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Probability Theory and Related Fields

Fluctuations of shapes of large areas under paths of random walks

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Summary. We discuss statistical properties of random walks conditioned by fixing a large area under their paths. We prove the functional central limit theorem (invariance principle) for these conditional distributions. The limiting Gaussian measure coincides with the conditional probability distribution of certain time-nonhomogeneous Gaussian random process obtained by an integral transformation of the white noise. From the point of view of statistical mechanics the studied problem is the problem of describing the fluctuations of the phase boundary in the one-dimensional SOS-model.

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1 Introduction

The problem of description of shapes of phase boundaries is a well-known problem of statistical mechanics. From the mathematical point of view it is equivalent to the investigation of the asymptotical behaviour of the corresponding sequence of probability measures describing the statistical properties of these boundaries (see the recent book [10] for a discussion of related questions in the case of the two-dimensional Ising model). The simplest variant of this problem arises in the one-dimensional Solid-On-Solid (SOS) model and has a nice probabilistic interpretation.

Consider a one-dimensional random walk $S_0 = 0$, $S_k = \sum_{i=1}^{k} \xi_i$, $k \geq 1$, where $\xi_1, \xi_2, \dots$ are independent identically distributed random variables having finite exponential moments. Assume that the variables ξ_i are integer-valued and the greatest common divisor of their values having non-vanishing probabilities is equal to 1. The random variable

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$$\eta_n = \sum_{k=0}^{n-1} S_k$$

presents the area under the trajectory $S_0, S_1, \dots, S_n$ of this random walk. Fix a real number q such that for some $\varepsilon > 0$ and all n the probabilities of events $\{\eta_n > n^2(q+\varepsilon)\}$ and $\{\eta_n < n^2(q-\varepsilon)\}$ do not vanish. Assume also that for all sufficiently large natural values n the events $\{\eta_n = [n^2 q]\}$ have positive probabilities (here $[n^2 q]$ denotes the integral part of the number $n^2 q$).

Let $x_n(t)$, $t \in [0,1]$, $n = 1, 2, \dots$, be the paths of the random process such that

$$x_n\left(\frac{k}{n}\right) = S_k, \qquad k = 0, 1, 2, \dots, n,$$

and $x_n(t)$ are linear on the intervals $[\frac{k}{n}, \frac{k+1}{n}]$. Define the conditional process

$$\theta_n(t) \equiv \big(x_n(t) | \, \eta_n = [n^2 q]\big).$$

The probability distributions of the normalized conditional processes $n^{-1}\theta_n(t)$ converge weakly to the probability distribution concentrated on some deterministic function $\bar{\mathbf{e}}(t)$. The limiting curve $\bar{\mathbf{e}}(t)$ is interpreted as the shape of the phase boundary in the one-dimensional SOS-model. The formulated result is known in the literature on mathematical physics [5]. From the probabilistic point of view it is a direct corollary from the known results of the sample paths large deviations theory (see, e.g., [8, Chap. 5]).

The limiting curve $\bar{\mathbf{e}}(t)$ can be calculated using the algorithm known in the physical literature as the Wulff construction [21]. To do this one needs to determine the surface tension (an angle-dependent function which can be explicitly calculated in our situation), then integrate it along any smooth curve $e(t)$, $t \in [0,1]$, with $e(0) = 0$, and finally, to minimize the value of this integral in the set of all such functions $e(t)$ with the fixed value of the integral $\int_0^1 e(t)\, dt = q$.

It is expected that a similar construction is also applicable to a wide class of more involved and physically more natural situations but the problem of mathematical justification of the Wulff construction in such situations becomes essentially more difficult. In the case of the two-dimensional Ising model at a sufficiently small temperature this problem was solved in the book [10]. Recently Ioffe ([15, 16]) extended this result to all subcritical temperatures.

The aim of the present paper is to study the asymptotics of fluctuations

$$\theta_n^*(t) \equiv n^{-1/2}\big(\theta_n(t) - n\bar{\mathbf{e}}(t)\big)$$

of the random process $\theta_n(t)$. We prove that the probability distributions of the processes $\theta_n^*(t)$ converge weakly to some Gaussian measure μ in the space $\mathbf{C}[0,1]$ of continuous functions on the segment $[0,1]$. The limiting measure μ presents the conditional distribution of certain inhomogeneous Gaussian process $\overline{\xi}(t)$ with independent increments conditioned by vanishing the value of integrals along its trajectories, $\overline{\eta} = \int_0^1 \overline{\xi}(t)\, dt = 0$.

To prove our main theorem we need to study the asymptotics of probabilities more accurately than it is usually done in the classical large deviations theory. Although there exist some interesting papers with the refinements of classical theory (see [2] and references there) we could not find a possibility to apply those general results to our situation. Instead of this we use a more elementary approach based on the multidimensional limit theorems for the tilted random variables (i.e., the variables obtained from the original ones via Cramèr transformation of their distributions).

There is another and physically even more natural variant of the problem, when the second end of the boundary is also fixed, i.e., when the process $\theta_n(t)$ is additionally conditioned by a condition like $\theta_n(1) = 0$. The results and the proofs for this modified variant of the problem are similar.

Combining the approach developed in the present paper with the methods of the book [10] one can obtain similar results for the fluctuations of the phase boundary in the Ising model but this is a topic for separate considerations.

2 Formulation of results

Let integer-valued random variables $\xi_1, \xi_2, \dots$ be independent and have the same probability distribution $\mathbf{P}(\cdot)$ with finite expectation and variance [1]

$$\mathbf{E}\,\xi \equiv a, \qquad \mathbf{D}\,\xi \equiv \mathbf{E}\,(\xi - \mathbf{E}\,\xi)^2 = \sigma^2 > 0. \tag{2.1}$$

Suppose that the greatest common divisor of the values of variable ξ having non-vanishing probabilities is equal to 1. Denote by $\mathscr{D}_\xi$ the set of real h such that

$$L(h) \equiv \ln \mathbf{E}\, \exp\{h\xi\} < \infty. \tag{2.2}$$

Assume that the set $\mathscr{D}_\xi$ is an interval (can be infinite or semi-infinite one) containing some neighbourhood of the origin.

Consider the random walk $S_0 = 0$, $S_k = \sum_{i=1}^{k} \xi_i, \dots$ generated by random variables ξ_i. For any natural number n define a random polygonal function $x_n(t)$, $t \in [0, 1]$:

$$x_n(t) \equiv S_{[nt]} + \{nt\}\xi_{[nt]+1}, \tag{2.3}$$

where $[a]$ denotes the integral part of a real number a and $\{a\} = a - [a]$ is its fractional part. Denote

$$Y_n = \frac{1}{n}(S_0 + S_1 + \dots + S_{n-1}) = \sum_{j=1}^{n} \left(1 - \frac{j}{n}\right)\xi_j. \tag{2.4}$$

Clearly, Y_n presents the "area under the graph" of the piecewise constant function of $t \in [0, 1)$ which equals S_i on the interval $[i/n, (i+1)/n)$. Our aim here is to investigate the asymptotical behaviour of random paths $x_n(t)$ with fixed "large" value of Y_n.

We start with the following definition.

[1] Here and in the following $\mathbf{E}$ and $\mathbf{D}$ denote the operators of mathematical expectation and of variance corresponding to their probability distribution.

Definition 2.1 *Let the random variable ξ be as described above. For any $h \in \mathscr{D}_\xi$ we consider the random variable ξ_h with the* (h-)**tilted probability distribution** $\mathbf{P}_h(\cdot)$,

$$\mathbf{P}_h(\xi = k) = \mathbf{P}(\xi_h = k) \equiv \exp\{kh - L(h)\}\mathbf{P}(\xi = k), \tag{2.5}$$

where $L(\cdot)$ is the logarithmic moment generating function from (2.2). This transformation of probabilities is called the **Cramèr transformation.**

Note that for every h from the interior $\mathscr{D}_\xi^\circ$ of the interval $\mathscr{D}_\xi$ we have

$$\mathbf{E}\,\xi_h = L'(h), \qquad \mathbf{D}\,\xi_h = L''(h) > 0. \tag{2.6}$$

Moreover, the function $L(h)$ is analytical in some complex neighbourhood $\mathscr{U}(\mathscr{D}_\xi^\circ)$ of the open interval $\mathscr{D}_\xi^\circ$.

Definition 2.2 *A real number r is called* ξ-**admissible** *if there exist $h \in \mathscr{D}_\xi^\circ$ such that*

$$\mathbf{E}\,\xi_h = r.$$

Comparing the last equality with (2.6) one can see that the set of all ξ-admissible numbers coincides with the interval $(\underline{R}, \overline{R})$ where

$$\underline{R} = \inf\{L'(h) : h \in \mathscr{D}_\xi^\circ\}, \qquad \overline{R} = \sup\{L'(h) : h \in \mathscr{D}_\xi^\circ\}.$$

In other words, $(\underline{R}, \overline{R})$ is the image of $\mathscr{D}_\xi^\circ$ under the strictly increasing mapping $h \mapsto L'(h)$.

Let $L_{Y_n}(h)$ be the logarithmic moment generating function corresponding to the random variable Y_n,

$$L_{Y_n}(h) \equiv \ln \mathbf{E}\,\exp\{hY_n\} = \sum_{j=1}^{n} L\Big(\Big(1 - \frac{j}{n}\Big)h\Big), \tag{2.7}$$

where the second equality above is due to the mutual independence of the variables ξ_i. Observe that the function $L_{Y_n}(h)$ is strictly convex in $\mathscr{D}_\xi$ and analytical in the neighbourhood $\mathscr{U}(\mathscr{D}_\xi^\circ)$ defined above. Denote also

$$L_{Y,\infty}(h) \equiv \lim_{n\to\infty} \frac{1}{n} L_{Y_n}(h) = \int_0^1 L(hx)\,dx. \tag{2.8}$$

Consider any sequence nq_n of real numbers such that $n^2 q_n$ are integer and $q_n \to q \neq a/2$ in such a way that

$$q_n - q = o\Big(\frac{1}{\sqrt{n}}\Big) \qquad \text{as} \qquad n \to \infty. \tag{2.9}$$

Definition 2.3 *Any sequence nq_n satisfying (2.9) is called Y_n-***regular** *if the following conditions hold:*

a) for any natural n the probability $\mathbf{P}(Y_n = nq_n)$ is positive;

b) for any natural n the value nq_n is Y_n-admissible, i. e., there exists a solution $h = h_n^0 \in \mathscr{D}_\xi^\circ$ of the equation

$$\frac{d}{dh} L_{Y_n}(h) \Big|_{h=h_n^0} = nq_n; \tag{2.10}$$

c) there exists a solution $\bar{h} = \bar{h}(q) \in \mathscr{D}_\xi^\circ$ of the equation

$$\frac{d}{dh} L_{Y,\infty}(h) \Big|_{h=\bar{h}} = \int_0^1 x\, L'(\bar{h}\, x)\, dx = q. \tag{2.11}$$

For future references we fix some Y_n-regular sequence nq_n. Then the random process

$$\theta_n(t) \equiv \big(x_n(t) | Y_n = nq_n\big) \tag{2.12}$$

is well defined.

Note that the variable Y_n has the mean value $\mathbf{E}\, Y_n = a(n-1)/2$ and the variance $\mathbf{D}\, Y_n = \sigma^2(n-1)(2n-1)/6n$ of order n. Therefore, the condition $2q \neq a$ corresponds to the situation of large values of Y_n. Moreover, in view of the strict monotonicity of the function $L'_{Y,\infty}(\cdot)$ in $\mathscr{D}_\xi^\circ$ the condition

$$\frac{d}{dh} L_{Y,\infty}(h) \Big|_{h=\bar{h}} = q \neq \frac{a}{2} = \frac{d}{dh} L_{Y,\infty}(h) \Big|_{h=0} \tag{2.13}$$

implies $\bar{h} \neq 0$. Define

$$\mathbf{e}_{\bar{h}}(t) \equiv \big(L(\bar{h}) - L(\bar{h} - \bar{h}t)\big) / \bar{h} \tag{2.14}$$

and consider normalized fluctuations of paths $\theta_n(t)$ around $n\mathbf{e}_{\bar{h}}(t)$,

$$\theta_n^*(t) \equiv \frac{1}{\sqrt{n}} \big(\theta_n(t) - n\mathbf{e}_{\bar{h}}(t)\big). \tag{2.15}$$

Let μ_n^* denotes the measure in the space $\mathbf{C}[0,1]$ of continuous functions on the segment $[0,1]$ induced by the probability distribution of the process $\theta_n^*(t)$. The following statement presents the main result of this paper.

Theorem 2.1 *Let a sequence $\xi_1, \xi_2, \ldots$ of integer-valued random variables and a sequence q_n be as described above.*

Then the sequence of measures μ_n^* converges weakly to some Gaussian measure μ^* in $\mathbf{C}[0,1]$. The limiting measure μ^* coincides with the conditional distribution of the random process $\overline{\xi}(t)$, $t \in [0,1]$, obtained by the integral transformation of the white noise dw_x,

$$\overline{\xi}(t) \equiv \int_0^t \big(L''(\bar{h} - \bar{h}x)\big)^{1/2}\, dw_x, \tag{2.16}$$

conditioned by fixing the value

$$\overline{\eta} \equiv \int_0^1 \overline{\xi}(t)\, dt = 0. \tag{2.17}$$

Remark 2.1.1 The limiting conditional Gaussian process $\overline{\theta}(t) = \left(\xi(t)|\overline{\eta} = 0\right)$ has zero mean and its correlation function $\overline{\mathscr{B}}_0(t,s) = \mathbf{E}\,\overline{\theta}(t)\overline{\theta}(s)$ can be easily calculated (see, e.g., [20, Chap. 2]),

$$\overline{\mathscr{B}}_0(t,s) = \int_0^{s\wedge t} L''\big(\bar{h}(1-x)\big)\,dx - \frac{\displaystyle\int_{1-s}^1 x\,L''\big(\bar{h}x\big)\,dx \int_{1-t}^1 x\,L''\big(\bar{h}x\big)\,dx}{\displaystyle\int_0^1 (1-x)^2 L''\big(\bar{h}(1-x)\big)\,dx}, \tag{2.18}$$

where $s \wedge t = \min(s,t)$.

Remark 2.1.2 Definition (2.15) and Theorem 2.1 together imply the law of large numbers for $\theta_n(t)$: distributions of random processes $n^{-1}\theta_n(t)$ converge weakly in $\mathbf{C}[0,1]$ to the distribution concentrated on the (deterministic) function $\mathbf{e}_{\bar{h}}(\cdot)$.

Remark 2.1.3 Using the inequalities for dual functions (see Property A.3 in Appendix below) one can easily obtain the estimates for moderate and large deviation probabilities of the process $\theta_n(t)$. Then arguments similar to those used in Theorems 5.2 and 5.4 below together with relations (5.5) and (2.9) give the following estimate

$$\frac{1}{n}\mathbf{E}\left(x_n(t)|Y_n = nq_n\right) - \mathbf{e}_{\bar{h}}(t) = o\left(\frac{1}{\sqrt{n}}\right).$$

As a result, the random process

$$\overline{\theta}_n^*(t) \equiv \frac{1}{\sqrt{n}}\big(\theta_n(t) - \mathbf{E}\,\theta_n(t)\big), \qquad t \in [0,1],$$

has the same asymptotical behaviour as $\theta_n^*(t)$.

