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Errata in "Integral formulation and fundamental solutions of dynamic poroelasticity and thermoelasticity"

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1 Corrections

Upon close inspection of their paper [1], the authors noticed a number of errors that have to be corrected. First of all, the coefficients a_{12} and a_{21} in equation (13) are not equal. This poses a problem with the reciprocal relation, because unwanted terms appear in equation (19) involving volume integrals of both solid displacements $\bar{u}_i$ and relative fluid displacements $\overline{w}_i$. To avoid this, the problem must be reformulated with the absolute fluid displacements U_i replacing w_i. In that case, equations (1) are replaced by Biot's equations in their original form

$$\tau_{ij,j} - b(\dot{u}_i - \dot{U}_i) + F_i = \varrho_{11}\ddot{u}_i + \varrho_{12}\ddot{U}_i, \tag{1.1}$$

$$\tau_{,i} + b(\dot{u}_i - \dot{U}_i) + X_i = \varrho_{12}\ddot{u}_i + \varrho_{22}\ddot{U}_i, \tag{1.2}$$

while equations (6) and (8) can be disregarded. Next it should be noted that σ_{ij} in (4.1) and (10.1) has to be replaced by τ_{ij}. These changes affect equations (13) which now take the corrected form

$$\begin{aligned} \bar{\tau}_{ij,j} + \overline{F}_i &= (\varrho_{11}s^2 + bs)\,\bar{u}_i + (\varrho_{12}s^2 - bs)\,\overline{U}_i - (\varrho_{11} + \varrho_{12})\,(s\mathring{u}_i + \mathring{v}_i) \\ &\quad - \varrho_{12}(s\mathring{w}_i + \mathring{\omega}_i) + b\mathring{w}_i = a_{11}\bar{u}_i + a_{12}\overline{U}_i - \bar{y}_i, \\ \bar{\tau}_{,i} + \overline{X}_i &= (\varrho_{12}s^2 - bs)\,\bar{u}_i + (\varrho_{22}s^2 + bs)\,\overline{U}_i - (\varrho_{12} + \varrho_{22})\,(s\mathring{u}_i + \mathring{v}_i) \\ &\quad - \varrho_{22}(s\mathring{w}_i + \mathring{\omega}_i) - b\mathring{w}_i = a_{12}\bar{u}_i + a_{22}\overline{U}_i - \bar{z}_i. \end{aligned} \tag{13}$$

Now, in order to leave all material from Sections 4, 5, 7 and 8 unaffected by the above changes except for the key substitution of $\overline{U}_i$ in place of $\overline{w}_i$, the tractions $\bar{t}_i$ should be replaced by $\bar{t}_i{}^* = \bar{\tau}_{ij}n_j$. Equation (48), however, remains as it is apart from changing $\overline{w}_{jk,j}$ to $\overline{U}_{jk,j}$.

There are also some typing errors. On the first line following equation (15), the term right-hand side must be replaced by the term left-hand side, while the opposite must happen on the first line following equation (19). Also, on the first integral in equations (16), (18) and (19) the primes must be moved from the stresses $\bar{\tau}_{ij}, \bar{\tau}$ to the strains $\bar{e}_{ij}, \bar{\varepsilon}$. A term was missing from equation (34) which now reads as

$$\begin{aligned} m_i = \pm\big\{&[(a_{11}b_{22} + a_{22}b_{11} - 2a_{12}b_{12}) \pm \big((a_{11}b_{22} + a_{22}b_{11} - 2a_{12}b_{12})^2 \\ &- 4(b_{11}b_{22} - b_{12}^2)\,(a_{11}a_{22} - a_{12}^2)\big)^{1/2}]\,[2(b_{11}b_{22} - b_{12}^2)]^{-1}\big\}^{1/2}. \end{aligned} \tag{34}$$

A parenthesis is missing in front of term $a_{12}\bar{\vartheta}^{(l)}$ in the second of equations (26). The term $1/4\pi r$ in equation (28.1) must read as $-1/4\pi r$, and the term $x_i x_j/r^3$ in equation (46) must read as $x_i x_j/r^2$. Finally, on the top line of page 97, equation (32) must be written instead of equation (33), and on the second line from the bottom of the same page $D^{(l)}$ must replace $D^{(1)}$.

