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Probability Theory and Related Fields

Martingale transforms and Hardy spaces

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Summary. Burkholder's martingale transforms are especially useful in studying "predictable" martingale Hardy spaces. "Characterizations" of such spaces via martingale transforms are provided. In particular, it is shown that for $0<p<\infty$, a martingale in $\mathbf{h}^p$, defined by the conditioned square function, is the martingale transform of a bmo_2 martingale with a multiplier sequence whose maximal function is in L^p.

1 Introduction and preliminaries

Let $(X, \mathscr{F}, \mu)$ be a probability space and let $\{\mathscr{F}_n\}_{n\geqq 1}$ be a nondecreasing sequence of sub-σ-fields of $\mathscr{F}$ such that $\mathscr{F}=\vee\mathscr{F}_n$. We consider martingales $f=\{f_n\}_{n\geqq 1}$ relative to $\{\mathscr{F}_n\}_{n\geqq 1}$ and use the convention that $f_0=0$. The maximal function, the square function and the conditioned square function of f are given, respectively, by $f^*=\sup_n|f_n|$, $S(f)=\left[\sum_{k=1}^{\infty}|d_k f|^2\right]^{1/2}$ and $s(f)=\left[\sum_{k=1}^{\infty}E(|d_k f|^2|\mathscr{F}_{k-1})\right]^{1/2}$ where $d_k f=f_k-f_{k-1}$, $k=1, 2, 3, \ldots$. We consider the following martingale Hardy spaces defined by these functions, $0<p\leqq\infty$:

$$H_*^p=\{f\colon \|f\|_{H_*^p}=\|f^*\|_p<\infty\};$$
$$H_S^p=\{f\colon \|f\|_{H_S^p}=\|S(f)\|_p<\infty\};$$
$$\mathbf{h}^p=\{f\colon \|f\|_{\mathbf{h}^p}=\|s(f)\|_p<\infty\}.$$

It is well-known that $H_*^p\approx H_S^p$ for $1\leqq p<\infty$, and in this case they are denoted by H^p. We also note that $H_S^p\subset\mathbf{h}^p$ when $2\leqq p<\infty$, and $\mathbf{h}^p\subset H_S^p$ when $0<p\leqq 2$. The spaces of martingales with bounded mean oscillation are defined by, for $1\leqq p<\infty$,

$$\mathrm{BMO}_p=\left\{f\colon \|f\|_{\mathrm{BMO}_p}=\sup_n\left\|\left[E\left(\left|\sum_{k=n}^{\infty}d_k f\right|^p\middle|\mathscr{F}_n\right)\right]^{1/p}\right\|_\infty<\infty\right\};$$
$$\mathrm{bmo}_p=\left\{f\colon \|f\|_{\mathrm{bmo}_p}=\sup_n\left\|\left[E\left(\left|\sum_{k=n+1}^{\infty}d_k f\right|^p\middle|\mathscr{F}_n\right)\right]^{1/p}\right\|_\infty<\infty\right\}.$$

All BMO_p spaces for $1 \leqq p < \infty$ are equivalent and are denoted by BMO, with the norm $\|\cdot\|_*$. However bmo_p form a decreasing family as the index p increases. Fefferman's duality theorem gives that $(H^1)' \approx \mathrm{BMO}$ and Herz [5] showed that $(\mathbf{h}^1)' \approx \mathrm{bmo}_2$. In addition, we consider the following "predictable" subspaces of H^p_*:

$$\mathscr{P}^p = \{f \in H^p_* : \exists \text{ nonnegative, nondecreasing adapted } \gamma = \{\gamma_n\}_{n \geqq 1}, \text{ such that } |f_n| \leqq \gamma_{n-1}, \forall n \geqq 1, \text{ and } \|f\|_{\mathscr{P}^p} = \inf_\gamma \|\gamma_\infty\|_p < \infty\}.$$

These martingale spaces have been studied by Garsia [4], Herz [5] and most recently by Weisz [6].

Burkholder's martingale transforms [1] are especially useful in studying "predictable" Hardy spaces such as $\mathscr{P}^p$ and $\mathbf{h}^p$. Denote, for $0 < p \leqq \infty$, the classes of adapted processes $v = \{v_n\}_{n \geqq 1}$,

$$V^p = \{v : \|v\|_{V^p} = \|v^*\|_p < \infty\}.$$

The martingale transform T_v for a given v is defined by $T_v = \sum_{n=1}^{\infty} v_{n-1} d_n f$. The following boundedness results were obtained by Chao-Long [3]:

Theorem A *Let* $0 < p \leqq \infty$, $v \in V^p$ *and* $\frac{1}{r} = \frac{1}{p} + \frac{1}{q}$.

