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Correction to Approximation with Restricted Spectra

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The purpose of this note is to correct the proof of Theorem 4 in [1]. The result in Theorem 4 is still correct. However, the application of Berberian's result in the proof is incorrect. In what follows we use the same notation as in [1].

Theorem 4. *Let C be a closed convex set of cardinality larger than one in the complex plane. Then any normal operator N has a unique best approximant $P_C(N)$ from $\mathfrak{N}(C)$ if and only if*

$$\hat{d}(\sigma(N), C) = d(s, C)$$

for all $s \in \sigma(N)$. Furthermore,

$$\hat{d}(\sigma(N), C) = \|N - P_C(N)\|.$$

In order to prove the above theorem, we need the following lemmas.

Lemma 1. *With the same notation as in Theorem 4, $\hat{d}(\sigma(N), C) = \|N - P_C(N)\|$.* This result is a distance formula of Halmos (cf. reference [6] in [1]).

Since $\sigma(N)$ is compact, we need only consider a compact restriction K of C. If K is a line segment, the proof would follow similarly and hence we assume K to be a compact convex body. Let $K_\alpha = \{x\colon d(x, K) = \alpha\}$ and y an interior point of K. Let $r_{n,i}$, $i = 1, \ldots, n+1$ ($r_{n,n+1} = r_{n,1}$), be rays emanating from y and terminated by K_α with each angle, l_i, between $r_{n,i}$ and $r_{n,i+1}$, equal to $2\pi/n$. Let $x_{n,i} = r_{n,i} \cap K_\alpha$ and $z_{n,i}$ the arc on K_α subtended by the angle l_i and the chord $y_{n,i}$ connecting the points $x_{n,i}$ and $x_{n,i+1}$. Finally, let $d_{n,i} = d(y_{n,k}, z_{n,k})$.

Lemma 2. *There is a sequence $\varepsilon_n \to 0$ such that $d_{n,k} \leqq \varepsilon_n/n$, $k = 1, \ldots, n$.*

To prove this lemma, consider the triangle with sides $r_{n,i}$, $r_{n,i+1}$ and $y_{n,i}$. Since K_α is a convex curve, there is a $\delta > 0$ such that the two angles facing $r_{n,i}$ and $r_{n,i+1}$ are between δ and $\pi - \delta$. Thus, using the law of sines,

$$|y_{n,i}| \leqq C_1 \sin \frac{2\pi}{n} \leqq C_2/n$$

for some constant C_2 independent of i and n. Now, using the fact that the radius of curvature for each point on K_α is at least α, it is easy to see that

$$d_{n,i} \leqq \varepsilon_n' |y_{n,i}| \leqq \varepsilon_n' C_2/n \equiv \varepsilon_n/n.$$

Returning to Theorem 4, let $x_i \in H_i$, $\|x_i\| = 1$, where H_i is a reducing subspace of N such that $\sigma(N|_{H_i}) = z_{n,i}$. If N' is a best approximant to N, then

$$\alpha \geqq \|(N-N')x_i\| \geqq |\langle (N-N')x_i, x_i\rangle| \geqq \alpha - \varepsilon_n/n.$$

Hence, $(N-N')x_i = (\alpha - \delta_{n,i}/n)x_i + f$, where $0 \leqq \delta_{n,1}, \ldots, \delta_{n,n} \leqq \delta_n \equiv \delta_n(\varepsilon_n, N') \to 0$, $\langle f, x_i\rangle = 0$ and $\|f\| \leqq \delta_n/\sqrt{n}$. Thus, if N_1 and N_2 are any two best approximants to N and $\|x\| = 1$, then

$$\begin{aligned}\|(N_1-N_2)x\| &\equiv \left\|\sum_{i=1}^{n} \alpha_i (N_1-N_2)x_i\right\| \leqq \sum_{i=1}^{n} |\alpha_i|\, \|(N_1-N_2)x_i\| \\ &\leqq \left(\sum_{i=1}^{n} \|(N_1-N_2)x_i\|^2\right)^{1/2} \leqq [n(\eta_n/\sqrt{n})^2]^{1/2} = \eta_n \to 0,\end{aligned}$$

where η_n comes from $\delta_n(\varepsilon_n, N_1)$ and $\delta_n(\varepsilon_n, N_2)$. Thus, $(N_1-N_2)x = 0$ for all x, or $N_1 = N_2$.

The proof of the converse is the same as in [1].

We also note that the reference to the Berberian embedding in the second line of the proof of Theorem 6 should be omitted.

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References

1. Chui, C. K., Smith, P. W., Ward, J. D.: Approximation with restricted spectra. Math. Z. **144**, 289–297 (1975)

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