Next, the use of the irrotationality condition on the fluid velocity vector as a means of producing a constraint equation between constants $A_k^{(l)}$, $B_k^{(l)}$, $C^{(l)}$ and $D^{(l)}$ is not correct. Instead, one can go back to the homogeneous version of equation (32) and make the substitutions $\bar{u}^{(l)} = A_k^{(l)} \exp(m_k r)$ and $\bar{v}^{(l)} = B_k^{(l)} \exp(m_k r)$ to find that

$$B_k^{(l)} = -\frac{a_{11} - m_k^2(\lambda_C + 2\mu)}{a_{12} - m_k^2 Q} A_k^{(l)}, \qquad k, l = 1, 2. \tag{40}$$

The above equation replaces original equations (40) and (41) and, along with original equations (37), forms a system of six equations in six unknowns ($A_1^{(l)}$, $A_2^{(l)}$, $B_1^{(l)}$, $B_2^{(l)}$, $C^{(l)}$ and $D^{(l)}$).

Finally, when the analogy between poroelasticity and thermoelasticity is explored, the term $-b(\dot{u}_i - \dot{U}_i)$ must be added to the left-hand side of equation (55) as a consequence of changing the governing equations (1) earlier on. Equation (56) is then replaced by

$$(\lambda + \mu)\, u_{j,ji} + \mu u_{i,jj} + \left(\frac{Q}{R} + 1\right) \tau_{,i} + (F_i + X_i) = (\varrho_{11} + \varrho_{12})\, \ddot{u}_i + (\varrho_{12} + \varrho_{22})\, \ddot{U}_i \tag{56}$$

which results from adding corrected original equation (56) to equation (1.2). As far as Table 1 is concerned, the original poroelastic F_i must be replaced by $F_i + X_i$, while Q/R must be replaced by $(Q/R) + 1$.

2 Additions

It is possible to use only four independent variables for the dynamic poroelastic problem, namely the solid displacements u_i and the fluid stress τ (or fluid pressure p since $\tau = -fp$). In that case, equations (49) and (50), both of which are now written in terms of the fluid displacements $\bar{U}_i$, $\bar{U}_n$, $\bar{U}_{ij}^{(1)}$ and $\bar{U}_{ij}^{(2)}$ instead of the fluid relative displacements, are substituted in the constitutive law (4.2), i.e.,

$$\bar{\tau}(\xi) = Q \frac{\partial \bar{u}_j(\xi)}{\partial \xi_j} + R \frac{\partial \bar{U}_j(\xi)}{\partial \xi_j} \tag{A.1}$$

and the resulting expression replaces equation (50). It should be noted here that since differentiation is with respect to point ξ, which was originally taken as the origin of the coordinate system, the radial distance between ξ and the integration point x is now $r = [(x_i - \xi_i)(x_i - \xi_i)]^{1/2}$ and consequently $\partial/\partial\xi_i = -\partial/\partial x_i$. The integral equation for the fluid stress becomes

$$\begin{aligned}\bar{\tau}(\xi, s) = \int_S \{&[Q\bar{t}_{ij,j}^{(1)} + R\bar{t}_{ij,j}^{(2)}]\, \bar{u}_i + [Q\bar{\tau}_{j,j}^{(1)} + R\bar{\tau}_{j,j}^{(2)}]\, \bar{U}_n \\ &- [Q\bar{u}_{ij,j}^{(1)} + R\bar{u}_{ij,j}^{(2)}]\, \bar{t}_i - [Q\bar{U}_{nj,j}^{(1)} + R\bar{U}_{nj,j}^{(2)}]\, \bar{\tau}\}\, dS(x) \\ + \int_V \{&[Q\bar{u}_{ij,j}^{(1)} + R\bar{u}_{ij,j}^{(2)}]\, (\bar{y}_i + \bar{F}_i) + [Q\bar{U}_{ij,j}^{(1)} + R\bar{U}_{ij,j}^{(2)}]\, (\bar{z}_i + \bar{X}_i)\}\, dV(x)\end{aligned} \tag{A.2}$$

where commas indicate spatial differentiation with respect to $\boldsymbol{x}$. This formulation has the advantage that four variables $(\bar{u}_i, \bar{\tau})$ are now involved instead of the previous six $(\bar{u}_i, \bar{U}_i)$, but the integral equation for $\bar{\tau}$ has higher order singularities than the one for $\bar{U}_i$, original equation (50). Given the relatively simple expressions (39) for fundamental solutions $\bar{u}_{ij}^{(1)}$ and $\bar{U}_{ij}^{(1)}$, the differentiations required to produce the integral equation (A.2) are straightforward.

