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The Martin Boundary and Completion of Markov Chains*

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Summary. In this paper we treat a time-symmetrical Martin boundary theory for continuous parameter Markov chains. This is done by reversing the time sense of a Markov chain X_t in such a way as to obtain a dual Markov chain $\tilde{X}_t$, and considering the two chains together. Various relations between the Martin exit boundaries $\mathscr{B}_0^*$ and $\tilde{\mathscr{B}}_0^*$ of these processes are studied. The exit boundary $\tilde{\mathscr{B}}_0^*$ of $\tilde{X}_t$ is in a sense an entrance boundary for X_t and vice versa. After a natural identification of certain points in $\mathscr{B}_0^*$ and $\tilde{\mathscr{B}}_0^*$ one can topologize $I \cup \mathscr{B}_0^* \cup \tilde{\mathscr{B}}_0^*$ in such a way that *both* X_t and $\tilde{X}_t$ have standard modifications in this space which are right continuous, have left limits, and are strongly Markov.

§ 1. Introduction

According to fundamental theorems of Knight [10], Ray [14], and more recently Kunita and Watanabe [11], it is possible under slight restrictions to complete the state space of a Markov process in such a way that an extension of the process is strongly Markov on the enlarged space. It is clearly of interest to see how such compactifications look in special cases. Doob [4] has given a treatment of the case of standard Markov chains.

In a different connection, Chung [2], Dynkin [5], Feller [6] and Williams [15] have considered compactifications of the state space of certain Markov chains based on the Feller and Martin exit boundaries. Chung has extensively analyzed the sample paths of the process in this case. Boundary states are used to describe the behavior at infinity, but the process is not regarded as taking its values in the enlarged space; in fact there are times – for example the last time the process hits certain boundary states – when the process can not be said to be either in the state space or in the boundary. A weak version of the strong Markov property holds [2] but the process is not strongly Markov.

In general the Martin boundaries of a Markov chain are too small to provide a strongly Markov extension. In this paper we show that under certain finiteness conditions, the Martin entrance and exit boundaries are sufficient.

Our methods rely on the basic time symmetry of the Markov property. We use both exit and entrance boundaries; the latter is defined by reversing the process in time. One of the advantages of this approach is that both the forward and reverse processes can be made strongly Markov in the same space, which is not necessarily true of previous compactifications.

We also find close relations between sticky and non-sticky boundary states – here we use Chung's terminology – and the non-branching and branching points introduced by Ray; the set of branching points is exactly the set of non-sticky exit boundary points. The relations between these types of points are also

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bound up with relations between the entrance and exit boundaries. For instance, in a certain sense one can say that a boundary point is sticky if and only if it is in both the entrance and exit boundaries.

A word is in order about the entrance boundary. We define it in terms of the reversed process rather than the dual process, so that it is not identical with the boundary defined by Hunt [7] and Doob [3], being in general smaller.

Let $(p_{ij}(t))$ be a standard substochastic transition matrix on the set I of positive integers. By adjoining a single extra state to I, we can and will assume $(p_{ij}(t))$ is stochastic.

Let p_0 be a probability measure on I putting strictly positive weight on each point. The Markov chain $\{Z(t), t \geqq 0\}$ associated with this transition function and having initial distribution p_0 can be chosen so that it is separable in the one-point compactification $I \cup \{\infty\}$ of I, and satisfies:

$$Z(t) = \liminf_{\substack{r \downarrow t \\ r \text{ rational}}} Z(r) \qquad \text{for all } t.$$

We assume that all states of the chain are stable, that is that each state is an exponential holding point for the process. Instead of dealing directly with the process Z, we deal with the process killed at S, where S is independent of Z and for all $t \geqq 0$, $P\{S > t\} = e^{-t}$. That is, we define processes $\{X(t), t \geqq 0\}$ and $\{\tilde{X}(t), t \geqq 0\}$ by

$$X(t) = \begin{cases} Z(t) & 0 \leqq t < S \\ \Delta & t \geqq S \end{cases}$$

$$\tilde{X}(t) = \begin{cases} \liminf_{r \downarrow t} Z(S-r) & 0 \leqq t < S \\ \tilde{\Delta} & t \geqq S, \end{cases}$$

where Δ and $\tilde{\Delta}$ are "death points" isolated from I. The process $\{X(t), t \geqq 0\}$ is again a Markov chain, with transition probabilities

$$\begin{aligned} P_{ij}(t) &= e^{-t} p_{ij}(t) && i, j \in I \\ P_{\Delta j}(t) &= \delta_{\Delta j} && j \in I \\ P_{j\Delta}(t) &= 1 - e^{-t} && j \in I. \end{aligned}$$

The $\tilde{X}(t)$-process is the $X(t)$-process reversed from the time S; it can be shown by direct calculation to be a separable Markov chain with transition probabilities $\tilde{P}_{ij}(t)$ where

$$\begin{aligned} \tilde{P}_{ij}(t) &= \frac{G(j)}{G(i)} P_{ji}(t) \qquad i, j \in I \\ \tilde{P}_{\tilde{\Delta}, j}(t) &= \delta_{\tilde{\Delta} j} \\ \tilde{P}_{j, \tilde{\Delta}}(t) &= 1 - e^{-t} \end{aligned}$$

and $G(j)$ is the expected time spent in state j by the $X(t)$-process.

We assume that the probability space $(\Omega, \mathscr{F}, \mathscr{P})$ on which this process is defined is equipped with shift operators θ_t and $\tilde{\theta}_t$, $t \geqq 0$, so that $X_s(\theta_t \omega) = X_{s+t}(\omega)$

and $\tilde{X}_s(\tilde{\theta}_t\omega)=\tilde{X}_{s+t}(\omega)$. For $t\geqq 0$, the fields $\mathscr{F}_t$ and $\tilde{\mathscr{F}}_t$ will denote the smallest completed Borel fields making measurable the families $\{X(s), 0\leqq s\leqq t\}$ and $\{\tilde{X}(s), 0\leqq s\leqq t\}$ respectively.

§ 2. The Martin Entrance and Exit Boundaries

Define α to be the infimum of all times $t>0$ such that either $X(t)=\Delta$ or the sample path of $X(\cdot)$ has infinitely many jumps in the interval $(\delta, t]$ for some $\delta>0$. Note $\alpha\leqq S<\infty$, w.p. 1. All states are stable, so if the process starts from a state in I, it stays in that state for some interval of time. In this case, for almost every sample path, one – and only one – of the following holds:

i) The sample path has a countable number of jumps $0<\tau_1<\tau_2<\cdots$ in $(0,\alpha]$, and $\alpha=\lim \tau_n$.

ii) The sample path has only finitely many jumps in $(0,\alpha]$, but infinitely many in $(0,\alpha+\delta]$ for each $\delta>0$.

iii) $X(\alpha)=\Delta$, in which case there are only finitely many jumps in $(0,\alpha]$.

We will also consider cases where the process does not start from the state space; then we have to take into account the additional possibility that $0<\delta<\alpha$ and that there are infinitely many jumps in $(0,\delta)$.

We can – and therefore will – introduce an auxiliary Markov chain X^M by:

$$\begin{aligned} X^M(t)&=X(t) && \text{if } t<\alpha \\ &=\Delta && \text{if } t\geqq\alpha \text{ and case i) or case iii) holds} \\ &=X(\alpha) && \text{otherwise.} \end{aligned}$$

It is easily verified that X^M is indeed a Markov chain with absorbing points Δ and ∞; the sample paths of X^M are step functions with jumps at times $0<\tau_1<\tau_2<\cdots$. This sequence terminates if the process strikes one of the absorbing points after finitely many jumps; to reduce the proliferation of special cases we set $\tau_{n+1}=\tau_n$ in case $X^M(\tau_n)$ is either Δ or ∞. There is a related discrete parameter jump chain $\{\chi_n\}$, called the *embedded jump chain*, defined by $\chi_n=X(\tau_n)$. Following Hunt [7][1] we can define a compactification I^* of I, called the *Martin exit space*, which is a compact metric space in which I is dense. The set $\mathscr{B}_0^*=I^*-I$ is called the *Martin exit boundary*. Then χ_n has a limit in I^* as $n\to\infty$; this limit is in fact in $\mathscr{B}_0^*$, as it follows from the construction of I^* that both Δ and ∞ can be identified with points of $\mathscr{B}_0^*$.