(i) T_v *is of types* (H^q_S, H^r_S) *and* $(\mathbf{h}^q, \mathbf{h}^r)$ *where* $0 < q \leqq \infty$.
(ii) T_v *is of type* (H^q, H^r_*) *where* $1 \leqq q < \infty$.

In the case that $q = \infty$, the spaces H^q_* and H^q_S could be (and are) replaced by BMO, and $\mathbf{h}^q$ by bmo_2:

Theorem B *Let* $0 < p < \infty$ *and* $v \in V^p$. *Then* T_v *is of types* (BMO, H^p_*), (BMO, H^p_S) *and* (bmo_2, $\mathbf{h}^p$).

Other related results are found in [3]. In the next section, we shall derive a boundedness result of T_v on $\mathscr{P}^q$ and use these boundedness properties to characterize the predictable spaces $\mathscr{P}^p$ and $\mathbf{h}^p$ via martingale transform T_v. When both v and f are martingales, $T(v, f) = T_v f$ is one version of paraproducts on martingales. We shall discuss various properties of the bilinear operator T in a sequel to this paper.

2 Martingale transforms and the spaces $\mathscr{P}^p$ and $\mathbf{h}^p$

We first note that the boundedness result of T_v on the maximal Hardy spaces H^q_* as stated in Theorem A(ii) is not completely satisfactory because of the restriction $1 \leqq q < \infty$. Nevertheless, we have the following result of T_v on the subspace $\mathscr{P}^q$ of H^q_*:

Theorem 1 *Let* $0 < p \leqq \infty$, $0 < q < \infty$ *and* $v \in V^p$. *Then* T_v *is of type* $(\mathscr{P}^q, \mathscr{P}^r)$ *with the bound* $C\|v\|_{V^p}$ *where* $\frac{1}{r} = \frac{1}{p} + \frac{1}{q}$.

Proof. For $f \in \mathscr{P}^q$, let $\gamma = \{\gamma_n\}_{n \geqq 1}$ be the (nondecreasing) least majorant of f. We have

$$(*) \qquad |d_n f| \leqq 2\gamma_{n-1}, \qquad |d_n(T_v f)| \leqq 2 v^*_{n-1} \gamma_{n-1} \equiv \rho_{n-1},$$

and

$$|(T_v f)| \leqq (T_v f)^*_{n-1} + \rho_{n-1}.$$

As in Burkholder [2], the pairs $(S(f), f^* + \gamma_\infty)$ and $((T_v g)^*, S(T_v g) + \rho_\infty)$ satisfy the good-λ inequality. Hence

$$\|S(f)\|_q \leqq C(\|f^*\|_q + \|\gamma_\infty\|_q) \leqq C\|f\|_{\mathscr{P}^q},$$

and

$$\begin{aligned}\|(T_v f)^*\|_r &\leqq C(\|S(T_v f)\|_r + \|\rho_\infty\|_r)\\ &\leqq C\|v\|_{V^p}(\|S(f)\|_q + \|\gamma_\infty\|_q)\\ &\leqq C\|v\|_{V^p}\|f\|_{\mathscr{P}^q}.\end{aligned}$$

Therefore, using $(*)$ again, we have

$$\begin{aligned}\|T_v f\|_{\mathscr{P}^r} &\leqq C(\|(T_v f)^*\|_r + \|\rho_\infty\|_r)\\ &\leqq C\|v\|_{V^p}\|f\|_{\mathscr{P}^q}.\end{aligned}$$

This proves Theorem 1.

We note here that the case $q = \infty$ (i.e., T_v with $v \in V^p$ is of type (BMO, $\mathscr{P}^p$) for $0 < p < \infty$), is contained in Theorem 3 below. This is a stronger result than Theorem B since $\mathscr{P}^p \subset h^p \subset H^p_S$ when $0 < p \leqq 2$, so that $\mathscr{P}^p \subset H^p_* \cap H^p_S$ for $0 < p < \infty$.

As indicated by A. Garsia [4], martingale transforms can be used to study the relations amongst the spaces $\mathscr{P}^p$, as well as amongst the spaces $\mathbf{h}^p$. He pointed out that for any given p and q, the elements in $\mathscr{P}^p$ (or $\mathbf{h}^p$) are martingale transforms of those in $\mathscr{P}^q$ (or $\mathbf{h}^q$, respectively). When one of the indices is more convenient to study certain problems (e.g. $q = 2$), such representations would be very useful. However, Garsia's discussion was incomplete in several respects. For instance, in the endpoint case (i.e. $q = \infty$), his argument failed to work for $p \leqq 1$. We shall extend Garsia's results to a sort of "characterization" of the spaces $\mathscr{P}^p$ and $\mathbf{h}^p$, and provide such a characterization for the endpoint case.

