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On a Characterization of Infinitely Divisible Characteristic Functionals on a Hilbert Space

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Summary. A characterization of infinitely divisible characteristic functionals on a Hilbert space, analogous to that of Johansen [1], is given.

1. Introduction

Let X be a separable Hilbert space, (x, y) denotes its inner product and $\|x\|$ the norm. Let μ be a probability measure on the σ-field of Borel subsets of X. The characteristic functional $\hat{\mu}(\cdot)$ is defined by

$$\hat{\mu}(y) = \int e^{i(x,y)} \, d\mu(x) \tag{1.1}$$

for $y \in X$.

Definition 1.1. A positive semi-definite Hermitian operator A on X is called an S-operator if it has finite trace. The class of sets $\{x: (Sx, x) < t\}$ where S runs over S-operators and t over positive numbers forms a neighborhood system at the origin for a certain topology on X which is called the S-topology. A net $\{x_\alpha\}$ converges to zero in S-topology if and only if $(S x_\alpha, x_\alpha)$ converges to zero for every S-operator S.

We have the following characterization of characteristic functionals on a Hilbert space due to Sazanov [3].

Theorem 1.1. *In order that a function $\hat{\mu}(\cdot)$ may be the characteristic functional of a distribution μ on X, it is necessary and sufficient that* (i) $\hat{\mu}(0) = 1$ (ii) $\hat{\mu}(\cdot)$ *be positive definite and* (iii) $\hat{\mu}(\cdot)$ *be continuous at* 0 *in the S-topology.*

Our aim in this paper is to give a similar characterization for infinitely divisible characteristic functionals on a Hilbert space. We shall state a few more definitions and the main theorem in the next section. Proof of the theorem is given in Section 3. For more details on probability measures and characteristic functionals on a Hilbert space, the reader is referred to either Parthasarathy [2], or Varadhan [4].

2. Infinitely Divisible Distributions

Definition 2.1. A probability measure μ on a Hilbert space X is said to be infinitely divisible if for every positive integer n,

$$\mu = \underbrace{\lambda_n * \lambda_n * \cdots * \lambda_n}_{n \text{ times}}$$

where λ_n is a probability measure on X and $*$ denotes the convolution operation.

Theorem 2.1. *If μ is an infinitely divisible distribution on X and $\hat{\mu}$ is its characteristic functional, then $\hat{\mu}(y) \neq 0$ for all $y \in X$.*

We refer the reader to Parthasarathy [2] for a proof of this theorem. We also note that a Hilbert space has no nontrivial compact subgroups and hence there are no nontrivial idempotent distributions. Further more, if $\hat{\mu}(\cdot)$ is the characteristic functional of an infinitely divisible distribution on X, then it has a unique representation of the form $\hat{\mu}(y) = e^{\chi(y)}$ for some complex valued function $\chi(\cdot)$ on X by Theorem 4.10 of Parthasarathy [2]. We shall call $\chi(\cdot)$ the logarithm of the characteristic functional $\hat{\mu}(\cdot)$. The main theorem of the paper will be stated now.

Theorem 2.2. *In order that $\chi(\cdot)$ be the logarithm of a characteristic functional of an infinitely divisible distribution μ on X it is necessary and sufficient that*

(i) $\chi(0) = 0, \chi(y) = \overline{\chi(-y)}$,

(ii) *$\chi(\cdot)$ is continuous at 0 in S-topology, and*

(iii) *for every choice y_i, $1 \leqq i \leqq N$ in X and complex numbers α_i, $1 \leqq i \leqq N$ such that $\sum_{i=1}^{N} \alpha_i = 0$,*

$$\sum_{i=1}^{N} \sum_{j=1}^{N} \chi(y_i - y_j)\, \alpha_i \bar{\alpha}_j \geqq 0.$$

Before we give a proof of the above theorem, we shall state a lemma. This lemma has been proved by Johansen [1] for complex valued functions on the real line. Proof of the lemma for complex valued functions on a Hilbert space is exactly similar to the proof given by Johansen [1] and hence it is omitted.