The above construction can be applied to the reversed process $\tilde{X}$ to get a process $\tilde{X}^M$ and embedded jump chain $\tilde{\chi}_n$ with a related compactification $\tilde{I}^*$, called the *Martin entrance space*. The *Martin entrance boundary* $\tilde{\mathscr{B}}_0^*$ is then just $\tilde{I}^*-I$. The Borel sets of I^* and $\tilde{I}^*$ are then the topological Borel sets with respect to the two Martin topologies, and the Borel sets of $\mathscr{B}_0^*$ and $\tilde{\mathscr{B}}_0^*$ are those inherited from the larger spaces. I^* and $\tilde{I}^*$ are distinct topological spaces, but we shall

[1] There is a slight difference in the boundaries defined by Doob [3] and Hunt [7]. Applied to our setting, Doobs definition allows the possibility that χ_n converges to a point of I as $n\to\infty$; under Hunt's definition, this is impossible.

often abuse notation by considering them apart from their topology, as when we write $I \cup I^* = I \cup \mathscr{B}_0^* \cup \tilde{\mathscr{B}}_0^*$.

The Martin topologies are adapted to the minimal processes X^M and $\tilde{X}^M$ in the sense that these processes have right and left limits in them, but they are not necessary compatible with the "whole" processes X and $\tilde{X}$. We will eventually replace these topologies with a single topology compatible with both X and $\tilde{X}$; for the present we use such phrases as "$X(t-)=a$" and "$X(t+)=b$" to mean, respectively, "X is in I for some interval $(t-\delta, t)$ and $\lim_{s \uparrow t} X(s)=a$ in the Martin exit topology," and "X is in I for some interval $(t, t+\delta)$ and $\lim_{s \downarrow t} X(s)=b$ in the Martin entrance topology." Because of the requirement that the process be in the state space for some time after t, $X(t+)$ may not be defined; similarly for $X(t-)$. Although the process $X(t)$ takes values in $I \cup \{\infty\} \cup \{\Delta\}$, $X(t+)$ and $X(t-)$ take values in the Martin extrance and exit spaces respectively. Thus it is possible that $X(t-)=a$, $X(t+)=b$ and $X(t)=+\infty$! We will use this convention for stopping times as well as fixed times.

If B is any set in $I^* \cup \tilde{I}^* \cup \{\infty\}$ define the sets $S_B(\omega)=\{t\colon X(t,\omega)\in B\}$, $S_B^+(\omega)=\{t\colon X(t+,\omega)\in B\}$ and $S_B^-(\omega)=\{t\colon X(t-,\omega)\in B\}$; $\tilde{S}_B$, $\tilde{S}_B^+$ and $\tilde{S}_B^-$ are the corresponding sets for $\tilde{X}$. A set A in $\mathscr{B}_0^*-\Delta$ or $\tilde{\mathscr{B}}_0^*-\tilde{\Delta}$ is said to be *negligible* if $P\{X(\alpha-)\in A\}=0$ or $P\{\tilde{X}(\tilde{\alpha}-)\in A\}=0$ respectively.

A second useful pair of topologies are the fine and cofine topologies on I^* and $\tilde{I}^*$ respectively; see for instance [9, Ch. 10]. These topologies are intrinsic topologies for X and $\tilde{X}$ respectively. The relevant property of these topologies is that if f is a function on I, then f has a fine limit at all but a negligible set of $\mathscr{B}_0^*$ iff $\lim_{s\uparrow\alpha} f(X(s))$ exists w.p. 1, and f has a cofine limit at all but a negligible set of $\tilde{\mathscr{B}}_0^*$ iff $\lim_{s\uparrow\tilde{\alpha}} f(\tilde{X}(s))$ exists w.p. 1.

§ 3. Extension of the Transition Functions to the Boundary

Much of our subsequent analysis depends on the existence of right and left limits for processes of the type $\{f(X(s)), s\geqq 0\}$. In deference to the fact that $f(X(s))$ may not be defined for all s, we will use the term "I-limit" to indicate the limit is taken only over s for which $X(s)\in I$. Since X is well-separable and is in I for each fixed s w.p. 1, this is equivalent to requiring the limit be taken over rational s only.

Our first theorem of this sort concerns the semi-group. It can be deduced from results in [4] or [12]. The proofs in these papers depend on a compactification of the state space; we provide a direct proof here.

Theorem 3.1. *Let f be bounded on I. Then for ω not in some exceptional null set, $s\to P_t f(X_s(\omega))$ has both left and right I-limits at all s for each $t>0$; these limits are continuous functions of t for $t>0$.*

Lemma 3.2. *Let $\tau_1\leqq\tau_2$ be stopping times, $A\subset I$ and let $\Lambda=\{X(s)\in A$, some s in $(\tau_1,\tau_2)\}$. Then for $\varepsilon>0$ there is a stopping time $\sigma\leqq\tau_2$ for which $P\{\sigma\in(\tau_1,\tau_2)$ and $X(\sigma)\in A\}\geqq P(\Lambda)-\varepsilon$.*

Proof. Since A is a countable set, there is some finite set $A'\subset A$ and $\delta>0$ such that, if $\tau=\inf\{t>\tau_1+\delta\colon X(t)\in A'\}$, then $P\{\tau\in(\tau_1,\tau_2)\}\geqq P(\Lambda)-\varepsilon$. By right continuity, $X(\tau)\in A'$. Just let $\sigma=\tau\wedge\tau_2$. q.e.d.

Proof of Theorem 3.1. Let $a<b$ and define

$$A=\{i:\ P_t f(i)<a\},\qquad B=\{i:\ P_t f(i)>b\}.$$

Let τ be any stopping time and let Λ_{ab} be the ω-set on which τ is a limit from the right of both $S_A(\omega)$ and $S_B(\omega)$. Let $\{\sigma_n\}$ be a decreasing sequence of stopping times such that $\sigma_0\equiv+\infty$, and for each $n\geqq 1$ σ_{2n} and σ_{2n-1} are chosen by Lemma 3.1 so that both

$$P\left\{\sigma_{2n-1}\in S_B\cap\left(\tau,\left(\tau+\frac{1}{n}\right)\wedge\sigma_{2n-2}\right);\ \Lambda_{ab}\right\}\geqq P(\Lambda_{ab})-\frac{1}{2^n},\tag{3.1}$$

$$P\{\sigma_{2n}\in S_A\cap(\tau,\sigma_{2n-1});\ \Lambda_{ab}\}\geqq P(\Lambda_{ab})-\frac{1}{2^n}.\tag{3.2}$$

By the Borel-Cantelli lemma, we have $X(\sigma_{2n})\in A$, $X(\sigma_{2n+1})\in B$, and $\sigma_n\downarrow\tau$ a.s. on Λ_{ab} for all large enough n.

Now $t>0$ so w.p. 1 $f(X(\tau+t))$ is continuous at t. It is a fact, observed by Hunt that if η,η_n are random variables, $\mathscr{F}_n$ σ-fields, such that $\mathscr{F}_n\downarrow\mathscr{F}$, $\eta_n\to\eta$ a.e. and all the η_n are dominated by an integrable random variable,

$$E\{\eta_n|\mathscr{F}_n\}\to E\{\eta|\mathscr{F}\}\qquad \text{a.e.}$$

Then:

$$\lim_{n\to\infty}E\{f(X(\sigma_n+t))|\mathscr{F}_{\sigma_n}\}=E\{f(X(\tau+t))|\bigwedge_n\mathscr{F}_{\sigma_n}\}.\tag{3.3}$$

Now $X(\sigma_n)\in I$ if $\sigma_n<\infty$ so the strong Markov property holds and the l.h.s. of (3.3) is $\lim_{n\to\infty}P_t f(X(\sigma_n))$. But this limit fails to exist a.e. on Λ_{ab} by the choice of the σ_n, so we must have $P(\Lambda_{ab})=0$. Then $P\{\bigcup_{a<b}\Lambda_{ab}\}=0$, which implies $P_t f(X(s))$ has an I-limit as $s\downarrow\tau$. By Doob's now-standard transfinite induction argument, $P_t f(X(s))$ must have right I-limits at all t w.p. 1. To show left limits exist for all t, define stopping times

$$\tau_0=0$$
$$\tau_{2n+1}=\inf\{t>\tau_{2n}:\ X(t)\in B\}$$
$$\tau_{2n+2}=\inf\{t>\tau_{2n+1}:\ X(t)\in A\}.$$

