\beta'' < 2$ and clearly converges at ∞ since $d > 0$. Consequently the double integral is finite. Thus $\sum P(D_n) < \infty$, and the conclusion follows from the Borel-Cantelli Lemma. $\square$

Acknowledgement. The first author thanks Jim Pitman for several stimulating discussions.

References

1. Adler, R.J.: The uniform dimension of the level sets of a Brownian sheet. Ann. Probab. **6**, 509–515 (1978)
2. Adler, R.J.: The geometry of random fields. New York: Wiley 1981
3. Cairoli, R., Walsh, J.B.: Stochastic integrals in the plane. Acta Math. **134**, 111–183 (1975)
4. Ciesielski, Z., Taylor, S.J.: First passage times and sojourn times for Brownian motion in space and the exact Hausdorff measure of the sample path. Trans. Am. Math. Soc. **103**, 434–450 (1962)
5. Dalang, R.C., Walsh, J.B.: The structure of a Brownian bubble. (Preprint)
6. Ernst, B.: The magic mirror of M.C. Escher. Toronto, Canada: Ballantine Books 1976
7. Garsia, A.M.: Continuity properties of Gaussian processes with multidimensional time parameter. In: Le Cam, L.M. et al. (eds.) Proc. Sixth Berkeley Symp. Math. Stat. and Prob., vol. II, pp. 369–374. Berkeley: University of California Press 1971
8. Hurewicz, W., Wallman, H.: Dimension theory. Princeton: Princeton University Press 1941
9. Kendall, W.: Contours of Brownian processes with several-dimensional time. Z. Wahrscheinlichkeitstheor. Verw. Geb. **52**, 269–276 (1980)
10. Knight, F.: Essentials of Brownian motion and diffusion. Providence, RI: Am. Math. Soc. 1981
11. Revuz, D., Yor, M.: Continuous martingales and Brownian motion. Berlin Heidelberg New York: Springer 1991
12. Walsh, J.B.: Martingales with a multidimensional parameter and stochastic integrals in the plane. In: delPino, G., Rebolledo, R. (eds.) Lectures in probability and statistics. (Lect. Notes Math., vol. 1215, pp. 329–491) Berlin Heidelberg New York: Springer 1986