Lemma 2.3. *Let $\chi(\cdot)$ be any complex valued function on X such that $\chi(0) = 0$, $\chi(y) = \overline{\chi(-y)}$. Then the following conditions are equivalent.*

(i) *For every choice y_i, $1 \leqq i \leqq N$ in X and complex numbers α_i, $1 \leqq i \leqq N$ and $\lambda \geqq 0$,*

$$\sum_{i=1}^{N} \sum_{j=1}^{N} e^{\lambda \chi(y_i - y_j)}\, \alpha_i \bar{\alpha}_j \geqq 0.$$

(ii) *For every choice y_i, $1 \leqq i \leqq N$ in X and complex numbers α_i, $1 \leqq i \leqq N$ such that*

$$\sum_{i=1}^{N} \alpha_i = 0, \qquad \sum_{i=1}^{N} \sum_{j=1}^{N} \chi(y_i - y_j)\, \alpha_i \bar{\alpha}_j \geqq 0.$$

(iii) *For every choice y_i, $1 \leqq i \leqq N$ in X and complex numbers α_i, $1 \leqq i \leqq N$,*

$$\sum_{i=1}^{N} \sum_{j=1}^{N} [\chi(y_i - y_j) - \chi(y_i) - \chi(-y_j)]\, \alpha_i \bar{\alpha}_j \geqq 0.$$

3. Proof of Theorem 2.2

Necessity. Suppose $\hat{\mu}(\cdot)$ is the characteristic functional of an infinitely divisible distribution μ on X. Let $\chi(\cdot)$ be the logarithm of $\hat{\mu}(\cdot)$. (i) follows from the definition of a characteristic functional. (ii) follows from Theorem 1.1. Since $\hat{\mu}$ is an infinitely divisible characteristic functional $\hat{\mu}^{1/n}$ is a uniquely determined characteristic functional for every n and hence $\hat{\mu}^r$ is a characteristic functional for every rational number r. Hence by Theorem 1.1,

$$\sum_{i=1}^{N}\sum_{j=1}^{N}[\hat{\mu}(y_i-y_j)]^r\alpha_i\bar{\alpha}_j\geqq 0 \tag{3.1}$$

for every choice of y_i, $1\leqq i\leqq N$ in X and complex numbers α_i, $1\leqq i\leqq N$. Let r_n be any sequence of rational numbers approaching $\lambda\geqq 0$ as n approaches ∞. By taking limits as $n\to\infty$ on both sides of (3.1), we get that

$$\sum_{i=1}^{N}\sum_{j=1}^{N}[\hat{\mu}(y_i-y_j)]^\lambda\alpha_i\bar{\alpha}_j\geqq 0 \tag{3.2}$$

for every choice of y_i, $1\leqq i\leqq N$ and complex numbers α_i, $1\leqq i\leqq N$, and $\lambda\geqq 0$. This proves (iii) in view of Lemma 2.3 since $\hat{\mu}(\cdot)=e^{\chi(\cdot)}$. This completes the proof of the necessity of the conditions (i), (ii) and (iii) of the theorem.

Sufficiency. Suppose $\chi(\cdot)$ is a complex valued functional which satisfies (i), (ii) and (iii) of the theorem. Define $\psi(y)=e^{\chi(y)}$ for any $y\in X$. Clearly $[\psi(0)]^\lambda=1$ for any $\lambda\geqq 0$ and ψ^λ is continuous at 0 in S-topology for any $\lambda\geqq 0$. Since

$$\sum_{i=1}^{N}\sum_{j=1}^{N}\chi(y_i-y_j)\alpha_i\bar{\alpha}_j\geqq 0 \tag{3.3}$$

for every choice of y_i, $1\leqq i\leqq N$ in X and complex numbers α_i, $1\leqq i\leqq N$ such that $\sum_{i=1}^{N}\alpha_i=0$, Lemma 2.3 implies that

$$\sum_{i=1}^{N}\sum_{j=1}^{N}e^{\lambda\chi(y_i-y_j)}\alpha_i\bar{\alpha}_j\geqq 0 \tag{3.4}$$

equivalently

$$\sum_{i=1}^{N}\sum_{j=1}^{N}[\psi(y_i-y_j)]^\lambda\alpha_i\bar{\alpha}_j\geqq 0 \tag{3.5}$$

for every choice of y_i, $1\leqq i\leqq N$ in X and every choice of complex numbers α_i, $1\leqq i\leqq N$. In other words ψ^λ is positive definite for every $\lambda\geqq 0$. This fact together with earlier remarks prove that ψ^λ is a characteristic functional for every $\lambda\geqq 0$ by Theorem 1.1. Hence ψ is an infinitely divisible characteristic functional which proves the sufficiency of the conditions (i), (ii) and (iii) of the theorem.

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