Plan of the proof of Theorem 2.1. First we consider the process

$$X_n(t) \equiv S_{[nt]}, \qquad t \in [0,1].$$

For every natural number k and any set of real numbers s_i, $0 < s_1 < s_2 < \ldots < s_k \le 1$ we form a random vector

$$\Omega_n \equiv \big(Y_n, X_n(s_1), \ldots, X_n(s_k)\big) \in R^{k+1}. \tag{2.19}$$

Then for every $\mathbf{M}_n = (m_n^0, m_n^1, \ldots, m_n^k)$ with $nm_n^0, m_n^1, \ldots, m_n^k \in Z^1$ we have

$$\mathbf{P}\big(X_n(s_1) = m_n^1, \ldots, X_n(s_k) = m_n^k | Y_n = m_n^0\big) = \frac{\mathbf{P}(\Omega_n = \mathbf{M}_n)}{\mathbf{P}(Y_n = m_n^0)}, \tag{2.20}$$

where $\mathbf{P}(\cdot|\cdot)$ denotes the conditional probability (provided the probability of the condition does not vanish).

To estimate the numerator and the denominator of the last fraction in the case of $m_n^0 = nq_n$ (see (2.12)) we build new random variables $Y_{n,h}$ and $\Omega_{n,\mathbf{H}}$ using the Cramèr transformation with parameters h and $\mathbf{H}$ of the original distributions of Y_n and Ω_n and then prove the corresponding local central limit theorems for these random variables. This makes possible to evaluate the asymptotics of conditional probabilities (2.20) by using the normal approximation for these tilted random variables. As a result, we obtain the central limit theorem for the finite-dimensional distributions of the conditional random process

$$\Theta_n(t) \equiv \left(X_n(t) \middle| Y_n = nq_n\right). \tag{2.21}$$

Noting that the interpolated conditional random process (recall (2.12))

$$\theta_n(t) \equiv \left(x_n(t) \middle| Y_n = nq_n\right)$$

is obtained from the process $\Theta_n(t)$ by adding the random variables uniformly bounded in probability as $n \to \infty$ we conclude that the finite-dimensional distributions of the process $\theta_n^*(t)$ tend to the same limiting distributions as for $\Theta_n^*(t)$ defined similarly to (2.15).

Finally, we establish the weak compactness of the sequence of measures μ_n^* in $\mathbf{C}[0, 1]$ by proving the inequality

$$\mathbf{E}\, |\theta_n^*(t) - \theta_n^*(s)|^4 \le C\,|t - s|^{7/4}$$

with some constant $C > 0$ uniformly in n and all $t, s \in [0, 1]$. It remains to apply Theorem 2.2 from [12, Chap. 9].

The detailed proof of Theorem 2.1 is given in Sect. 4–6 of this paper. □

The explicit limit formula (2.14) for the function $\mathbf{e}_{\bar{h}}(t)$ can be explained by using well-known facts from the theory of large deviations. Let $\mathbf{C}_0$ denotes the space of all absolutely continuous functions $f(t), t \in [0, 1]$, such that $f(0) = 0$. Define

$$I(f) = \int_0^1 L^*\left(f'(t)\right) dt, \qquad f \in \mathbf{C}_0, \tag{2.22}$$

where $L^*(x)$ is the Legendre transformation [2]

$$L^*(x) = \sup_h \left(xh - L(h)\right)$$

of the logarithmic moment generating function $L(h)$ from (2.2) and $f'(t)$ is the derivative of the function f. The functional (2.22) is known in the literature as the *rate function of the sample paths large deviation principle* for the random walk $S_0 = 0$, $S_k = \sum_{i=1}^k \xi_i$, $k \ge 1$, (see, e.g., [8, Chap. 5]). The infimum of this rate function governs the asymptotics of large deviation probabilities for the considered random walk. The direct check assures us that the function $\mathbf{e}_{\bar{h}}(t)$ from (2.14) gives the solution of the following variational problem

[2] Here and in the following we omit restrictions near the signs like upper bounds, sums, integrals etc. when the appropriate operation is going over the whole set of possible values of parameters, summation indices, integration variables respectively.

$$I(f) \to \inf; \quad f \in \mathbf{C_0}, \quad \int_0^1 f(t)\,dt = q. \tag{2.23}$$

Observe however that results concerning the paths large deviation principle are usually formulated in the so-called integral form when the integral condition $Y_n \geq nq_n$ is considered instead of the local one, $Y_n = nq_n$. Nevertheless, these results are valid also in our case since the local condition $Y_n = nq_n$ can be understood as a limiting case of the integral one, $nq_n \leq Y_n \leq n(q_n + \epsilon)$, when first $n \to \infty$ and then $\epsilon \searrow 0$.

A similar result holds also in the case of non-lattice random variables. Namely, let $\xi_1, \xi_2, \ldots$ be a sequence of independent identically distributed random variables satisfying conditions (2.1) and (2.2). Assume that these variables have a bounded continuous probability density. Define the random polygon $x_n(t)$, $t \in [0, 1]$, and the area Y_n by (2.3) and (2.4) respectively. Then the variables $x_n(t)$ and Y_n also have bounded continuous probability densities.

Fix any sequence of real numbers q_n satisfying (2.9) with $2q \neq a$. We call this sequence *Y_n-regular* if the density of Y_n does not vanish at the point nq_n and conditions b), c) of Definition 2.3 hold true. Then for every natural number k and any set of real numbers $s_i, 0 < s_1 < s_2 < \ldots < s_k \leq 1$ the mutual conditional probability densities of the random variables $x_n(s_1), x_n(s_2), \ldots, x_n(s_k)$ under the condition $Y_n = nq_n$ are well defined. Now we can define the conditional random process (2.12) as the random process with finite dimensional distributions having these conditional densities.

Theorem 2.2 *Let a sequence $\xi_1, \xi_2, \ldots$ of random variables having a bounded continuous probability density and a sequence q_n be as described above. Then the statement of Theorem 2.1 holds true.*

Remark 2.2.1 The condition of the existence of a continuous probability density used in this theorem is essential since otherwise there are no natural definition of the conditional distributions under the condition $\{Y_n = nq_n\}$. The results of this paper can also be extended to a more wide class of non-lattice random variables, if we change the condition $\{Y_n = nq_n\}$ to the condition $\{|Y_n - nq_n| < \varepsilon_n\}$ where $\varepsilon_n = o(n^{1/2})$ as $n \to \infty$.

Plan of the proof of Theorem 2.2. The proof of this theorem is very similar to that of Theorem 2.1. The only essential difference is that we need now to prove the corresponding local central limit theorems for the probability densities of random elements $Y_{n,h}$ and $\Omega_{n,\mathbf{H}}$ instead of the probabilities of their values. But since such theorems for densities are very similar to their analogs in the discrete case (see, e.g., [18]) we omit the proof of Theorem 2.2. □

As mentioned in Sect. 1 there is another interesting variant of the main problem when one considers the conditional distributions of random walks $x_n(t)$ with fixed value of area Y_n and value $x_n(1) = S_n$ at the terminating point. Problems of this kind arise in statistical mechanics (see, e.g., [5], [17]) in the context of the so-called SOS-models (see discussion of this and related questions in Sect. 3 of the present paper). We formulate the corresponding results.

Consider the random vector $\Lambda_n = (Y_n, S_n)$ and denote its logarithmic moment generating function by $L_{\Lambda_n}(H)$, $H = (h_0, h_1)$,

$$L_{\Lambda_n}(H) \equiv \ln \mathbf{E} \exp\{h_0 Y_n + h_1 S_n\} = \sum_{j=1}^{n} L\Big(\Big(1 - \frac{j}{n}\Big)h_0 + h_1\Big).$$

Let $\mathscr{D}_{\Lambda_n}$ be the set $\{H = (h_0, h_1) \in R^2 : L_{\Lambda_n}(H) < \infty\}$. Similarly to (2.8) define

$$L_{\Lambda,\infty}(H) \equiv \lim_{n\to\infty} \frac{1}{n} L_{\Lambda_n}(H) = \int_0^1 L\big((1-x)h_0 + h_1\big)\, dx.$$

Clearly, $L_{\Lambda,\infty}(H)$ is finite for all H from the set

$$\mathscr{D}_\Lambda = \Big\{H = (h_0, h_1) : h_1, h_0 + h_1 \in \mathscr{D}_\xi^\circ\Big\}.$$

Let $A_n = (nq_n, nb_n)$ be any sequence of real numbers such that $n^2 q_n$ and nb_n are integer and $n^{-1}A_n \to A = (q, b)$, $2q \neq b$, in such a way that

$$n^{-1}A_n - A = o\Big(\frac{1}{\sqrt{n}}\Big) \qquad \text{as} \qquad n \to \infty. \tag{2.24}$$

Definition 2.4 *Any sequence A_n satisfying (2.24) is called Λ_n-**regular** if the following conditions hold:*

a) for any natural n the probability $\mathbf{P}(\Lambda_n = A_n)$ is positive;

b) for any natural n the pair A_n is Λ_n-admissible, i. e., there exists a solution $H = H_n \in \mathscr{D}_\Lambda$ of the equation

$$\nabla_H L_{\Lambda_n}(H) \Big|_{H=H_n} = A_n;$$

c) there exists a solution $\widehat{H} = \widehat{H}(A) \in \mathscr{D}_\Lambda$ of the equation

$$\nabla_H L_{\Lambda,\infty}(H) \Big|_{H=\widehat{H}} = A. \tag{2.25}$$

In the last two equalities ∇_H denotes the gradient with respect to $H = (h_0, h_1)$.

Fix any Λ_n-regular sequence A_n. Then the random process

$$\hat{\theta}_n(t) \equiv \big(x_n(t) \mid \Lambda_n = A_n\big)$$

is well defined. Determine the quantities $\hat{h}_0 = \hat{h}_0(q, b)$ and $\hat{h}_1 = \hat{h}_1(q, b)$ from the following system of two equations (cf. (2.25)):

$$\begin{cases} \displaystyle\int_0^1 L'\big(\hat{h}_1 + y\hat{h}_0\big)\, dy = b, \\ \displaystyle\int_0^1 y\, L'\big(\hat{h}_1 + y\hat{h}_0\big)\, dy = q. \end{cases} \tag{2.26}$$

Using convexity property of $L(\cdot)$ it is possible to prove that there exists a unique pair of such quantities (see, e. g., [14, Appen.]). Moreover, they satisfy the condition $\hat{h}_0 \neq 0$ and so one can define the function

$$\begin{aligned}\hat{\mathbf{e}}_{\hat{h}_0,\hat{h}_1}(t) &\equiv \int_0^t L'\left(\hat{h}_1 + (1-y)\hat{h}_0\right) dy \\ &= \left(L(\hat{h}_1 + \hat{h}_0) - L(\hat{h}_1 + (1-t)\hat{h}_0)\right) / \hat{h}_0.\end{aligned} \tag{2.27}$$

(Note that the integral expression in (2.27) was obtained earlier in [5], although the simple formula in the last part of (2.27) was not presented there.) Consider normalized fluctuations of the paths $\hat{\theta}_n(t)$ around the function $n\hat{\mathbf{e}}_{\hat{h}_0,\hat{h}_1}(t)$,

$$\hat{\theta}_n^*(t) \equiv \frac{1}{\sqrt{n}} \left(\hat{\theta}_n(t) - n\hat{\mathbf{e}}_{\hat{h}_0,\hat{h}_1}(t)\right),$$

and denote by $\hat{\mu}_n^*$ the corresponding measure in the space $\mathbf{C}[0,1]$.

Theorem 2.3 *Let a sequence $\xi_1, \xi_2, \ldots$ of integer-valued random variables and sequences q_n and b_n be as described above.*

Then the sequence of measures $\hat{\mu}_n^*$ converges weakly to some Gaussian measure $\hat{\mu}^*$ in $\mathbf{C}[0,1]$. The limiting measure $\hat{\mu}^*$ coincides with the conditional probability distribution of the random process $\hat{\xi}(t)$, $t \in [0,1]$, obtained by the integral transformation of the white noise dw_x,

$$\hat{\xi}(t) \equiv \int_0^t \left(L''(\hat{h}_1 + (1-x)\hat{h}_0)\right)^{1/2} dw_x, \qquad t \in [0,1],$$

conditioned by the conditions

$$\hat{\eta} \equiv \int_0^1 \hat{\xi}(t)\, dt = 0 \qquad \textit{and} \qquad \hat{\xi}(1) = 0.$$

Remark 2.3.1 Note, that the equations (2.26) can be rewritten in the equivalent form

$$\begin{cases} \dfrac{1}{\hat{h}_0}\left(L(\hat{h}_1 + \hat{h}_0) - L(\hat{h}_1)\right) = b, \\ \dfrac{1}{\hat{h}_0} L(\hat{h}_1 + \hat{h}_0) - \dfrac{1}{\hat{h}_0^2} \displaystyle\int_0^{\hat{h}_0} L(\hat{h}_1 + y)\, dy = q \end{cases}$$

obtained in [17, Theorem 3] in a similar situation.

Proof of Theorem 2.3 follows the same scenario as that of Theorem 2.1. For details see [14, Chap. 1]. □

Observe that the function $\hat{\mathbf{e}}_{\hat{h}_0,\hat{h}_1}(\cdot)$ presents the solution of the following variational problem (cf. (2.23))

$$I(f) \to \inf; \quad f \in \mathbf{C_0}, \quad \int_0^1 f(t)\, dt = q, \quad f(1) = b. \tag{2.28}$$

Clearly, a result similar to Theorem 2.3 holds also in the non-lattice case under the conditions analogous to that of Theorem 2.2. But since our "Formulation of results" is rather long yet, we will not make it longer and leave the details for the reader.

3 Physical interpretation

The situation considered in the present paper is known in statistical physics as 1-dimensional SOS-model (Solid-On-Solid model), which is the simplest interface model. In view of its simplicity the 1D SOS-model is used for investigating the properties of phase boundaries and so it was studied in the literature (see, e.g., [6], [5], [17]). Our aim here is to discuss formulated results from the physical point of view.

Let us recall some notions needed (cf. [1], [6]). The SOS-model consists of the interfaces without overhangs and therefore its configurations of the horizontal length N are represented by sets of heights $\{r_i\}_{i=0}^N$, $r_0 = 0$. The energy of the configuration $R = \{r_i\}_{i=0}^N$ is determined by the Hamiltonian

$$\mathscr{H}_N(R) = \sum_{i=0}^{N-1} U(r_{i+1} - r_i),$$

where $U(\cdot)$ is a real-valued function. There are many possible natural choices for $U(\cdot)$ (see, e.g., [6] for a list of examples). For the sake of simplicity we restrict ourselves to the case of integer-valued heights r_i (though the generalization to the non-integer case is straightforward).