By the fact that right-hand I-limits of $P_t f(X(s))$ exist, $\tau_{n+1}>\tau_n$ on $\{\tau_n<\infty\}$. By the lemma we can find a sequence $\{\sigma_n\}$ of stopping times such that $\sigma_n\in[\tau_n,\tau_{n+1}]$ and $P\{\sigma_{2n}\notin S_A,\sigma_{2n+1}\notin S_B,\sigma_{2n+1}<\infty\}<1/2^n$. Let $\tau=\sup\sigma_n$ and $\Gamma_{ab}=\{\tau<\infty\}$. Again, as $s\to f(X(\tau+s))$ is continuous at t a.s. on Γ_{ab}, we have as before:

$$\lim_{n\to\infty}E\{f(X(\sigma_n+t))|\mathscr{F}_{\sigma_n}\}=E\{f(X(\sigma+t)|\bigvee_n\mathscr{F}_{\sigma_n}\}\qquad\text{on }\Gamma_{ab}.$$

As the l.h.s. is $\lim_{n\to\infty}P_t f(X(\sigma_n))$, which diverges a.e. on Γ_{ab}, $P(\Gamma_{ab})=0$.

Before proving continuity of the transition functions we need the following lemma.

Lemma 3.3. *Let* $g_j(s) = I\text{-}\lim_{u \downarrow s} P_{X(u)j}(t)$ *and* $h_j(s) = I\text{-}\lim_{u \uparrow s} P_{X(u)j}(t)$. *Then for a.e.* ω *and for all* s, $\sum g_j(s) = \sum h_j(s) = 1$.

Proof. Let $\{A_n\}$ be a sequence of finite sets increasing to $I \cup \{\Delta\}$, and let $B_n = \{i: \sum_{j \in A_n} P_{ij}(t) \leqq 1 - \varepsilon\}$. The sets B_n decrease to the empty set. Suppose T_n is the first time X hits B_n. We claim $T = \lim T_n$ is infinite. If not, there are two cases: either $T = T_n$ for all sufficiently large n or $T > T_n$, $\forall n$. In the former case, we can use Lemma 3.2 to find a sequence $\{\sigma_n\}$ of stopping times decreasing to T such that on $\{T < \infty\}$, $X(\sigma_n) \in B_n$ for all sufficiently large n. Then for each fixed N:

$$1 - \varepsilon \geqq \lim_{n \to \infty} \sum_{j \in A_N} P_{X(\sigma_n)j}(t) = \lim_{n \to \infty} E\{X(t + \sigma_n) \in A_N | \mathscr{F}_{\sigma_n}\} \\ = E\{X(t + T) \in A_N | \bigwedge_n \mathscr{F}_{\sigma_n}\}. \tag{3.4}$$

As $N \to \infty$ the right hand side approaches $E\{X(t+T) \in I \cup \{\Delta\} | \mathscr{F}_{\sigma_n}\} = 1$, a contradiction. On the other hand, if $T > T_n$, all n, we can find an increasing sequence $\{\sigma_n\}$ of stopping times with $T_n \leqq \sigma_n < T$ such that $X(\sigma_n) \in B_n$ for all sufficiently large n a.e. on $\{T < \infty\}$. Then the argument (3.4) for this sequence shows $T = \infty$ a.e. But this is true simultaneously for a sequence of ε going to zero, which proves the lemma. q.e.d.

The proof of Theorem 3.1 is readily completed. By Dini's theorem, if, for each k, $a_{nk} \to a_k$, where a_{nk} and a_k are positive, and if $\sum_k a_{nk} = \sum_k a_k = 1$ for all n, then $\sum_k a_{nk}$ converges uniformly in n. Thus, if f is bounded

$$P_{s+t} f(X(u)) = \sum_j P_{X(u)j}(t) P_s f(j). \tag{3.5}$$

As u either decreases or increases to u_0 through values for which $X(u) \in I$, by the lemma and the above remark, (3.5) converges uniformly. Since $s \to P_s f(j)$ is continuous for all j, $s \to I\text{-}\lim_{u \to u_0} P_{s+t} f(X(u))$ is also continuous. q.e.d.

By virtue of the fact that S is independent of the original process Z and that Z is actually continuous w.p. 1 for each fixed t, we have

$$\tilde{X}(t) = X(S - t) \qquad \text{w.p. } 1$$

simultaneously for all rational t. Consequently, statements about right I-limits of X imply corresponding statements about left I-limits of $\tilde{X}$ and vice-versa. With this remark and the symmetry of X and $\tilde{X}$ we have:

Corollary 3.4. *Let* f *be bounded on* I *and* $t > 0$. *For a.e.* ω, *the following have right and left I-limits, and these limits are continuous in* t *for* $t > 0$.

a) $s \to P_t f(X_s)$, b) $s \to P_t f(\tilde{X}_s)$,

c) $s \to \tilde{P}_t f(X_s)$, d) $s \to \tilde{P}_t f(\tilde{X}_s)$.

As a consequence of this we can extend both semigroups P_t and $\tilde{P}_t$ to $\mathscr{B}_0^* \cup \tilde{\mathscr{B}}_0^*$. We indicate this for P_t, as the extension of $\tilde{P}_t$ is similar.

Let $f=I_j$ in Corollary 3.4. Then $\lim_{s\uparrow\alpha} P_{X(s)j}(t)$ exists a.e. and is continuous in t; this implies $i\to P_{ij}(t)$ has a fine limit at all but a negligible subset N_0 of $\mathscr{B}_0^*$, where N_0 is independent of t. Similarly, by (b) of Corollary 3.4, $\lim_{s\uparrow\tilde{\alpha}} P_{\tilde{X}(s)j}(t)$ exists, so that $i\to P_{ij}(t)$ has a continuous cofine limit at all but a negligible subset $\tilde{N}_0$ of $\tilde{\mathscr{B}}_0^*$, where $\tilde{N}_0$ is also independent of t[2]. Thus we define

$$\begin{aligned} P_{bj}(t) &= \text{fine}\lim_{i\to b} P_{ij}(t) && \text{if } b\in\mathscr{B}_0^*-N_0 \\ &= \text{cofine}\lim_{i\to b} P_{ij}(t) && \text{if } b\in\tilde{\mathscr{B}}_0^*-\tilde{N}_0-\tilde{\Delta} \\ &= \delta_{bb} && \text{if } b\in\tilde{N}_0\cup N\cup\tilde{\Delta}. \end{aligned} \tag{3.6}$$

A consequence of Lemma 3.1 is that if b is not in some negligible subset $N_1\subset\mathscr{B}_0^*$ or $\tilde{N}_1\subset\tilde{\mathscr{B}}_0^*$, $\sum_{j\in I\cup\Delta\cup\tilde{\Delta}} P_{bj}(t)=1$.

Note that for $t>0$

$$P\{X(\alpha+t)=j\,|\bigvee_n \mathscr{F}_{\tau_n}\}=\lim_{n\to\infty} P_{X(\tau_n)j}(t) \qquad \text{a.e.}$$

This is almost surely $P_{X(\alpha-)j}(t)$; this gives us a limited version of the strong Markov property due to Chung in case $\mathscr{B}_0^*$ is countable. If $\Lambda\in\bigvee_n\mathscr{F}_{\tau_n}$ then

$$P\{X(\alpha+t)=j;\Lambda\}=E\{P_{X(\alpha-)j};\Lambda\}. \tag{3.7}$$

A second limited form of the strong Markov property comes from consideration of right limits. If T is a stopping time for the forward process, then

$$P\{X(T+t)=j\,|\mathscr{F}_T\}=\lim_{n\to\infty} P_{X(T+1/n)j}(t) \qquad \text{a.e. on } \{T<\infty\}.$$

This limit is almost surely $P_{X(T+)j}$ a.e. on $\{X(T+)\in\tilde{\mathscr{B}}_0^*\cup I\}$[3], so for $\Lambda\in\mathscr{F}_T$, we have

$$P\{X(T+t)=j,\,X(T+)\in\tilde{\mathscr{B}}_0^*\cup I;\Lambda\}=E\{P_{X(T+)j}(t);\,X(T+)\in\tilde{\mathscr{B}}_0^*\cup I;\Lambda\}. \tag{3.8}$$

Both (3.7) and (3.8) will be strengthened later.

