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On the Existence of Equivalent Finite Invariant Measures*

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Summary. Necessary and sufficient conditions are given for the existence of a finite measure which is equivalent to a given measure and invariant with respect to each transformation in a given commutative semigroup of measurable null-invariant point transformations. This result was already known for denumerably generated semigroups. A complementary result is proved which states that if one such equivalent measure exists, then there exists a unique equivalent measure which agrees with the original measure on the invariant sets.

Let (Ω, Σ, P) be a probability space, let T be a non-empty collection of point transformations τ mapping Ω into Ω which are measurable ($A\in\Sigma$ implies $\tau^{-1}A\in\Sigma$) and null-invariant ($P(A)=0$ implies $P(\tau^{-1}A)=0$), let T^* be the collection of finite products of members of T, and for $A\in\Sigma$ define

$$r(A)=\inf\{P(\tau^{-1}A)\,|\,\tau\in T^*\}. \tag{1}$$

The main purpose of this paper is to prove:

Theorem 1. *If the transformations in T commute, then a necessary and sufficient condition for the existence of a finite measure which is equivalent to P and invariant with respect to all the transformations in T is:*

(C) *If $A\in\Sigma$ and $P(A)>0$, then $r(A)>0$.*

Theorem 1 generalizes early work of Calderon [2] and Dowker [3] in which T contains only one transformation, and also generalizes more recent work of Blum and Friedman [1] in which T is denumerable and the transformations in T commute.

Let I be the collection of measurable sets which are invariant with respect to every τ in T in the sense that $A\in I$ if $A\in\Sigma$ and $P(\tau^{-1}A\,\triangle\,A)=0$ for every τ in T. The following result strengthens Theorem 1.

Theorem 2. *If there exists a finite measure m which is equivalent to P and invariant with respect to all the transformations in T, then there is such a measure which agrees with P on I, and it is unique.*

In the proof of Theorem 1 we use the following lemma which occurs as Theorem 3 of [2]:

Lemma. *If m is a finite and finitely additive measure which is equivalent to P and invariant with respect to each τ in T, and if*

$$\mu(A)=\inf\left\{\sum_{n=1}^{\infty} m(A_n): A_n\in\Sigma,\ A\subset\bigcup_{n=1}^{\infty} A_n\right\}$$

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for each $A \in \Sigma$, then μ is a finite measure which is equivalent to P and is invariant with respect to each τ in T.

Proof of Theorem 1. Suppose m is a finite measure equivalent to P and invariant with respect to each $\tau \in T$. If $P(A) > 0$ and $r(A) = 0$, then $P << m$ implies $m(A) > 0$, and $m << P$ implies $\inf_{\tau \in T^*} \{m(\tau^{-1} A)\} = 0$ which contradicts the invariance of m. Thus (C) is necessary.

Now suppose (C) holds. For each $A \in \Sigma$ let G_A be the topological space consisting of [0, 1] with its standard topology, and let $G = \prod_{A \in \Sigma} G_A$ with the standard product topology; by Tychonoff's Theorem G is compact. Note that each $g \in G$ is a set function on Σ taking values in [0, 1]. Let F_o be the collection of elements $g \in G$ such that

i) g is finitely additive,

ii) $A \in \Sigma$ and $P(A) = 0$ imply $g(A) = 0$, and

iii) $g(A) \geqq r(A)$ for all $A \in \Sigma$;

note that ii), iii), and the fact that P satisfies (C) imply that g and P are equivalent, and that iii) implies $g(\tau^{-1} A) \geqq r(\tau^{-1} A) \geqq r(A)$ for all $A \in \Sigma$ and all $\tau \in T^*$. For each non-empty subset S of T define $F_S = \{g \mid g \in F_o$ and $g(\tau^{-1} A) = g(A)$ for all $\tau \in S$ and all $A \in \Sigma\}$. Note that F_o and F_S are closed sets. Our objective is to show that F_T is non-empty. Then an application of the lemma completes the proof of the theorem.

The original proof of the authors involved well ordering T and using an induction argument to show that whenever S is an initial segment of T the set F_S is non-empty. The argument given here is due to Professor U. Krengel and is much simpler.

