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Brownian confinement and pinning in a Poissonian potential. I

Alain-Sol Sznitman

Departement Mathematik, ETH-Zentrum, CH-8092 Zürich, Switzerland
(e-mail: sznitman@math.ethz.ch)

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Summary. We study the behavior of a d dimensional Brownian motion in a soft repulsive Poissonian potential over long time intervals $[0, t]$. We introduce certain t and configuration dependent scales, which grow almost linearly with t. For typical configurations with probability tending to 1 as t goes to ∞, the size of displacements of the process is bounded above by these scales, (confinement effect). The proof involves calculations beyond "leading order". To this end we use a coarse grained picture of the environment (method of enlargement of obstacles) and of the path (a backbone of excursions between clearings and forest parts in the environment). These coarse grained pictures are also used in the sequel [11] to the present article, when proving the pinning effect.

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0. Introduction

We study in the present article and its sequel [11], the behavior of a canonical Brownian motion $Z.$, in dimension $d \geq 1$, under the influence of random soft obstacles. These obstacles are given by means of a Poissonian potential which is the sum of translates at the points x_i of a Poisson cloud, of a non-negative, bounded measurable, compactly supported shape function $W(\cdot)$. We also assume that $W(\cdot)$ is not a.e. equal to 0. We let P_x, $x \in \mathbb{R}^d$, stand for the Wiener measure starting from x and $\mathbb{P}$ for the law of the Poisson cloud with constant intensity $\nu > 0$, on the space Ω of simple pure point measures on $\mathbb{R}^d$. We are specifically interested in the behavior of the path $Z.$, for large t, under the "quenched" path measure on $C(\mathbb{R}_+, \mathbb{R}^d)$:

$$(0.1) \qquad Q_{t,\omega} = \frac{1}{S_{t,\omega}} \exp\left\{ - \int_0^t V(Z_s, \omega) ds \right\} P_0 \, .$$

Here $\omega = \sum_i \delta_{x_i}$ stands for the typical cloud configuration, $V(x,\omega) = \sum W(x - x_i) = \int W(x-y)\,\omega(dy)$ is the Poissonian potential, and $S_{t,\omega}$ the normalizing constant. In a slightly formal way $Z.$ under $Q_{t,\omega}$, is a Brownian motion feeling a time non homogeneous drift:

$$
(0.2) \qquad \left\{ \begin{aligned} dZ_s &= d\beta_s + \frac{\nabla u}{u}\,(t-s,\, Z_s)ds,\ 0 \le s \le t\ , \\ Z_0 &= 0 \end{aligned} \right.
$$

where β is a d-dimensional Brownian motion and u is the solution of the random parabolic equation:

$$
\left\{ \begin{aligned} \partial_t\, u &= \frac{1}{2}\ \Delta u - V(\cdot,\omega)\, u \\ u_{t=0} &= 1\ . \end{aligned} \right.
$$

Various models of random motion in typical landscapes of random potentials can be found in the physical literature of disordered media. We refer for instance to Nattermann-Renz [5], Krug-Halpin Healy [4] for polymer models in columnar disorder, to Zeldovitch-Molchanov-Ruzmaikin-Sokolov [2] for questions of intermittency, and on the mathematical side to Gärtner-Molchanov [2], Carmona-Molchanov [1], Hanin-Mazel-Shlosman-Sinai [3]. Models with Gaussian potentials are common but models with potentials bounded below are studied as well, see [3] and [4]. Our Poissonian potential should be viewed as an example of a translation invariant potential with independence properties, which is bounded from below. In this respect the assumption $W \ge 0$ is crucial.

In our previous work [6], [7], we showed that for typical configurations, the particle under $Q_{t,\omega}$ settles at time t "near points of low local eigenvalue". They typically lie at distance "almost t" from the origin and correspond to certain big holes or "clearings" in the cloud configuration of size of order const $(\log\, t)^{1/d}$ and volume of order const$(d,\nu)(\log\, t)$.

The object of the present work and its sequel [11] is to conduct a more detailed study of the path behavior, and in particular of the mechanism through which the particle settles down. This last point is intimately connected to a certain variational problem which we now motivate by a heuristic argument. If we use "an eigenfunction expansion", then

$$
(0.4) \qquad S_{t,\omega} = e^{t(\frac{1}{2}\,\Delta - V)}1(0) \text{ ``=''} \sum_i \varphi_i(0) < \varphi_i,\, 1 > e^{-\lambda_i\, t}\ ,
$$

where $\lambda_i \ge 0$ are the "eigenvalues" of $-\frac{1}{2}\,\Delta + V$ and φ_i are the corresponding "eigenfunctions". For large t only λ_i "near 0" should matter. The corresponding eigenfunctions should be "localized near a point" and exhibit an exponential decay "at rate α". In particular $\varphi_i(0)$ should be viewed in (0.4) as exponentially small in the distance from the origin to the point around which φ_i is localized and $< \varphi_i,\ 1 >$ should be viewed as a "constant". The "preferred location" of the particle at time t should therefore be the site where the eigenfunction corresponding to the dominating term in the right member of (0.4) localizes.

This location should come as a locus of minimum of the formal variational principle

$$\min_x (\alpha\,|x| + t\,\lambda_i(x)) \,. \tag{0.5}$$

x being the "localization point" corresponding to $\lambda_i(x)$.

It turns out that the (very) heuristic variational principle (0.5) is not the most appropriate to begin a rigorous mathematical investigation, and a suitable reformulation of it will only be studied in the sequel [11] of the present article.

The object corresponding to the above mentioned heuristic coefficient α is the 0-th Lyapounov coefficient introduced in [8]. We know from [8] that there exists a norm $\alpha_0(\cdot)$ on $\mathbb{R}^d$, such that $\mathbb{P}$-a.s.:

$$\lim_{y\to\infty} \frac{1}{|y|}\,|-\log e_0(0,y,\omega) - \alpha_0(y)| = 0 \,, \text{ where} \tag{0.6}$$

$$\begin{aligned} e_0(x,y,\omega) &= E_x\Big[\exp\Big\{-\int_0^{H(y)} V(Z_s,\omega)ds\Big\},\, H(y)<\infty\Big], \\ & \quad x,y\in\mathbb{R}^d,\ \omega\in\Omega\,, \end{aligned} \tag{0.7}$$

if $H(y)$ stands for the entrance time of $Z.$ in $B(y) \overset{\text{def}}{=} \bar{B}(y,1)$. It is also shown in [8] that $\alpha_0(\cdot)$ governs as well the $\mathbb{P}$-a.s. directional exponential decay of $g(0,\cdot,\omega)$ if $g(\cdot,\cdot,\omega)$ stands for the 0-Green function of $-\frac{1}{2}\Delta+V(\cdot,\omega)$, and that when $W(\cdot)$ is rotationally invariant, $\alpha_0(\cdot)$ is a multiple of the Euclidean norm.

Our starting point is a variational principle which turns out to govern the scale in which the particle $Z.$ evolves:

$$\mu_t = \min_{u\geq 0}\,\{u + t\lambda(B_{\alpha_0}(0,u))\}\,, \tag{0.8}$$

where $B_{\alpha_0}(0,u)$ denotes the open ball in the α_0 norm with center 0, radius u and $\lambda(B_{\alpha_0}(0,u))$ its principal Dirichlet eigenvalue for $-\frac{1}{2}\Delta + V$.

From our work [6], we know that $\mathbb{P}$-a.s.

$$\lambda(B_{\alpha_0}(0,u)) \sim \bar{c}(d,\nu)/(\log u)^{2/d}, \text{ as } u\to\infty\,, \tag{0.9}$$

see (1.4), (1.6) below. It is easily deduced, see Lemma 1.2, that for typical cloud configurations, $\mu_t \sim \bar{c}(d,\nu)\,t(\log t)^{-2/d}$ and the loci of minima in (0.8) occur in a scale which is $o(t/(\log t)^{2/d})$ but grows faster than t^γ, for $\gamma<1$, as t tends to infinity. We define in (1.8) a family of scales $S_\eta(t)$, $\eta\in(0,1]$, decreasing with η and approximating from above the loci of minima in (0.8):

$$S_\eta(t) = \inf\{v\geq 0,\ \eta u + t\lambda(B_{\alpha_0}(0,u)) \geq \mu_t, \text{ for all } u\geq v\}\,.$$

Much of the work in the present article is in a sense preparatory to the derivation of the "pinning effect" in [11]. However we prove here in Theorem 4.1 a "confinement result" which shows that for typical configurations, with $Q_{t,\omega}$ probability tending to 1 as t tends to infinity the path $Z_.$ does not leave $B_{\alpha_0}(0, S_\eta(t))$ (with the notations of (0.8)) up to time t, when $\eta \in (0,1)$.

One substantial difficulty in the derivation of the confinement result stems from the fact that the scales $S_\eta(t)$ are negligible in comparison with $t(\log t)^{-2/d}$ which governs the leading behavior of $-\log S_{t,\omega}$, and of μ_t in (0.8). This forces us to conduct estimates "beyond leading order".

Let us finally explain how the article is organized.

Section I introduces and derives some first properties of the various random spatial scales which are relevant to our study.

In Sect. II, we present lowerbounds on $S_{t,\omega}$ and similar expressions, see Lemma 2.1 and Theorem 2.2. In contrast to [7], we do not seek to express our results in terms of deterministic rates. This turns out to be more convenient when we later take advantage of cancellations.

Section III deals in essence with upperbounds, and is the backbone in the construction of the present article. These upperbounds rely on both a coarse grained picture of the Poissonian cloud and of the Brownian path. The coarse grained picture of the cloud stems from the "method of enlargement of obstacles", see [6], [7], and produces "clearings" and "forest" areas in the cloud. The coarse grained picture of the path uses a finite covering of path space of low enough combinatorial complexity which keeps track of the skeleton of excursions of the process in and out of the clearings. Our main results are Theorem 3.2 and Lemma 3.3. The main novelty when for instance compared to [9], is that exponential bounds are derived with exponential precision finer than the principal rate $t(\log t)^{-2/d}$. As mentioned above this is forced upon us because we are trying to track the trajectory in scales which are $o(t(\log t)^{-2/d})$.

In Sect. IV, we combine upperbounds of Sect. III and lowerbounds of Sect. II and derive the above mentioned confinement property.

Section V investigates more closely the behavior of the various spatial scales in the one dimensional situation. In particular, we show in Theorem 5.1 that for large t with high $\mathbb{P}$-probability these various scales are "comparable" to $t/(\log t)^3$. Similar results in higher dimension would very much be of interest. Even the correct scale replacing $t/(\log t)^3$ when $d > 1$, is unclear. On this last issue see however [4].

I. The spatial scales

The goal of this section is to introduce the collection of spatial scales in terms of which the confinement result (see section IV) will later be expressed. We keep here the notations from the introduction. Throughout the sequel for A a closed subset of $\mathbb{R}^d$,

$$H_A = \inf\{s \geq 0, Z_s \in A\} ,$$

denotes the entrance time of $Z.$ in A, so that $H(y) = H_{\bar{B}(y,1)}$ in the notations of (0.7). For U an open subset of $\mathbb{R}^d$,

$$T_U = \inf\{s \geq 0, Z_s \notin U\} ,$$

denotes the exit time of $Z.$ from U. We let

$$a = a(W) > 0 , \tag{1.1}$$

be the smallest number such that $W(\cdot) = 0$ outside $\bar{B}(0,a)$.

We will need the following result. It was shown in [10] (0.6), when $d \geq 2$, and when $d = 1$ the same claim follows from Theorem 2.6 of [8] and the ergodic theorem, that one has a "non degenerate velocity" to reach $B(y)(= \bar{B}(y,1)$, see (0.7)) under the weighted measure $e_0(0,y,\omega)^{-1} \exp\left\{-\int_0^t V(Z_s,\omega)ds\right\} P_0$: there is a $K(d,\nu,W) \in (1,\infty)$ such that $\mathbb{P}$-a.s.:

$$\overline{\lim_{y\to\infty}} \frac{1}{|y|} E_0\Big[H(y) \exp\Big\{-\int_0^{H(y)} V(Z_s,\omega)ds\Big\}, H(y) < \infty\Big]/e_0(0,y,\omega) < K(d,\nu,W), \tag{1.2}$$

where we recall that $H(y) = H_{B(y)}$.