§ 4. Reflecting Atoms

Following Chung we divide boundary points into two classes, according to whether they are regular for themselves or not. Chung called the former "sticky"; because of the analogy with diffusion processes, where "sticky" has a different meaning, and because of some of the following results, in particular the role of these points in the exit and entrance boundaries, we call such points "reflecting". More precisely if $A\subset I^*\cup\tilde{I}^*$, define

$$\begin{aligned} \gamma_A &= \inf\{t>0\colon X(t-) \text{ or } X(t+)\in A\} \\ &= \infty \qquad \text{if there is no such } t, \end{aligned}$$

and

$\tilde{\gamma}_A$ the corresponding time for the $\tilde{X}(t)$-process,

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[2] The point ∞ is negligible for both X and $\tilde{X}$, and can thus be included in both N_0 and $\tilde{N}_0$.

[3] $X(T+)$ and $X(T-)$ for stopping times are defined as for fixed times. See Section 2.

where the convention of section 2 on $X(t+)$ and $X(t-)$ is still in effect. A point $a \in \mathscr{B}_0^*$ is *reflecting* if $P\{\gamma_a \circ \theta_{\gamma_a} = 0\} > 0$ and *non-reflecting* otherwise. Similarly, $a \in \tilde{\mathscr{B}}_0^*$ is *reflecting* if $P\{\tilde{\gamma}_a \circ \theta_{\tilde{\gamma}_a} = 0\} > 0$[4].

Chung has shown that for any point $a \in \mathscr{B}_0^*$, and a.e. ω: the set $S_a^-(\omega) = \{t: X(t-, \omega) = a\}$ is countable; if a is non-reflecting $S_a^-(\omega)$ has no finite accumulation points, and if a is reflecting then each t in $S_a^-(\omega)$ is a limit from the right of points of $S_a^-(\omega)$. Applying this first to the forward and then to the reverse process we see that a point a in either $\mathscr{B}_0^*$ or $\tilde{\mathscr{B}}_0^*$ is reflecting iff

$$P\{\gamma_a \circ \theta_{\gamma_a} = 0 \mid \gamma_a < \infty\} = P\{\tilde{\gamma}_a \circ \tilde{\theta}_{\tilde{\gamma}_a} = 0 \mid \tilde{\gamma}_a < \infty\} = 1. \tag{4.1}$$

This zero-one law for reflecting atoms can be extended considerably – the following proposition is the first step in this extension.

Proposition 4.1. *Let* $a \in \mathscr{B}_0^* \cup \tilde{\mathscr{B}}_0^*$ *be reflecting, and define σ-fields* $G_a(t) = \mathscr{B}(X(\gamma_a + s); 0 < s \leq t)$ *if* $t > 0$, *and* $G_a(0) = \bigcap_{t>0} G_a(t)$. *Then the trace of* $G_a(0)$ *on* $\{\gamma_a < \infty\}$ *is trivial.*

Proof. If a reflecting, the process hits a infinitely often in any neighborhood of γ_a; thus, if f is bounded and $t > 0$, $I\text{-}\lim_{s \downarrow \gamma_a} P_t f(X(s))$, since it exists, must be $P_t f(a)$.

Let $\Lambda \in G_a(0)$. Then

$$\begin{aligned} &E\{f(X(\gamma_a + t));\ \Lambda, \gamma_a < \infty \mid G_a(0)\} \\ &= \lim_{n \to \infty} E\left\{f\left(\gamma_a + \frac{1}{n} + t\right);\ \Lambda, \gamma_a < \infty \mid G_a\left(\frac{1}{n}\right)\right\}. \end{aligned}$$

But $X(\gamma_a + (1/n))$ is almost surely in I so the strong Markov property is valid; noting that $\Lambda \in G_a(1/n)$ we have

$$= \lim_{n \to \infty} P_t f\left(X\left(\gamma_a + \frac{1}{n}\right)\right) I_{\{\Lambda, \gamma_a < \infty\}} = P_t f(a)\, I_{\{\gamma_a < \infty, \Lambda\}}.$$

This implies Λ and $X(\gamma_a + t)$ are independent. A similar calculation shows Λ is independent of any finite family $X(\gamma_a + t_j)$, $0 < t_1 < \cdots < t_n$. It follows that Λ is independent of $G_a(0)$, and so must have probability zero or one. q.e.d.

If two atoms a and b of $\mathscr{B}_0^*$ are reflecting, it follows from this proposition that $P\{\gamma_b \circ \theta_{\gamma_a} = 0 \mid \gamma_a < \infty\}$ is either zero or one, and is equal to $P\{\gamma_a \circ \theta_{\gamma_b} = 0 \mid \gamma_b < \infty\}$. The same is true of $\tilde{\mathscr{B}}_0^*$, so we define an equivalence relation R: a is *equivalent* (R) to b if they are both reflecting atoms, either both in $\mathscr{B}_0^*$ or both in $\tilde{\mathscr{B}}_0^*$, and $P\{\gamma_b \circ \theta_{\gamma_a} = 0 \mid \gamma_a < \infty\} = 1$. R is easily seen to be an equivalence relation. Then define

$$\mathscr{B}_0 = \mathscr{B}_0^*/R$$

$$\tilde{\mathscr{B}}_0 = \tilde{\mathscr{B}}_0^*/R,$$

so that equivalent reflecting atoms are identified.

[4] Recalling the convention of § 2 on $X(t+)$, $X(t-)$, it is not difficult to show a reflecting atom must be non-negligible for either X or $\tilde{X}$.

§5. Harmonic Measures

In order to relate the boundary behavior of X and $\tilde{X}$ to quantities defined in terms of the minimal processes, we introduce the harmonic measures. This section is devoted to the development of their properties, the most important for this paper being Theorem 5.4 which tells us that harmonic measures have no oscillatory discontinuities along the sample paths.

If A and B are Borel subsets of $\mathscr{B}_0$ and $\tilde{\mathscr{B}}_0$ respectively, define

$$H_A(i) = P\{\lim_{n\to\infty} \chi_n \in A \mid \chi_0 = i\}$$

$$\tilde{H}_B(i) = P\{\lim_{n\to\infty} \tilde{\chi}_n \in B \mid \tilde{\chi}_0 = i\},$$

where χ_n is the jump process of X and $\tilde{X}_n$ the jump process of $\tilde{X}$. H_A and $\tilde{H}_B$ are called *harmonic measures*. The following elementary facts hold for both H_A and $\tilde{H}_B$, though we state them for the forward chain only.

H1) For fixed i, $H_A(i)$ is a measure on $\mathscr{B}_0$. $H_\Delta(i) > 0$, all i, and $H_{\mathscr{B}_0 - \Delta}(i) < 1$.

H2) $H_B(i)$ is regular for the jump chain.

H3) w.p. 1, $\lim_{n\to\infty} H_B(\chi_n) = 1$ if $\lim \chi_n \in B$

$= 0$ if $\lim \chi_n \notin B$.

H4) Let T be any stopping time. Then the processes Y_1 and Y_2 are a submartingale and a supermartingale respectively, where

$$\begin{aligned} Y_1(t) &= H_B(X(T+t)) && 0 < t < \alpha \circ \theta_T \\ &= 1 && t \geqq \alpha \circ \theta_T,\ t > 0 \end{aligned}$$

$$\begin{aligned} Y_2(t) &= H_B(X(T+t)) && 0 < t < \alpha \circ \theta_T \\ &= 0 && t \geqq \alpha \circ \theta_T,\ t > 0. \end{aligned}$$

The point $t = 0$ can be included in the parameter set in case $X(T) \in I$.

The fact that $H_\Delta(i) > 0$ is a consequence of the stability of the point i and the fact that there is positive probability of jumping from i to Δ in any time interval. (H2) and (H3) follow directly from the fact that

$$H_B(\chi_n) = P\{\lim_{m\to\infty} \chi_m \in B \mid \chi_1, \ldots, \chi_n\}$$

and Lévy's martingale convergence theorem.