Introducing a positive parameter β called an *inverse temperature* and assuming the finiteness of the *partition function*

$$Z_{N,\beta} = \sum_{r_1 \in Z^1} \cdots \sum_{r_N \in Z^1} e^{-\beta \mathscr{H}_N(R)}$$

we define the Gibbs probability distribution in the ensemble of surfaces $\{r_i\}_{i=0}^N$ by

$$\mathbf{P}_{N,\beta}(R) = Z_{N,\beta}^{-1} e^{-\beta \mathscr{H}_N(R)}.$$

Rewriting the last expression in terms of jumps $k_i \equiv r_i - r_{i-1}$, $i = 1, \dots, n$, we see that this Gibbs distribution coincides with the probability distribution of random walk $r_0 = 0$, $r_j = \sum_{i=1}^{j} k_i$, $j \geq 1$, generated by the sequence of independent (integer-valued) jumps k_i having the same distribution

$$\mathbf{P}_\beta(k) = \frac{e^{-\beta U(k)}}{Z_\beta}, \qquad \text{where} \qquad Z_\beta = \sum_{k \in Z^1} e^{-\beta U(k)}, \tag{3.1}$$

i.e., with the situation considered in the present paper.

Consequently, theorems formulated in Sect. 2 describe the statistical properties of the interfaces in 1D SOS-model: Theorem 2.3 studies the interfaces with fixed endpoints and the area under the interface, Theorem 2.1 describes the similar situation but with the free right end, and Theorem 2.2 generalizes the previous situation to the case of non-integer values r_i.

There is a well-known general approach in statistical mechanics called the Wulff construction [21], which is used for describing the asymptotical shapes of interfaces. It is based on a fundamental notion of the surface tension and consists in minimizing the total surface tension along interfaces with given constraints.

To discuss it more explicitly we fix a sequence of real numbers b_N such that $b_N \to b = \tan\varphi$ (with $\varphi \in (-\pi/2, \pi/2)$), nb_N is an integer number, and consider the constrained interfaces $R_{b_N} = \{r_i\}_{i=0}^N$ satisfying the condition $r_N = Nb_N$. Denote the corresponding partition function by $Z_{N,\beta}(Nb_N)$,

$$Z_{N,\beta}(Nb_N) = \sum_{R_{b_N}} e^{-\beta\mathcal{H}_N(R_{b_N})}.$$

The quantity (see, e. g., [17])

$$T(\varphi) = -\frac{\cos\varphi}{\beta} \lim_{N\to\infty} \frac{\ln Z_{N,\beta}(Nb_N)}{N} \tag{3.2}$$

is called the *surface tension* of the inclined interface with the slope angle φ. The last limit can be evaluated explicitly if we assume in addition that for all ε in some neighbourhood of the origin the following sum is finite

$$Z_\beta(\varepsilon) = \sum_{k\in Z^1} e^{\varepsilon k} e^{-\beta U(k)} < \infty.$$

(Note that this condition holds true in all situations listed in [6]). Then the known Cramèr theorem (see, e.g., [9], [7], [4]) is applicable and one obtains

$$\begin{aligned} T(\varphi) &= -\frac{\cos\varphi}{\beta}\Big(\lim_{N\to\infty} \frac{\ln \mathbf{P}_{N,\beta}(r_N = Nb_N)}{N} + \ln Z_\beta\Big) \\ &= \frac{\cos\varphi}{\beta}\big(L_\beta^*(\tan\varphi) - L_\beta(0)\big), \end{aligned} \tag{3.3}$$

where $L_\beta^*(\cdot)$ is the Legendre transformation of the function $L_\beta(\cdot) \equiv \ln Z_\beta(\cdot)$. Observe, that instead of the integral form of the Cramèr theorem describing the asymptotics of the probability $\mathbf{P}_{N,\beta}(r_N \geq Nb_N)$ we use here the local one (for $\mathbf{P}_{N,\beta}(r_N = Nb_N)$), which is also true (see, e.g., [11]).

Consider the space $\mathbf{C_0}$ of all absolutely continuous functions $f(t)$, $t \in [0,1]$, such that $f(0) = 0$. Every $f \in \mathbf{C_0}$ is rectifiable and so it is possible to introduce the natural parameterization along the graph $\gamma(f)$ of the function $f(\cdot)$ denoting by $s = s(t)$ the arc length from the starting point $(0,0)$ of $\gamma(f)$ till the point $(t, f(t))$. Let φ_s and ds denote the slope angle of the tangent and the length element at the point s under this parameterization. The *Wulff functional* is defined by

$$\mathcal{W}(f) \equiv \int_{\gamma(f)} T(\varphi_s)\, ds, \tag{3.4}$$

where $T(\cdot)$ is the surface tension introduced in (3.2).

Consider the collection of one-dimensional SOS interfaces satisfying the restrictions from Theorem 2.3. According to the Wulff principle the limiting shape $\mathbf{e}_{q,b}(t)$ of the phase boundary under such constraints presents the solution of the following variational problem:

$$\mathscr{W}(f) \to \inf; \qquad f \in \mathbf{C}_0, \quad \int_0^1 f(t)\,dt = q, \quad f(1) = b. \tag{3.5}$$

On the other hand, the Wulff functional $\mathscr{W}(\cdot)$ and the rate function $I(\cdot)$ for the sample paths large deviation principle defined as in (2.22) for the random walk with the single step distribution given by (3.1) are closely related. Namely, substituting (3.3) into (3.4) and changing the variables one easily obtains the relation

$$\mathscr{W}(f) = \frac{1}{\beta}\int_0^1 L^*\big(f'(t)\big)\,dt - \frac{L_\beta(0)}{\beta} = \frac{1}{\beta}I(f) - \frac{L_\beta(0)}{\beta}$$

that immediately implies the equivalence of variational problems (2.28) and (3.5). Consequently, the function $\hat{\mathbf{e}}_{\hat{h}_0,\hat{h}_1}(t)$ from (2.26)–(2.27) coincides with the Wulff shape $\mathbf{e}_{q,b}(t)$ (i. e., the solution of (3.5)) and thus one comes to the following conclusion: *in the case of 1D SOS-model the Wulff principle coincides with the standard approach of large deviation theory.*

One can expect that with the appropriate definition of the surface tension the Wulff principle is applicable to much more general situations. As it was mentioned in Sect. 1, the Wulff principle has been proved for the two-dimensional ferromagnetic Ising model ([10], [15, 16]). Its mathematical justification for two-dimensional SOS-models describing the surfaces separating three-dimensional phases is a very difficult open mathematical problem.

4 Limit theorems for tilted distributions

Let $\xi_1, \xi_2, \ldots$ be a sequence of i. i. d. random variables introduced in Sect. 2 and Y_n be the area from (2.4), $Y_n = \sum_{j=1}^n \left(1 - \frac{j}{n}\right)\xi_j$. For any $h \in \mathscr{D}_\xi$ consider the tilted variable $Y_{n,h}$ with the distribution

$$\mathbf{P}\big(Y_{n,h} = x\big) = \exp\{xh - L_{Y_n}(h)\}\,\mathbf{P}\big(Y_n = x\big), \tag{4.1}$$

where $L_{Y_n}(\cdot)$ is the logarithmic moment generating function for the random variable Y_n and x is a real number of the kind m/n with integer m. Clearly, the mean value and the variance of $Y_{n,h}$ can be calculated from the equalities

$$\mathbf{E}\,Y_{n,h} = \frac{d}{dh}L_{Y_n}(h), \qquad \mathbf{D}\,Y_{n,h} = \frac{d^2}{dh^2}L_{Y_n}(h). \tag{4.2}$$

The probability of interest $\mathbf{P}(Y_n = x)$ can be expressed as

$$\mathbf{P}\,(Y_n = x) = \exp\{-\big(xh - L_{Y_n}(h)\big)\}\,\mathbf{P}\,\big(Y_{n,h} = x\big)\,. \tag{4.3}$$

Therefore, if for given x one can evaluate the probability $\mathbf{P}(Y_{n,h} = x)$ from the left hand side of (4.1) explicitly, say can prove the local limit theorem for it, then the classical large deviation estimate in the strong form ([11, sec. 2]) for the probability $\mathbf{P}(Y_n = x)$ will be available. Note that the left hand side in (4.3) does not depend on h and so one is free to choose any possible value of h. As usually in the large deviation theory the best value $h(x)$ of the tuning parameter h can be determined from the condition

$$\mathbf{E}\, Y_{n,h} = x$$

that prescribes to choose such a tilted distribution first moment of which coincides with x. Recall that due to Definition 2.3 b) in the case $x = nq_n$ such $h = h_n^0$ always exists. Observe also that in view of the first relation in (4.2) the function $h \mapsto xh - L_{Y_n}(h)$ in the exponent in (4.3) attains its maximum at the point $h = h(x)$, i. e., equals the value of the Legendre transformation of L_{Y_n} at the point x,

$$L^*_{Y_n}(x) \equiv \sup_h \bigl(xh - L_{Y_n}(h)\bigr). \tag{4.4}$$

In this case (4.3) boils down to

$$\mathbf{P}\,(Y_n = x) = \exp\left\{-L^*_{Y_n}(x)\right\}\, \mathbf{P}\left(Y_{n,h(x)} = x\right). \tag{4.5}$$

Consider now the Y_n-regular sequence nq_n fixed in Sect. 2. Without loss of generality we may assume that the limiting value q of q_n satisfies the condition $2q > a$ which in its turn implies $\bar{h} > 0$ (recall (2.13)). For future references we fix a segment $\mathscr{H} \subset \mathscr{D}_\xi^\circ$ such that

$$\left[0, \bar{h}\right] \subset \mathscr{H}^\circ, \tag{4.6}$$

where $\mathscr{H}^\circ$ denotes the interior of $\mathscr{H}$.

In what follows we will need the estimates on the rate of convergence of h_n^0 calculated from (2.10) to $\bar{h}$. For this reason we observe that convergence of the kind (2.8) is valid for all derivatives. Moreover, for any $k = 0, 1, \ldots$ one has

$$\frac{1}{n}\frac{d^k}{dh^k} L_{Y_n}(h) = \frac{d^k}{dh^k} L_{Y,\infty}(h) + O(n^{-1}), \tag{4.7}$$

where the estimate $O(\cdot)$ is uniform in h from any fixed compact subset of $\mathscr{D}_\xi^\circ$. Then, applying the last relation to $k = 1$ and $h = h_n^0$ and using the implicit function theorem for $L'_{Y,\infty}(\cdot)$ one easily obtains

$$h_n^0 - \bar{h} = O(q_n - q) + O(n^{-1}), \tag{4.8}$$

where the remainder terms $O(n^{-1})$ and $O(q_n - q)$ are uniform in $\bar{h} \in \mathscr{H}$ and $q_n \in L'_{Y,\infty}(\mathscr{H}^\circ)$ respectively. Here $L'_{Y,\infty}(\mathscr{H}^\circ)$ denotes the image of $\mathscr{H}^\circ$ under the map $L'_{Y,\infty}(\cdot)$. Consequently, $h_n^0 \to \bar{h}$ as $n \to \infty$ and without loss of generality we may assume that every number h_n^0 belongs to the interior $\mathscr{H}^\circ$ of the compact set $\mathscr{H} \subset \mathscr{D}_\xi^\circ$.

Analogous constructions can be drawn for the random vector Ω_n from (2.19),

$$\Omega_n = (Y_n, S_{[ns_1]}, S_{[ns_2]}, \ldots, S_{[ns_k]}),$$

defined for any fixed integer number $k = 0, 1, \ldots$ and a collection $\mathscr{S}$ of real numbers $s_1, s_2, \ldots, s_k$ satisfying the condition $0 < s_1 < s_2 < \ldots < s_k \leq 1$. For any vector $\mathbf{H} = (h^0, h^1, \ldots, h^k) \in R^{k+1}$ we put

$$L_{\Omega_n}(\mathbf{H}) \equiv \ln \mathbf{E} \exp\{(\mathbf{H}, \Omega_n)\} = \sum_{j=1}^{n} L(h_{j,n}), \tag{4.9}$$

where $(\cdot,\cdot)$ is the usual inner product in R^{k+1},

$$h_{j,n} = h_{j,n}(\mathbf{H}) \equiv \left(1 - \frac{j}{n}\right) h^0 + \sum_{l=1}^{k} h^l \, \chi_{\{j \leq [ns_l]\}}, \tag{4.10}$$

and $\chi_{\{j \leq [ns_l]\}}$ denotes the indicator function of the relation $\{j \leq [ns_l]\}$. Put

$$\mathscr{D}_{\Theta_n} \equiv \{\mathbf{H} \in R^{k+1} : L_{\Omega_n}(\mathbf{H}) < \infty\}.$$

The sets $\mathscr{D}_{\Theta_n}$ depend essentially on n and on the collection $\mathscr{S}$. However, any of these sets contains the following region

$$\mathscr{D}_{k+1} \equiv \{\mathbf{H} \in R^{k+1} : -d < h^0 < \bar{h} + d, |h^l| < d, l = 1, 2, \ldots, k\}, \tag{4.11}$$

provided the constant $d = d(k, \bar{h}, \mathscr{H}) > 0$ is sufficiently small. Moreover, we can choose d in such a way that for any $\mathbf{H} \in \mathscr{D}_{k+1}$ all $h_{j,n}$, $n = 1, 2, \ldots$, $j = 1, 2, \ldots, n$, defined in (4.10) belong to the set $\mathscr{H}^\circ$. For future references we fix such d.

Similarly to (4.1) we consider the random vector $\Omega_{n,\mathbf{H}}$ with the tilted distribution

$$\mathbf{P}\left(\Omega_{n,\mathbf{H}} = \mathbf{M}\right) = \exp\{(\mathbf{M}, \mathbf{H}) - L_{\Omega_n}(\mathbf{H})\}\, \mathbf{P}(\Omega_n = \mathbf{M}), \tag{4.12}$$

with $\mathbf{M} = (m^0, m^1, \ldots, m^k) \in R^{k+1}$ such that the numbers $nm^0, m^1, \ldots, m^k$ are integers. Then the vector $\mathbf{E}\,\Omega_{n,\mathbf{H}}$ of mean values and the covariance matrix $\mathbf{Cov}\,\Omega_{n,\mathbf{H}}$ can be found from (cf. (4.2))

$$\mathbf{E}\,\Omega_{n,\mathbf{H}} = \nabla_{\mathbf{H}} L_{\Omega_n}(\mathbf{H}), \qquad \mathbf{Cov}\,\Omega_{n,\mathbf{H}} = \mathbf{Hess}\, L_{\Omega_n}(\mathbf{H}), \tag{4.13}$$

where $\nabla_{\mathbf{H}}$ denotes the gradient and $\mathbf{Hess}\, L_{\Omega_n}(\mathbf{H})$ is the Hessian matrix (the matrix of the second derivatives) of L_{Ω_n} as a function of the variables $h^0, \ldots, h^k$. Consider also the Legendre transformation (cf. (4.4))

$$L^*_{\Omega_n}(\mathbf{M}) \equiv \sup_{\mathbf{H}} \big((\mathbf{M}, \mathbf{H}) - L_{\Omega_n}(\mathbf{H})\big).$$

If vectors $\mathbf{H}$ and $\mathbf{M}$ satisfy the condition

$$\mathbf{M} = \nabla_{\mathbf{H}} L_{\Omega_n}(\mathbf{H}),$$

then one obtains (cf. (4.5), (A.1))

$$\mathbf{P}(\Omega_n = \mathbf{M}) = \exp\{-L^*_{\Omega_n}(\mathbf{M})\}\, \mathbf{P}\left(\Omega_{n,\mathbf{H}} = \mathbf{M}\right). \tag{4.14}$$

The functions $n^{-1}L_{\Omega_n}(\mathbf{H}), n \geq 1$, are analytical and uniformly bounded provided $\mathbf{H}$ belongs to some complex neighbourhood $\mathscr{U}(\mathscr{D}_{k+1})$ of the set $\mathscr{D}_{k+1}$ from (4.11). As in (2.8) we define

$$L_{\Omega,\infty}(\mathbf{H}) \equiv \lim_{n\to\infty} \frac{1}{n} L_{\Omega_n}(\mathbf{H}) = \int_0^1 L\big(\widetilde{h}(x)\big)\, dx,$$

where (cf. (4.10))

$$\widetilde{h}(x) = (1-x)h^0 + \sum_{l=1}^{k} h^l\, \chi_{\{x<s_l\}} \tag{4.15}$$

with $\chi_{\{x<s_l\}}$ denoting the indicator function of the set $\{x < s_l\}$. Note that the analog of (4.7) is also true here with the estimate $O(n^{-1})$ uniform in $\mathbf{H} \in \mathscr{U}(\mathscr{D}_{k+1})$.