It is also interesting to look at two-dimensional dynamic poroelasticity. The integral equations (51) are still valid with indices i, j ranging from 1 to 2. The fundamental solution (39), however, needs to be integrated along the z-axis. We focus on the case $l = 1$ since the case $l = 2$ is similar. At first, one has to go back to the forcing function (22) and introduce a unit force $F = 1$ so that $F_{ij}^{(1)} = F\delta_{ij}\delta(\boldsymbol{x})$. Then, a constant force density $f = F/2L$ equally spaced along length $2L$ about both sides of the origin and along the z-axis is inintroduced. The contribution to the fundamental solution $\bar{u}_{ij}^{(l)}$ from each point source $\Delta F = f\,dz$ is then integrated along z from L to $-L$ and then L is allowed to go to infinity through a limiting process. It should be noted that one must first integrate potentials $\bar{\theta}(r)$ and $\bar{\varphi}(r)$ given by equations (29) and (33) and then go to the first of equations (24) to reconstitute the fundamental solution through differentiation of the two potentials. This way, there are two types of integrals to evaluate, namely

$$\begin{aligned} &\int\limits_{-\infty}^{\infty} \frac{\exp\left(n\sqrt{z^2+\varrho^2}\right)}{\sqrt{z^2+\varrho^2}}\,dz = 2K_0(-n\varrho), \\ &\lim_{L\to\infty} \int\limits_{-L}^{L} \frac{f\,dz}{\sqrt{z^2+\varrho^2}} = -2f\ln\varrho \end{aligned} \tag{A.3}$$

where $\varrho^2 = x_1{}^2 + x_2{}^2 = r^2 - z^2$ and K_0 is the modified Bessel function of zero order. The following expression is finally obtained for the two-dimensional fundamental solution $\bar{u}_{ij}^{(l)}$ by letting $f = 1$:

$$\begin{aligned} \bar{u}_{ij}^{(l)} = {} & \frac{x_i x_j}{\varrho^2}\left[A_1{}^{(l)} m_1{}^2 K_2(-m_1\varrho) + A_2{}^{(l)} m_2{}^2 K_2(-m_2\varrho) - C^{(l)} n^2 K_2(-n\varrho)\right] \\ & + \delta_{ij} C^{(l)} n^2 K_2(-n\varrho) - \frac{\delta_{ij}}{\varrho}\left[A_1{}^{(l)} m_1 K_1(-m_1\varrho) + A_2{}^{(l)} m_2 K_1(-m_2\varrho)\right. \\ & \left. + C^{(l)} n K_1(-n\varrho)\right]. \end{aligned} \tag{A.4}$$

In the above, K_1 and K_2 are the modified Bessel functions of order one and two, respectively.

Recently, we have become aware of two more publications on this subject, namely those of Norris [2] and Bonnet [3]. The approach we have followed is very similar to that followed by Norris [2], except that he works in the frequency domain and neglects the effect of the initial conditions. Other than that, Norris [2] uses the same governing equations and our parameters a_{ij} are equal to his terms $-\omega^2\bar{\varrho}_{ij}$. As a matter of fact, it is easy to show that the complex wave numbers ik_T, ik_F and ik_S in [2] are the same as our parameters n, m_1 and m_2 (see equations (34) and (35)), respectively, provided $-\omega^2$ is replaced by s^2. Based on that, formal equivalence between the fundamental solutions for the solid displacements u_{ij} given in [2] and here (equation (39)) can be demonstrated. Bonnet [3] works with the solid displacements u_i and fluid pressure p in the frequency domain and using a poroelasticity-thermoelasticity analogy derives the two-dimensional fundamental solution.

His expression for $\bar{u}_{ij}$ is in form equivalent to our expression prior to differentiating the expressions of the integrated potentials $\bar{\theta}(r)$ and $\bar{\varphi}(r)$ if we take into account that $2K_0(-n\varrho) = -i\pi H_0^{(2)}(in\varrho)$, where $H_0^{(2)}$ is the Hankel function of zero order. Although we previously demonstrated that we can revert to the (u_i, p) type of formulation from our original one (u_i, U_i), it is difficult to show that, after appropriate rearrangements are done, Bonnet's [3] fundamental solution for p is the same as ours due to the large amount of algebra involved.

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