Theorem 2 *Let* $0 < p < q < \infty$ *and* $\alpha = \dfrac{pq}{q-p}$.

(i) *For* $f \in \mathscr{P}^p$, *there exists a* $g \in \mathscr{P}^q$ *such that* $f = T_v g = \Sigma v_{n-1} d_n g$ *where* v *is an adapted process in* V^α, *with* $\|v\|^\alpha_{V^\alpha} \leqq \|f\|^p_{\mathscr{P}^p}$, *and*

$$\|g\|_{\mathscr{P}^q} \leqq C_{p,q}\|f\|^{p/q}_{\mathscr{P}^p}.$$

Conversely, for any $v \in V^\alpha$ *and* $g \in \mathscr{P}^q$, $f = T_v g$ *is in* $\mathscr{P}^p$ *and* $\|f\|_{\mathscr{P}^p} \leqq C\|v\|_{V^\alpha}\|g\|_{\mathscr{P}^q}$.
(ii) *Same statement as in* (i) *when* $\mathscr{P}$ *is replaced by* $\mathbf{h}$.

Proof. The first parts of statements (i) and (ii) are both due to Garsia [4] by constructing the process v. (In fact, v is constructed as a nondecreasing process.) The converse parts of (i) and (ii) follow from Theorem 1 and Theorem A.

Now for the endpoint case, $q=\infty$, $\mathscr{P}^\infty \subset L^\infty$ and $\mathbf{h}^\infty$ are replaced as usual by BMO and bmo_2, respectively.

Theorem 3 *Let* $0<p<\infty$. *Given* $f\in\mathscr{P}^p$, *there exist* $g\in\mathrm{BMO}$ *and a (nonnegative, nondecreasing)* $v\in V^p$ *with* $\|g\|_{\mathrm{BMO}_1}\leqq 4$ *and* $\|v\|_{V^p}\leqq C\|f\|_{\mathscr{P}^p}$, *such that* $f=T_v g$. *Conversely, each* $f=T_v g$, *with* $v\in V^p$ *and* $g\in\mathrm{BMO}$, *must be in* $\mathscr{P}^p$ *and* $\|f\|_{\mathscr{P}^p} \leqq C\|v\|_{V^p}\|g\|_*$.

Proof. For the converse assertion, we know from Theorem B that $\|T_v g\|_{H^p_*} \leqq C\|v\|_{V^p}\|g\|_*$. It remains to show that $f=T_v g$ is actually in $\mathscr{P}^p$. In fact, since

$$|(T_v g)_n| \leqq (T_v g)^*_{n-1} + v^*_{n-1}\|g\|_*,$$

we have

$$\begin{aligned}\|T_v g\|_{\mathscr{P}^p} &\leqq C\|(T_v g)^*\|_p + C\|v^*\|_p\|g\|_* \\ &\leqq C\|v\|_{V^p}\|g\|_*.\end{aligned}$$

Now suppose $f\in\mathscr{P}^p$. Choose $0<p_0<p$ and define $v_0=1$ and

$$v_k = \sup_{m\leqq k} E(\gamma_\infty^{p_0}|\mathscr{F}_m)^{\frac{1}{p_0}} \qquad k\geqq 1,$$

where $\gamma=\{\gamma_n\}_{n\geqq 1}$ is the nondecreasing least majorant of f. Consider

$$g_n = \sum_{k=1}^{n} v_{k-1}^{-1}\, d_k f, \qquad n\geqq 1.$$

We have

$$\begin{aligned} g_N - g_{n-1} &= \sum_{n}^{N} v_{k-1}^{-1}\, d_k f \\ &= v_{N-1}^{-1}(f_N - f_{n-1}) + \sum_{n}^{N-1} (f_k - f_{n-1})(v_{k-1}^{-1} - v_k^{-1}). \end{aligned}$$

So

$$\begin{aligned} |g_N - g_{n-1}| &\leqq 2v_{N-1}^{-1}\gamma_{N-1} + 2\sum_{n}^{N-1} \gamma_{k-1}(v_{k-1}^{-1} - v_k^{-1}) \\ &= 2v_{n-1}^{-1}\gamma_{n-1} + 2\sum_{n}^{N-1} v_k^{-1}(\gamma_k - \gamma_{k-1}). \end{aligned}$$