Define the matrix

$$B_n(\mathbf{H}) \equiv \frac{1}{n}\mathbf{Hess}\, L_{\Omega_n}(\mathbf{H}) = \left\| b^{(n)}_{l,m} \right\|^k_{l,m=0}. \tag{4.16}$$

In view of the analog of (4.7) for the functions $L_{\Omega_n}(\mathbf{H})$ one easily obtains the relation

$$B_n(\mathbf{H}) = B(\mathbf{H}) + O(n^{-1}), \tag{4.17}$$

where the matrix $B(\mathbf{H}) = \|b_{l,m}\|^k_{l,m=0}$ has the elements

$$b_{l,m} = \frac{\partial^2}{\partial h^l \partial h^m} L_{\Omega,\infty}(\mathbf{H}) \tag{4.18}$$

and the term $O(n^{-1})$ in (4.17) is uniform in $\mathbf{H} \in \mathscr{U}(\mathscr{D}_{k+1})$. For any vector $\mathbf{T} = (t_0, t_1, \ldots, t_k) \in R^{k+1}$ define the quadratic form

$$\begin{aligned} \mathscr{B}_{n,\mathbf{H}}(\mathbf{T}) &= \sum_{l,m=0,1,\ldots,k} t_l t_m b^{(n)}_{l,m} \\ &= \frac{1}{n}\sum_{j=1}^{n} L''(h_{j,n})\Big(\Big(1-\frac{j}{n}\Big)t_0 + \sum_{l=1}^{k} \chi_{\{j\leq[ns_l]\}} t_l\Big)^2 \geq 0 \end{aligned} \tag{4.19}$$

and the quadratic form

$$\begin{aligned} \mathscr{B}_{\mathbf{H}}(\mathbf{T}) &= \sum_{l,m=0,1,\ldots,k} t_l t_m b_{l,m} \\ &= \int_0^1 L''\big(\widetilde{h}(x)\big)\Big((1-x)t_0 + \sum_{l=1}^{k} \chi_{\{x<s_l\}} t_l\Big)^2 dx \geq 0. \end{aligned} \tag{4.20}$$

Clearly, the strict convexity of $L(\cdot)$ (recall (2.6)) implies the strict positive definiteness of the symmetric matrices $B_n(\mathbf{H})$ and $B(\mathbf{H})$, $\mathbf{H} \in \mathscr{D}_{k+1}$, (in the case of $B_n(\mathbf{H})$ at least for all sufficiently large n, $n \geq n_0(\mathscr{S})$).

Theorem 4.1 *Let a sequence of vectors* $\mathbf{H}_n \in \mathscr{D}_{k+1}$ *be given such that* $\mathbf{H}_n \to \mathbf{H} \in \mathscr{D}_{k+1}$ *with* $\mathscr{D}_{k+1}$ *defined in (4.11). Put*

$$\Omega_n^* = \frac{1}{\sqrt{n}}\left(\Omega_{n,\mathbf{H}_n} - \mathbf{E}\,\Omega_{n,\mathbf{H}_n}\right). \tag{4.21}$$

Then the distribution of the vector Ω_n^* *tends weakly to the distribution of the Gaussian random vector* $\overline{\Theta}$ *with zero mean and covariance matrix* $B(\mathbf{H})$.

Remark 4.1.1 The distribution of the random vector $\overline{\Theta}$ is nondegenerate. We denote its density by $\overline{p}_{\mathbf{H}}(\mathbf{X})$, $\mathbf{X} \in R^{k+1}$, and its characteristic function by $\overline{\Phi}_{\mathbf{H}}(\mathbf{T})$,

$$\overline{\Phi}_{\mathbf{H}}(\mathbf{T}) = \exp\left\{-\frac{1}{2}\left(B(\mathbf{H})\mathbf{T},\mathbf{T}\right)\right\}, \quad \mathbf{T} \in R^{k+1}. \tag{4.22}$$

Proof. Fix any $\mathbf{T} = (t_0, t_1, \ldots, t_k) \in R^{k+1}$. Using the Taylor expansion for the logarithm of the characteristic function $\widehat{\Phi}_n(\cdot)$ of the vector Ω_n^* we get

$$\begin{aligned} \ln \widehat{\Phi}_n(\mathbf{T}) &= L_{\Omega_n}\left(\mathbf{H}_n + in^{-1/2}\mathbf{T}\right) - L_{\Omega_n}(\mathbf{H}_n) - \frac{i}{\sqrt{n}}\left(\mathbf{T}, \mathbf{E}\,\Omega_{n,\mathbf{H}_n}\right) \\ &= -\frac{1}{2}\left(B_n(\mathbf{H}_n)\mathbf{T},\mathbf{T}\right) + R_n, \end{aligned} \tag{4.23}$$

where the matrix $B_n(\cdot)$ is determined in (4.16) and the remainder term R_n equals

$$R_n = -\frac{i}{6n^{3/2}} \sum_{l,m,p=0}^{k} t_l t_m t_p \frac{\partial^3}{\partial h^l \partial h^m \partial h^p} L_{\Omega_n}\left(\mathbf{H}_n + i\omega n^{-1/2} T\right)$$

with some $\omega = \omega(\mathbf{H}_n, T)$, $0 \le \omega \le 1$. The uniform boundedness of the family of analytical functions $n^{-1}L_{\Omega_n}(\mathbf{H})$, $n = 1, 2, \ldots$, $\mathbf{H} \in \mathscr{U}(\mathscr{D}_{k+1})$, implies the uniform boundedness of their third derivatives. Consequently, $R_n = O(n^{-1/2})$ as $n \to \infty$ uniformly in $\mathbf{T}$ from any fixed compact set in R^{k+1}. Finally, (4.17) implies the relation

$$\lim_{n\to\infty} \ln \widehat{\Phi}_n(\mathbf{T}) = -\frac{1}{2}\left(B(\mathbf{H})\mathbf{T},\mathbf{T}\right) = \ln \overline{\Phi}_{\mathbf{H}}(\mathbf{T}) \tag{4.24}$$

which finishes the proof. □

Our next step consists in the evaluation of the probabilities $\mathbf{P}(\Omega_n = \mathbf{M}_n)$ and $\mathbf{P}(Y_n = m_n^0)$ entering the right-hand side of (2.20).

Fix any integer $k \ge 0$ and assume that a sequence of vectors $\mathbf{H}_n \in R^{k+1}$, $\mathbf{H}_n = (h_n^0, h_n^1, \ldots, h_n^k)$, satisfy the assumption of Theorem 4.1. According to the choice of the value d in (4.11) all points $h_{j,n} = h_{j,n}(\mathbf{H}_n)$, $j = 1, 2, \ldots, n$; $n = 1, 2, \ldots$, calculated as in (4.10) belong to $\mathscr{H}^\circ$ (see (4.6)). Without loss of generality (taking d slightly smaller if necessary) we may assume also that $\widetilde{h}(x) \in \mathscr{H}^\circ$ (recall (4.15)) for any $x \in [0, 1]$.

Denote

$$\mathscr{M}_n^{k+1} = \left\{\left(m^0, m^1, \ldots, m^k\right) \in R^{k+1} : \{nm^0, m^1, \ldots, m^k\} \subset Z^1\right\}. \tag{4.25}$$

Let $\mathbf{M}_n = (m_n^0, m_n^1, \ldots, m_n^k)$ be any sequence of vectors such that $\mathbf{M}_n \in \mathscr{M}_n^{k+1}$ for all n. Put (cf. (4.13))

$$\mathbf{E}_n = \mathbf{E}\,\Omega_{n,\mathbf{H}_n}$$

and define the vector $\mathbf{X}_n \in R^{k+1}$ by

$$\mathbf{X}_n = \frac{1}{\sqrt{n}}\left(\mathbf{M}_n - \mathbf{E}_n\right). \tag{4.26}$$

Theorem 4.2 *Let $k \geq 0$ be any integer number and vectors $\mathbf{H}_n$ be as in Theorem 4.1. Then uniformly in $\mathbf{M}_n \in \mathscr{M}_n^{k+1}$ one has*

$$n^{\frac{k+3}{2}}\mathbf{P}\left(\Omega_{n,\mathbf{H}_n} = \mathbf{M}_n\right) - \overline{p}_{\mathbf{H}}(\mathbf{X}_n) \to 0 \quad as \quad n \to \infty, \tag{4.27}$$

where $\overline{p}_{\mathbf{H}}(\cdot)$ is the density of the Gaussian random vector $\overline{\Theta}$ with the characteristic function $\overline{\Phi}_{\mathbf{H}}(\mathbf{T})$ from (4.22).

The following corollary specifies the statement of Theorem 4.2 in the one-dimensional case. Let nq_n be the Y_n-regular sequence fixed in Sect. 2, h_n^0 and $\bar{h}$ be the solutions of (2.10) and (2.11) correspondingly. Denote

$$b^2 = b^2(\bar{h}) \equiv \int_0^1 L''\left(\bar{h}(1-x)\right)(1-x)^2\,dx.$$

For any real $x \in \mathscr{M}_n^1$ (recall (4.25)) put $z = (x - nq_n)/\sqrt{n}$.

Corollary 4.3 *Uniformly in $x \in \mathscr{M}_n^1$ one has*

$$n^{3/2}\mathbf{P}\left(Y_{n,h_n^0} = x\right) - \frac{1}{\sqrt{2\pi}b}e^{-z^2/2b^2} \to 0 \qquad as \quad n \to \infty. \tag{4.28}$$

To prove Theorem 4.2 we need the following simple observation.

Lemma 4.4 *Fix any real numbers δ, δ_0, $0 < \delta \leq \delta_0 \leq 1/16$. Denote by $\mathscr{O}_\delta(m)$, $m \in Z^1$, the δ-neighborhood of the integer number m in the real line R^1 and put*

$$\mathscr{O}_\delta = \bigcup_{m \in Z^1} \mathscr{O}_\delta(m). \tag{4.29}$$

Consider an arithmetic progression $a_0, a_1, \ldots, a_n$ of the length $n+1$ with the step v and denote by $\mathscr{N}_\delta$ the number of elements a_i located outside of the set $\mathscr{O}_\delta$. Then the following statements hold true.

A. *If $8\delta n^{-1} \leq |v| \leq 1/2$, then*

$$\mathscr{N}_\delta \geq \left(\frac{1}{2} - 4\delta_0\right)n - 3. \tag{4.30}$$

B. *Let $|v| \leq 8\delta n^{-1}$ and assume additionally that the element a_0 is located outside of the set $\mathscr{O}_\delta$ at a positive distance ρ from it, $\rho = \mathbf{dist}\left(a_0, \mathscr{O}_\delta\right) > 0$. Then*

$$\mathscr{N}_\delta \geq \frac{\rho}{2\delta_0 + \rho}\,n. \tag{4.31}$$

Proof. **A**. First observe that at most $\left[|a_n - a_0|\right] + 2 \le |v|n + 2$ intervals $\mathcal{O}_\delta(m)$ can contain the points a_i. Moreover, every interval $\mathcal{O}_\delta(m)$ contains no more than $\left[2\delta|v|^{-1}\right] + 1$ elements of the progression. Hence, the number $\mathcal{M}_\delta = n + 1 - \mathcal{N}_\delta$ of the points a_i belonging to the set $\mathcal{O}_\delta$ satisfies the following inequality

$$\mathcal{M}_\delta \le (|v|n + 2)(2\delta|v|^{-1} + 1) = 2\delta n + 4\delta|v|^{-1} + |v|n + 2. \qquad (4.32)$$

To estimate the right-hand side of (4.32) we consider two possibilities:

a). If $1/2 \ge |v| \ge 2/n$, then $4\delta|v|^{-1} \le 2\delta n$, $|v|n \le n/2$, and so $\mathcal{M}_\delta \le (4\delta + 1/2)n + 2$.

b). If $2/n \ge |v| \ge 8\delta/n$, then $4\delta|v|^{-1} \le n/2$, $|v|n \le 2$, and so $\mathcal{M}_\delta \le (2\delta + 1/2)n + 4$.

Obtained inequalities together with the definition of $\mathcal{M}_\delta$ imply (4.30).

B. Note that $|v|n \le 8\delta \le 1/2$. Three cases are possible.

a). If $|v|n \le \rho$, then $\mathcal{N}_\delta = n + 1$.

b). If $\rho < |v|n < \rho + 2\delta$, then $\mathcal{N}_\delta \ge \left[\rho/|v|\right] + 1 \ge \rho/|v| \ge \rho n/(2\delta_0 + \rho)$.

c). If $\rho + 2\delta \le |v|n \le 1/2$, then at most one interval $\mathcal{O}_\delta(m)$ can contain points a_i and so $\mathcal{M}_\delta \le \left[2\delta/|v|\right] + 1 \le 2\delta/|v| + 1$. Consequently, $\mathcal{N}_\delta \ge n + 1 - \left(2\delta/|v| + 1\right) \ge \rho n/(2\delta + \rho)$. Collecting all three estimates one obtains (4.31). □

Proof of Theorem 4.2 Assume first that $k \ge 1$.

For any $h \in \mathcal{K}$ (recall (4.6)) we denote by $\varphi_h(t)$ the characteristic function of the random variable ξ_h defined in (2.5),

$$\varphi_h(t) \equiv \mathbf{E} \exp\{it\xi_h\} = \exp\{L(h + it) - L(h)\}.$$

Let us collect some properties of the function $\varphi_h(t)$ which will be used in the following. First of all, for any $h \in \mathcal{K}$ and $t \in R^1$

$$|\varphi_h(t)| \le \varphi_h(0) = 1. \qquad (4.33)$$

Since the probability distribution of the random variable ξ_h is concentrated on the integer lattice Z^1, the function $\varphi_h(t)$ is a 2π-periodical function of t, i. e., $\varphi_h(t + 2\pi) \equiv \varphi_h(t)$ for every $h \in \mathcal{K}$. Then, for any δ, $0 < \delta < \pi$, there exists a constant $C = C(\mathcal{K}, \delta) > 0$ such that for every $h \in \mathcal{K}$ and any t, $\delta \le t \le 2\pi - \delta$, one has

$$|\varphi_h(t)| \le e^{-C}. \qquad (4.34)$$

And finally, there exists a constant $\alpha = \alpha(\mathcal{K}) > 0$ such that for all $h \in \mathcal{K}$ and any t, $|t| \le \pi$, the following inequality holds

$$|\varphi_h(t)| \le \exp\{-\alpha t^2 L''(h)\}. \qquad (4.35)$$

The last two inequalities follow easily from the known properties of the characteristic functions of lattice random variables and from the compactness of the set $\mathcal{K}$.

