Integr. equ. oper. theory 27 (1997) 497 – 501
0378-620X/97/040497-05 $ 1.50+0.20/0
© Birkhäuser Verlag, Basel, 1997

Integral Equations and Operator Theory

ERRATA FOR: POLAR DECOMPOSITIONS IN FINITE DIMENSIONAL INDEFINITE SCALAR PRODUCT SPACES: SPECIAL CASES AND APPLICATIONS

Yuri Bolshakov, Cornelis V. M. van der Mee*, André C. M. Ran,
Boris Reichstein, Leiba Rodman**

In Theorem 1.1 of [1] a mistake was made in the statement of condition (ii). The theorem should read as follows.

THEOREM 1.1. *($F = \mathbb{C}$ or $F = \mathbb{R}$) An $n \times n$ matrix X admits an H-polar decomposition if and only if all the conditions* (i), (ii), *and* (iii) *below are satisfied.*

(i) *For each negative eigenvalue λ of $X^{[*]}X$ the part of the canonical form of $\{X^{[*]}X, H\}$ corresponding to λ can be presented in the form*

$$\{\text{diag } (A_i)_{i=1}^m, \quad \text{diag } (H_i)_{i=1}^m\},$$

where, for $i = 1, \ldots, m$,

$$A_i = J_{k_i}(\lambda) \oplus J_{k_i}(\lambda), \qquad H_i = Q_{k_i} \oplus -Q_{k_i}.$$

(ii) *The part of the canonical form of $\{X^{[*]}X, H\}$ corresponding to the zero eigenvalue can be presented in the form*

$$\{\text{diag } (B_i)_{i=0}^m, \quad \text{diag } (H_i)_{i=0}^m\},$$

where $B_0 = O_{k_0}$, $H_0 = I_{p_0} \oplus -I_{n_0}$, $p_0 + n_0 = k_0$ and, for each $i = 1, \ldots, m$, the pair $\{B_i, H_i\}$ is of one of the following two forms:

$$B_i = J_{k_i}(0) \oplus J_{k_i}(0), \qquad H_i = Q_{k_i} \oplus -Q_{k_i}, \quad k_i \geq 1,$$

or

$$B_i = J_{k_i}(0) \oplus J_{k_i-1}(0), \qquad H_i = \varepsilon_i(Q_{k_i} \oplus Q_{k_i-1}),$$

* The work of this author was performed under the auspices of C.N.R.–G.N.F.M. and partially supported by the research project, "Nonlinear problems in analysis and its physical, chemical, and biological applications: Analytical, modelling and computational aspects," of the Italian Ministry of Higher Education and Research (M.U.R.S.T.).

** The work of this author partially supported by the NSF grant DMS 9123841 and by an NSF International Cooperation Grant.

with $\varepsilon_i = \pm 1$, *and* $k_i > 1$.
Assume that (ii) *holds and denote the corresponding basis in* $\mathrm{Ker}\,(X^{[*]}X)^n$ *in which this is achieved by*

$$\{e_{i,j}\}_{i=0}^{m}{}_{j=1}^{l_i}$$

where $l_0 = k_0$ *and* $l_i = 2k_i$ *in case* B_i *is an even size matrix, and* $l_i = 2k_i - 1$ *in case* B_i *is an odd size matrix.*

(iii) *There is a choice of basis* $\{e_{i,j}\}_{i=0}^{m}{}_{j=1}^{l_i}$ *such that* (ii) *holds and*

$$\begin{aligned}\mathrm{Ker}\,X = \mathrm{span}\{e_{i,1} + e_{i,\ k_i+1} \mid l_i = 2k_i,\ i = 1,\ldots,m\} \oplus \\ \oplus\ \mathrm{span}\{e_{i,1} \mid l_i = 2k_i - 1,\ i = 1,\ldots,m\} \oplus\ \mathrm{span}\{e_{0,j}\}_{j=1}^{k_0}.\end{aligned}$$

The mistake is in the fact that for both cases in condition (ii) it is stated in Theorem 1.1 of [1] that $k_i > 1$; it is essential that $k_i = 1$ is allowed in the first case. Note that, in part (ii), pairs $\{B_i, H_i\}$ with $B_i = J_1(0) \oplus J_1(0) = O_2$ and $H_i = Q_1 \oplus -Q_1 = \mathrm{diag}(1, -1)$ could in principle be subsumed under the pair $\{B_0, H_0\}$. However, the condition (iii) involving the interplay between the basis $\{e_{i,j}\}_{i=0}^{m}{}_{j=1}^{l_i}$ and $\mathrm{Ker}\,X$ necessitates distinguishing between $\{B_0, H_0\}$ and such $\{B_i, H_i\}$.