To prove (H4) let $0 < s < t$; we prove only that Y_2 is a supermartingale; the proof for Y_1 is similar. The supermartingale inequality is clear on $\{\alpha \circ \theta_T < s\}$, as $Y(s)$ and $Y(t)$ are both zero. On $\{\alpha \circ \theta_T > s\}$, it is enough to prove it for the case $T = s = 0$ and $X(0) = i$.

$$\begin{aligned} E_i\{Y_2(t)\} &= E_i\{H_B(X(t)), \alpha > t\} = P_i\{\lim \chi_n \circ \theta_t \in B, \alpha > t\} \\ &= P_i\{\lim \chi_n \in B, \alpha > t\} \leqq P_i\{\lim \chi_n \in B\} = H_B(i). \end{aligned}$$

Before proceeding to the properties of $H_B(X(t))$ we state an awkward-appearing but useful lemma which will come up several times below. It is a type of local version of the converse Borel-Cantelli lemma.

Lemma 5.1. *Let $A \in \mathscr{F}_\alpha$ and let $h(i) = P\{A | X(0) = i\}$. Let τ be any stopping time and let Λ be the set where τ is a limit point from the right of times t for which either $X(t-) \in \mathscr{B}_0$ or $X(t+) \in \tilde{\mathscr{B}}_0$. Then for a.e. ω in the set $\Lambda \cap \{I\text{-}\limsup_{t \downarrow \tau} h(X(t)) > 0\}$, τ is also a limit point from the right of times t for which $\theta_t \omega \in A$. If τ is accessible*[5]*, the analogous statement holds for left limits.*

Proof. It is enough to prove the lemma for the set $\Gamma = \Lambda \cap \{\limsup_{t \downarrow \tau} h(X(t)) > \varepsilon\}$. Assume $P(\Gamma) > 0$ and, using Lemma 3.2, define a decreasing sequence $\{\sigma_n\}$ of stopping times such that $\sigma_n = \infty$ off Γ, $\sigma_n \downarrow \tau$ and $\sigma_n > \tau$ on Γ, and

$$P\{\Gamma_n\} \geqq \left(1 - \frac{1}{2^n}\right) P(\Gamma)$$

where

$$\Gamma_n = \Gamma \cap \{X(\sigma_n) \in \{i \colon h(i) > \varepsilon\}; \alpha \circ \theta_{\sigma_n} < \sigma_{n-1} - \sigma_n\}.$$

Let $A_n = \theta_{\sigma_n}^{-1} A$. We claim A_n occurs for infinitely many n.

Let $\mathscr{G}$ be the field generated by the events $\Gamma, A_{n+1}, \ldots, A_{n+k}$ and consider

$$P\{A_n | \mathscr{G}\} \geqq P\{A_n | \mathscr{G}, \Gamma_n\} P\{\Gamma_n | \mathscr{G}\}. \tag{5.1}$$

For $j \geqq 1$, $\Gamma_n \cap A_{n+j} \in \mathscr{F}_{\sigma_n}$ so by the strong Markov property – which holds if $X(\sigma_n) \in I$ – the r.h.s. of (5.1) is

$$\geqq \varepsilon P\{\Gamma_n | \mathscr{G}\}.$$

If $H_n \subset \Gamma$ is the subset for which $P\{\Gamma_n | \mathscr{G}\} \leqq \frac{1}{2}$, then

$$P\{A_n^c | \mathscr{G}\} \leqq 1 - \frac{\varepsilon}{2}(1 - I_{H_n}),$$

and for each k

$$P\{A_n^c \cap \cdots \cap A_{n+k}^c\} \leqq E\left\{\prod_{j=n}^{n+k} \left[1 - \frac{\varepsilon}{2}(1 - I_{H_j})\right]\right\}. \tag{5.2}$$

But now, if we integrate $P(\Gamma_n | \mathscr{G})$ over $\Gamma = H_n \cup (\Gamma - H_n)$ and note this integral is bounded by $P(\Gamma)(1 - 1/2^n)$, we see

$$P\{H_n\} \leqq \frac{1}{2^{n-1}} P(\Gamma).$$

By the Borel-Cantelli lemma, $I_{H_j} = 0$ for all but a finite number of j. The r.h.s. of (5.2) accordingly goes to zero as $k \to \infty$.

It τ is accessible, then $\Omega = K_1 \cup K_2 \cup \cdots$ where for each i there is a sequence $\{\tau_n\}$ such that on K_i, $\tau_n < \tau$, $\tau_n \uparrow \tau$. The lemma can be proved in the case of left-hand limits by choosing the times σ_n as before, except now $\sigma_n \uparrow$ and $\sigma_n \in (\tau_n, \tau)$ with high probability on $K_i \cap \{\limsup_{t \uparrow \tau} h(i) > \varepsilon\}$. q.e.d.

---

[5] A stopping time T is accessible if there exist countably many sets $K_n \subset \Omega$ with $P\left\{\bigcup_{n=1}^{\infty} K_n\right\} = 1$ and stopping times S_{nj} such that for each n we have $S_{nj} < T$ and $S_{nj} \uparrow T$ as $j \to \infty$ a.e. on K_n. T is predictable if we can choose K_1 such that $P(K_1) = 1$.

This will usually be used in the situation where $A=\{X(\alpha-)\in B\}$ where $B\in\mathscr{B}_0$. In this case $\theta_t^{-1}A=\{X(\alpha\circ\theta_t-)\in B\}$ and we have:

Corollary. *Let A and τ be as above, and let $B\subset\mathscr{B}_0$ be a Borel set; then for a.e. ω in the set $A\cap\{I\text{-}\limsup_{t\downarrow\tau} H_B(X(t))>0\}$, τ is a limit point from the right of times t for which $X(t-)\in B$. If τ is accessible, the analogous statement holds for left limits.*

At this point we abandon the time symmetry in our hypotheses. In order to be able to describe the process completely in terms of the Martin boundaries it is necessary to introduce some type of finiteness assumptions on either the entrance or exit boundary. We make the customary choice, that is, to restrict the exit boundary. The corresponding results for a finite entrance boundary will be gotten by reversing the process.

From now on, we will assume:

(A) All states of $I\cup\Delta$ are conservative.

(B) There exists a finite subset $\mathscr{B}$ of $\mathscr{B}_0$ with the property that $P\{X(\alpha-)\in\mathscr{B}_0-\mathscr{B}\}=0$, and for each $a\in\mathscr{B}$, $P\{X(\alpha-)=a\}>0$.

A word is in order about assumption (A); the reason for introducing it is that non-conservative states can act as boundary atoms, so that allowing infinitely many non-conservative states subverts the finiteness assumption on $\mathscr{B}$. As Pittinger [13] has shown, one can always allow non-conservative states as long as there are only finitely many of them. Under these assumptions, almost every sample path of the X process is known to have the following character:

(S1) The set $S_I(\omega)$ is the union of a countable number of disjoint right-open intervals $\{\mathscr{I}_n\}$ satisfying:

$$\text{if}\quad \mathscr{I}_n=[a_n,b_n)\quad\text{or}\quad(a_n,b_n),\quad\text{and}\quad b_n<\gamma_\Delta,\quad\text{then}$$

i) $X(b_n-)\in\mathscr{B}$, $X(a_n+)\in I\cup\tilde{\mathscr{B}}_0$,

ii) $\xi\in(a_n,b_n)\Rightarrow X(\xi), X(\xi+), X(\xi-)\in I$.

(S2) Let $\mathscr{S}$ be the set of reflecting atoms of $\mathscr{B}$. The set $S_{\mathscr{S}}^-(\omega)$ is countable and dense in itself, and the set $S_{\mathscr{B}-\mathscr{S}}^-(\omega)$ is discrete. The set $S_I(\omega)$ is dense in $[0,\infty)$.

Except for the statement about left-hand limits, which is obvious upon consideration of the reversed process, the above is due to Chung.