Using relations (4.9) and (4.10) we rewrite the characteristic function $\Phi_n(\mathbf{T})$ of the vector $\Omega_{n,\mathbf{H}_n}$ in terms of functions $\varphi_h(t)$,

$$\Phi_n(\mathbf{T}) = \mathbf{E}\,\exp\{i\left(\mathbf{T},\Omega_{n,\mathbf{H}_n}\right)\} = \prod_{j=1}^{n} \varphi_{h_{j,n}}\left(t_{j,n}\right), \tag{4.36}$$

where the numbers $t_{j,n}$ are calculated from $\mathbf{T} = \left(t^0, t^1, \dots, t^l\right)$ via (cf. (4.10))

$$t_{j,n} \equiv \left(1 - \frac{j}{n}\right) t^0 + \sum_{l=1}^{k} t^l \chi_{\{j \le [ns_l]\}}. \tag{4.37}$$

Note that (recall (4.23))

$$\widehat{\Phi}_n(\mathbf{T}) = \Phi_n(n^{-1/2}\mathbf{T}) \exp\left\{-\frac{i}{\sqrt{n}}\left(\mathbf{T}, \mathbf{E}\,\Omega_{n,\mathbf{H}_n}\right)\right\} \tag{4.38}$$

is the characteristic function of the random vector Ω_n^* defined in (4.21).

Using the well-known inversion formula for the Fourier transformation and definition (4.22) we rewrite the left-hand side

$$R_n = n^{\frac{k+3}{2}} \mathbf{P}\left(\Omega_{n,\mathbf{H}_n} = \mathbf{M}_n\right) - \overline{p}_{\mathbf{H}}(\mathbf{X}_n)$$

of (4.27) in the form

$$\begin{aligned} R_n = \frac{1}{(2\pi)^{k+1}} & \int_{\mathbf{A}} \widehat{\Phi}_n(\mathbf{T}) e^{-i(\mathbf{T},\mathbf{X}_n)}\, d\mathbf{T} \\ & - \frac{1}{(2\pi)^{k+1}} \int_{R^{k+1}} \overline{\Phi}_{\mathbf{H}}(\mathbf{T}) e^{-i(\mathbf{T},\mathbf{X}_n)}\, d\mathbf{T}, \end{aligned} \tag{4.39}$$

where

$$\mathbf{A} = \left\{\mathbf{T} = (t_0, t_1, \dots, t_k) \in R^{k+1} : |t_0| \le \pi n^{3/2}, |t_l| \le \pi\sqrt{n}, l = 1, 2, \dots, k\right\}.$$

Following the standard proof of the local limit theorem (see, e.g., [13, Sect. 43]) we evaluate the left-hand side of (4.39) as the sum of four terms,

$$(2\pi)^{-(k+1)} \left(J_1 + J_2 + J_3 + J_4\right),$$

where, for some positive constants A and Δ,

$$\begin{aligned} J_1 &= \int_{\mathbf{A_1}} |\widehat{\Phi}_n(\mathbf{T}) - \overline{\Phi}_{\mathbf{H}}(\mathbf{T})|\, d\mathbf{T}, \qquad \mathbf{A_1} = [-A, A]^{k+1}, \\ J_2 &= \int_{\mathbf{A_2}} \overline{\Phi}_{\mathbf{H}}(\mathbf{T})\, d\mathbf{T}, \qquad \mathbf{A_2} = R^{k+1} \setminus \mathbf{A_1}, \\ J_p &= \int_{\mathbf{A_p}} |\widehat{\Phi}_n(\mathbf{T})|\, d\mathbf{T}, \qquad p = 3, 4, \end{aligned}$$

with

$$\begin{aligned} \mathbf{A_3} &= \{\mathbf{T} \in R^{k+1} : |t_l| \le \Delta\sqrt{n}, l = 0, 1, \dots, k\} \setminus \mathbf{A_1}, \\ \mathbf{A_4} &= \mathbf{A} \setminus (\mathbf{A_1} \cup \mathbf{A_3}). \end{aligned} \tag{4.40}$$

Fix an arbitrary $\varepsilon > 0$. We will show in the following that for a convenient choice of the constants $A = A(\varepsilon)$ and $\Delta = \Delta(\mathscr{S})$ one has the bounds $J_i < \varepsilon/4$,

$i = 1,2,3,4$, for all sufficiently large n. This will imply the assertion of the theorem. It remains to evaluate all J_i.

First of all, due to Central Limit Theorem 4.1, for every finite $A > 0$ one obtains the convergence $J_1 \to 0$ as $n \to \infty$.

Secondly, it is evident that

$$J_2 = \int_{\mathbf{A}_2} \overline{\Phi}(\mathbf{T})\, d\mathbf{T} \to 0 \qquad \text{as} \qquad A \to \infty. \tag{4.41}$$

To estimate J_3 we fix any $\mathbf{T} \in \mathbf{A}_3$ and put

$$\Delta = \frac{\pi}{k+1}. \tag{4.42}$$

Then all the numbers $t_{j,n}$ defined in (4.37) satisfy the condition $|t_{j,n}| \le \pi\sqrt{n}$. Hence, evaluating each factor in (4.36) by the help of (4.35) one obtains the bound (recall (4.38), (4.19) and (4.20))

$$|\widehat{\Phi}_n(\mathbf{T})| \le \exp\{-\alpha \mathscr{B}_{n,\mathbf{H}_n}(\mathbf{T})\} \le \exp\{-\alpha c \mathscr{B}_{\mathbf{H}}(\mathbf{T})\},$$

where the last inequality follows from estimate (4.17), the convergence $\mathbf{H}_n \to \mathbf{H}$ and the positive definiteness of the quadratic forms $\mathscr{B}_{n,\mathbf{H}_n}(\mathbf{T})$ and $\mathscr{B}_{\mathbf{H}}(\mathbf{T})$ provided $c > 0$ is sufficiently small. As a result,

$$J_3 = \int_{\mathbf{A}_3} |\widehat{\Phi}_n(\mathbf{T})|\, d\mathbf{T} \le \int_{\mathbf{A}_2} \exp\{-\alpha c \mathscr{B}_{\mathbf{H}}(\mathbf{T})\}\, d\mathbf{T} \to 0 \quad \text{as} \quad A \to \infty.$$

To evaluate J_4 put

$$\delta = \frac{1}{17(k+1)^2}. \tag{4.43}$$

For any $\mathbf{T} \in \mathbf{A}_4$ denote by $N_n(\mathbf{T})$ the number of indexes $j = 1,2,\ldots,n$ such that $\tau_{j,n} \notin \mathscr{O}_\delta$ (recall (4.29)), where

$$\tau_{j,n} \equiv \frac{1}{2\pi\sqrt{n}}\, t_{j,n}. \tag{4.44}$$

Using (4.33) and (4.34) to estimate factors in the representation (4.36) one has

$$|\widehat{\Phi}_n(\mathbf{T})| = \prod_{j=1}^{n} \left|\varphi_{h_{j,n}}\left(\frac{1}{\sqrt{n}}\, t_{j,n}\right)\right| \le \exp\{-CN_n(\mathbf{T})\}.$$

Our aim here is to prove that for $\mathbf{T} \in \mathbf{A}_4$ and all sufficiently large n

$$N_n(\mathbf{T}) \ge \beta n, \tag{4.45}$$

were $\beta > 0$ is a constant depending only on the set $\mathscr{S} = (s_1, s_2, \ldots, s_k)$. Then

$$J_4 = \int_{\mathbf{A}_4} |\widehat{\Phi}_n(\mathbf{T})|\, d\mathbf{T} \le (2\pi)^{k+1} n^{\frac{k+3}{2}} \exp\{-C\beta n\} \to 0$$

as $n \to \infty$ and we obtain the needed estimate for J_4.

It remains only to prove (4.45). First observe that in view of its definition (see (4.44) and the relation (4.37)) the sequence $\tau_{j,n}, j = 1, 2, \dots, n$, splits into $k+1$ arithmetic progressions (k progressions if $s_k = 1$) with the same step v,

$$\tau_{j,n} - \tau_{j-1,n} = v = -\frac{t_0}{2\pi n^{3/2}} \tag{4.46}$$

for $[ns_l] < j \le [ns_{l+1}]$, $l = 0, 1, \dots, k+1$. Here we put $s_0 = 0$, $s_{k+1} = 1$.

Fix any $\mathbf{T} \in \mathbf{A_4}$. It follows from the definition (4.40) that for any $\mathbf{T} \in \mathbf{A_4}$ there exists a number $l \in \{0, 1, \dots, k\}$ such that $|t_l| > \Delta\sqrt{n}$. Let $l_\mathbf{T}$ denotes the minimal such l. Two cases are possible: $l_\mathbf{T} = 0$ and $l_\mathbf{T} > 0$.

Consider first the case $l_\mathbf{T} = 0$. Then $\Delta\sqrt{n} < |t_0| \le \pi n^{3/2}$ and so, in view of (4.42) and (4.46), the step v satisfies the condition

$$\frac{1}{2} \ge v > \frac{\Delta}{2\pi n} = \frac{1}{2n(k+1)}.$$

On the other hand, at least one of the progressions mentioned above consists of no less than $\lfloor (n+1)/(k+1) \rfloor$ elements. Then for all sufficiently large n one has the inequality $v\ \lfloor (n+1)/(k+1) \rfloor \ge 8\delta$ and so Lemma 4.4.**A** can be applied. As a result, for all sufficiently large n

$$N_n(\mathbf{T}) \ge \left(\frac{1}{2} - \frac{4}{17(k+1)^2}\right) \left[\frac{n+1}{k+1}\right] - 3 \ge \frac{n}{4(k+1)}. \tag{4.47}$$

In the case $l_\mathbf{T} > 0$ the statement **B** of Lemma 4.4 is applicable. Indeed, put $j_l = \lfloor ns_{l_\mathbf{T}} \rfloor$ and consider the difference (recall (4.44), (4.37))

$$R = \tau_{j_l,n} - \tau_{j_l+1,n} = \frac{1}{2\pi\sqrt{n}} \left(\frac{t_0}{n} + t_{l_\mathbf{T}}\right).$$

(for $s_k = 1$, $l_\mathbf{T} = k$ we put $\tau_{n+1,n} = 0$.) Using the inequalities $|t_0| \le \Delta\sqrt{n}$ and $\Delta\sqrt{n} < |t_{l_\mathbf{T}}| \le \pi\sqrt{n}$ it is easy to verify that for all sufficiently large n

$$6\delta \le |R| < 3/5 < 1 - 6\delta.$$

Consequently, at least one of the points $\tau_{j_l,n}$, $\tau_{j_l+1,n}$ ($\tau_{n,n}$ in the case $s_k = 1$, $l_\mathbf{T} = k$) is located outside of the set $\mathscr{O}_\delta$ on the distance $\rho \ge 2\delta$.

To apply the statement **B** of Lemma 4.4 it remains to observe that for the fixed set $\mathscr{S}$ there exists a constant $\gamma = \gamma(\mathscr{S}) > 0$ such that for all sufficiently large n the length of every progression obtained above is no less than γn. Hence,

$$N_n(\mathbf{T}) \ge \frac{\rho}{2\delta + \rho} \gamma n \ge \frac{\gamma}{2} n. \tag{4.48}$$

Obviously, (4.47) and (4.48) imply (4.45).

Collecting all the estimates of the integrals J_p obtained above we finish the proof for $k \ge 1$.

The case $k = 0$ can be treated in the same way with obvious simplifications in the formulas (4.10), (4.42), (4.43), and in the evaluation of J_4. □

Observe that in the arguments above the Gaussian density $\overline{p}_\mathbf{H}(\cdot)$ can be replaced by the density of zero-mean Gaussian distribution with the covariance matrix $B_n(\mathbf{H}_n)$ (recall (4.23), (4.24)). In particular, one has

Corollary 4.5 *There exist positive constants n_0, c_0 and C_0 such that for all $n \geq n_0$*

$$\frac{c_0}{n}\left(L''_{Y_n}(h_n^0)\right)^{-1/2} \leq \mathbf{P}\left(Y_{n,h_n^0} = nq_n\right) \leq \frac{C_0}{n}\left(L''_{Y_n}(h_n^0)\right)^{-1/2}, \tag{4.49}$$

where h_n^0 is determined from (2.10).

For the future references we make also the following simple observation.

Corollary 4.6 *Let all $\mathbf{X}_n$ in (4.26) be uniformly bounded. Then in the conditions of Theorem 4.2 one has*

$$\mathbf{P}\left(\Omega_{n,\mathbf{H}_n} = \mathbf{M}_n\right) = n^{-\frac{k+3}{2}}\overline{p}_{\mathbf{H}}(\mathbf{X}_n)(1 + o(1)),$$

where the estimate $o(1)$ is uniform in such $\mathbf{X}_n$. In particular, this probability is positive for all sufficiently large n, $n \geq n_0$, and therefore there exist positive constants c_i, C_i, $i = 1, 2$, such that uniformly in such $\mathbf{X}_n$ and $n \geq n_0$ one has

$$c_2 \leq c_1\overline{p}_{\mathbf{H}}(\mathbf{X}_n) \leq n^{\frac{k+3}{2}}\mathbf{P}\left(\Omega_{n,\mathbf{H}_n} = \mathbf{M}_n\right) \leq C_1\overline{p}_{\mathbf{H}}(\mathbf{X}_n) \leq C_2. \tag{4.50}$$

5 Convergence of finite-dimensional distributions

We prove here the convergence of finite-dimensional distributions of the random process $\theta_n^*(t)$ from (2.15) to the corresponding distributions of the conditional random process $\overline{\theta}(t) = \left(\overline{\xi}(t)|\overline{\eta} = 0\right)$ (recall (2.16), (2.17)). To do this we check such a convergence for the random process $\Theta_n^*(t)$ (cf. (2.15), (2.21), (2.14))

$$\Theta_n^*(t) \equiv \frac{1}{\sqrt{n}}\left(\Theta_n(t) - n\mathbf{e}_{\bar{h}}(t)\right)$$

and prove the convergence $\Theta_n^*(t) - \theta_n^*(t) \to 0$ in probability as $n \to \infty$.

As before, let nq_n be the Y_n-regular sequence fixed in Sect. 2, h_n^0 and $\bar{h}$ be the solutions of equations (2.10) and (2.11) correspondingly. Then the sequence of vectors

$$\mathbf{H}_n^0 \equiv \left(h_n^0, 0, \ldots, 0\right) \in R^{k+1} \tag{5.1}$$

converges to the vector (recall (4.8))

$$\mathbf{H}^0 \equiv \left(\bar{h}, 0, \ldots, 0\right) \in R^{k+1} \tag{5.2}$$

and all $\mathbf{H}_n^0$ belong to the region $\mathscr{D}_{k+1}$ from (4.11).