The mistake has had effect in the statement of Theorem 3.4 of [1]. This should read as follows (the proof remains the same as in [1]):

THEOREM 3.4 *Let* X *be an* H*-plus matrix. Then* X *has an* H*-polar decomposition if and only if the following conditions are satisfied:*

(a) $X^{[*]}X$ *is invertible, or* $0 \in \sigma(X^{[*]}X)$ *and there are at least as many linearly independent positive eigenvectors corresponding to the zero eigenvalue as there are Jordan blocks of order* 2,

(b) *In case* $0 \in \sigma(X^{[*]}X)$, *the part of the canonical form of* $\{X^{[*]}X, H\}$ *corresponding to the zero eigenvalue of* $X^{[*]}X$ *can be presented in the form*

$$\{O_k \oplus B \oplus \cdots \oplus B \oplus C \oplus \cdots \oplus C,\ G \oplus K \oplus \cdots \oplus K \oplus L \oplus \cdots \oplus L\}, \qquad (3.4)$$

where $G = I_p \oplus -I_q$ $(p + q = k)$, $B = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$, $K = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, *the summands* B *and* K *are repeated* m *times each in* (3.4), $C = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \oplus (0)$, $L = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \oplus (1)$, *the summands* C *and* L *are repeated* r *times each in* (3.4), *and this form is achieved with respect to a basis* $\{e_{ij}\}_{i=0}^{m+r}{}_{j=1}^{l_i}$ $(l_0 = k, l_i = 2$ *for* $i = 1,\ldots,m$ *and* $l_i = 3$ *for* $i = m+1,\ldots,m+r)$ *in* $\mathrm{Ker}\,(X^{[*]}X)^2$ *such that* $\mathrm{Ker}\,X$ *is given by*

$$\mathrm{span}\,\{e_{01},\ldots,e_{0k}\} \oplus \mathrm{span}\,\{e_{11} + e_{12},\ldots,e_{m1} + e_{m2}\} \oplus \mathrm{span}\,\{e_{m+1\,1},\ldots,e_{m+r\,1}\}.$$

(c) $X^{[*]}X$ *does not have negative eigenvalues.*

In particular, a strict H*-plus matrix has an* H*-polar decomposition.*

Note that the original versions of Theorems 1.1 and 3.4 in [1] are valid if the additional assumption that $\operatorname{Ker}(X^{[*]}X)$ is H-semidefinite is made, i.e., there do not exist vectors $x, y \in \operatorname{Ker}(X^{[*]}X)$ such that $[x,x]\cdot[y,y] < 0$.

The correction made in Theorem 3.4 has some impact on the subsequent sections of [1]. First of all, the two paragraphs following the proof of Theorem 3.4 are incorrect. In fact, if H has exactly one positive eigenvalue and X is an H-plus matrix, $X^{[*]}X$ and $XX^{[*]}$ are not necessarily similar, contrary to the claim made in these two paragraphs. A counterexample is given by the pair of matrices $X = \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}$ and $H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ for which $X^{[*]}X = \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}$ and $XX^{[*]} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$. Note that in this example $X^{[*]}$ is an H-plus matrix as well but neither X nor $X^{[*]}$ admit H-polar decompositions.

The two paragraphs following Theorem 3.4 of [1] have subsequently been employed to prove Proposition 6.3 and Theorem 6.4. Proposition 6.3 is easily corrected and its statement should be as follows.