Theorem 5.2. *Let $B\subset\mathscr{B}$. Almost every sample path of the process $\{H_B(X(t)), t>0\}$ has right and left I-limits for all $t>0$.*

Proof. Let $0<a<b<1$ and let

$$A'=\{i\colon H_B(i)\leq a\},\qquad B'=\{i\colon H_B(i)\geq b\}.$$

Define a sequence T_n of stopping times by:

$$\begin{aligned} T_1&=\gamma_{A'}\\ T_{2n}&=\gamma_{B'}\circ\theta_{T_{2n-1}} \qquad (5.3)\\ T_{2n+1}&=\gamma_{A'}\circ\theta_{T_{2n}}. \end{aligned}$$

Let $T=\lim_{n\to\infty} T_n$, and suppose $T<\infty$ with positive probability. There are two cases:

Case 1. $T_n=T$ for all sufficiently large n.

Then $\alpha\circ\theta_T$ is zero, for if not, by (H 4) the process

$$Y(t)=\begin{cases} H_B(X(T+t), & 0<t<\alpha\circ\theta_t \\ 1, & t\geqq\alpha\circ\theta_t \end{cases} \tag{5.4}$$

is a positive sub-martingale, and therefore has a limit as $t\downarrow 0$; but it cannot have a limit on the set where both $\alpha\circ\theta_T=0$ and $T_n=T$, for on this set,

$$\overline{\lim}\, H_B(X(t))\geqq b, \quad \underline{\lim}\, H_B(X(t))\leqq a \qquad \text{as } t\downarrow T.$$

Notice that T is a limit from the right of $S_{\mathscr{B}}^-$, and

$$\overline{\lim_{t\downarrow T}}\, H_B(X(t))\geqq b\geqq 0 \quad \text{and} \quad \overline{\lim_{t\downarrow T}}\, H_{\mathscr{B}-B}(X(t))\geqq 1-a>0.$$

By the corollary to Lemma 5.1 T is a limit from the right of both $S_{\mathscr{B}-B}^-$ and S_B^-. But B and $\mathscr{B}-B$ are finite so T is a limit from the right of both S_c^- and S_d^- for some $c\in B$, $d\in\mathscr{B}-B$; but this means $c\sim d$, a contradiction since c and d are distinct points of $\mathscr{B}$.

Case 2. $T_n<T<\infty$ all n.

If we replace T by T_n in (5.4) we see that submartingale convergence requires $\alpha\circ\theta_{T_n}<T-T_n$ a.e. for all n. Thus T is a limit from the left of $S_{\mathscr{B}}^-$; applying the corollary to Lemma 5.1 as in case 1, we see T must be a limit from the left of both S_c^- and S_d^- where $c\in B$, $d\in\mathscr{B}-B$. This is again a contradiction, implying $T=+\infty$, w.p. 1. Thus w.p. 1 $H_B(X(t))$ has only finitely many crossings of (a, b). This is true simultaneously for all rational a and b. q.e.d.

We remark that $H_B(\tilde{X}(t))$ also has right and left I-limits; this follows from the theorem and the fact that $\tilde{X}(r)=X(S-r)$ a.s. simultaneously for all rational $r<S$, so that right and left I-limits are interchanged.

Thus H_B must have a fine limit at every point of $\mathscr{B}$ and a cofine limit at all but a negligible set in $\tilde{\mathscr{B}}_0$. We thus extend the definition of H_B to $\mathscr{B}\cup\tilde{\mathscr{B}}_0$ by fine and cofine continuity at those points where the limits exist, setting it equal to zero where they do not.

Since we have placed no restrictions on the entrance boundary, the previous theorems, which relied on finiteness restrictions on $\mathscr{B}$, cannot be immediately translated into theorems on $\tilde{H}_B$. However, for the most part these theorems can be proved using the same ideas as before.

If $a\in\mathscr{B}$, let $\Lambda(a)=\{b\in\tilde{\mathscr{B}}_0\colon H_a(b)=1\}$ and let $\Gamma_\delta=\{b\in\tilde{\mathscr{B}}_0\colon H_{\mathscr{S}}(b)<1-\delta\}$, $\delta>0$. The sets $\Lambda(a)$ are disjoint since if a and a' are distinct points of $\mathscr{B}$, we have for all $i\in I$ and thus for all $i\in\tilde{\mathscr{B}}_0$: $H_a(i)+H_{a'}(i)\leqq 1$. The set $\Lambda(a)$ is the set of points in $\tilde{\mathscr{B}}_0$ which are in a sense equivalent to a, and the set $\Gamma=\bigcup_{\delta>0}\Gamma_\delta$ is the analogue of the set of non-reflecting points, as the following shows.

Theorem 5.3. *If $\delta>0$, the set $S_{\Gamma_\delta}^+$ is almost surely discrete.*

Proof. We show no point is a limit from the right of $S_{\Gamma_\delta}^+$. A minor modification of the argument shows no point is a left-limit of $S_{\Gamma_\delta}^+$ either.

Suppose t is a right-hand limit of $S_{\Gamma_\delta}^+(\omega)$; if ω is not in some null set N_1, $I\text{-}\lim_{s \downarrow t} H_{\mathscr{S}}(X(s))$ exists and is bounded above by $1-\delta$. But now if ω is not in a second null-set N_2, it must also be a limit from the right of $S_{\mathscr{B}}^-$, and hence, as $S_{\mathscr{B}-\mathscr{S}}^-$ is discrete, of $S_{\mathscr{S}}^-$. Thus if ω is not in some third null set N_3, $\lim_{s \downarrow t} H_{\mathscr{S}}(X(s)) = 1$. Clearly ω cannot be in the complement of $N_1 \cup N_2 \cup N_3$. q.e.d.

With this established, we can extend Theorem 5.2 to harmonic measures on the entrance boundary.

Theorem 5.4. *Let $B \subset \mathscr{B}$, and let C be a Borel set in Γ_δ for some $\delta > 0$. Then w.p. 1 the following processes have right and left I-limits at all t:*

a) $\{H_B(X_t); t \geqq 0\}$, b) $\{H_B(\tilde{X}_t), t \geqq 0\}$,

c) $\{\tilde{H}_C(X_t), t \geqq 0\}$, d) $\{\tilde{H}_C(\tilde{X}_t), t \geqq 0\}$.

Proof. Part (a) is just Theorem 5.2; b) and c) follow from a) and d) respectively by noting that left limits with respect to X_t become right limits with respect to $\tilde{X}_t$ and vice versa. It remains to prove d). This can be proved in much the same way as Theorem 5.2, this time considering $\tilde{X}$ rather than X. First observe that for a.e. ω and any t, $\tilde{\alpha} \circ \tilde{\theta}_t = 0$ implies either $\tilde{X}(t) = \tilde{\Delta}$ or t is a limit point from the right of $\tilde{S}_{\mathscr{B}}^+$.

Let $a < b$ and let $A' = \{i: \tilde{H}_C(i) < a\}$, $B' = \{i: \tilde{H}_C(i) > b\}$. Define stopping times $T_1, T_2, \ldots$ by

$$
\begin{aligned}
T_1 &= \tilde{\gamma}_{A'} \\
T_{2n} &= \tilde{\gamma}_{B'} \circ \tilde{\theta}_{T_{2n-1}} \\
T_{2n+1} &= \tilde{\gamma}_{A'} \circ \tilde{\theta}_{T_{2n}}, \qquad \text{and let } T = \lim T_n.
\end{aligned}
\tag{5.5}
$$

If $T < \infty$ with positive probability, then there is positive probability that either $T = T_n$ for all large enough n or $T_n < T$ for all n. In the first case, T must be a limit from the right of both $\tilde{S}_{A'}$ and $\tilde{S}_{B'}$; but this means $\tilde{\alpha} \circ \tilde{\theta}_T = 0$, since otherwise the submartingale $Y(t)$ would fail to have a limit at the origin, where

$$
Y(t) = \begin{cases} \tilde{H}_B(\tilde{X}(T+t)) & 0 < t < \tilde{\alpha} \circ \tilde{\theta}_T \\ 1 & t \geqq \tilde{\alpha} \circ \tilde{\theta}_T. \end{cases}
\tag{5.6}
$$

By our remark above, T is a limit from the right of $\tilde{S}_{\mathscr{B}}^+$; as we have seen $\overline{\lim_{t \downarrow s}} H_C(\tilde{X}(t)) \geqq b > 0$, so by Lemma 5.1, T is a limit from the right of $\tilde{S}_C^- \subset \tilde{S}_{\Gamma_\delta}^-$. This contradicts the discrete nature of $\tilde{S}_{\Gamma_\delta}^-$. The other possibility is that $T_n < T$ for all n. Replacing T by T_n in (5.6), we see $\tilde{\alpha} \circ \theta_{T_n} < T - T_n$, hence T must be a limit from the left of $\tilde{S}_{\mathscr{B}}^+$ almost surely on $\{T < \infty\}$. Apply Lemma 5.1 once more: as above, T is almost surely a limit from the left of $S_{\Gamma_\delta}^-$, again a contradiction. q.e.d.