From Jensen's inequality, we get

$$v_k^{-1} \leqq E(\gamma_\infty^{p_0}|\mathscr{F}_k)^{-\frac{1}{p_0}} \leqq E(\gamma_\infty^{-1}|\mathscr{F}_k), \qquad k\geqq 1.$$

Thus

$$E(|g_N - g_{n-1}||\mathscr{F}_n) \leqq 2 + 2E\left(\sum_{n}^{N-1} E\left(\frac{\gamma_k - \gamma_{k-1}}{\gamma_\infty}\middle|\mathscr{F}_k\right)\middle|\mathscr{F}_n\right) \leqq 4.$$

Hence $g \in \mathrm{BMO}$ with $\|g\|_{\mathrm{BMO}_1} \leqq 4$. Moreover,

$$\|v\|_{V^p}^p = \|v^*\|_p^p = E((\gamma_\infty^{p_0})^{*\frac{p}{p_0}})$$
$$\leqq CE(\gamma_\infty^p) = C\|f\|_{\mathscr{P}^p}^p.$$

This completes the proof of the theorem.

Theorem 4 *Let $0<p<\infty$. Given $f \in \mathbf{h}^p$, there exist $g \in \mathrm{bmo}_2$ with $\|g\|_{\mathrm{bmo}_2} \leqq 1$, and a (nonnegative, nondecreasing) $v \in V^p$ with $\|v\|_{V^p} \leqq C\|f\|_{\mathbf{h}^p}$ such that $f = T_v g$. Conversely, each $f = T_v g$, with $v \in V^p$ and $g \in \mathrm{bmo}_2$, must be in $\mathbf{h}^p$ and $\|f\|_{\mathbf{h}^p} \leqq C\|v\|_{V^p}\|g\|_{\mathrm{bmo}_2}$.*

Proof. The converse assertion follows from Theorem B. Given $f \in \mathbf{h}^p$. Choose p_0 with $0<p_0<p$, and define $v_0 = 1$ and for $n \geqq 1$,

$$v_n = \sup_{m \leqq n} E(s(f)^{p_0} | \mathscr{F}_m)^{\frac{1}{p_0}};$$

$$g_n = \sum_1^n v_{k-1}^{-1} d_k f.$$

Thus $f = T_v g$. Moreover, we have

$$E(|g_N - g_n|^2 | \mathscr{F}_n) = \sum_{k=n+1}^{N} E(v_{k-1}^{-2} E(|d_k f|^2 | \mathscr{F}_{k-1}) | \mathscr{F}_n).$$

Since

$$v_{k-1}^{-2} \leqq E(s(f)^{p_0} | \mathscr{F}_{k-1})^{-\frac{2}{p_0}}$$
$$\leqq E(s(f)^{-2} | \mathscr{F}_{k-1}), \qquad k \geqq 1,$$

we get

$$E(|g_N - g_n|^2 | \mathscr{F}_n) \leqq \sum_{k=n+1}^{N} E(s(f)^{-2} E(|d_k f|^2 | \mathscr{F}_{k-1}) | \mathscr{F}_n)$$
$$\leqq 1.$$

Hence $g \in \mathrm{bmo}_2$ and $\|g\|_{\mathrm{bmo}_2} \leqq 1$.

In addition, since $p_0 < p$, we have

$$\|v\|_{V^p} \leqq E((s(f)^{p_0})^{*\frac{p}{p_0}})^{\frac{1}{p}}$$
$$\leqq C\|s(f)\|_p = C\|f\|_{\mathbf{h}^p}.$$

This completes the proof.

Finally, we remark that a similar "characterization" for the spaces H^p is not obtainable. The reason is as follows. For any $v \in V^p$ $(1 \leqq p < \infty)$ and $g \in \mathrm{BMO}$, we have $T_v g \in \mathscr{P}^p$. Since $\mathscr{P}^p$ is a proper subspace of H^p in general, we know $f \in H^p$ could not be represented as $f = T_v g$. Likewise, the statement: "Given $f \in \mathbf{h}^r$, there exist $v \in V^p$ and $g \in \mathscr{P}^q$ with $\frac{1}{r} = \frac{1}{p} + \frac{1}{q}$, such that $f = T_v g$" is false, because, from Theorem 1, we know that such a $T_v g$ is in $\mathscr{P}^r$ and $\mathscr{P}^r$ is a proper subspace of $\mathbf{h}^r$ in general.

The argument used here has been extended by F. Weisz to treat two-parameter martingale spaces [7].

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