PROPOSITION 6.3 *A matrix M satisfying the Stokes criterion allows an H-polar decomposition, unless one of the following two cases occurs:*

(a) *All eigenvalues of $M^{[*]}M$ vanish and $M^{[*]}M$ has one Jordan block of order* 2, *or*
(b) $M^{[*]}M = 0$, *and* $\operatorname{Ker} M$ *is* NOT *a three-dimensional subspace having an H-orthogonal basis consisting of one H-neutral vector and two H-negative vectors.*

The original statement of Theorem 6.4 turns out to be correct, although we have rephrased part (4) (see below). In its proof one should replace the last two paragraphs, since they are based on the two incorrect paragraphs following the proof of Theorem 3.4 in [1]. For the sake of convenience we now state Theorem 6.4 and give a new proof for the part that has to be replaced. We recall that $\mathcal{W}$ is the class of finite linear combinations with nonnegative coefficients of real H-unitary matrices in the connected component of the identity; here $H = \operatorname{diag}[1,-1,-1,-1]$. Note that $\mathcal{W}$ is closed (see [3]).

THEOREM 6.4 *Let M be a real 4×4-matrix satisfying $M_{11} \geq 0$. Let σ be the sign ± 1 of the product of the nonzero eigenvalues of M. Then $M \in \mathcal{W}$ if and only if one of the following four situations occurs:*

(1) *$M^{[*]}M$ has the positive eigenvalue λ_0 corresponding to an H-positive eigenvector and a positive and two nonnegative eigenvalues λ_1, λ_2 and λ_3 (with $\lambda_1 \geq \lambda_2 \geq \lambda_3$) corresponding to H-negative eigenvectors, and $\sqrt{\lambda_0} \pm \sqrt{\lambda_1} \geq |\sqrt{\lambda_2} \pm \sigma\sqrt{\lambda_3}|$;*
(2) *$M^{[*]}M$ is diagonalizable with one positive and three zero eigenvalues, and $\sigma = +1$;*
(3) *$M^{[*]}M$ has the positive eigenvalue λ and the nonnegative eigenvalues μ and ν but is not diagonalizable. The eigenvectors corresponding to μ and ν are H-negative, whereas to the double eigenvalue λ there corresponds one Jordan block of order* 2 *with the positive sign in the sign characteristic of $\{M^{[*]}M, H\}$. Moreover, $\sigma = +1$, $\mu = \nu$ and $\lambda \geq \mu$;*
(4) *$M^{[*]}M$ has only zero eigenvalues. The matrix M has at most rank one and* $\operatorname{Im} M$ *and* $\operatorname{Im} M^{[*]}$ *are both H-nonnegative.*

PROOF.

If M satisfies the stokes criterion and either is invertible or is singular having an H-polar decomposition, then the proof given in [1] applies without changes.

If M satisfies the Stokes criterion and $M^{[*]}M$ is nilpotent, then M is a non-strict H-plus matrix. (Observe that every matrix that satisfies the Stokes criterion is H-plus; see, e.g., Theorem 2.3 in [2].) Thus $[Mu, Mu] \geq 0$ for any $u \in \mathbb{R}^4$ and therefore $\operatorname{Im} M$ is an H-nonnegative subspace. Since H has only one positive eigenvalue, the rank of M must be zero or one. Let us assume it is one. Then M has the form

$$M = [\cdot, \eta]\, \xi,$$

where ξ and η are H-nonnegative vectors with positive first entry. We easily compute

$$M^{[*]} = [\cdot, \xi]\, \eta, \qquad M^{[*]}M = [\cdot, \eta]\, [\xi, \xi]\, \eta, \qquad MM^{[*]} = [\cdot, \xi]\, [\eta, \eta]\, \xi.$$

Then the largest eigenvalue of $M^{[*]}M$ equals $[\xi, \xi]\, [\eta, \eta]$ with corresponding eigenvector η. Hence if both ξ and η are H-positive vectors, then M admits an H-polar decomposition by Proposition 6.3, and the proof is reduced to the case in which the proof in [1] applies.

We now restrict ourselves to the following three cases in which at least one of ξ, η is H-neutral:

(a) $[\xi, \xi] = [\eta, \eta] = 0$;
(b) $[\xi, \xi] > 0$ and $[\eta, \eta] = 0$;
(c) $[\xi, \xi] = 0$ and $[\eta, \eta] > 0$.