Let us also recall some results on the Poissonian potential and on the principal Dirichlet eigenvalues $\lambda(U)$ of $-\frac{1}{2}\Delta + V$ in large boxes $U = (-t,t)^d$. From [6], we know that $\mathbb{P}$-a.s.:

$$\sup_{(-t,t)^d} V(\cdot,\omega) = o(\log t), \text{ as } t \to \infty, \text{ and} \tag{1.3}$$

$$\lambda((-t,t)^d) \sim \bar{c}(d,\nu)/(\log t)^{2/d}, \text{ as } t \to \infty, \text{ provided} \tag{1.4}$$

$$\bar{c}(d,\nu) = \lambda_d/R_0^2, \text{ with } R_0 = (d/\nu\omega_d)^{1/d} , \tag{1.5}$$

if λ_d stands for the principal Dirichlet eigenvalue of $-\frac{1}{2}\Delta$ in $B(0,1)$ and ω_d for the volume $|B(0,1)|$ of $B(0,1)$.

We now define Ω_1 to be a set of full measure where (0.6) (the "shape theorem"), (1.2) - (1.4) hold. The family of scales we want to introduce involves certain variational problems where the principal Dirichlet eigenvalue $\lambda(B_{\alpha_0}(0,u))$ of $-\frac{1}{2}\Delta + V$ in the open ball of radius u centered at 0 for the $\alpha_0(\cdot)$ norm (see (0.6)) plays an important role. The following lemma will be useful:

Lemma 1.1: *$\lambda(B_{\alpha_0}(0,u))$ is a measurable function of ω, continuous decreasing in $u \in (0,\infty)$, tending to $+\infty$ as $u \to 0$. For $\omega \in \Omega_1$:*

$$\lambda(B_{\alpha_0}(0,u)) \sim \bar{c}(d,\nu)(\log u)^{-2/d}, \text{ as } u \to \infty . \tag{1.6}$$

Proof: The first part of the statement is fairly standard, in fact very much in the same vein as Lemma 1.2 of [7]. As for (1.6), observe that for a suitable constant $C \geq 1$:

$$\left(-\frac{u}{C}, \frac{u}{C}\right)^d \subseteq B_{\alpha_0}(0,u) \subseteq (-Cu, Cu)^d, \text{ for } u > 0 .$$

The claim (1.6) now follows from (1.4). □

Our basic variational formula comes with the definition of μ_t via: for $t > 0, \omega \in \Omega$,

$$(1.7) \qquad \begin{aligned} \mu_t &= \inf_{u>0} \cdot \{u + t\lambda(B_{\alpha_0}(0,u))\} \in (0,\infty) \\ &= t \inf_{u>0} \cdot \{u \cdot \frac{1}{t} + \lambda(B_{\alpha_0}(0,u))\} . \end{aligned}$$

The second formula shows that μ_t/t is a finite concave, and therefore continuous, function of $1/t$.

We now want to introduce a family of scales which approximates from above the loci of minima of $u \to u + t\lambda(B_{\alpha_0}(0,u))$. For $\eta \in (0,1], t > 0, \omega \in \Omega$, we define the "upper scales":

$$(1.8) \qquad S_\eta(t,\omega) = \inf\{v > 0; \eta u + t\lambda(B_{\alpha_0}(0,u)) \geq \mu_t, \text{ for all } u \geq v\} .$$

It is immediate to argue that $S_\eta(t,\omega) \in (0,\infty)$ and that it is a decreasing function of η. We also introduce one "lower scale":

$$(1.9) \qquad s_1(t,\omega) = \inf\{u > 0, u + t\lambda(B_{\alpha_0}(0,u)) \leq \mu_t\} .$$

Obviously:

$$(1.10) \qquad s_1(t,\omega) \leq S_\eta(\nu,\omega), \text{ for } \eta \in (0,1], \ t > 0, \ \omega \in \Omega .$$

We record some elementary properties of these quantities.

Lemma 1.2: *When* $\omega \in \Omega_1$*:*

$$(1.11) \qquad \mu_t \sim \bar{c}(d,\nu)\, t(\log t)^{-2/d}, \text{ as t} \to \infty ,$$

$$(1.12) \quad \text{for } \eta \in (0,1], \ S_\eta(t,\omega) = o(t/(\log t)^{2/d}) \text{ as } t \to \infty, \text{ and for } \gamma \in (0,1) ,$$

$$(1.13) \qquad t^\gamma = o(s_1(t,\omega)), \text{ as } t \to \infty .$$

Proof: Observe that (1.11) is an elementary consequence of (1.6), (1.7). Observe also that

$$(1.14) \qquad \eta S_\eta(t,\omega) + t\lambda(B_{\alpha_0}(0, S_\eta(t,\omega))) = \mu_t ,$$

and $\overline{\lim}\, S_\eta(t,\omega)/t(\log t)^{-2/d} > 0$, would contradict (1.6), (1.11). This proves (1.12). Analogously

$$(1.15) \qquad s_1(t,\omega) + t\lambda(B_{\alpha_0}(0, s_1(t,\omega))) = \mu_t ,$$

which together with (1.6), (1.11) forces $\underline{\lim}\, s_1(t,\omega)t^{-\gamma} \geq 1$, when $\gamma \in (0,1)$, and therefore (1.13). □

As a result of (1.14), whenever $\omega \in \Omega_1$, and $0 < \eta < \eta' \leq 1$,

$$\eta\, S_\eta(t,\omega) \geq \eta'\, S_{\eta'}(t,\omega), \quad \text{for } t > 0 . \tag{1.16}$$

In particular $S_\eta(t,\omega) - S_{\eta'}(t,\omega) \to \infty$, as $t \to \infty$.

Finally we introduce another scale which roughly describes the mutual distance between points of low local eigenvalues. This last scale will play an important role in the study of "pinning of trajectories" in the sequel of the present paper (see [11], section II). We define the local eigenvalue at a point via:

$$\bar{\lambda}_t(x,\omega) = \lambda(B(x,t^{1/3})),\ x \in \mathbb{R}^d,\ t > 0,\ \omega \in \Omega . \tag{1.17}$$

The scale controlling the "mutual distance between points of low local eigenvalues" is defined as:

$$\begin{aligned} D_t(\omega) \; &= \; \inf\{|x - x'|;\ x, x' \in B_{\alpha_0}(0, S_{\frac{1}{2}}(t,\omega)),\ |x - x'| > 3\,t^{1/3}\ , \\ & \qquad \bar{\lambda}_t(x,\omega),\ \bar{\lambda}_t(x',\omega) \leq (\mu_t + S_{\frac{1}{2}}(t))/t\} . \end{aligned} \tag{1.18}$$

Observe that for $\omega \in \Omega_1$, $(\mu_t + S_{\frac{1}{2}}(t))/t \sim \bar{c}(d,\nu)(\log t)^{-2/d}$. It follows from the method of enlargement of obstacles, see the proof of Theorem 3.1 of [7], especially (3.20) - (3.22) that there exist a set $\Omega_2 \subseteq \Omega_1$ of full $\mathbb{P}$-measure such that:

$$\text{for } \omega \in \Omega_2,\ \text{for all } \gamma \in (0,1),\ t^\gamma = o(D_t(\omega)) . \tag{1.19}$$

II. Lower bounds

We now develop lower bounds on the norming constant $S_{t,\omega}$ (see (0.1)). In contrast to the strategy used in [7], section II, we do not try here to express our lower bounds in terms of "deterministic functions".

Throughout this section ω is a fixed element of Ω_2, and the dependence on ω in the notation is dropped whenever this causes no confusion.

Let B stand for some bounded open connected subset of $\mathbb{R}^d$ and φ be the unique non negative principal Dirichlet eigenfunction of $-\frac{1}{2}\Delta + V$ in B with unit L^2-norm. For $s > 0$, $x, y \in \mathbb{R}^d$:

$$\exp\{-\lambda(B)s\}\,\varphi(x) = \int r_B(s,x,y)\,\varphi(y)dy\ , \tag{2.1}$$

where for U an open subset of $\mathbb{R}^d$ $r_U(s,x,y)$, $s > 0, x, y \in \mathbb{R}^d$ is defined as:

$$\begin{aligned} r_U(s,x,y) = \; & (2\pi s)^{-d/2}\, \exp\Big\{ - \frac{(y-x)^2}{2s} \Big\} \\ & E^s_{x,y}\Big[\exp\Big\{ - \int_0^s V(Z_u,\omega)du \Big\},\ T_U > s\Big]\ , \end{aligned} \tag{2.2}$$

provided $E^s_{x,y}$ stands for the Brownian bridge expectation in time s from x to y (and $r_U(s,x,y) = 0$ if x or y does not belong to U). In other words, r_U is the kernel of $e^{t(\frac{1}{2}\Delta - V)}$ with Dirichlet conditions on U.

Let $\widetilde{B}$ stand for the $\sqrt{d}$ neighborhood B. If we cover B by disjoint boxes $q + [0,1)^d$, $q \in \mathbb{Z}^d$, it is easy to argue that for some $y \in B$,

$$\int_{y+(-1,1)^d} \varphi^2(z)dz \geq \frac{1}{2\,|\widetilde{B}|}\,. \tag{2.3}$$

In fact the factor $\frac{1}{2}$ is unnecessary here, but is present in view of applications given in section I of the sequel to the present article [11]. We begin with a general lemma:

Lemma 2.1: *There exists $c(d) > 0$, such that for $x \in \mathbb{R}^d$, B a bounded connected open set of R^d, and y satisfying (2.3), we have for $t \geq 2$:*

$$\begin{aligned} E_x\Big[\exp\Big\{-\int_0^t V(Z_s,\omega)ds\Big\}\Big] &\geq \frac{c(d)}{|\widetilde{B}|}\exp\Big\{-\lambda(B)t - 2\sup_{y+(-2,2)^d} V(\cdot,\omega)\Big\} \\ &\cdot E_x\Big[\exp\Big\{-\int_0^{H(y)} V(Z_s,\omega)ds\Big\},\ H(y) < t\Big]\,. \end{aligned} \tag{2.4}$$

Proof: The left hand side of (2.4) is bigger than (integrate only over $\{H(y) < t\}$):

$$E_x\Big[\exp\Big\{-\int_0^{H(y)} V(Z_s,\omega)ds\Big\},\ H(y) < t,\ E_{Z_{H(y)}}\Big[\exp\Big\{-\int_0^t V(Z_s,\omega)ds\Big\}\Big]\Big]\,,$$

and letting C stand for the "box" $y + (-1,1)^d \supset B(y)$,

$$\begin{aligned} &E_{Z_{H(y)}}\Big[\exp\Big\{-\int_0^t V(Z_s,\omega)ds\Big\}\Big] \geq \\ &E_{Z_{H(y)}}\Big[Z_2 \in C,\ \exp\Big\{-\int_0^2 V(Z_s,\omega)ds\Big\}\int r_B(t-2,Z_2,z)dz\Big]\,. \end{aligned} \tag{2.5}$$

Observe that

$$\begin{aligned} 0 \leq \varphi(z) &= \exp\{\lambda(B)\}\int r_B(1,z,z')\,\varphi(z')dz' \\ &\leq \exp\{\lambda(B)\}\Big(\int r_B^2(1,z,z')dz'\Big)^{1/2}\Big(\int \varphi^2 dz'\Big)^{1/2} \\ &\leq \exp\{\lambda(B)\}\,, \end{aligned} \tag{2.6}$$

where we used that $r_B^2(1,\cdot,\cdot) \leq r_B(1,\cdot,\cdot)$ (see (2.2)). Consequently, the right hand side of (2.5) is bigger than:

$$\exp\{-\lambda(B)\}\, E_{Z_{H(y)}}\Big[Z_2 \in C,\ \exp\Big\{-\int_0^2 V(Z_s,\omega)ds\Big\}$$
$$\int r_B(t-2,Z_2,z)\,\varphi(z)dz\Big]$$
$$= \exp\{-\lambda(B)(t-1)\}\, E_{Z_{H(y)}}\Big[Z_2 \in C,\ \exp\Big\{-\int_0^2 V(Z_s,\omega)ds\Big\}\,\varphi(Z_2)\Big]$$
$$\geq \exp\{-\lambda(B)(t-1)\}\ \inf_{C\times C} r_{\mathbb{R}^d}(2,\cdot,\cdot)\ \int_C \varphi(z)dz\ .$$

Observe now that $\inf\limits_{C\times C}\ r_{\mathbb{R}^d}(2,\cdot,\cdot) \geq 2\ \exp\{-2 \sup\limits_{y+(-2,2)^d} V(\cdot,\omega)\}\cdot c(d)$, if

$$c(d) = \frac{1}{2}\ \inf_{(-1,1)^d\times(-1,1)^d}\ r_{(-2,2)^d,V\equiv 0}\,(2,\cdot,\cdot) > 0\ , \tag{2.7}$$

with obvious notations. Using (2.6) once more, the right hand side of (2.5) is bigger than:

$$2c(d)\ \exp\Big\{-\lambda(B)\,t - 2 \sup_{y+(-2,2)^d} V(\cdot,\omega)\Big\}\ \int_C \varphi^2(z)dz$$
$$\geq \frac{c(d)}{|\widetilde{B}|}\ \exp\Big\{-\lambda(B)\,t - 2 \sup_{y+(-2,2)^d} V(\cdot,\omega)\Big\}$$

using (2.3).