An immediate consequence of this theorem is that $\tilde{H}_C$ has a fine limit at each point of $\mathscr{B}$, so that we can extend $\tilde{H}_C$ to $\mathscr{B}$ by fine continuity. To be definite, we set $\tilde{H}_C = I_C$ on $\tilde{\mathscr{B}}_0$; then $\tilde{H}_C$ is cofine continuous except for a negligible subset of $\tilde{\mathscr{B}}_0$.

Proposition 5.5. *Let C be a Borel set in Γ_δ. If $a\in\mathscr{B}$, $\tilde{H}_C(a)<1$, and if $a\in\mathscr{S}$, $\tilde{H}_C(a)=0$.*

Proof. If $a\in\mathscr{S}$ and $\tilde{H}_C(a)>0$, Lemma 5.1 applied to $\tilde{H}_C(\tilde{X}(t))$ would imply $\tilde{S}_C^-$, and therefore $\tilde{S}_{\Gamma_\delta}^-$, is not discrete. If $a\in\mathscr{B}-\mathscr{S}$, the set $\tilde{S}_a^+$ is discrete so that if $T=\inf\tilde{S}_a^+$, then $\tilde{X}(T+)=a$ and $\tilde{\alpha}\circ\tilde{\theta}_T>0$ a.e. on the set $\{T<\infty\}$. The conclusion then follows immediately from Lemma 5.6 below.

Lemma 5.6. *Let T be a stopping time. Then*

$$P\{\alpha\circ\theta_T>0, \limsup_{t\downarrow 0} H_{\mathscr{B}-\Delta}(X(T+t))=1\}=0; \tag{5.7}$$

(5.7) *remains valid if α, θ_T, $H_{\mathscr{B}-\Delta}$, and X are replaced by $\tilde{\alpha}$, $\tilde{\theta}_T$, $\tilde{H}_{\tilde{\mathscr{B}}-\tilde{\Delta}}$, and $\tilde{X}$ respectively.*

Proof. The process

$$Y(t)=\begin{cases} H_{\mathscr{B}-\Delta}(X(T+t)) & \text{if } \alpha\circ\theta_T>t>0\\ 1 & t\geqq\alpha\circ\theta_T\end{cases}$$

is a bounded submartingale, and has a limit Y_0 at $t=0$. If $\Lambda=\{Y_0=1\}$, then for $t>0$:

$$\int_\Lambda Y_t\geqq\int_\Lambda Y_0=P(\Lambda);$$

but $Y_t\leqq 1$ so $Y_t=1$ a.e. on Λ. By separability, $Y_t\equiv 1$ a.e. on Λ; since $H_{\mathscr{B}-\Delta}<1$ on E, $\alpha\circ\theta_T=0$ a.e. on Λ. q.e.d.

A second application of this lemma is to show $\Lambda(a)$ is negligible if $a\in\mathscr{B}-\mathscr{S}$ — hence $H_a<1$ on $\tilde{\mathscr{B}}$ except for a negligible set. Let $\gamma=\gamma_{\Lambda(a)}$ be the first time $X(t+)\in\Lambda(a)$. As $\Lambda(a)\subset\Gamma_\delta$ for any $\delta>0$, $S_{\Lambda(a)}^+$ is discrete, so $X(\gamma+)\in S_{\Lambda(a)}^+$ a.e. on $\{\gamma<\infty\}$. Thus $\alpha\circ\theta_\gamma>0$ and $\lim_{t\downarrow\gamma} H_a(X(t))=1$ on $\{\gamma<\infty\}$; by Lemma 5.6, $P\{\gamma<\infty\}=0$.

§ 6. The Boundary

Now and for the remainder of this paper assumptions (A) and (B) are in effect. We have shown in § 5 that the harmonic measures H_A and $\tilde{H}_B$, where $A\in\mathscr{B}$ and $B\subset\Gamma_\delta$ for some $\delta>0$, are defined on $I\cup\mathscr{B}\cup\tilde{\mathscr{B}}_0$, are fine continuous at all points of $\mathscr{B}$ and cofine continuous at $\tilde{\mathscr{B}}_0$ excepting possibly a negligible set. Further $H_A|_{\mathscr{B}}=I_A$ and $\tilde{H}_B|_{\tilde{\mathscr{B}}_0}=I_B$.

Let $\{G_n\}$ be a countable basis for the Martin entrance boundary and let $B_n=G_n\cap\Gamma_{1/n}$. Let $\{u_n, n=1,2,\ldots\}$ be some ordering of the functions H_a, $a\in\mathscr{B}$ and $\tilde{H}_{B_n}$, $n=1,2,\ldots$ and let

$$u_0(i)=\begin{cases}\dfrac{1}{i} & i\in I\\ 0 & i\in B\cup\tilde{\mathscr{B}}_0.\end{cases}$$

Let $\mathscr{H}$ be the topology on $I\cup\mathscr{B}\cup\tilde{\mathscr{B}}_0$ generated by the sets of the form

$$\{x\in I\cup\mathscr{B}\cup\tilde{\mathscr{B}}_0\colon |u_{i_j}(x)-u_{i_j}(a)|<\varepsilon, j=1,\ldots,n\}$$

where $i_1,\ldots,i_n$ are non-negative integers and $a\in I\cup\mathscr{B}\cup\tilde{\mathscr{B}}_0$. The restriction of $\mathscr{H}$ to I is then the discrete topology and $\mathscr{H}$ makes all functions $u_0, u_1, \ldots$ continuous in $I\cup\mathscr{B}\cup\tilde{\mathscr{B}}_0$. Notice that the topology $\mathscr{H}$ identifies a reflecting atom a of $\mathscr{B}$ with the set $\Lambda(a)\subset\tilde{\mathscr{B}}_0$; thus we may consider only the set $\mathscr{A}=\mathscr{B}\cup\Gamma$, which is identical to $\mathscr{B} \cup \tilde{\mathscr{B}}_0$ as far as the topology $\mathscr{H}$ is concerned. Further $\mathscr{H}$ is a metric topology; $\mathscr{H}$ is generated by the distance

$$d(x, y) = \sum_{n=0}^{\infty} \frac{1}{2^{n+1}} |u_n(x) - u_n(y)|.$$

This is easily seen to be a pseudo-metric on $I \cup \mathscr{A}$. The only question is whether it separates points, or equivalently, whether $\{u_n\}$ separates points. But $\{u_n\}$ includes u_0, which separates points of I from each other and from points of $\mathscr{A}$, and it includes the functions H_a, $a \in \mathscr{B}$, which separate $\mathscr{B}$. The functions $\tilde{H}_{B_n}$ separate points of Γ since $\tilde{H}_{B_n}|_\Gamma = I_{B_n}$, and also separate points of Γ from $\mathscr{B}$ by Proposition 5.5. Thus $\mathscr{H}$ is a Hausdorff topology on $I \cup \mathscr{A}$, and a sequence $\{x_n\}$ converges to x in this topology iff $u_k(x_n) \to u_k(x)$ for all k.