Denote (recall (4.13), (4.9) and (2.10))

$$\mathbf{E}_n^0 \equiv \mathbf{E}\,\Omega_{n,\mathbf{H}_n^0} = (nq_n, \mathbf{e}_n^1, \ldots, \mathbf{e}_n^k). \tag{5.3}$$

It follows from (4.13), (4.9) and (4.10) that

$$\mathbf{e}_n^i = \mathbf{e}_n(s_i) \equiv \frac{\partial}{\partial h^i}L_{\Omega_n}(\mathbf{H})\Big|_{\mathbf{H}=\mathbf{H}_n^0} = \sum_{j=1}^{[ns_i]} L'\left(\left(1 - \frac{j}{n}\right)h_n^0\right). \tag{5.4}$$

Similarly to (4.7) we find that

$$\frac{1}{n}\mathbf{e}_n(s_i) = \int_0^{s_i} L'(h_n^0 - h_n^0 t)\,dt + O(n^{-1}),$$

where the estimate $O(n^{-1})$ is uniform in $\mathscr{K}$ (recall (4.6)). Moreover, the analytical dependence of the function $\mathbf{e}_h(s) = \big(L(h) - L(h - hs)\big)/h$ on h in some neighbourhood of $\bar{h}$ as well as relation (4.8) imply that

$$\begin{aligned}\frac{1}{n}\mathbf{e}_n(s_i) &= \mathbf{e}_{\bar{h}}(s_i) + s_i O(h_n^0 - \bar{h}) + s_i O(n^{-1})\\ &= \mathbf{e}_{\bar{h}}(s_i) + s_i O(q_n - q) + s_i O(n^{-1}),\end{aligned} \tag{5.5}$$

where the estimates $O(\cdot)$ are uniform in $s_i \in [0,1]$.

Consider an arbitrary vector $\mathbf{M}_n \in \mathscr{M}_n^{k+1}$ of the kind

$$\mathbf{M}_n = \big(nq_n, m_n^1, \ldots, m_n^k\big) \tag{5.6}$$

and define $x_n^i = n^{-1/2}(m_n^i - \mathbf{e}_n^i)$, $i = 1, \ldots, k$. Denote by $\overline{p}_k(\cdot)$ the probability density of the Gaussian random vector $\overline{\Theta} = \big(\overline{\eta}, \overline{\xi}_1, \ldots, \overline{\xi}_k\big)$ with the characteristic function $\overline{\Phi}_{\mathbf{H}^0}(\mathbf{T})$ (recall (4.22), (5.2)). Then

$$\tilde{p}_k(x^1, \ldots, x^k|0) \equiv \frac{\overline{p}_k(\mathbf{X}^0)}{\overline{p}_0(0)}, \qquad \mathbf{X}^0 = (0, x^1, \ldots, x^k), \tag{5.7}$$

gives the probability density of the conditional distribution $\big(\overline{\xi}_1, \ldots, \overline{\xi}_k | \overline{\eta} = 0\big)$.

Lemma 5.1 *Let x_n^i be uniformly bounded. Then*

$$\mathbf{P}\left(\Theta_n(s_1) = m_n^1, \ldots, \Theta_n(s_k) = m_n^k\right) = n^{-\frac{k}{2}}\tilde{p}_k(x_n^1, \ldots, x_n^k|0)\,(1 + o(1)) \tag{5.8}$$

as $n \to \infty$; the estimate $o(1)$ is uniform in such x_n^i.

Proof. It follows from (5.1), (5.6), (4.9), (4.10) and (2.7) that

$$\left(\mathbf{M}_n, \mathbf{H}_n^0\right) = nq_n h_n^0, \quad L_{\Omega_n}\left(\mathbf{H}_n^0\right) = L_{Y_n}\left(h_n^0\right).$$

Hence, applying (4.12) to $\mathbf{M} = \mathbf{M}_n$ and $\mathbf{H} = \mathbf{H}_n^0$ and using (4.1) with $x = nq_n$ and $h = h_n^0$ we obtain

$$\frac{\mathbf{P}(\Omega_n = \mathbf{M}_n)}{\mathbf{P}(Y_n = nq_n)} = \frac{\mathbf{P}(\Omega_{n,\mathbf{H}_n^0} = \mathbf{M}_n)}{\mathbf{P}(Y_{n,h_n^0} = nq_n)}.$$

In view of (4.27) with $\mathbf{H} = \mathbf{H}^0$, $\mathbf{H}_n = \mathbf{H}_n^0$, (4.28) with $h_n = h_n^0$, $x = nq_n$, and the definition of $\overline{p}_k(\cdot)$ we rewrite the last ratio in the form

$$\frac{\mathbf{P}(\Omega_{n,\mathbf{H}_n^0} = \mathbf{M}_n)}{\mathbf{P}(Y_{n,h_n^0} = nq_n)} = n^{-\frac{k}{2}}\frac{\overline{p}_k(\mathbf{X}_n^0)}{\overline{p}_0(0)}\,(1 + o(1)), \tag{5.9}$$

where the estimate $o(1)$ is uniform in the case of uniformly bounded x_n^i. Finally, substituting (5.7) into (5.9) we get (5.8). □

Theorem 5.2 *Fix any natural k, a set of real numbers* $0 < t_1 < t_2 < \ldots < t_k \leq 1$ *and denote (cf. (2.15))*

$$\Theta_n^*(t) = \frac{1}{\sqrt{n}}\big(\Theta_n(t) - n\mathbf{e}_{\bar{h}}(t)\big), \tag{5.10}$$

where the process $\Theta_n(t)$ *is defined in (2.21).*

Then the distribution of the random vector $\big(\Theta_n^*(t_1), \ldots, \Theta_n^*(t_k)\big)$ *tends weakly to the Gaussian distribution with the probability density* $\tilde{p}_k(\cdot|0)$ *from (5.7). This limiting distribution coincides with the corresponding finite-dimensional distribution of the measure* μ^* *from Theorem 2.1.*

Proof. In view of (5.5) and (2.9) one has $\mathbf{e}_n(t) - n\mathbf{e}_{\bar{h}}(t) = o\big(\sqrt{n}\big)$ uniformly in $t \in [0, 1]$ and so it is enough to prove the statement of the theorem for the random vector

$$\frac{1}{\sqrt{n}}\big(\Theta_n(t) - \mathbf{e}_n(t)\big).$$

For $\mathbf{T} = (t_1, \ldots, t_k)$ put $\Theta_n(\mathbf{T}) \equiv \big(\Theta_n(t_1), \ldots, \Theta_n(t_k)\big)$. If $\mathbf{x} = (x_1, \ldots, x_k) \in R^k$, $\mathbf{y} = (y_1, \ldots, y_k) \in R^k$, we will write $\mathbf{y} \geq \mathbf{x}$ instead of $\{y_1 \geq x_1, \ldots, y_k \geq x_k\}$. Denote also (cf. (5.3))

$$\mathbf{e_n} = (\mathbf{e}_n^1, \ldots, \mathbf{e}_n^k).$$

According to Theorem 2.2 from [3, Chap. 1] it is sufficient to prove the asymptotical smallness of the difference

$$R_n = \mathbf{P}\big(\mathbf{y}\sqrt{n} \leq \Theta_n(\mathbf{T}) - \mathbf{e_n} \leq \mathbf{z}\sqrt{n}\big) - \int_{\mathbf{y}\leq\mathbf{x}\leq\mathbf{z}} \tilde{p}_k(\mathbf{x}|0)\, d\mathbf{x} \tag{5.11}$$

for every $\mathbf{y}, \mathbf{z} \in R^k$, $\mathbf{y} \leq \mathbf{z}$.

To do this we recall that the estimate $o(\cdot)$ in (5.8) is uniform in $\mathbf{x}$ belonging to every fixed compact set in R^k and so one can rewrite

$$\mathbf{P}\Big(\mathbf{y} \leq \frac{\Theta_n(\mathbf{T}) - \mathbf{e_n}}{\sqrt{n}} \leq \mathbf{z}\Big) = \sum_{\mathbf{m}\in Z^k :\, \mathbf{y}\sqrt{n}\leq\mathbf{m}-\mathbf{e_n}\leq\mathbf{z}\sqrt{n}} \mathbf{P}\left(\Theta_n(\mathbf{T}) = \mathbf{m}\right)$$
$$= (1 + o(1)) \sum_{\substack{\mathbf{x}:\, \mathbf{x}=(\mathbf{m}-\mathbf{e_n})/\sqrt{n},\\ \mathbf{m}\in Z^k,\ \mathbf{y}\sqrt{n}\leq\mathbf{m}-\mathbf{e_n}\leq\mathbf{z}\sqrt{n}}} \tilde{p}_k(\mathbf{x}|0)n^{-\frac{k}{2}}. \tag{5.12}$$

It remains to observe that the last sum in (5.12) presents the Riemannian sum for the integral expression in (5.11).

To prove the last assertion of the theorem we note that in view of definitions (2.16) and (2.17) one easily obtains

$$\overline{\eta} \equiv \int_0^1 \int_0^t \left(L''(\bar{h} - \bar{h}x)\right)^{1/2} dw_x dt = \int_0^1 (1-x)\left(L''(\bar{h} - \bar{h}x)\right)^{1/2} dw_x$$

and so

$$\mathbf{E}\,\bar{\xi}(t)\,\bar{\xi}(s) = \int_0^{s\wedge t} L''(\bar{h} - \bar{h}x)\,dx,$$

$$\mathbf{E}\,\bar{\xi}(t)\,\bar{\eta} = \int_0^{t} (1-x)L''(\bar{h} - \bar{h}x)\,dx,$$

$$\mathbf{E}\,\bar{\eta}\,\bar{\eta} = \int_0^{t} (1-x)^2 L''(\bar{h} - \bar{h}x)\,dx.$$

Since the last expressions coincide with the appropriate elements of the covariance matrix $B(\mathbf{H}^0)$ of the Gaussian random vector $\overline{\Theta} = \left(\bar{\eta}, \bar{\xi}_1, \dots, \bar{\xi}_k\right)$ with the characteristic function $\overline{\Phi}_{\mathbf{H}^0}(\mathbf{T})$ (recall (4.18), (4.15), (4.22), (5.2)) the last assertion of the theorem follows from the first part of the theorem, definition (5.7) and well-known properties of conditional distributions [20, Chap. 2]. □

To study the finite-dimensional distributions of the process $\theta_n(t)$ from (2.12) we observe that due to (2.15), (5.10), (2.12), (2.21), and (2.3) one has

$$\theta_n^*(t) - \Theta_n^*(t) = \frac{\{nt\}}{\sqrt{n}}\left(\xi_{[nt]+1} \mid Y_n = nq_n\right). \tag{5.13}$$

Since $\xi_{[nt]+1}$ depends weakly from Y_n for large n it is naturally to expect that the last expression vanishes in probability as $n \to \infty$. The next lemma contains the key result in this direction.

Lemma 5.3 *Fix any number $\rho > 0$ such that the real 2ρ-neighbourhood $\mathscr{K}_{2\rho}$ of the segment $\mathscr{K}$ lies inside the set $\mathscr{D}_\xi^\circ$. Then there exist constants $C = C(\mathscr{K}_{2\rho}) > 0$ and $n_0 = n_0(\{q_n\})$ such that uniformly in $n \geq n_0$ and $j = 1, \dots, n$ the following inequality holds*

$$\mathbf{E}\left(e^{\rho|\xi_j|} \mid Y_n = nq_n\right) < C. \tag{5.14}$$

Proof. Let a be any number such that $\mathbf{P}(\xi = a) > 0$. Then

$$\mathbf{P}\left(\xi_j = a | Y_n = nq_n\right) = \frac{\mathbf{P}\left(Y_n = nq_n | \xi_j = a\right)}{\mathbf{P}(Y_n = nq_n)}\mathbf{P}(\xi_j = a).$$

We prove below that for all sufficiently large n the following inequality holds

$$\frac{\mathbf{P}\left(Y_n = nq_n | \xi_j = a\right)}{\mathbf{P}(Y_n = nq_n)} \leq C \exp\left\{\left(1 - \frac{j}{n}\right)ah_n^0 + \rho|a|\right\}, \tag{5.15}$$

where h_n^0 gives the solution of (2.10) and $C > 0$ is some absolute constant. Then for all integer a one has

$$\mathbf{P}\left(\xi_j = a | Y_n = nq_n\right) \leq C \exp\left\{\left(1 - \frac{j}{n}\right)ah_n^0 + \rho|a|\right\}\mathbf{P}\left(\xi_j = a\right)$$

and so (5.14) follows easily from the inequality

$$\mathbf{E}\left(e^{\rho|\xi_j|} \mid Y_n = nq_n\right) \leq C \sum_{k=-\infty}^{+\infty} \exp\left\{\left(1 - \frac{j}{n}\right)kh_n^0 + 2\rho|k|\right\}\mathbf{P}\left(\xi_j = k\right)$$

and the inclusion $\left[(1-j/n)\,h_n^0-2\rho,\,(1-j/n)\,h_n^0+2\rho\right]\subset \mathcal{K}_{2\rho}$.

It remains to verify estimate (5.15). Defining

$$\widetilde{Y}_n=\widetilde{Y}_n(j)\equiv Y_n-\left(1-\frac{j}{n}\right)\xi_j,\qquad n\tilde{q}_n\equiv nq_n-\left(1-\frac{j}{n}\right)a$$

we obtain

$$\mathbf{P}\left(Y_n=nq_n\,|\,\xi_j=a\right)\equiv\mathbf{P}\big(\widetilde{Y}_n=n\tilde{q}_n\big). \tag{5.16}$$

The logarithmic moment generating function $L_{\widetilde{Y}_n}(\cdot)$ of the random variable $\widetilde{Y}_n$,

$$L_{\widetilde{Y}_n}(h)\equiv\ln\mathbf{E}\,\exp\{h\widetilde{Y}_n\}=L_{Y_n}(h)-L\Big(\Big(1-\frac{j}{n}\Big)h\Big), \tag{5.17}$$

satisfies the relation of the kind (4.7) with the estimate $O(n^{-1})$ that is uniform in $j=1,\dots,n$ and h belonging to any fixed compact subset of $\mathcal{D}_\xi^\circ$. Therefore, determining $\tilde{h}_n^0$ from the equation

$$\frac{d}{dh}L_{\widetilde{Y}_n}(h)\,\Big|_{h=\tilde{h}_n^0}=nq_n \tag{5.18}$$

one easily gets (recall (2.10))

$$\frac{d}{dh}L_{Y,\infty}(h)\,\Big|_{h=h_n^0}-\frac{d}{dh}L_{Y,\infty}(h)\,\Big|_{h=\tilde{h}_n^0}=O(n^{-1}).$$

As a result, the implicit function theorem gives the estimate $\tilde{h}_n^0-h_n^0=O(n^{-1})$ as $n\to\infty$ which is uniform in $j=1,\dots,n$. Thus,

$$|\tilde{h}_n^0-h_n^0|\le\rho/2 \tag{5.19}$$

for all sufficiently large n.