If we insert this estimate in our first lower bound on the left member of (2.4), we obtain our claim. □

We now give an application of the previous lemma to the derivation of a lower bound on $S_{t,\omega}$. To this end we introduce

$$\psi(y,\omega) = E_0\Big[H(y) < 2K\,|y|,\ \exp\Big\{-\int_0^{H(y)} V(Z_s,\omega)ds\Big\}\Big]\Big/ e_0(0,y,\omega)\ , \tag{2.8}$$

where $K(d,\nu,W)$ is the constant appearing in (1.2). It is easily seen that $\psi(\cdot,\omega)$ is bounded away from 0 on compact sets of $\mathbb{R}^d$ and since $\omega \in \Omega_2$, by (1.2):

$$\underline{\lim}_{y\to\infty}\ \psi(y,\omega) > \frac{1}{2}\ . \tag{2.9}$$

Theorem 2.2: *Assume $\omega \in \Omega_2$, for large t, for any bounded connected open set B included in $B(0,t/2K)$ (K the constant from (1.2)) and y satisfying (2.3),*

$$S_{t,\omega} \geq \frac{c(d)}{|\widetilde{B}|}\ \exp\{-\lambda(B)\,t - \log t\}\ e_0(0,y,\omega)\,\psi(y,\omega)\ . \tag{2.10}$$

Proof: Observe that $y \in B \subset B(0,t/2K)$, and therefore $H(y) < 2K\,|y|$ implies $H(y) < t$. Consequently:

$$e_0(0,y,\omega)\,\psi(y,\omega) \leq E_0\Big[H(y) < t,\ \exp\Big\{-\int_0^{H(y)} V(Z_s,\omega)ds\Big\}\Big]\ .$$

Our claim (2.10) now follows from (2.4) and (1.3). □

Remark 2.3: One can combine (0.6), (1.4), (2.9) and (2.10) to show that for $\omega \in \Omega_2$,

$$\varliminf_{t\to\infty} \frac{(\log t)^{2/d}}{t} \log S_{t,\omega} \geq -\bar{c}(d,\nu) \tag{2.11}$$

and if one also uses the upper bound (1.17) of [7] one easily sees that for $\omega \in \Omega_2$,

$$\lim_{t\to\infty} \frac{(\log t)^{2/d}}{t} \log S_{t,\omega} = -\bar{c}(d,\nu)\,.$$

This was shown to be $\mathbb{P}$-a.s. the case in [6]. □

III. Partitioning of the path space

The goal of this section is to introduce for each $\eta \in [\frac{1}{2}, 1)$, $\beta \in (0,1)$, $t > 0$, and ω in a set of full $\mathbb{P}$-measure a suitable covering $\mathscr{G}_{\eta,\beta,t,\omega}$ of the path space $C(\mathbb{R}_+, \mathbb{R}^d)$. This covering will on the one hand cover "most of the space $C(\mathbb{R}_+, \mathbb{R}^d)$" as far as the measure $Q_{t,\omega}$ is concerned. On the other hand the covering will not have too high complexity, in the sense that the number of elements of the covering will be $\exp\{o(S_1(t,\omega))\}$ for large t.

We shall also derive certain uniform upper bounds on terms like

$$E_0\Big[G, \exp\Big\{-\int_0^t V(Z_s,\omega)ds\Big\}\Big]\,, \tag{3.1}$$

for G an arbitrary element of the covering, see Theorem 3.2 below. This together with the lower bounds on $S_{t,\omega}$ from Theorem 2.2 will provide uniform upper bounds on quantities like $Q_{t,\omega}(G)$, $G \in \mathscr{G}_{\eta,\beta,t,\omega}$.

As a consequence of the "low complexity" of the family $\mathscr{G}_{\eta,\beta,t,\omega}$ it will follow that the union of any subcollection of G's for which $Q_{t,\omega}(G)$ is known to have a uniform decay at exponential rate $S_1(t,\omega)$ has $Q_{t,\omega}$-probability tending to 0 as t tends to infinity.

The construction of this covering will involve on the one hand a suitable coarse grained picture of the cloud of obstacles. This picture stems from the method of enlargement of obstacles and singles out certain "big holes" in the cloud, called the clearings and "dense parts" called the "forest". The typical scale of the clearings will be $(\log t)^{1/d}$.

On the other hand we shall construct a suitable coarse grained picture of the excursions of the process $Z_.$ in and out of the clearings.

The fact that we aim at working with an increased precision ($\exp\{o(S_1(t))\}$) will force us to define excursions in a new way, departing for instance from our work in [9], where the precision was only $\exp\{o(t/(\log t)^{2/d})\}$.

We first recall notations concerning "clearings" and "forest". We define the boxes $C_{n,m}$, $n \geq 1$, $m \in \mathbb{Z}^d$ via

(3.2) $C_{n,m} = \{z \in \mathbb{R}^d,\ m_i(\log 2^n)^{1/d} \le z_i < (m_i+1)(\log 2^n)^{1/d},\ i = 1, \dots, d\}$.

The coarse grained description of the cloud will depend on three parameters $r > 0$, $b > a(W)$, $0 < \delta < 1$, which will be chosen below in term of $\eta \in [\frac{1}{2}, 1)$. The integer n is viewed as a function of $t \ge 2$, via:

$$2^{n(t)} \le t < 2^{n(t)+1} \ . \tag{3.3}$$

Before explaining how r, b, δ are picked for a given η, we fist introduce the coarse grained picture of the cloud corresponding to r, b, δ.

For a given $\omega = \sum_j \delta_{x_j} \in \Omega$, we say that $x_i \in C_{n,m}$ is good at level n if for all balls $C = \bar{B}(x_i, 10^{\ell+1} b)$, with $0 \le \ell$ and $10^{\ell+1} b < \frac{1}{2} (\log 2^n)^{1/d}$,

$$\Big| C_{n,m} \cap C \cap \Big(\bigcup_{x_j \in C_{n,m}} \bar{B}(x_j, b) \Big) \Big| \ge \frac{\delta}{3^d} \, |C_{n,m} \cap C| \ . \tag{3.4}$$

The parameter b describes the size of the enlarged obstacles, whereas δ determines whether a point is "good" in the sense that it is well surrounded in scales going from unit up to $(\log 2^n)^{1/d}$, in the sense of (3.4). We let $\mathrm{Good}_{n,m}$ stand for the set of good points in $C_{n,m}$ and $G_n = \bigcup_{m \in \mathbb{Z}^d} \mathrm{Good}_{n,m}$.

We then chop identically each segment $[m_i(\log 2^n)^{1/d}, (m_i+1)(\log 2^n)^{1/d}]$ in at most $[\frac{\sqrt{d}}{b} (\log 2^n)^{1/d}] + 1$ segments of length $\frac{b}{\sqrt{d}}$ except may be for the "last one".

The third parameter $r > 0$, enables to define forest boxes and clearing boxes. To this end we introduce the event $\mathscr{C}\ell_{n,m}$, "there is a clearing of size $r(\log 2^n)^{1/d}$ in the box $C_{n,m}$" via

$$\mathscr{C}\ell_{n,m} = \{\omega \in \Omega,\ |\widetilde{U}_{n,m}| \ge 2^{-d} \, |B(0, r(\log 2^n)^{1/d})|\} \ , \tag{3.5}$$

provided $\widetilde{U}_{n,m}$ denotes the open subset of $\overset{\circ}{C}_{n,m}$ obtained by taking the complement in the interior of $C_{n,m}$ of closed subboxes which receive a good point of $C_{n,m}$. We then introduce $\mathscr{A}_n(\omega)$ the closed subset of $\mathbb{R}^d$, union of the $\bar{C}_{n,m}$, $m \in \mathbb{Z}^d$, where "there is a clearing of size $r(\log 2^n)^{1/d}$":

$$1_{\mathscr{A}_n(\omega)}(z) = \sum_m 1_{\overline{C}_{n,m}}(z) \, 1_{Cl_{n,m}}(\omega) \ . \tag{3.6}$$

Dropping the subscript n, we let $\mathscr{A}^1$ stand for the open set of points in $\mathbb{R}^d$ at distance strictly less than $(\log 2^n)^{1/d}$ from $\mathscr{A}$, if $\mathscr{A}$ is empty, so is $\mathscr{A}^1$.

Let us recall some results from [6], Lemma 2.2, Proposition 2.3, which yield some control over the size of clearings. Define for $r > 0$, $b > a$, $0 < \delta < 1$,

$$\alpha = (\nu 2^{-d} \, |B(0,r)| - \nu\delta - 2^d \log 2(\sqrt{d}/b)^d)/2d \ , \tag{3.7}$$

and when $\alpha > 0$,

$$m_0 = [\alpha^{-1}] + 2 \ . \tag{3.8}$$

When $\alpha > 0$, $\mathbb{P}$-a.s.

$$\text{(3.9)}\quad \begin{array}{l} \text{there exist } n_0(\omega) \text{ such that for } n \geq n_0(\omega), \text{ for any } m \in [-2^n - 1, 2^n]^d, \\ \text{among the } C_{n,m'} \text{ with } \sup_i |m_i - m_i'| \stackrel{\text{def}}{=} \|m - m'\| \leq 2^{n\alpha - 1}, \text{ there are} \\ \text{at most } m_0 \text{ clearing boxes, and if } C_{n,m} \text{ is a clearing box,} \\ m \in [-2^n - 1, 2^n]^d, \text{ the connected component of } \mathscr{A}^1 \text{ containing } C_{n,m} \\ \text{is included in } V_{n,m} \stackrel{\text{def}}{=} \Big(\bigcup_{\|m' - m\| \leq 3m_0} \bar{C}_{n,m} \Big). \end{array}$$

Moreover, from Lemma 3.2 of [7], we know that for r, b, δ satisfying $\alpha > 0$, $\mathbb{P}$-a.s. for any $\gamma \in (0,1] \cap \mathbb{Q}$,

$$\text{(3.10)}\quad \begin{array}{l} \text{for large } n, \text{ for } m \in [-2^{n\gamma}, 2^{n\gamma}]^d, \\ |\widetilde{\mathscr{U}}_{n,m}| \leq \frac{1}{\nu} \, [\gamma d + N_0(\nu\delta + 2^d \log 2\,(\sqrt{d}/b)^d)]\,(\log 2^n)\,. \end{array}$$

$$\text{(3.11)}\quad \begin{array}{l} \text{for large } n, \text{ for } m, m' \in [-2^n - 1, 2^n]^d, \ \|m - m'\| \leq 2^{n\gamma} \text{ and} \\ V_{n,m} \cap V_{n,m'} = \emptyset, \ |\widetilde{\mathscr{U}}_{n,m}| + |\widetilde{\mathscr{U}}_{n,m'}| \leq \frac{1}{\nu} \, [d(1+\gamma) + 2N_0(\nu\delta \\ +2^d \log 2\,(\sqrt{d}/b)^d)]\,(\log 2^n)\,, \end{array}$$

provided $N_0 = (6m_0 + 1)^d$, and $\widetilde{\mathscr{U}}_{n,m}$ denotes the complement in $V_{n,m}$ of closed subboxes of $\bar{C}_{n,m'}$, $\|m' - m\| \leq 3m_0$ which receive a good point of $C_{n,m'}$.