Writing $\zeta = (1, 1, 0, 0)^T$ and $\theta = (1, 0, 0, 0)^T$ so that ζ is H-neutral and θ is H-positive, we first find the H-unitary matrices U and V in the connected component of the identity and the positive numbers c and d such that $\xi = cU\zeta$ and $\eta = dV\zeta$ in case (a), $\xi = cU\theta$ and $\eta = dV\zeta$ in case (b), and $\xi = cU\zeta$ and $\eta = dV\theta$ in case (c). Such U, V, c and d exist. Since multiplication by cU from the left and by dV from the right pertains to a bijective transformation of $\mathcal{W}$ onto itself, it suffices to prove that the following three matrices belong to $\mathcal{W}$:

$$\begin{cases} [\cdot, \zeta]\zeta = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix} \oplus \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \\[2ex] [\cdot, \zeta]\theta = \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} \oplus \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \\[2ex] [\cdot, \theta]\zeta = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} \oplus \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}. \end{cases}$$

This is indeed the case. To see this introduce

$$\begin{aligned} X(\varepsilon) &= \begin{bmatrix} 1+\varepsilon^2 & -1+\varepsilon^2 \\ 1-\varepsilon^2 & -1-\varepsilon^2 \end{bmatrix} \oplus \begin{bmatrix} 2\varepsilon & 0 \\ 0 & -2\varepsilon \end{bmatrix}, \\ Y(\varepsilon) &= \begin{bmatrix} 1+\varepsilon^2 & -1+\varepsilon^2 \\ -1+\varepsilon^2 & 1+\varepsilon^2 \end{bmatrix} \oplus \begin{bmatrix} 2\varepsilon & 0 \\ 0 & 2\varepsilon \end{bmatrix}, \\ Z(\varepsilon) &= \begin{bmatrix} 1+\varepsilon^2 & 1-\varepsilon^2 \\ 1-\varepsilon^2 & 1+\varepsilon^2 \end{bmatrix} \oplus \begin{bmatrix} 2\varepsilon & 0 \\ 0 & 2\varepsilon \end{bmatrix}. \end{aligned}$$

Then one checks that

$$X(\varepsilon)^{[*]}X(\varepsilon) = Y(\varepsilon)^{[*]}Y(\varepsilon) = Z(\varepsilon)^{[*]}Z(\varepsilon) = 4\varepsilon^2 I.$$

From this it easily follows that $X(\varepsilon)$, $Y(\varepsilon)$ and $Z(\varepsilon)$ are in $\mathcal{W}$. Further, we have that

$$\begin{cases} \lim_{\varepsilon\downarrow 0} \quad X(\varepsilon) = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix} \oplus \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \\ \lim_{\varepsilon\downarrow 0} \frac{1}{2}(Y(\varepsilon) + X(\varepsilon)) = \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} \oplus \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \\ \lim_{\varepsilon\downarrow 0} \frac{1}{2}(Z(\varepsilon) + X(\varepsilon)) = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} \oplus \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}. \end{cases}$$

The proof is then completed by observing that $\mathcal{W}$ is a closed set. ∎

REFERENCES

[1] Y. Bolshakov, C.V.M. van der Mee, A.C.M. Ran, B. Reichstein and L. Rodman: Polar decompositions in finite dimensional indefinite scalar product spaces: special cases and applications, in: *Recent Developments in Operator Theory and its Applications*, OT 87 (I. Gohberg, P. Lancaster, P.N. Shivakumar, eds.) Birkhäuser, Basel, 1996, 61–94.

[2] R. Loewy and H. Schneider: Positive operators on the n-dimensional ice-cream cone, *J. of Mathematical Analysis and Appl.*, 49 (1975), 375–392.

[3] C. V. M. van der Mee: An eigenvalue criterion for matrices transforming Stokes parameters, *J. Math. Phys.*, 34 (1993), 5072–5088.

Yuri Bolshakov
Department of Mathematics
Yaroslavl State University
Yaroslavl, Russia

Cornelis V.M. van der Mee
Dipartimento di Matematica
Università di Cagliari
Via Ospedale 72, 09124 Cagliari, Italy

André C.M. Ran
Faculteit Wiskunde en Informatica
Vrije Universiteit Amsterdam
De Boelelaan 1081a
1081 HV Amsterdam, The Netherlands

Boris Reichstein
Department of Mathematics
The Catholic University of America
Washington, DC 20064, USA

Leiba Rodman
Department of Mathematics
The College of William and Mary
Williamsburg, VA 23187-8795, USA