The convergence $\tilde{h}_n^0-h_n^0\to 0$ as $n\to\infty$ and the analog of (4.7) for $L_{\widetilde{Y}_n}(\cdot)$ mean that the asymptotical behaviour of $\widetilde{Y}_n$ is close to that of Y_n, i. e., all the statements of Sect. 4 hold true for $\widetilde{Y}_n$. In particular, the probability $\mathbf{P}\big(\widetilde{Y}_n=nq_n\big)$ is positive for all sufficiently large n. Then (recall (2.10), (5.18), (A.1), and (5.17))

$$\begin{aligned}L_{Y_n}^*(nq_n)-L_{\widetilde{Y}_n}^*(nq_n)&=h_n^0nq_n-L_{Y_n}(h_n^0)-\tilde{h}_n^0nq_n+L_{\widetilde{Y}_n}(\tilde{h}_n^0)\\&=\big(h_n^0-\tilde{h}_n^0\big)nq_n+L_{Y_n}(\tilde{h}_n^0)-L_{Y_n}(h_n^0)-L\Big(\Big(1-\frac{j}{n}\Big)\tilde{h}_n^0\Big)\end{aligned}$$

and therefore this expression is bounded uniformly in $j=1,\dots,n$. Finally, (4.5) and (4.50) together imply the inequality

$$\frac{\mathbf{P}\big(\widetilde{Y}_n=nq_n\big)}{\mathbf{P}\big(Y_n=nq_n\big)}\le\frac{\exp\{-L_{\widetilde{Y}_n}^*(nq_n)\}\widetilde{C}_2n^{-3/2}}{\exp\{-L_{Y_n}^*(nq_n)\}c_2n^{-3/2}}\le C_3$$

which in view of (5.16) gives for all sufficiently large n the estimate

$$\frac{\mathbf{P}\left(Y_n=nq_n\,|\,\xi_j=a\right)}{\mathbf{P}\left(Y_n=nq_n\right)}\le C_3\frac{\mathbf{P}\big(\widetilde{Y}_n=n\tilde{q}_n\big)}{\mathbf{P}\big(\widetilde{Y}_n=nq_n\big)}. \tag{5.20}$$

To evaluate the last fraction we apply analog of (4.5) for $\widetilde{Y}_n$ to obtain

$$\frac{\mathbf{P}(\widetilde{Y}_n = n\tilde{q}_n)}{\mathbf{P}(\widetilde{Y}_n = nq_n)} = \exp\{L^*_{\widetilde{Y}_n}(nq_n) - L^*_{\widetilde{Y}_n}(n\tilde{q}_n)\}\frac{\mathbf{P}(\widetilde{Y}_{n,\tilde{h}^0_n} = n\tilde{q}_n)}{\mathbf{P}(\widetilde{Y}_{n,\tilde{h}^0_n} = nq_n)}. \tag{5.21}$$

Observe that (5.18) and (A.3)–(A.4) imply

$$L^*_{\widetilde{Y}_n}(n\tilde{q}_n) - L^*_{\widetilde{Y}_n}(nq_n) \geq \frac{d}{dx}L^*_{\widetilde{Y}_n}(nq_n)(n\tilde{q}_n - nq_n) = \tilde{h}^0_n\Big(1 - \frac{j}{n}\Big)a \tag{5.22}$$

and so it remains to evaluate the last fraction in (5.21).

Let first $|a| \leq D\sqrt{n}$ with some constant $D > 0$. Then the analog of (4.50) for $\widetilde{Y}_n$ imply

$$\frac{\mathbf{P}(\widetilde{Y}_{n,\tilde{h}^0_n} = n\tilde{q}_n)}{\mathbf{P}(\widetilde{Y}_{n,\tilde{h}^0_n} = nq_n)} \leq \frac{\tilde{C}_2 n^{-3/2}}{\tilde{c}_2 n^{-3/2}} = C_4. \tag{5.23}$$

In the case $|a| \geq D\sqrt{n}$ we find a constant C_5 such that

$$\frac{\mathbf{P}(\widetilde{Y}_{n,\tilde{h}^0_n} = n\tilde{q}_n)}{\mathbf{P}(\widetilde{Y}_{n,\tilde{h}^0_n} = nq_n)} \leq \frac{1}{\tilde{c}_2 n^{-3/2}} \leq C_5 e^{\rho|a|/2} \tag{5.24}$$

for all n. Finally, the estimates (5.20)–(5.24) and (5.19) together imply (5.15). □

Theorem 5.4 *Fix any natural k, a set of real numbers $0 < t_1 < \ldots < t_k \leq 1$ and consider the process $\theta^*_n(t)$ from (2.15).*

Then the distribution of the random vector $\big(\theta^*_n(t_1), \ldots, \theta^*_n(t_k)\big)$ tends to the Gaussian distribution with the density $\tilde{p}_k(\cdot|0)$ defined in (5.7).

Proof. Denote

$$\Upsilon_n = \big(\Theta^*_n(t_1), \ldots, \Theta^*_n(t_k)\big), \qquad \upsilon_n = \big(\theta^*_n(t_1), \ldots, \theta^*_n(t_k)\big).$$

In view of the result of Theorem 5.2 it is enough to show that the difference $\Upsilon_n - \upsilon_n$ vanishes in probability as $n \to \infty$.

Let $C_1 = C_1(\rho) > 0$ be a constant such that $|x| \leq C_1 \exp\{\rho|x|\}$ for all real x. Then (5.13) and (5.14) imply the estimate

$$\mathbf{E}\,|\theta^*_n(t) - \Theta^*_n(t)| \leq \frac{1}{\sqrt{n}}\mathbf{E}\left(|\xi_{[nt]+1}|\,|\,Y_n = nq_n\right) < \frac{CC_1}{\sqrt{n}} \searrow 0 \qquad \text{as} \qquad n \to \infty$$

and so the difference $\theta^*_n(t) - \Theta^*_n(t)$ vanishes in probability as $n \to \infty$. Clearly, the same is true for $\Upsilon_n - \upsilon_n$. □

6 Weak compactness of the sequence μ_n^*

To complete the proof of the Theorem 2.1 it remains to prove the weak compactness of the sequence of measures μ_n^*. We obtain it here as an implication of Theorem 2.2 from [12, Chap. 9] which presents a sufficient condition for the weak compactness of sequences of measures in $\mathbf{C}[0,1]$. The following statement verifies the assumption of the mentioned theorem.

Theorem 6.1 *There exists a constant $C>0$ such that*

$$\mathbf{E}\,|\theta_n^*(t)-\theta_n^*(s)|^4 \le C|t-s|^{7/4}$$

uniformly in all $n \ge n_0$ and all segments $[s,t] \subseteq [0,1]$, $s<t$.

The remaining part of this section is devoted to the proof of Theorem 6.1. Two cases, $\Delta \equiv |t-s| \le n^{-8/9}$ and $\Delta > n^{-8/9}$, are treated separately.

Lemma 6.2 *There exists a constant $C'>0$ such that for all $n \ge n_1$ and all $[s,t] \subseteq [0,1]$, $\Delta \le n^{-8/9}$, the following inequality holds true*

$$\mathbf{E}\,|\theta_n^*(t)-\theta_n^*(s)|^4 \le C'|\Delta|^{7/4}. \tag{6.1}$$

Proof. Denote $\bar{e}_{j,n} \equiv L'\big((1-j/n)h_n^0\big)$ and consider the function (cf. (5.4))

$$\bar{\mathbf{e}}_{\mathbf{n}}(t) = \sum_{j=1}^{[nt]} \bar{e}_{j,n} + \{nt\}\bar{e}_{[nt]+1,n}. \tag{6.2}$$

According to estimates (5.5) and (2.9) one has the following relation

$$\bar{\mathbf{e}}_{\mathbf{n}}(t) - n\mathbf{e}_{\bar{h}}(t) = to(\sqrt{n}) \qquad \text{as} \qquad n \to \infty$$

that is uniform in $t \in [0,1]$. Consequently, it is enough to prove the assertion of the lemma for the random process

$$\theta_n^{**}(t) \equiv \frac{1}{\sqrt{n}}\big(\theta_n(t) - \bar{\mathbf{e}}_{\mathbf{n}}(t)\big). \tag{6.3}$$

Observe that due to Jensen inequality for the function $y=x^4$,

$$\Big(\sum_{i=1}^{l} a_i\Big)^4 \le l^3 \sum_{i=1}^{l} a_i^4, \tag{6.4}$$

and Lemma 5.3 one has the estimate

$$\mathbf{E}\left(|\xi_j - \bar{e}_{j,n}|^4 | Y_n = nq_n\right) \le 8C_4\mathbf{E}\left(e^{\rho|\xi_j|} \mid Y_n = nq_n\right) + 8|\bar{e}_{j,n}|^4 < C_5 \tag{6.5}$$

provided n is sufficiently large. Here the constant C_4 is such that the inequality $|x|^4 \le C_4 \exp\{\rho|x|\}$ holds for all real x.

Define

$$n_t \equiv [nt], \qquad n_s \equiv [ns]. \tag{6.6}$$

For $n_t = n_s$ the assertion of the lemma follows easily from the observation

$$\begin{aligned}\theta_n^{**}(t) - \theta_n^{**}(s) &= \frac{\{nt\} - \{ns\}}{\sqrt{n}}\big(\xi_{n_s} - \bar{e}_{n_s,n}|Y_n = nq_n\big) \\ &= \sqrt{n}\Delta\big(\xi_{n_s} - \bar{e}_{n_s,n}|Y_n = nq_n\big),\end{aligned} \tag{6.7}$$

relation (6.5), and the condition $n\Delta \le 1$. Otherwise $n_t > n_s$ and we consider two cases, $\Delta \le \frac{1}{n}$ and $\Delta > \frac{1}{n}$, separately. The following formula is the starting point in our reasoning (recall (6.3), (6.2), (2.12), and (2.3))

$$\theta_n^{**}(t) - \theta_n^{**}(s) = \sum_{j=n_s+1}^{n_t+1} \frac{\alpha_{s,t}^{(n)}(j)}{\sqrt{n}}\big(\xi_j - \bar{e}_{j,n}|Y_n = nq_n\big), \tag{6.8}$$

where

$$\alpha_{s,t}^{(n)}(j) = \begin{cases} \{nt\}, & \text{if } \quad j = n_t + 1, \\ 1, & \text{if } \quad n_s + 1 < j < n_t + 1, \\ 1 - \{ns\}, & \text{if } \quad j = n_s + 1, \\ 0, & \text{otherwise.} \end{cases} \tag{6.9}$$

Let $\Delta \le \frac{1}{n}$. Then $n_t = n_s + 1$ and one easily obtains (cf. (6.5))

$$\begin{aligned}\mathbf{E}\ |\theta_n^*(t) - \theta_n^*(s)|^4 &\le 8C_5\frac{(1 - \{ns\})^4}{n^2} + 8C_5\frac{\{nt\}^4}{n^2} \\ &\le 8C_5 n^2|t-s|^4 \le 8C_5|t-s|^2\end{aligned} \tag{6.10}$$

for all sufficiently large n. In the second inequality above we use the simple relation $a^4 + b^4 \le (a+b)^4$ and the equality $1 - \{ns\} + \{nt\} = n\Delta$.

In the case $\Delta > \frac{1}{n}$ we use Jensen inequality (6.4), the expansion (6.8), the estimate $|\alpha_{s,t}^{(n)}(j)| \le 1$, and the simple observation $n_t + 1 - n_s < 2 + n\Delta \le 2 + n^{1/9}$ following from (6.6) and the condition $\Delta \le n^{-8/9}$ to obtain

$$\begin{aligned}\mathbf{E}\,|\theta_n^*(t) - \theta_n^*(s)|^4 &\le \frac{(n_t + 1 - n_s)^3}{n^2}\sum_{j=n_s+1}^{n_t+1}\mathbf{E}\,\Big(\big(\xi_j - \bar{e}_{j,n}\big)^4|Y_n = nq_n\Big) \\ &\le C_5\frac{(n\Delta + 2)^{7/4}(2 + n^{1/9})^{9/4}}{n^2} \le C'|t-s|^{7/4}\end{aligned} \tag{6.11}$$

for all sufficiently large n. Combining (6.7), (6.10), and (6.11) we get (6.1). □

Lemma 6.3 *There exists a constant $C'' > 0$ such that for all $n \ge n_2$ and all $[s,t] \subseteq [0,1]$, $\Delta > n^{-8/9}$, one has*

$$\mathbf{E}\ |\theta_n^*(t) - \theta_n^*(s)|^4 \le C''|\Delta|^2. \tag{6.12}$$

Proof. As before, it is enough to prove (6.12) for the random process $\theta_n^{**}(\cdot)$ from (6.3). Define the random variable ζ_n (recall (6.9)),

$$\zeta_n \equiv x_n(t) - x_n(s) = \sum_{j=n_s+1}^{n_t+1}\alpha_{s,t}^{(n)}(j)\xi_j,$$

and consider the random vector $\Lambda_n = \left(Y_n, \zeta_n/\sqrt{\Delta}\right)$. Let $L_{\Lambda_n}(\mathbf{H})$, $\mathbf{H} = (h_0, h_1)$, be its logarithmic moment generating function,

$$L_{\Lambda_n}(\mathbf{H}) \equiv \ln \mathbf{E}\, \exp\{(\mathbf{H}, \Lambda_n)\} = \sum_{j=1}^{n} L\Big(\Big(1 - \frac{j}{n}\Big)h_0 + \frac{\alpha_{s,t}^{(n)}(j)}{\sqrt{\Delta}} h_1\Big). \tag{6.13}$$

For $\mathbf{H}_n^0 = (h_n^0, 0)$ with h_n^0 determined from (2.10) denote

$$\mathbf{E}_n^{\Delta} \equiv \nabla_{\mathbf{H}} L_{\Lambda_n}(\mathbf{H}) \Big|_{\mathbf{H}=\mathbf{H}_n^0} = (nq_n, \mathbf{e}_{\Delta}), \tag{6.14}$$

where

$$\mathbf{e}_{\Delta} \equiv \frac{\partial}{\partial h_1} L_{\Lambda_n}(\mathbf{H}) \Big|_{\mathbf{H}=\mathbf{H}_n^0} = \sum_{j=n_s+1}^{n_t+1} \frac{\alpha_{s,t}^{(n)}(j)}{\sqrt{\Delta}} \bar{e}_{j,n}.$$

Observe that in view of the estimate

$$\mathbf{E}\Big(\frac{\theta_n^{**}(t) - \theta_n^{**}(s)}{\sqrt{\Delta}}\Big)^4 \leq \sum_{k \geq 0} (k+1)^4 \mathbf{P}\Big(\frac{|\zeta_n - \mathbf{e}_{\Delta}\sqrt{\Delta}|}{\sqrt{n\Delta}} > k \,|\, Y_n = nq_n\Big) \tag{6.15}$$

it is sufficient to prove the finiteness of the last sum for all $n \geq n_2$. We prove below the following estimate

$$\mathbf{P}\big(|\zeta_n - \mathbf{e}_{\Delta}\sqrt{\Delta}| > k\sqrt{n\Delta} |\, Y_n = nq_n\big) \leq g_n(k),$$

where for some positive constants C_1, C_2, α_1, α_2 and δ

$$g_n(k) = \begin{cases} C_1 \exp\{-\alpha_1 k^2\}, & \text{if} \quad |k| \leq \delta\sqrt{n\Delta}, \\ C_2 \exp\{-\alpha_2 n^{1/18}|k|\}, & \text{if} \quad |k| > \delta\sqrt{n\Delta}. \end{cases} \tag{6.16}$$

Then the convergence in (6.15) follows immediately.