The point of the construction we have just explained is that when n is large, it provides good lower bounds on $\lambda(O)$, for O an open subset of $\bigcup_{m \in [-2^n, 2^n - 1]^d} C_{n,m}$ in terms of the principal Dirichlet eigenvalue of $-\frac{1}{2}\Delta$ in a ball having volume equal to the maximal $|\widetilde{\mathscr{U}}_{n,m}|$, $m \in [-2^n - 1, 2^n]^d$, such that $V_{n,m} \cap O \cap \mathscr{A}_1 \neq \emptyset$. For instance see (3.11), (3.19) of [7].

We define the open set

$$\text{(3.12)}\qquad \widetilde{\mathscr{T}_n} = (-2^n(\log 2^n)^{1/d},\ 2^n(\log 2^n)^{1/d})^d \ \text{ and } \ \mathscr{T}_n = \frac{1}{2}\,\widetilde{\mathscr{T}_n}\,.$$

To control excursions of the process, we recall that from Lemma 1.3 of [9], there exist positive constants $c_1(d, \nu, W) \in (0,1)$, $c_2(d, \nu, W)$ such that $\mathbb{P}$-a.s.

$$\text{(3.13)}\qquad \text{for large } n, \ \sup_{x \in \widetilde{\mathscr{T}_n}} E_x\Big[\exp\Big\{-\int_0^{T_v} V(Z_s, \omega)ds\Big\}\Big] \leq \exp\{-c_1 v\}\,,$$

as soon as $v \geq c_2(\log 2^n)$, with the notation:

$$\text{(3.14)}\qquad T_v = \inf\{s \geq 0,\ |Z_s - Z_0| \geq v\}\,.$$

We now define Ω_3 as a set of full $\mathbb{P}$-measure on which (3.9), (3.10), (3.11), (3.13) hold for all rationals $r > 0$, $b > a(W)$, $0 < \delta < 1$, with $\alpha > 0$.

We now explain how r, b, δ are picked for each $\eta \in [\frac{1}{2}, 1)$. For each such given η, we choose $r > 0$, $b > a(W)$, $0 < \delta < 1$, rationals satisfying $\alpha > 0$, and

(3.15) for $\omega \in \Omega_3$ and large $t, \lambda(\mathscr{A}^c) \geq 2\,\bar{c}(d,\nu)/\{(1-\eta)(\log t)^{2/d}\}$,

(3.16) for $\omega \in \Omega_3$, $\epsilon > 0$, when t is large for any open sets O_1, $O_2 \subseteq \mathscr{T}_n$, with diam $(O_1 \cup O_2) \leq 3t^{3/4}$ and dist $(O_1 \cap \mathscr{A}_1, O_2 \cap \mathscr{A}_1) \geq (\log t)^{1+\epsilon}$, $\lambda(O_1) \leq (\frac{16}{15})^{2/d}\ \bar{c}(d,\nu)(\log t)^{-2/d}$ implies $\lambda(O_2) \geq c_3(d)\ \lambda(O_1)$,

where $c_3(d)$ is some dimension dependent constant belonging to $(1, \infty)$.

(3.17) for $\omega \in \Omega_3$, $\epsilon > 0$, when t is large if O_1 and O_2 are open subsets of $(-2t, 2t)^d$ such that dist $(\mathscr{A}^1 \cap O_1, \mathscr{A}^1 \cap O_2) \geq (\log t)^{1+\epsilon}$, diam $(O_i) \leq t^{1/3}$, and $\lambda(O_i) \leq (\frac{16}{15})^{2/d}\ \bar{c}(d,\nu)(\log t)^{-2/d}$, $i = 1, 2$, then dist $(O_1, O_2) \geq t^{1/2}$.

Let us give here a word of comment on (3.15) - (3.17), which are applications of the method of enlargement of obstacles.

In fact the estimate (3.15) can be realized uniformly in $\omega \in \Omega$. It follows for instance from (A.12) of [6] where one takes $\mathscr{T} = \mathbb{R}^d$ and $M > 2\bar{c}(d,\nu)/(1-\eta)$. Heuristically it corresponds to the fact that when $x \in \mathscr{A}^c$, the ball $B(x, 2r(\log 2^n)^{1/d})$ has a non vanishing fraction of its volume covered by subboxes receiving a good point and the process has a positive chance of being destroyed before moving to distance $r(\log 2^n)^{1/d}$ when its starts in a box receiving a good point.

As for (3.16) it is proven basically as (3.26) of [7]. Heuristically it comes from the fact that when $\omega \in \Omega_3$, for large n we can find some $V_{n,m}$ intersecting $O_1 \cap \mathscr{A}_1$ such that

$$|\widetilde{\mathscr{U}}_{n,m}| \text{“}\geq\text{”} \frac{15}{16}\ \frac{d}{\nu}\ \log 2^n\ .$$

and by (3.11) for all m' with $V_{n,m'} \cap O_2 \cap \mathscr{A}_1 \neq \emptyset$,

$$|\widetilde{\mathscr{U}}_{n,m'}| \text{“}\leq\text{”} \left(1 + \frac{3}{4} - \frac{15}{16}\right)\ \frac{d}{\nu}\ (\log 2^n) = \frac{13}{16}\ \frac{d}{\nu}\ (\log 2^n)\ ,$$

and now the method of enlargement of obstacles forces

$$\lambda(O_2) \text{“}\geq\text{”} \lambda_d \cdot \left\{\left(\frac{13}{16}\right)\ \frac{d}{\nu}\ \log 2^n\right\}^{-2/d}\ .$$

In fact we can pick here $c_3(d) \in (1,\ (\frac{15}{13})^{2/d})$. The estimate (3.17) is a simple consequence of (3.16).

Our next objective is to describe the excursions of the process $Z_{\cdot}$ in and out of the clearings. Indeed the coarse grained picture of the path that we are soon going to consider will keep track of a skeleton of such excursions. We aim at producing uniform estimates on $Q_{t,\omega}(G)$, $G \in \mathscr{G}$ (the covering), which detect exponential decays with a precision of order $S_1(t)$. This forces us to pick a neighborhood of $\mathscr{A}$ of size $\Delta(t,\omega)$ much larger than $(\log 2^n)^{1/d}$ (see (3.18) below), when defining excursions in and out of $\mathscr{A}$. This departs from our choice in [9], and reduces the complexity of excursions to be considered (see the constraint on N_t in the set E defined after (3.28)).

From now on for the rest of section III we pick a fixed $\omega \in \Omega_3$, $\eta \in [\frac{1}{2}, 1)$ and $\beta \in (0,1)$. By our previous discussion η determines the value of the parameters r, b, δ so that (3.15) - (3.17) hold. Our size of neighborhood is

$$\Delta(t,\omega) = t\,(\log t)^{-2/d} \cdot (\log t)^{1+\beta}/S_1(t) \tag{3.18}$$

Since ω is fixed, it will usually be dropped from the notations. From (1.12), (1.13) we know that for any $\rho > 0$:

$$\Delta_t = o(t^\rho) \text{ and } \log t = o(\Delta_t) \text{ as } t \to \infty\,. \tag{3.19}$$

We now introduce

$$\widetilde{\mathscr{A}} = \text{the } \Delta_t\text{-open neighborhood of } \mathscr{A}. \tag{3.20}$$

It is important for the sequel to control the size of the connected components of $\widetilde{\mathscr{A}}$, very much in the same spirit as (3.9). To this end we have

Lemma 3.1: *For $\omega \in \Omega_3$, $\eta \in [\frac{1}{2}, 1)$, $\beta \in (0,1)$, for large enough t, whenever $C_{n,m} \subseteq \mathscr{A}$ and dist $(C_{n,m}, \mathscr{T}_n) \le \Delta_t$, the connected component of $\widetilde{\mathscr{A}}$ containing $C_{n,m}$ is included in*

$$\widetilde{V}_{n,m} = \Big(\bigcup_{\|m'-m\| \le M(t,\omega)} C_{n,m}\Big)^0\,, \text{ provided} \tag{3.21}$$

$$M_t = ([2\,\Delta_t(\log 2^n)^{-1/d}] + 2)\,m_0 \tag{3.22}$$

(see (3.8) for the definition of m_0).

Proof: The proof is quite similar to that of Lemma 2.3 of [6] which yields (3.9) here. Pick n large enough so that

$$2^{n\alpha-1} > M_t + \Delta_t$$

where α defined in (3.7) is positive by our choice in (3.15) - (3.17). Then define

$$W_{n,m} = \bigcup_{\|m'-m\| \le [2^{n\alpha-1} - \Delta_t]} \bar{C}_{n,m'}$$

and consider the connected component of $\widetilde{\mathscr{A}} \cap W_{n,m}$ containing $C_{n,m}$. Any point in $\widetilde{\mathscr{A}} \cap W_{n,m}$ lies within distance Δ_t of some clearing box $C_{n,m'}$ with $\|m'-m\| \le 2^{n\alpha-1}$, provided n is large.

By (3.9), for large n there are at most m_0 such boxes. Therefore the projection of the connected component of $\widetilde{\mathscr{A}} \cap W_{n,m}$ containing $C_{n,m}$ on each coordinate axis is a segment of length at most $(2\,\Delta_t + (\log 2^n)^{1/d})\,m_0$ which also contains the projection of $C_{n,m}$.

It follows that the connected component of $\widetilde{\mathscr{A}} \cap W_{n,m}$ containing $\widetilde{C}_{n,m}$ does not intersect $\partial W_{n,m}$, and therefore coincides with the component of $\widetilde{\mathscr{A}}$ containing $C_{n,m}$. It now follows that this component is included in $\widetilde{V}_{n,m}$. □

An important consequence of the lemma is that for large t, any component of $\widetilde{\mathscr{A}}$ intersecting $\mathscr{T}_n$ has diameter smaller than:

$$\sqrt{d}(2M_t + 1)(\log 2^n)^{1/d} \leq c_4(d, \nu, W, \eta)\,\Delta_t\,. \tag{3.23}$$

We are now ready to introduce the excursions of the process $Z.$ between $\mathscr{A}$ and $\widetilde{\mathscr{A}}^c$. We let θ_t, $t \geq 0$, stand for the canonical shift on $C(\mathbb{R}_+, \mathbb{R}^d)$, and define

$$\begin{aligned} R_1 &= \inf\{u \geq 0,\ Z_u \in \mathscr{A}\} \leq \infty\,, \\ D_1 &= \inf\{u \geq R_1,\ Z_u \notin \widetilde{\mathscr{A}}\} \leq \infty, \text{ and by induction for } k \geq 1 \\ R_{k+1} &= R_1 \circ \theta_{D_k} + D_k,\ k \geq 1\,, \\ D_{k+1} &= D_1 \circ \theta_{D_k} + D_k,\ k \geq 1\,, \text{ so that} \end{aligned} \tag{3.24}$$

$0 \leq R_1 \leq D_1 \leq R_2 \leq \ldots \leq R_k \leq D_k \leq \ldots \leq \infty$, and all these inequalities, with the exception may be of the first one, are strict if the left member of the inequality is finite. We then define

$$N_t = \sum_{i \geq 1} 1\{R_i \leq t\}\,, \tag{3.25}$$

which measures in some sense the number of excursions between $\mathscr{A}$ and $\widetilde{\mathscr{A}}^c$ up to time t. To control displacements at distance t^α, we also introduce

$$\begin{aligned} H^1 &= T_{t^\alpha} \text{ (see (3.14) for the notation), and for } i \geq 1 \\ H^{i+1} &= H^1 \circ \theta_{H^i} + H^i, \text{ as well as} \end{aligned} \tag{3.26}$$

$$I_t = \sum_{i \geq 1} 1\{H_i \leq t\}\,, \tag{3.27}$$

which measures the number of successive displacements at distance t^α performed up to time t. We define $c_5(d, \nu, W)$ via:

$$c_5\, c_1 = 2\,\bar{c}(d, \nu) \text{ (see (3.13) for the definition of } c_1\text{)}, \tag{3.28}$$

and the event

$$E = \{T_{\mathscr{T}_n} > t,\ 1 \leq N_t \leq c_5\, S_1(t)(\log t)^{-1-\beta},\ I_t < [t^{1-\alpha}]\}\,.$$