Let $\mathscr{S}$ be the collection of finite non-empty subsets of T. If S is in $\mathscr{S}$, then the argument of Theorem 3 of [1] shows that $F_S \neq \phi$. (In fact it shows that F_S contains a probability measure.) Now $\{F_S \mid S \in \mathscr{S}\}$ is a collection of compact sets having the finite intersection property (since a finite intersection of sets in $\{F_S \mid S \in \mathscr{S}\}$ is again in $\{F_S \mid S \in \mathscr{S}\}$). Since G is compact $\phi \neq \bigcap_{S \in \mathscr{S}} F_S = F_T$.

Proof of Theorem 2. First note that I is a σ-algebra. Let m^* and P^* be the restrictions to I of m and P respectively. They are equivalent so by the Radon-Nikodym theorem there exists a non-negative I-measurable function f such that

a) $P^*(A) = \int_A f \, dm^*$ for each $A \in I$, and

b) $m^* \{\omega \mid f(\omega) = 0\} = 0$.

For $A \in \Sigma$ define $Q(A) = \int_A f \, dm$ and note that Q and P^* agree on I; the fact that Q is invariant with respect to each $\tau \in T$ follows from the same fact for m and the I-measurability of f. If $A \in \Sigma$ and $P(A) = 0$, then $m(A) = 0$ so $Q(A) = 0$; thus $Q << P$. If $A \in \Sigma$ and $Q(A) = 0$, then from the definition of Q we have

$$m[A \cap \{\omega \mid f(\omega) > 0\}] = 0.$$

But
$$0 \leqq m(A) = m[A \cap \{\omega | f(\omega) > 0\}] + m[A \cap \{\omega | f(\omega) = 0\}]$$
$$\leqq 0 + m\{\omega | f(\omega) = 0\} = m^*\{\omega | f(\omega) = 0\} = 0 \text{ so } m(A) = 0$$

and thus $P(A)=0$. Therefore $P << Q$ so P and Q are equivalent.

Now suppose Q_1 and Q_2 are two measures which agree on I and are invariant with respect to each $\tau \in T$. Then $\mu = Q_1 - Q_2$ is a signed measure which is zero on I and is invariant with respect to each $\tau \in T$. From the Hahn decomposition for μ there is a measurable set A such that A is a positive set for μ and A^c is a negative set for μ. If A were not τ-invariant, then either $\mu(A \cap \tau^{-1} A) > \mu(A)$ or $\mu(A \cup \tau^{-1} A) > \mu(A)$ contradicting the maximality of A. Thus A (and consequently A^c) is in I. It follows that $\mu(A) \equiv 0$ so $Q_1 = Q_2$.

Remark 1. It would be interesting to know whether the Q of Theorem 2 satisfies $Q(A) \geqq r(A)$ for all $A \in \Sigma$. This is, of course, true if T is finite.

Remark 2. The authors tried to generalize Theorem 5 of [1] but became convinced instead that a rigorous proof of Theorem 5 of [1] has not yet been published.

If $T = \{\tau_1, \tau_2\}$, (C) holds, and both τ_1 and τ_2 are of period 2, the authors of [1] show on p. 303 how to obtain the desired finite measure equivalent to P and invariant with respect to both τ_1 and τ_2. However, suppose τ_1 and τ_2 are both of period 3. Let S_k be the collection of all finite products of the form $\tau_{i_1} \tau_{i_2} \ldots \tau_{i_k}$ such that $i_\alpha = 1$ or 2 for $\alpha = 1, \ldots, k$ and such that neither $\tau_1 \tau_1 \tau_1$ nor $\tau_2 \tau_2 \tau_2$ appear anywhere in the product. Let c_k be the number of terms in S_k. The authors of [1] seem to suggest using a Banach limit of the sequences
$$\left\{\left[\sum_{k=1}^{N} c_k\right]^{-1} \sum_{k=1}^{N} \sum_{\tau \in S_k} P(\tau^{-1} A)\right\}.$$

If that is the case, the number of terms in
$$\sum_{k=1}^{N} \sum_{\tau \in S_k} P(\tau^{-1} A) - \sum_{k=1}^{N} \sum_{\tau \in S_k} P[\tau^{-1}(\tau_i^{-1} A)]$$
divided by $\sum_{k=1}^{N} c_k$ does not go to zero as $N \to \infty$ for $i=1$ or for $i=2$. Thus the finite measure obtained from the Banach limit need be neither τ_1-invariant nor τ_2-invariant.

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