To prove estimate (6.16) we consider the vector (cf. (6.14))

$$\mathbf{Z}_n^{\Delta} \equiv (nq_n, \mathbf{e}_{\Delta} + k\sqrt{n}) = \mathbf{E}_n^{\Delta} + (0, k\sqrt{n}) \tag{6.17}$$

and determine $\overline{\mathbf{H}}_n = \overline{\mathbf{H}}_n(k) = (\bar{h}_n^0(k), \bar{h}_n^1(k))$ from the condition

$$\nabla_{\mathbf{H}} L_{\Lambda_n}(\mathbf{H}) \Big|_{\mathbf{H}=\overline{\mathbf{H}}_n} = \mathbf{Z}_n^{\Delta}, \tag{6.18}$$

It follows from (6.13) and the implicit function theorem that provided k in (6.17) is of order $\sqrt{n\Delta}$ the quantities $\bar{h}_n^0(k) - h_n^0$ and $\bar{h}_n^1(k)\sqrt{\Delta}$ are of order Δ. Therefore, there exist $\delta = \delta(\rho) > 0$ and $n_3 > 0$ such that for all k, $|k| \leq \delta\sqrt{n\Delta}$, and $n \geq n_3$ the following inequalities hold true

$$|\bar{h}_n^0(k) - h_n^0| \leq \rho\Delta, \qquad |\bar{h}_n^1(k)| \leq \rho\sqrt{\Delta}. \tag{6.19}$$

Consequently, if ρ is the same as in Lemma 5.3, then the function $L_{\Lambda_n}(\overline{\mathbf{H}}_n)$ as well as all its derivatives are uniformly bounded. For future references we fix such value $\delta(\rho)$.

Assuming that $\zeta_n - \mathbf{e}_\Delta\sqrt{\Delta} \geq 0$ (in the opposite case the estimates are similar) we rewrite

$$\begin{aligned}
&\mathbf{P}\big(\zeta_n > \mathbf{e}_\Delta\sqrt{\Delta} + k\sqrt{n\Delta} \,|\, Y_n = nq_n\big) \\
&\quad = \frac{\mathbf{P}\big(Y_n = nq_n, \zeta_n > \mathbf{e}_\Delta\sqrt{\Delta} + k\sqrt{n\Delta}\big)}{\mathbf{P}(Y_n = nq_n)} \\
&\quad = \frac{e^{-L^*_{\Lambda_n}(\mathbf{Z}_n^\Delta)}}{e^{-L^*_{Y_n}(nq_n)}} \frac{\mathbf{P}_{\overline{\mathbf{H}}_n}\big(Y_n = nq_n, \zeta_n > \mathbf{e}_\Delta\sqrt{\Delta} + k\sqrt{n\Delta}\big)}{\mathbf{P}(Y_{n,h_n^0} = nq_n)},
\end{aligned} \tag{6.20}$$

where $\overline{\mathbf{H}}_n$ was determined in (6.18), h_n^0 in (2.10) and $\mathbf{P}_{\overline{\mathbf{H}}_n}(\cdot,\cdot)$ denotes the tilted distribution of Λ_n (cf. (4.12)) with the fixed value $\mathbf{H} = \overline{\mathbf{H}}_n$.

Our aim here is to evaluate the last expression in (6.20). Let first $|k| \leq \delta\sqrt{n\Delta}$. It follows from (6.13), (6.14), (2.10) and duality relations (A.1) that

$$L^*_{\Lambda_n}(\mathbf{E}_n^\Delta) = L^*_{Y_n}(nq_n) \qquad \text{and} \qquad \partial_1 L^*_{\Lambda_n}(\mathbf{E}_n^\Delta) = 0, \tag{6.21}$$

where $\partial_1 L^*_{\Lambda_n}(\mathbf{E}_n^\Delta)$ denotes the derivative of the function $L^*_{\Lambda_n}(x_0, x_1)$ with respect to x_1. Hence, applying (A.5) one obtains

$$L^*_{\Lambda_n}(\mathbf{Z}_n^\Delta) - L^*_{Y_n}(nq_n) = \int_0^{k\sqrt{n}} \big(k\sqrt{n} - y\big)\partial_1{}^2 L^*_{\Lambda_n}(nq_n, \mathbf{e}_\Delta + y)\,dy. \tag{6.22}$$

To bound the derivative $\partial_1{}^2 L^*_{\Lambda_n}(\cdot)$ from below we determine $\mathbf{H}_n^y = \big(h_n^{0,y}, h_n^{1,y}\big)$ from the condition

$$\nabla_{\mathbf{H}} L_{\Lambda_n}(\mathbf{H})\Big|_{\mathbf{H}=\mathbf{H}_n^y} = \mathbf{E}_n^y \equiv \big(nq_n, \mathbf{e}_\Delta + y\big),$$

where $|y| \leq |k|\sqrt{n} \leq \delta\sqrt{n\Delta}$. The same estimates as in Sect. 4 (see (4.19)) prove that the matrix $\mathbf{Hess}L_{\Lambda_n}(\mathbf{H}_n^y)$ is positively definite provided n is sufficiently large. Then

$$0 < \det \mathbf{Hess}L_{\Lambda_n}(\mathbf{H}_n^y) \leq \frac{\partial^2}{\partial {h_0}^2} L_{\Lambda_n}(\mathbf{H}_n^y)\,\frac{\partial^2}{\partial {h_1}^2} L_{\Lambda_n}(\mathbf{H}_n^y)$$

and applying duality relation (A.1) one obtains

$$\frac{\partial^2}{\partial {x_1}^2} L^*_{\Lambda_n}(\mathbf{E}_n^y) = \frac{\frac{\partial^2}{\partial {h_0}^2} L_{\Lambda_n}(\mathbf{H}_n^y)}{\det \mathbf{Hess}L_{\Lambda_n}(\mathbf{H}_n^y)} \geq \Big(\frac{\partial^2}{\partial {h_1}^2} L_{\Lambda_n}(\mathbf{H}_n^y)\Big)^{-1} \geq \frac{C_3}{n},$$

since for all $\mathbf{H}_n^y$ under consideration the derivative $\frac{\partial^2}{\partial {h_1}^2} L_{\Lambda_n}(\mathbf{H}_n^y)$ is bounded from above. Substituting the last estimate into (6.22) one gets

$$L^*_{\Lambda_n}(\mathbf{Z}_n^\Delta) - L^*_{Y_n}(nq_n) \geq \frac{C_3}{n}\int_0^{k\sqrt{n}} \big(k\sqrt{n} - y\big)\,dy = \frac{C_3 k^2}{2} \tag{6.23}$$

provided $|k| \leq \delta\sqrt{n\Delta}$.

In the opposite case, $|k| \geq \delta\sqrt{n\Delta}$, we apply Property A.2 to obtain

$$L^*_{\Lambda_n}(\mathbf{Z}_n^{\Delta}) - L^*_{\Lambda_n}(\mathbf{E}_n^{\Delta}) \geq \frac{C_3}{2}\delta\sqrt{n\Delta}|k| \geq \frac{C_3\delta n^{1/18}}{2}|k|. \tag{6.24}$$

It remains only to estimate the last fraction in (6.20). Consider first the case $|k| < \delta\sqrt{n\Delta}$. Let $L_{Y_{n,\overline{\mathbf{H}}_n}}(h)$ be the logarithmic moment generating function of the first component $Y_{n,\overline{\mathbf{H}}_n}$ of the tilted random vector $\Lambda_{n,\overline{\mathbf{H}}_n}$,

$$L_{Y_{n,\overline{\mathbf{H}}_n}}(h) \equiv \ln\Big(\sum_{k_0\in Z^1} e^{hk_0}\mathbf{P}_{\overline{\mathbf{H}}_n}(Y_n = k_0)\Big) = L_{\Lambda_n}(\bar{h}_n^0 + h, \bar{h}_n^1) - L_{\Lambda_n}(\bar{h}_n^0, \bar{h}_n^1).$$

In view of the choice $\delta > 0$ in (6.19) the quantity $n^{-1}L''_{Y_{n,\overline{\mathbf{H}}_n}}(h)$ is bounded from below by $\inf_{h\in\mathscr{H}_{2p}} L''(h)$ for all sufficiently large n. Therefore, all the considerations of Sect. 4 are true for $Y_{n,\overline{\mathbf{H}}_n}$ and so (cf. (4.49))

$$\mathbf{P}_{\overline{\mathbf{H}}_n}\big(Y_n = nq_n\big) \leq \frac{\overline{C}_0}{n}\Big(\frac{\partial^2}{\partial h_0{}^2}L_{\Lambda_n}(\overline{\mathbf{H}}_n)\Big)^{-1/2}.$$

As a result, one obtains (recall (6.21))

$$\frac{\mathbf{P}_{\overline{\mathbf{H}}_n}\big(Y_n = nq_n\big)}{\mathbf{P}(Y_{n,h_n^0} = nq_n)} \leq \frac{\overline{C}_0}{c_0}\left(\frac{\frac{d^2}{dh^2}L_{Y_n}(h_n^0)}{\frac{\partial^2}{\partial h_0{}^2}L_{\Lambda_n}(\overline{\mathbf{H}}_n)}\right)^{1/2} \leq C_4, \tag{6.25}$$

where the last inequality follows from the boundedness of $n^{-1}L''_{Y_n}(h)$ in $\mathscr{H}^{\circ}$.

In the opposite case, $|k| \geq \delta\sqrt{n\Delta}$, one easily gets (recall (4.50))

$$\frac{\mathbf{P}_{\overline{\mathbf{H}}_n}\big(Y_n = nq_n\big)}{\mathbf{P}(Y_{n,h_n^0} = nq_n)} \leq \frac{1}{\mathbf{P}(Y_{n,h_n^0} = nq_n)} \leq \frac{n^{3/2}}{c_2} \leq C_5 \exp\{\frac{C_3\delta n^{1/18}}{4}|k|\}. \tag{6.26}$$

Finally, (6.16) follows from (6.23)–(6.26) with $C_1 = C_4$, $C_2 = C_5$, $\alpha_1 = C_3/2$, and $\alpha_2 = C_3\delta/4$. □

The assertion of Theorem 6.1 follows immediately from Lemmas 6.2 and 6.3.

Proof of Theorem 2.1 The statement of Theorem 2.1 is a simple implication of Theorems 5.2, 5.4, 6.1, and Theorem 2.2 from [12, Chap. 9]. □

Appendix

We collect here some properties of convex functions used above.

Property A.1 *Let $f(\cdot)$ be a strictly convex twice continuously differentiable real function defined in a region $U \subset R^m$ $(m \geq 1)$ and $f^*(p)$ be its Legendre transformation, $f^*(p) \equiv \sup_x\big((x,p) - f(x)\big)$, $p \in R^m$. Assume that the values $x \in U$ and $p \in R^m$ are related via $\nabla f(x) = p$. Then the following relations hold*

$$\begin{aligned} f^*(p) &= (x,p) - f(x), \\ \nabla f^*(p) &= x, \\ \mathbf{Hess} f^*(p) &= \big(\mathbf{Hess} f(x)\big)^{-1}. \end{aligned} \tag{A.1}$$

Observe that in the considered case the matrix $\mathbf{Hess} f(x)$ of the second derivatives $f(x)$ as a function of $x \in R^m$ is strictly positive definite at x.

This duality property of the Legendre transformation can be verified directly or induced from the known facts ([19, Chap. 5]).

Property A.2 *Let f_γ, $\gamma \in \Gamma$, be a family of convex functions satisfying the condition $f_\gamma(x) \geq f_\gamma(0) = 0$. Assume that for some $b > 0$ and all x, $|x| \leq D$, with some positive D the following inequality holds true*

$$f_\gamma(x) \geq bx^2.$$

Then for all x, $|x| \geq D$, one has

$$f_\gamma(x) \geq bD|x|.$$

Proof. Evidently. □

Finally, we prove here the following property of the Legendre transformation. Let $f_n(\cdot)$ be a sequence of strictly convex twice continuously differentiable functions defined in some δ-neighbourhood $U_\delta(x_*)$ of the point $x_* \in R^1$. Assume that there exists a strictly convex twice continuously differentiable real function $f(\cdot)$ defined in $U_\delta(x_*)$ such that

$$\frac{d^k}{dx^k} f_n(x) = \frac{d^k}{dx^k} f(x) + O(n^{-1}), \qquad k = 0, 1, 2, \tag{A.2}$$

where the estimate $O(\cdot)$ is uniform in some fixed segment $\mathscr{K} \subset U_\delta(x_*)$. Let p_n be a sequence of real numbers such that $p_n - p = o(n^{-1/2})$ as $n \to \infty$ for some p and for any natural n there exists the solution x_n of the equation $f_n'(x_n) = p_n$ belonging to the interior $\mathscr{K}^\circ$ of the compact set $\mathscr{K}$. Suppose also that the limiting point x of the sequence x_n also belongs to $\mathscr{K}^\circ$. Let $f_n^*(\cdot)$, $f^*(\cdot)$ be the Legendre transformations of the functions $f_n(\cdot)$ and $f(\cdot)$ correspondingly.

Property A.3 *There exist positive constants $D_0 = D_0(\{p_n\}, \mathscr{K})$ and $\alpha = \alpha(\mathscr{K})$ such that:*

a) for all real s satisfying the condition $|s| < D_0$ one has

$$f_n^*(p_n + s) \geq f_n^*(p_n) + x_n s + \alpha s^2. \tag{A.3}$$

b) for any $D = D_n$, $0 < D \leq D_0$, and any s, $|s| \geq D$, one has:

$$f_n^*(p_n + s) \geq f_n^*(p_n) + x_n s + \alpha D|s|. \tag{A.4}$$

In the proof of Property A.3 we will use the following simple formula that can be verified directly using the integration by parts. Let $f \in \mathbf{C}^2(a,b)$, $x_0 \in (a,b)$ and $f'(x_0) = 0$. Then for any $x \in (a,b)$ one has

$$f(x) - f(x_0) = \int_{x_0}^{x} (x - y) f''(y)\, dy. \tag{A.5}$$

Proof. Let $\mathscr{K}_1 \subset \mathscr{K}^\circ$ be the smallest segment containing all the sequence x_n and its limit as well. Clearly, the difference $\mathscr{K}^\circ \setminus \mathscr{K}_1$ consists of two intervals. Every function $f_n'(\cdot)$ is strictly increasing in $\mathscr{K}^\circ$, therefore for any n the image $f_n'(\mathscr{K}^\circ \setminus \mathscr{K}_1)$ also consists of two intervals. Denote by D' the length of minimal of them (over all n). Observe that (A.2) implies the estimate $D' > 0$. We put $D_0 = D'/2$. Then for any s, $|s| \leq D_0$, and any n one has

$$p_n + s \in \{f_n'(x) : x \in \mathscr{K}^\circ\}.$$

Denote $g_n(s) = f_n^*(p_n+s) - f_n^*(p_n) - x_n s$. Since every $f_n'(\cdot)$ is strictly increasing the equation $f_n'(y_n) = p_n + r$ has a unique solution $y_n = y_n(r)$ for any r, $|r| \leq D_0$, and so (A.1) implies

$$g_n''(r) = f_n^{*''}(p_n + r) = \left(f_n''(y_n)\right)^{-1} \geq \inf_n \inf_{x \in \mathscr{K}^\circ} \left(f_n''(x)\right)^{-1} = 2\alpha,$$

where in view of (A.2) the constant α is positive. Applying (A.5) we obtain

$$g_n(s) = \int_0^s (s-r) g_n''(r)\, dr \geq 2\alpha \int_0^s (s-r)\, dr = \alpha s^2$$

which coincides with (A.3). Finally, relation (A.4) follows easily from (A.3) and Property A.2. □

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