As we will shortly see, E carries most of the mass of $Q_{t,\omega}$ and we will cover E by a family $\mathscr{G}$ of events of complexity $\exp\{o(S_1(t))\}$ as $t \to \infty$. We now come to the description of the covering $\mathscr{G}_{\eta,\beta,t,\omega}$. When this causes no ambiguity, we will write $\mathscr{G}$. We define for $i \geq 1$:

$$\begin{aligned} R_i^- &= [R_i],\ R_i^+ = [R_i] + 1\ , \\ D_i^- &= [D_i],\ D_i^+ = [D_i] + 1\ . \end{aligned} \tag{3.29}$$

The collection $\mathscr{G}$ consists of two types of events, depending on whether the time t occurs during an interval $[D_k, R_{k+1})$ (first type) or $[R_k, D_k)$ (second type). Events G of the first type have the form

$$\begin{aligned} G = \ &\{R_{N+1} \wedge T_{\mathscr{T}_n} > t \geq D_N,\ Z_{R_1} \in B(x_1),\ Z_{D_1} \in B(y_1)\ , \dots, \\ &Z_{R_N} \in B(x_N),\ Z_{D_N} \in B(y_N),\ Z_t \in B(x_{N+1}),\ R_1^- = r_1,\ D_1^- = d_1, \dots \\ &R_N^- = r_N,\ D_N^- = d_N\},\ \text{where} \end{aligned}$$

$$1 \leq N \leq c_5\ S_1(t)(\log t)^{-1-\beta}\ , \tag{3.30}$$

$$x_i, y_j \in \frac{1}{\sqrt{d}}\ \mathbb{Z}^d \cap \mathscr{T}_n,\ \text{for}\ 1 \leq i \leq N+1,\ 1 \leq j \leq N\ , \tag{3.31}$$

(3.32) for $1 \leq i \leq N$, $B(x_i)$ intersects $\mathscr{A}$, $B(y_i)$ intersects the connected component of $\widetilde{\mathscr{A}}$ containing $B(x_i)$,

(3.33) the number of distinct connected components of $\widetilde{\mathscr{A}}$ containing some $B(x_i)$, is smaller than $m_0\ [t^{1-\alpha}]$,

$$0 \leq r_1 \leq d_1 \leq \dots \leq r_N \leq d_N \leq t\ \text{ are integers.} \tag{3.34}$$

In close analogy, events of the second type have the form:

$$\begin{aligned} G \ = \ &\{D_N \wedge T_{\mathscr{T}_n} > t \geq R_N,\ Z_{R_1} \in B(x_1),\ Z_{D_1} \in B(y_1)\ , \dots, Z_{R_N} \in B(x_N)\ , \\ &R_1^- = r_1,\ D_1^- = d_1, \dots, R_N^- = r_N\}\ , \end{aligned} \tag{3.35}$$

and now the parameters $N, x_1, y_1, \dots, x_N, y_N, r_1, d_1, \dots, d_{N-1}, r_N$, satisfy entirely analogous constraints to (3.30) - (3.34), with obvious modifications.

The above defined events G keep track of a coarse grained information on the structure of excursions of the path $Z_{\cdot}$ between clearings and forest.

To each above defined element G of the covering $\mathscr{G}$, we associate its "cost" defined for G of the first type through:

$$\text{(3.36)}\quad \begin{aligned} \text{cost}(G) = & -\log e_0(0,x_1) \vee (\lambda_f\, \tau_0) + \sum_{k=1}^{N} a(y_k, x_{k+1}) \vee (\lambda_f\, \tau_k) \\ & + \sum_{k=1}^{N} t_k(\lambda(\mathscr{C}(x_k)) \wedge 1)\,, \end{aligned}$$

where we used the notations:
$\lambda_f = 2\,\bar{c}(d,\nu)/\{(1-\eta)(\log t)^{2/d}\}$ (f stands for forest, see (3.15)), $\mathscr{C}(x)$ when $B(x) \cap \widetilde{\mathscr{A}} \neq \emptyset$, is the component of $\widetilde{\mathscr{A}}$ containing $B(x)$

$$\text{(3.37)}\quad a(z,z',\omega) = -\inf_{x\in B(z)} \log e_0(x,z',\omega),\ z,z' \in \mathbb{R}^d$$

$$\text{(3.38)}\quad \begin{aligned} \tau_k &= (r_{k+1} - d_k - 1)_+, \ \text{ for } 1 \le k < N\,, \\ \tau_N &= (t - d_N - 1)_+,\ \tau_0 = r_1, \ \text{ and} \end{aligned}$$

$$\text{(3.39)}\quad t_k = (d_k - r_k - 1)_+,\ 1 \le k \le N\,.$$

Analogously when G is of the second type we define

$$\text{(3.40)}\quad \begin{aligned} \text{cost}(G) = & -\log e_0(0,x) \vee (\lambda_f\, \tau_0) + \sum_{k=1}^{N-1} a(y_k, x_{k+1}) \vee (\lambda_f\, \tau_k) \\ & + \sum_{k=1}^{N} t_k(\lambda(\mathscr{C}(x_k)) \wedge 1) \end{aligned}$$

the only difference being that now in the definition of t_N, in (3.39), d_N is to be replaced by t. In other words, $t_N = (t - r_N - 1)_+$ in (3.40).

To unify notations we also set $\tau_N = 0$, for G of the second type.

Since (3.30) holds, the τ_k, $0 \le k \le N$, t_k, $1 \le k \le N$, always satisfy

$$\text{(3.41)}\quad t - 2\,c_5\,S_1(t)(\log t)^{-1-\beta} - 2 \le \sum_{0}^{N} \tau_i + \sum_{1}^{N} t_k \le t\,.$$

It is may be useful to mention that condition (3.33) plays an important role in making cost (G) a useful quantity. The true implications of (3.33) will become apparent in Lemma 3.3 below. The main object of our rather lengthy construction is

Theorem 3.2: *For* $\omega \in \Omega_3$, $\eta \in [\frac{1}{2}, 1)$, $\beta \in (0,1)$,

$$\text{(3.42)}\quad \overline{\lim_{t\to\infty}}\ \frac{(\log t)^{2/d}}{t}\ \log\, Q_{t,\omega}(E^c) < 0\,,$$

(3.43) *for large enough* t, $E \subset \bigcup_{G\in\mathscr{G}} G$ *and* $|\mathscr{G}| = \exp\{o(S_1(t))\}$ *as* $t \to \infty$,

$$\text{(3.44)} \qquad \varlimsup_{t\to\infty} S_1(t)^{-1} \sup_{G\in\mathscr{G}} \Big\{ \log E_0\Big[G, \exp\Big\{ - \int_0^t V(Z_s,\omega)ds\Big\}\Big] + \text{cost}(G)\Big\} \le 0\,.$$

Proof: We begin with (3.42). From standard Brownian estimates and (2.11), we know that $\varlimsup_{t\to\infty} (\log t)^{2/d}\, t^{-1} \log Q_{t,\omega}(T_{\mathscr{T}_n} \le t) < 0$. From the strong Markov property and (3.13), for large t:

$$\begin{aligned}
&E_0\Big[T_{\mathscr{T}_n} > t,\, N_t \ge c_5\, S_1(t)(\log t)^{-1-\beta},\, \exp\Big\{ - \int_0^t V(Z_s,\omega)ds\Big\}\Big] \le \\
&E_0\Big[R_M < T_{\mathscr{T}_n},\, \exp\Big\{ - \int_0^{R_M} V(Z_s,\omega)ds\Big\}\Big],\ \text{ with } M = [c_5\, S_1(t)(\log t)^{-1-\beta}] \\
&\le \exp\{-c_1(M-1)\,\Delta_t\}\,,
\end{aligned}$$

since $\Delta_t \ge c_2(\log 2^n)$, and Z_{D_i} and $Z_{R_{i+1}}$ lie at distance $\ge \Delta_t$ for $i \ge 1$. Observe now that

$$c_1\, c_5\, S_1(t)(\log t)^{-1-\beta}\, \Delta_t = 2\,\bar{c}(d,\nu)\, t(\log t)^{-2/d}\,.$$

It now easily follows from the previous upper bound and the asymptotic behavior of the normalizing constant $S_{t,\omega}$ that

$$\varlimsup_{t\to\infty} \frac{(\log t)^{2/d}}{t} \log Q_{t,\omega}(T_{\mathscr{T}_n} > t,\, N_t \ge c_5\, S_1(t)(\log t)^{-1-\beta}) < 0\,.$$

By similar arguments, see also (1.32) of [9]:

$$\varlimsup_{t\to\infty} \frac{(\log t)^{2/d}}{t} \log Q_{t,\omega}(T_{\mathscr{T}_n} > t,\, I_t \ge [t^{1-\alpha}]) < 0\,.$$

The last observation is that for large t:

$$\begin{aligned}
&Q_{t,\omega}(N_t = 0) \\
&\le Q_{t,\omega}(T_{\mathscr{A}^c} > t) \le S_{t,\omega}^{-1} \times c(d)(1+t^d)\, \exp\Big\{ - \frac{2\,\bar{c}(d,\nu)}{1-\eta}\, \frac{t}{(\log t)^{2/d}}\Big\}\,,
\end{aligned}$$

by (3.15) and (1.17) of [7] (with $B = 1$). This finishes the proof of (3.42).

Let us now prove (3.43). The argument is somewhat in the spirit of [9], see after (1.43). Let us first bound $|\mathscr{G}|$. The number of points in $\frac{1}{\sqrt{d}}\,\mathbb{Z}^d \cap \mathscr{T}_n$ is no larger than $c(d)\,2^{nd}\,\log 2^n \le c'(d)\,t^d(\log t)$, and for fixed N between 1 and $c_5\, S_1(t)(\log t)^{-1-\beta}$ the number of possibilities for $x_1, y_1, \dots, x_N, y_N$ (and possibly x_{N+1}) is no larger than:

$$\begin{aligned}
\{c'(d)\,t^d(\log t)\}^{2N+1} &\le \exp\{3c_5\, S_1(t)(\log t)^{-1-\beta} \log\{c'(d)\,t^d(\log t)\}\} \\
&= \exp\{o(S_1(t))\}\,.
\end{aligned}$$

Similarly the number of possibilities for $r_1, d_1, \dots, r_N, d_N$ is no larger than $(t+1)^{2N} \le \exp\{2c_5\, S_1(t)(\log t)^{-1-\beta}\log(1+t)\} = \exp\{o(S_1(t))\}$.
It then easily follows that $|\mathscr{G}| = \exp\{o(S_1(t))\}$.

Let us now explain why for large t, $E \subset \cup_{\mathcal{G}} G$. For large t the requirement $T_{\mathcal{T}_n} > t$, $I_t < [t^{1-\alpha}]$ implies by (3.9) that the sequence Z_{R_j}, $1 \le j \le N_t$, does not meet more than $m_0[t^{1-\alpha}]$ distinct clearing boxes $\bar{C}_{n,m}$. Once this remark is made this easily yields that E is covered by the union of sets G of the first type and second type which in particular fulfill (3.33).

Let us now prove (3.44). We will restrict the estimates to the case of events G of the first type. The case of events of the second type being entirely analogous. For any such G as in (3.36)

$$
\begin{aligned}
& E_0\Big[G, \exp\Big\{-\int_0^t V(Z_s,\omega)ds\Big\}\Big] \le \\
& E_0\Big[T_{\mathcal{T}_n} \wedge t \ge D_N,\ Z_{R_1} \in B(x_1),\ Z_{D_1} \in B(y_1), \dots, Z_{R_N} \in B(x_N)\ , \\
& Z_{D_N} \in B(y_N)\ , \\
& R_1^- = r_1, D_1^- = d_1, \dots, R_N^- = r_N, D_N^- = d_N, \exp\Big\{-\int_0^{D_N} V(Z_s,\omega)ds\Big\} \\
& E_{Z_{D_N}}\Big[R_1 \wedge T_{\mathcal{T}_n} > t - D_N,\ Z_{t-D_N} \in B(x_{N+1})\ , \\
& \exp\Big\{-\int_0^{t-D_N} V(Z_u,\omega)du\Big\}\Big]\Big]\ .
\end{aligned} \tag{3.45}
$$

The term in the inner expectation can be bounded in two ways. When t is sufficiently large, using (1.3), it is on the one hand smaller than

$$
e_0(Z_{D_N},\ x_{N+1},\omega) \le \exp\{-a(y_N,x_{N+1}) + \log t\}\ ,
$$

on the set $\{Z_{D_N} \in B(y_N)\}$ with the notations of (3.37), using the upperbound following (1.28) of [8], see also (1.51) of [9]. On the other hand, by (1.17) of Lemma 1.3 of [7], applied with $B = 1$, and $U = \mathcal{T}_n \cap \mathcal{A}^c$, so that $\lambda(U) \ge \lambda_f$, by (3.15), the inner expectation can also be bounded by

$$
c(d)\,(1+t^d)\ \exp\{-\lambda_f(t - D_N)\} \le c(d)\,(t^d+1)\ \exp\{-\lambda_f\,\tau_N\}
$$

on the set $\{D_N^- = d_N\}$. Here we implicitly assume that t is large enough so that $\lambda_f \le 1$. It now follows that the left member of (3.45) is smaller than:

$$
\begin{aligned}
& E_0\Big[T_{\mathcal{T}_n} \ge D_N,\ Z_{R_1} \in B(x_1), \dots, Z_{D_N} \in B(y_N),\ R_1^- = r_1, \dots, D_N^- = d_N\ , \\
& \exp\Big\{-\int_0^{D_N} V(Z_s,\omega)ds\Big\}\Big] \\
& \times \exp\Big\{-(\lambda_f\,\tau_N) \vee a(y_N,x_{N+1}) + \log c(d) + \log(1+t^d)\Big\}\ .
\end{aligned}
$$

The expectation in the above expression is smaller than:

$$
\begin{aligned}
& E_0\Big[T_{\mathcal{T}_n} > R_N,\ Z_{R_1} \in B(x_1), \dots, Z_{R_N} \in B(x_N),\ R_1^- = r_1, \dots, R_N^- = r_N\ , \\
& \exp\Big\{-\int_0^{R_N} V(Z_u,\omega)du\Big\}\ E_{Z_{R_N}}\Big[D_1 \wedge T_{\mathcal{T}_n} \ge (d_N - R_N)_+\ , \\
& \exp\Big\{-\int_0^{(d_N-R_N)_+} V(Z_u,\omega)du\Big\}\Big]\Big]\ .
\end{aligned}
$$

Now applying (1.17) of [7] again, we see that the inner expectation is smaller than:

$$c(d)(1+t^d)\ \exp\{-\{1\wedge\lambda(\mathscr{C}(x_N))\}\,(d_N-R_N)_+\}$$
$$\le c(d)(1+t^d)\ \exp\{-\{1\wedge\lambda(\mathscr{C}(x_N))\}\,t_N\},\quad \text{on}\ \{R_N^-=r_N\}\,.$$

If we now proceed by induction, we see that for G of the first type and t large enough so that $\lambda_f\le 1$:

$$\log\Big(E_0\Big[G,\ \exp\Big\{-\int_0^t V(Z_s,\omega)ds\Big\}\Big]\Big)\le$$
$$-\Big\{-\log e_0(0,x_1,\omega)\vee(\lambda_f\,\tau_0)\ +\ \sum_1^N(\lambda_f\,\tau_k)\vee a(y_k,x_{k+1})\tag{3.46}$$
$$+\sum_1^N t_k(1\wedge\lambda(\mathscr{C}(x_k)))\Big\}\ +\ (2N+1)(\log c(d)+\log(1+t^d))\,.$$

In case of G of second type we obtain a quite analogous estimate. Now since $N\le c_5\,S_1(t)(\log t)^{-1-\beta}$, the last term $(2N+1)(\log c(d)+\log(1+t^d))$ is clearly $o(S_1(t))$ and our claim (3.44) follows. □

We will now derive a lower bound on cost (G) which will also highlight the role of condition (3.33).

Lemma 3.3: *For $\omega\in\Omega_3$, $\eta\in[\frac{1}{2},1)$, $\beta\in(0,1)$,*

$$\underline{\lim}_{t\to\infty}\ \frac{(\log t)^{1+\beta}}{S_1(t)}\ \inf_{G\in\mathscr{G}}\Big\{\text{cost}(G)-\Big(\eta\ \sup_i\{-\log e_0(0,x_i)\}\tag{3.47}$$
$$+t\min_{1\le k\le N}\lambda(\mathscr{C}(x_k))\wedge\frac{2\bar{c}(d,\nu)}{(\log t)^{2/d}}\Big)\Big\}>-\infty,$$

where for G of first type i varies between 1 and $N+1$, and for G of second type i varies between 1 and N.

Proof: Observe that for $a,b\ge 0$, $a\vee b\ge\eta a+(1-\eta)b$. Therefore

$$\text{cost}(G)\ge\eta\cdot(-\log e_0(0,x_1)+\sum_k a(y_k,x_{k+1}))+$$
$$\frac{2\bar{c}(d,\nu)}{(\log t)^{2/d}}(\sum_k\tau_k)+\min_k(\lambda(\mathscr{C}(x_k))\wedge 1)\cdot(\sum_k t_k)$$

Now for large t, using (3.41) and $2\bar{c}(\log t)^{-2/d}\le 1$, we find that for $G\in\mathscr{G}$:

$$\text{cost}(G)\ge\eta\cdot(-\log e_0(0,x_1)+\sum_k a(y_k,x_{k+1}))$$
$$+t\min_{1\le k\le N}\lambda(\mathscr{C}(x_k))\wedge\frac{2\bar{c}}{(\log t)^{2/d}}-4\bar{c}(\log t)^{-2/d}(c_5\,S_1(t)(\log t)^{-1-\beta}+1)\,.$$

We will now give a lower bound on the expression

(3.48) $$-\log e_0(0,x_1)+\sum_k a(y_k,x_{k+1})\,.$$

The key observation is that for any $2\le i(\le N+1$ if G is of the first type or $\le N$, if G is of the second type), we can extract a sequence $x_{k(j)}$, $0\le j\le \ell$, of at most $m_0\,[t^{1-\alpha}]+2$ points such that

(3.49) $$0=x_{k(0)},\ x_1=x_{k(1)} \text{ and for } 2\le j\le \ell,\ x_{k(j)-1} \text{ and } x_{k(j-1)}$$

belong to the same component of $\widetilde{\mathscr{A}}$, and $x_{k(\ell)}=x_i$ or belongs to the same component of $\widetilde{\mathscr{A}}$.

The construction of this sequence follows the same idea as in [9] (see before (1.44)). We first pick 0 and x_1. If $x_1=x_i$ or lie in the same component of $\widetilde{\mathscr{A}}$ we are finished. Otherwise we look for the last $k\le i$ such that x_k belongs to the component $\mathscr{C}(x_1)$ and define $k(2)=k+1$. If $x_{k(2)}=x_i$ or lie in the same component of $\widetilde{\mathscr{A}}$, we are finished, otherwise we proceed as before. Observe that (3.48) is bigger than:

$$\begin{aligned}&-\log e_0(0,x_1)+\sum_{j=2}^{\ell} a(y_{k(j)-1},x_{k(j)})=\\&-\log e_0(0,x_1)+\sum_{j=2}^{\ell} a(x_{k(j-1)},x_{k(j)})+a(x_{k(\ell)},x_i)\\&+\sum_{j=2}^{\ell}\{a(y_{k(j)-1},x_{k(j)})-a(x_{k(j-1)},x_{k(j)})\}-a(x_{k(\ell)},x_i)\,.\end{aligned}$$

If we now make use of the natural subadditive property of $a(\cdot,\cdot)$ (see [8], (1.10)):

$$\begin{aligned}&-\log e_0(0,y)\le -\log e_0(0,x)+a(x,y),\ \text{ and}\\&a(x,z)\le a(x,y)+a(y,z),\ \text{ for } x,y,z\in\mathbb{R}^d\,,\end{aligned}$$

we see that the right member of the above equality is bigger than

(3.50) $$-\log e_0(0,x_i)-\sum_{j=2}^{\ell} a(x_{k(j-1)},\,y_{k(j)-1})-a(x_{k(\ell)},x_i)\,.$$

Now by (1.11) of [8] and (1.3) we have the following bound on $a(\cdot,\cdot)$ for large t and any $z,z'\in\widetilde{\mathscr{T}_n}$:

(3.51) $$a(z,z')\le c(d,\nu,W)\,(1+|z-z'|)\,(\log t)\,.$$

By construction, $B(y_{k(j)-1})$ intersects $\mathscr{C}(x_{k(j-1)})$ and $x_{k(\ell)}=x_i$ or lie in the same component of $\widetilde{\mathscr{A}}$, and all these components intersect $\mathscr{T}_n$, and have diameter smaller than $c_4\,\Delta_t$ (see (3.23)). Since $\ell\le m_0\,t^{1-\alpha}+1$, our lower bound (3.50) on (3.48) together with (3.51) enable us to conclude that (3.47) holds. □

We now develop a variant of Theorem 2.3, Lemma 3.3 which will be of use in the next section.

Proposition 3.4: *For* $\omega \in \Omega_3$, $\eta \in [\frac{1}{2}, 1)$, $\beta \in (0,1)$,

$$\overline{\lim}\, S_1(t)^{-1} \sup_{G\in\mathscr{G}, z\in\widetilde{\mathscr{T}_n}} \Big\{ \log \Big(E_0\Big[G \cap \{H(z) \le t\},\ \exp\Big\{ -\int_0^t V(Z_s,\omega)ds\Big\}\Big]\Big)$$

$$+\eta \times -\log e_0(0,z) + t \min_{1\le k\le N} \lambda(\mathscr{C}(x_k)) \wedge \frac{2\,\bar{c}(d,\nu)}{(\log t)^{2/d}} \Big\} \le 0\,. \tag{3.52}$$

Proof: The proof is analogous to (3.44), (3.46). For large t consider some $G \in \mathscr{G}_{t,\omega}$ and $z \in \widetilde{\mathscr{T}_n}$. If $B(z)$ intersects some $\mathscr{C}(x_k)$, $k \le N$, $-\log e_0(0,z) \le -\log e_0(0,x_k) + c(d,\nu,W)[1+|z_k - x_k|](\log t)$.

If we restrict in (3.52), $G \in \mathscr{G}$ and $z \in \widetilde{\mathscr{T}_n}$ by the condition $B(z)$ intersects some $\mathscr{C}(x_k)$, $k \le N$, the resulting statement is a direct consequence of (3.44), (3.47) and our bound (3.23) on the size of components of $\widetilde{\mathscr{A}}$ meeting $\mathscr{T}_n$.

If $B(z)$ does not intersect any $\mathscr{C}(x_k)$, $k \le N$, then

$$G \cap \{H(z) \le t\} \subseteq \bigcup_{k_0} G \cap \{D_{k_0} < H(z) \le R_{k_0+1} \wedge t\}$$

with the convention $D_0 = -1$, and k_0 runs between 0 and N when G is of first type and 0 and $N-1$ when G is of second type.

Our claim (3.52) will now follow thanks (3.30), once we prove:

$$\overline{\lim_{t\to\infty}}\ S_1(t)^{-1} \sup_{G,z,k_0} \Big\{ \log E_0\Big[G \cap \{D_{k_0} < H(z) \le R_{k_0+1} \wedge t\},$$

$$+ \exp\Big\{ -\int_0^t V(Z_s,\omega)ds\Big\}\Big] + \eta \times -\log e_0(0,z) \tag{3.53}$$

$$+ t \min_{1\le k\le N} \lambda(\mathscr{C}(x_k)) \wedge \frac{2\bar{c}(d,\nu)}{(\log t)^{2/d}} \Big\} \le 0\,,$$

where now $G \in \mathscr{G}$, $z \in \widetilde{\mathscr{T}_n}$ and $B(z)$ does not intersect any $\mathscr{C}(x_k)$, and k_0 runs between 0 and N or 0 and $N-1$ depending on whether G is of first or second type.

Now the analogous estimate to (3.46) can be performed, and the term $a(y_{k_0}, x_{k_0+1})$ if $k_0 \ge 1$, or $-\log e_0(0,x_1)$ if $k_0 = 0$, can be replaced by $a(y_{k_0}, z)$ when $k_0 \ge 1$, or $-\log e_0(0,z)$ if $k_0 = 0$. So now the argument used in Lemma 3.3 with x_i replaced by z, immediately yields (3.53). □

IV. Confinement property

We are now going to show in this section that for typical cloud configurations with $Q_{t,\omega}$ probability tending to 1 as t tends to infinity the path $Z_.$ does not exit the ball $B_{\alpha_0}(0, S_\eta(t))$, up to time t, this for each $\eta \in (0,1)$. This result will be our first application of the (long) construction of section III. Further applications will come in the sequel [11] to the present article.

Theorem 4.1: *For* $\omega \in \Omega_3$, $\eta \in (0,1)$,

$$\lim_{t\to\infty} Q_{t,\omega}\,(T_{B_{\alpha_0}(0,S_\eta(t))} \le t) = 0\,. \tag{4.1}$$

Proof: We pick some fixed $\omega \in \Omega_3$. With no loss of generality we can assume $\eta \in [\frac{1}{2}, 1)$. We choose $\eta_1 \in (\eta, 1)$. We now apply the construction of section III with η_1 and $\beta = \frac{1}{2}$. From the lowerbound (2.10) applied to $B = B_{\alpha_0}(0, s_1(t))$, (2.9) and (0.6) and (1.15):

$$\varliminf_{t\to\infty}\ (\log\, S_{t,\omega} + \mu_t)/s_1(t) \ge 0\,. \tag{4.2}$$

Now from (3.42), (3.43) it suffices to show that

$$\begin{aligned}&\varlimsup_{t\to\infty}\ S_{\eta_1}(t)^{-1}\ \sup_{G\in\mathcal{G}}\Big\{\log\, E_0\Big[G\cap\{T_{B_{\alpha_0}(0,S_\eta)}\le t\}\,,\\ &\exp\Big\{-\int_0^t V(Z_s,\omega)ds\Big\}\Big] + \mu_t\Big\} < 0\,.\end{aligned} \tag{4.3}$$

We can cover $\widetilde{\mathcal{T}_n}\backslash B_{\alpha_0}(0, S_\eta(t))$ for large t by balls $B(z)$, $z \in D = \frac{1}{\sqrt{d}}\ \mathbb{Z}^d \cap (\widetilde{\mathcal{T}_n}\backslash B_{\alpha_0}(0, S_\eta(t)))$. The number of these balls is polynomially growing in t.

Consider some $G \in \mathcal{G}$. If some x_i which comes in the definition of G, and by construction belongs to $\widetilde{\mathcal{T}_n}\cap\frac{1}{\sqrt{d}}\ \mathbb{Z}^d$, is in D, therefore satisfies $\alpha_0(x_i) \ge S_\eta(t)$, then

$$G\cap\Big\{T_{B_{\alpha_0}(0,S_\eta(t))}\le t\Big\}\subseteq G\cap\{H(x_i)\le t\},\ \text{otherwise}$$
$$G\cap\Big\{T_{B_{\alpha_0}(0,S_\eta(t))}\le t\Big\}\subseteq \bigcup_{z\in D}\ G\cap\{H(z)\le t\}\,,$$

and $\alpha_0(z) \ge \alpha_0(x_i)$ for all i. Therefore (4.3) follows from:

$$\begin{aligned}&\varlimsup_{t\to\infty}\ S_{\eta_1}(t)^{-1}\\ &\times\Big\{\sup_{G,z}\ \log\, E_0\Big[G\cap\{H(z)\le t\},\ \exp\Big\{-\int_0^t V(Z_s,\omega)ds\Big\}\Big] + \mu_t\Big\} < 0\,,\end{aligned} \tag{4.4}$$

where G runs over $\mathcal{G}$, z over D with $\alpha_0(z) \ge \alpha_0(x_i)$, for all x_i coming in the definition of G. Now by (3.52) it suffices to show that

$$\begin{aligned}&\varlimsup_{t\to\infty}\ S_{\eta_1}(t)^{-1}\\ &\times\sup_{G,z}\Big\{\mu_t + \eta_1\ \log\, e_0(0,z) - t\ \min_{1\le k\le N}\ \lambda(\mathcal{C}(x_k))\wedge\frac{2\,\bar{c}(d,\nu)}{(\log\, t)^{2/d}}\Big\} < 0\end{aligned} \tag{4.5}$$

where G, z vary over the same set as before. Now if we use (3.23), for large t, and any G and z as above

$$\bigcup_{k=1}^{N} \mathscr{C}(x_k) \subset B_{\alpha_0}(0,R), \text{ provided}$$
$$R = \alpha_0(z) + \text{const } \Delta_t \ .$$

If we use (0.6) we see the expression in (4.5) is smaller than

$$\begin{aligned} &\overline{\lim_{t\to\infty}}\ S_{\eta_1}(t)^{-1} \sup_{G,z} \Big\{\mu_t - \frac{(\eta+\eta_1)}{2}\ \alpha_0(z) - t\ \lambda(B_{\alpha_0}(0,R))\Big\} \\ = &\overline{\lim_{t\to\infty}}\ S_{\eta_1}(t)^{-1} \sup_{G,z} \Big\{\mu_t - \frac{(\eta+\eta_1)}{2}\ R - t\ \lambda(B_{\alpha_0}(0,R))\Big\} \ . \end{aligned}$$

Since by construction $R \geq S_\eta(t)$, using (1.14) this last expression is smaller than:

$$\overline{\lim_{t\to\infty}}\ S_{\eta_1}(t)^{-1} \Big\{ - \frac{(\eta_1-\eta)}{2}\ S_\eta(t)\Big\} < 0 \ ,$$

which completes our proof of (4.1). □

V. More on the scales in the one dimensional situation

We now come back to the study of the various scales introduced in section I, in the case of a one dimensional medium. Our goal is to show that for large t with probability close to 1, the scales $D_t(\omega)$, $s_1(t,\omega)$, $S_\eta(t,\omega)$ are "comparable" to the scale $s(t) = t/(\log t)^3$.

The reason why we are able to push further our estimates in the one dimensional case, is roughly that for large ℓ the distribution of $\lambda((0,\ell))^{-1/2}$ under $\mathbb{P}$ is comparable to that of a maximum of i.i.d. exponential variables, see (5.4), (5.5) below.

One first naive reason leading to the scale $s(t) = t(\log t)^{-3}$, is to consider the function

(5.1) $$u > 0 \longrightarrow u + t\ \lambda(B_{\alpha_0}(0,u)) \ ,$$

coming in the variational problem defining μ_t, in (1.7). If one replaces $\lambda(B_{\alpha_0}(0,u))$ by the equivalent function $\tilde{c}(d,\nu)(\log u)^{-2/d} (d = 1)$, the new variational problem reaches a minimum at a $u(t) \sim s(t)$. A somewhat deeper heuristic argument can be given along the following line. One breaks space into "independent blocks of size $s(t)$" and then argues that $s(t) \sim t$ size of fluctuations under $\mathbb{P}$ of $\lambda(B_{\alpha_0}(0,s(t)))$.

Let us now introduce some more notations and assumptions. We assume in this section that 0 is a point of density of the shape function $W(\cdot)$, that is

(5.2) $$\underline{\lim_{x\to 0_+}}\ \frac{1}{x} \int_0^x W(y)dy > 0 \ \text{ and } \ \underline{\lim_{x\to 0_+}}\ \frac{1}{x} \int_{-x}^0 W(y)dy > 0 \ .$$

This represents no loss of generality but simply a possible translation by a deterministic amount of the given Poissonian point process and of the given shape function.

We consider the subspace $\widetilde{\Omega} \subset \Omega$ of full $\mathbb{P}$-measure consisting of cloud configurations ω putting infinite mass in $\mathbb{R}_-$ and $\mathbb{R}_+$ and no mass in 0. For $\omega \in \widetilde{\Omega}$, we define the sequence of strictly increasing positive random variables $(T_k^{\pm})_{k\geq 1}$, such that:

$$\omega = \sum_{k\geq 1} \delta_{T_k^+(\omega)} + \sum_{k\geq 1} \delta_{-T_k^-(\omega)}, \text{ for } \omega \in \widetilde{\Omega} .$$

We also introduce the variables:

$$(5.3) \quad \begin{aligned} \ell_k^{\pm} &= T_{k+1}^{\pm} - T_k^{\pm},\ k \geq 1,\ \ell_0 = T_1^+ + T_1^- , \\ M_k^{\pm} &= \max(\ell_{k'}^{\pm},\ 1 \leq k' \leq k),\ M_k = \max(M_k^+, M_k^-) . \end{aligned}$$

Our results rely on the following eigenvalue estimates. On the one hand, since $V(x,\omega) = 0$ when $|x - T_k^{\pm}| > a$ for all $k \geq 1$, we have for $i,j \geq 1$:

$$(5.4) \quad \lambda((-T_{i+1}^-,\ T_{j+1}^+)) \leq \frac{\pi^2}{2} / (\max(M_i^-, M_j^+, \ell_0) - 2a)_+^2 .$$

On the other hand we know from (5.5), (5.7) in lemma 5.2 of [7] that

$$(5.5) \quad \begin{aligned} &\text{for } i,j \geq 1 \text{ with } \max(M_i^+, M_j^-, \ell_0) \geq L , \\ &\lambda((-T_{i+1}^-,\ T_{j+1}^+)) \geq \frac{\pi^2}{2} / (\max(M_i^-,\ M_j^+,\ \ell_0) + K)^2 , \end{aligned}$$

where L are K are positive constants depending on $W(\cdot)$. Our main result is

Theorem 5.1: $(d = 1)$

$$(5.6) \quad \begin{aligned} &\lim_{\rho\to 0} \varliminf_{t\to\infty} \mathbb{P}\Big[\rho t(\log t)^{-3} < s_1(t) < S_{\frac{1}{2}}(t) < \frac{1}{\rho}\, t(\log t)^{-3} , \\ &D_t > \rho\, t(\log t)^{-3}\Big] = 1 . \end{aligned}$$

Proof: We pick $n(t)$ via

$$(5.7) \quad 2^n(\log 2^n)^3 \leq t < 2^{n+1}(\log 2^{n+1})^3, \text{ so that as } t \to \infty$$

$$(5.8) \quad s(t) \stackrel{\text{def}}{=} t(\log t)^{-3} \text{ (which is of order } 2^{n(t)}) .$$

From the law of large numbers, we know that $\mathbb{P}$-a.s

$$(5.9) \quad T_m^+/m \longrightarrow 1/\nu \text{ and } T_m^- \longrightarrow 1/\nu \text{ as } m \longrightarrow +\infty .$$

Our claim (5.6) will follow, once we show:

$$(5.10) \quad \lim_{k_0\to\infty} \varliminf_{t\to\infty} \mathbb{P}[s_1(t) \geq \alpha_0 \min(T_{2^n-k_0}^+\ T_{2^n-k_0}^-)] = 1 ,$$

$$(5.11) \quad \lim_{k_0\to\infty} \varliminf_{t\to\infty} \mathbb{P}[S_{\frac{1}{2}}(t) \leq \alpha_0 \max(T_{2^{n+k_0}}^+,\ T_{2^{n+k_0}}^-)] = 1 ,$$

and for $k_0 \geq 1$,

$$\text{(5.12)} \qquad \begin{aligned} &\lim_{k_1\to\infty}\ \overline{\lim_{t\to\infty}}\ \mathbb{P}[D_t < 2^{n-k_1};\\ &S_{\frac{1}{2}}(t) + 2\alpha_0\, t^{1/3} < \alpha_0 \min(T^+_{2^{n+k_0}},\ T^-_{2^{n+k_0}})] = 0\ , \end{aligned}$$

where α_0 stands for $\alpha_0(1)$. Let us start the proof of (5.10). For $n \geq 1$, $k \in \mathbb{Z}$ with $n + k \geq 0$, we define:

$$I_{n,k} = (-T^-_{2^{n+k}+1},\ T^+_{2^{n+k}+1})\ .$$

Observe that the inequality,

$$\lambda(I_{n,-k}) > \lambda(I_{n,0}) + \frac{\alpha_0}{t}\ \max(T^-_{2^n+1},\ T^+_{2^n+1})\ ,$$

with $k > 0$, $n - k \geq 0$, implies that

$$s_1(t) \geq \alpha_0 \min(T^-_{2^n-k},\ T^+_{2^n-k})\ .$$

Moreover, as follows from (5.8), (5.9), $\mathbb{P}$-a.s. as $t \to \infty$:

$$\text{(5.13)} \qquad \frac{\alpha_0}{t}\ \max(T^-_{2^n+1},\ T^+_{2^n+1}) \sim \frac{\alpha_0}{\nu t}\ 2^n \leq \frac{\alpha_0}{\nu}\ (\log\ 2^n)^{-3}\ .$$

Therefore (5.10) will follow if we show, with $c = 2\alpha_0/\nu$:

$$\lim_{k_0\to\infty}\ \underline{\lim_{t\to\infty}}\ \mathbb{P}[\lambda(I_{n,-k_0}) > \lambda(I_{n,0}) + c(\log\ 2^n)^{-3}] = 1\ .$$

If we now use (5.5) on $\lambda(I_{n,-k_0})$ and (5.4) on $\lambda(I_{n,0})$, it suffices to show

$$\text{(5.14)}\ \lim_{k_0\to\infty}\ \underline{\lim_{t\to\infty}}\ \mathbb{P}\Big[\frac{\pi^2}{2}\ (M_{2^{n-k_0}}+K)^{-2} > \frac{\pi^2}{2}\ (M_{2^n}-2a)_+^{-2}+c(\log\ 2^n)^{-3}\Big] = 1\ .$$

Now M_{2^m} is distributed as the maximum of 2^{m+1} independent exponential variables with parameter ν, for $m \geq 0$, and

$$\text{(5.15)} \qquad \lim_{\ell\to\infty}\ \underline{\lim_{m\to\infty}}\ \mathbb{P}\Big[\frac{1}{\nu}\ \log\ 2^m - \ell \leq M_{2^m} \leq \frac{1}{\nu}\ \log\ 2^m + \ell\Big] = 1\ .$$

Using the inequality $(1-x)_+^{-2} \geq 1 + 2x$ for $x \geq 0$, (5.14) will follow from

$$\lim_{k_0\to\infty}\ \underline{\lim_{t\to\infty}}\ \mathbb{P}\Big[\frac{\pi^2}{2}\ (M_{2^{n-k_0}} + K)^{-2} > \frac{\pi^2}{2}\ \Big(M_{2^n} - 2a - \frac{c}{\pi^2}\frac{(M_{2^n} - 2a)^3}{(\log\ 2^n)^3}\Big)^{-2}\Big] = 1\ .$$
(5.16)

Then (5.15) easily implies

$$\lim_{k_0\to\infty}\ \underline{\lim_{t\to\infty}}\ \mathbb{P}\Big[M_{2^n} - 2a - \frac{c}{\pi^2}\ \frac{(M_{2^n} - 2a)^3}{(\log\ 2^n)^3} > M_{2^{n-k_0}} + K\Big] = 1\ .$$

Our claim (5.16) and therefore (5.10) follows.

Let us prove (5.11). Observe now that when $k_0 \geq 1$, the inequalities

$$\lambda(I_{n,0}) + \tfrac{\alpha_0}{t}\ \max(T^+_{2^n+1}, T^-_{2^n+1}) \leq \lambda(I_{n,k+1}) + \frac{\alpha_0}{2t}\ \min(T^-_{2^{n+k}}, T^+_{2^{n+k}}),\ \text{for all } k \geq k_0\ ,$$

imply that $S_{\frac{1}{2}}(t) \le \alpha_0 \max(T^-_{2^{n+k_0}}, T^+_{2^{n+k_0}})$. With the help of (5.7), (5.9), our claim (5.11) will therefore follow if we show that for $c > 0$:

$$\lim_{k_0\to\infty} \varliminf_{t\to\infty} \mathbb{P}\Big[\bigcap_{k\ge k_0} \{\lambda(I_{n,0}) \le \lambda(I_{n,k+1}) + c\,2^k(\log 2^{n+1})^{-3}\}\Big] = 1\,,$$

Using once more (5.4), (5.5) and the inequality $(1+x)^{-2} \ge 1-2x$ for $x \ge 0$, it suffices to prove that:

$$\lim_{k_0\to\infty} \varliminf_{t\to\infty} \mathbb{P}\Big[\bigcap_{k\ge k_0} \Big\{\frac{\pi^2}{2}\Big/\Big(M_{2^n} - 2a + c\,\frac{2^k}{\pi^2}\,\frac{(M_{2^n}-2a)^3}{(\log 2^{n+1})^3}\Big)^2 < \frac{\pi^2}{2}\Big/(M_{2^{n+k+1}}+K)^2\Big\}\Big] = 1\,,$$

and thanks to (5.15), it is enough to prove that for $c_1, c_2 > 0$

$$\lim_{k_0\to\infty} \varliminf_{t\to\infty} \mathbb{P}\Big[\bigcap_{k\ge k_0} \{M_{2^{n+k+1}} \le a_{n,k}\}\Big] = 1\,,$$

where $a_{n,k} = \dfrac{\log 2^n}{\nu} + c_1\,2^k - c_2$. To this end, notice that

$$\sum_{k\ge k_0} \mathbb{P}[M_{2^{n+k+1}} \ge a_{n,k}] = \sum_{k\ge k_0} 1 - (1-\exp\{-\nu a_{n,k}\})^{2^{n+k+2}}$$

$$\le \sum_{k\ge k_0} 2^{n+k+2} \log[(1-e^{-\nu a_{n,k}})^{-1}]\,.$$

Finally when $n(t)$ is large

$$\le 2\sum_{k\ge k_0} 2^{n+k+2}\, e^{-\nu a_{n,k}} = 2\sum_{k\ge k_0} 2^{k+2}\, e^{-\nu(c_1 2^k - c_2)}\,,$$

which is as small as desired if we pick k_0 large. Notice by the way that $n(t)$ does not appear in the final expression. This shows (5.11).

Let us finally prove (5.12). Thanks to (5.15) it suffices to prove that for $k_0 \ge 1$ and $A > 0$:

(5.18) $$\lim_{k_1\to\infty} \varlimsup_{t\to\infty} \mathbb{P}[D_t < 2^{n-k_1},\ \mathscr{E}] = 0\,, \text{ provided}$$

(5.19) $$\begin{aligned}\mathscr{E} = {} & \{S_{\frac{1}{2}}(t) + 2\alpha_0\, t^{1/3} < \alpha_0 \min\{T^+_{2^{n+k_0}}, T^-_{2^{n+k_0}}\}\} \\ & \cap\{\max\{T^+_{2^{n+k_0+1}}, T^-_{2^{n+k_0+1}}\} < 2^{n+k_0+1}/\nu\} \\ & \cap\Big\{\frac{\log 2^n}{\nu} - A \le M_{2^{n+k_0}} \le \frac{\log 2^n}{\nu} + A\Big\}\,.\end{aligned}$$

Observe that when $(\mu_t + S_{\frac{1}{2}}(t))/t$ is small enough, on the event $\mathscr{E}$, if $x \in B_{\alpha_0}(0, S_{\frac{1}{2}}(t))$ is such that $\bar\lambda(x) \overset{\text{def}}{=} \lambda((x - t^{1/3}, x + t^{1/3})) \le (\mu_t + S_{\frac{1}{2}}(t))/t$, then by Lemma 5.1 of [7], $(x - t^{1/3}, x + t^{1/3})$ intersects some interval determined by consecutive points of the cloud with length $\ell = \ell_k^+$, or ℓ_k^- or ℓ_0 with $1 \le k \le 2^{n+k_0}$ such that

$$\frac{\pi^2}{2}\,(\ell+K)^{-2} \le (\mu_t + S_{\frac{1}{2}}(t))/t \le \frac{\pi^2}{2}\left(\frac{\log 2^n}{\nu} - A - 2a\right)^{-2} +$$
$$\frac{2\alpha_0}{t}\,\max(T^+_{2^{n+k_0}+1}, T^-_{2^{n+k_0}+1}) < \frac{\pi^2}{2}\left(\frac{\log 2^n}{\nu} - A - 2a\right)^{-2}$$
$$+\,\alpha_0\, 2^{n+k_0+2}/\nu\, t \le \frac{\pi^2}{2}\left(\frac{\log 2^n}{\nu} - A - 2a\right)^{-2} + 2c(\log 2^n)^{-3}$$

where $c = 2^{k_0+1}\,\alpha_0/\nu$. Using $(1-x)^{-2} \ge 1+2x$, $x \in (0,1)$, for large t the above last expression is smaller than:

$$\frac{\pi^2}{2}\left(\frac{\log 2^n}{\nu} - A - 2a - \frac{2c}{\pi^2}\left(\frac{\frac{1}{\nu}\log 2^n - A - 2a}{\log 2^n}\right)^3\right)^{-2}$$
$$\le \frac{\pi^2}{2}\left(\frac{\log 2^n}{\nu} - A - 2a - \frac{2c}{\pi^2\,\nu^3}\right)^{-2}.$$

Consequently we see that

$$\ell \ge \frac{\log 2^n}{\nu} - c', \quad \text{with } c' = A + 2a + \frac{2^{k_0+2}}{\pi^2\,\nu^4}\,\alpha_0 + K\,. \tag{5.20}$$

It is straightforward to argue that the labels corresponding to intervals $\ell_k^\pm$, ℓ_0 satisfying (5.20) are well separated, in the sense that:

$$\lim_{k_2\to\infty}\ \overline{\lim_{t\to\infty}}\ \mathbb{P}\Big[M_{2^{n-k_2}} \ge \frac{\log 2^n}{\nu} - c'\Big] = 0\,, \text{ and}$$
$$\lim_{k_2\to\infty}\ \overline{\lim_{t\to\infty}}\ \mathbb{P}\Big[\exists\, k,k' :\ 1 \le k < k' \le 2^{n+k_0},\ k'-k < 2^{n-k_2} \text{ and}$$
$$\ell_k^+ \wedge \ell_{k'}^+ \ge \frac{\log 2^n}{\nu} - c'\Big] = 0$$

together with a similar estimate on $\ell_k^-\ \ell_{k'}^-$.

Finally observe that "well separated labels" correspond to "well separated points of the cloud", in the sense that for given $k_2 \ge 1$:

$$\overline{\lim_{t\to\infty}}\ \mathbb{P}\Big[\Big\{T^+_{2^{n-k_2}} < \frac{2^{n-k_2}}{2\nu}\Big\} \cup \bigcup_{\substack{1\le k<k'\le 2^{n+k_0}\\ k'-k\ge 2^{n-k_2}}} \Big\{T^+_{k'} - T^+_{k+1} < \frac{2^{n-k_2}}{2\nu}\Big\}\Big] = 0\,,$$

as well as a similar estimate on the T^-.

It follows from the above discussion that our claim (5.18) will be proven if for any $k_2 \ge 1$

$$\lim_{k_1\to\infty}\ \overline{\lim_{t\to\infty}}\ \mathbb{P}\Big[\{D_t < 2^{n-k_1}\} \cap \mathscr{E} \cap \Big\{D_t > \frac{2^{n-k_2}}{2\nu} - 4t^{1/3}\Big\}\Big] = 0\,,$$

but this last point is immediate and this finishes the proof of (5.12) and therefore of our claim (5.6). □

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