Recall the notation of §1: $Z(t)$ is a Markov chain with all states stable, $X(t)$ is the subprocess of Z whose lifetime is an exponential random variable independent of $Z(t)$, and $\tilde{X}(t)$ is the reverse process of $X(t)$. The Martin exit boundary is supposed to satisfy the finiteness conditions (A) and (B). With the set of $\mathscr{A}$ as defined above, we have

Theorem 6.1. *There exist standard modifications of the processes X and $\tilde{X}$ which are right-continuous, and have left limits in the space $I \cup \mathscr{A}$ with topology $\mathscr{H}$. The right limits of X and the left limits of $\tilde{X}$ are in $I \cup \mathscr{S} \cup \Gamma$ while the left limits of X and right limits of $\tilde{X}$ are in $I \cup \mathscr{B}$.*

We already know that $u_n(x(t))$ has right and left I-limits with probability one so if $I \cup \mathscr{A}$ were compact, the theorem would be a triviality. However, $I \cup \mathscr{A}$ is not compact. We could easily complete it using the metric d, but this would introduce extraneous points and would still leave us the task of proving the process lives in the original space.

Proof. We define a standard modification $\hat{X}$ of X by $\hat{X}(t) = I\text{-}\lim_{s \downarrow t} X_t$, where the limit is taken in the topology $\mathscr{H}$. Now $I\text{-}\lim_{s \downarrow t} u_n(X(s))$ exists for all t and n; we must show that there exists $a \in I \cup \mathscr{A}$ such that this limit is $u_n(a)$.

The interval $[0, \infty)$ can be written as the union of S_I and S_I^c. The first set is a union of countably many disjoint right-open intervals $\{\mathscr{I}_n\}$ by $(S\ 1)$, and is dense in $[0, \infty)$.

Case 1. t is in the interior of one of the $\mathscr{I}_n$. Then $\hat{X}(t+) = \hat{X}(t) = X(t)$ and $\hat{X}(t-) = X(t-)$ in I. This immediately shows $\hat{X}$ is a standard modification of X since for each t, $X(t)$ is in the interior of some $\mathscr{I}_n$ w.p. 1.

If t is not in the interior of an $\mathscr{I}_n$, it is either a limit of endpoints of the $\mathscr{I}_n$, an endpoint itself, or both. If the former is true, it must even be a limit of right-hand endpoints of the $\mathscr{I}_n$. There are several cases to be distinguished.

Case 2. t is a left-hand endpoint of some $\mathscr{I}_n$. Then we claim $\hat{X}(t) = X(t+) \in I \cup \mathscr{S} \cup \Gamma$. For each n,

$$\lim_{s \downarrow t} u_n(\hat{X}(s)) = \lim_{s \downarrow t} u_n(X(s)) = u_n(X(t+)),$$

and $X(t+) \in I \cup \tilde{\mathscr{B}}$. If $X(t+) \in I \cup \Gamma$, we are done; if $X(t+) \in \mathscr{B} - \Gamma$, then w.p. 1 $X(t+) \in \Lambda(a)$ for some $a \in \mathscr{B} \subset \mathscr{A}$, and $u_n(X(t+)) = u_n(a)$ for all n.

Case 3. t is a right hand endpoint of some $\mathscr{I}_n$ and $t<\gamma_\Delta$. Then $\hat{X}(t-)=X(t-)\in\mathscr{B}$ since, for each n, $\lim_{s\uparrow t} u_n(\hat{X}(s))=\lim_{s\uparrow t} u_n(X(s))=u_n(X(t-))$, and by (i) of (S 1), $X(t-)\in\mathscr{B}$. Since there are only countably many $\mathscr{I}_n$, the above statements hold simultaneously w.p. 1 for all n.

Case 4. t is neither an interior point nor left-hand endpoint of any $\mathscr{I}_n$. We claim $\hat{X}(t)\in\mathscr{S}$. There exists a sequence $t_n\downarrow t$ such that t_n is a right-hand endpoint of some $\mathscr{I}_m$. Thus by (S 1) there must be a unique $a\in\mathscr{B}$, for which $X(t_n-)=a$ for all large enough n. But for each k, $I\text{-}\lim_{s\downarrow t} u_k(X(s))$ exists, and thus must equal $\lim_{n\to\infty} u_k(X(t_n-))=u_k(a)$, all k. Thus $\hat{X}(t)=a$.

Case 5. It remains to show that $\hat{X}(t)$ has a left limit in $\mathscr{S}\subset\mathscr{A}$ in case t is neither an interior point nor a right-hand endpoint of an $\mathscr{I}_n$. In this case there must be a sequence $t_n\uparrow t$ of right-hand endpoints of the $\mathscr{I}_n$, and hence there is an $a\in\mathscr{S}$ for which $\hat{X}(t_n-)=a$ for all large enough n. Then for each k, $I\text{-}\lim_{s\uparrow t} u_k(X(s))$, since it exists, must be $u_k(a)$; this proves the assertion. The desired standard modification of $\tilde{X}$ is just $\hat{\hat{X}}(t)=\lim_{s\uparrow t}\hat{X}(S-s)$, the limit being taken in the topology $\mathscr{H}$. q.e.d.

§ 7. The Strong Markov Property

The following theorem is true even without finiteness restrictions on $\mathscr{B}$.

Theorem 7.1. *Let $a\in\mathscr{B}$. Then for $t>0$, $P_{ij}(t)\to P_{aj}(t)$ as $H_a(i)\to 1$.*

Proof. By Theorem 5.1 of [2]:

$$P_i\{X_t=j\}=\int_0^t p_i(a,ds)\,P_{aj}(t-s)+\sum_{b\in\mathscr{B}-\{a\}}\int_0^t p_i(b,ds)\,P_{bj}(t-s)$$
$$+P_i\{X_t=j|\alpha>t\}\int_t^\infty p_i(\mathscr{B},ds)$$

where $p_i(a,ds)$ is the joint distribution of $(X(\alpha^-),\alpha)$ given $X(0)=i$. If $p_i^z(a,ds)$ is the joint distribution of $(Z(\alpha^-),\alpha)$ where Z is the original process, then $p_i(a,ds)=e^{-s}p_i^z(a,ds)$ if $a\neq\Delta$. Then

$$H_a(i)=\int_0^\infty e^{-s}p_i^z(a,ds).$$

As $H_a(i)\to 1$, p_i^z, and therefore p_i, must tend weakly to the unit mass at $(a,0)$. The conclusion now follows from continuity of the $P_{bj}(\cdot)$. q.e.d.

This means that for each $t>0$, $i\to P_{ij}(t)$ has an $\mathscr{H}$-limit at each point a of $\mathscr{B}$; this limit must be $P_{aj}(t)$, which was defined in Section 3. It is not necessarily true that $i\to P_{ij}(t)$ is $\mathscr{H}$-continuous at Γ; however, it is cofine-continuous there at all but a possibly negligible set. Inasmuch as $S^+(\Gamma_\delta)$ is discrete for each $\delta>0$, this is enough to assure that $\lim_{s\downarrow u} P_{\hat{X}(s)j}(t)=P_{\hat{X}(u)j}(t)$ for all $u\in S^+(\Gamma_\delta)$, and hence for all $u\in S^+(\Gamma)$. Thus we see $s\to P_{\hat{X}(s)j}(t)$ is right-continuous for all $t>0$, $j\in I$. This is sufficient to prove by the usual method that:

Theorem 7.2. *$\hat{X}$ is a strong Markov process.*

§ 8. Branching Points and Quasi Left Continuity

A point $b \in \mathscr{A} \cup I$ is said to be a *nonbranching* point if for every bounded continuous function f on $\mathscr{A} \cup I$, $\lim_{t \downarrow 0} P_t f(b) = f(b)$, and is a *branching* point otherwise.

It is clear that all points of I are nonbranching, since $P_{ij}(t) \to \delta_{ij}$ as $t \to 0$ for all j. Furthermore, all points of Γ, other than possibly a negligible set, are nonbranching; if we let T be $\inf S^+_{\Gamma_\delta}$, then $\hat{X}(t) \in \Gamma_\delta$ on $\{T < \infty\}$ and $\hat{X}$ is right-continuous there, so that if f is continuous, $f(\hat{X}(t)) \to f(\hat{X}(T))$ as $t \downarrow T$. If f is bounded, the strong Markov property implies $P_t f(b) \to f(b)$ for all except possibly a negligible subset of Γ_δ. As δ is arbitrary, we see that the set of branching points in Γ is negligible.

Theorem 8.1. *Except for possibly a negligible subset of Γ, the set of branching points of $\mathscr{A} \cup I$ is $\mathscr{B} - \mathscr{S}$. Further if $\{T_n\}$ is an increasing sequence of stopping times with limit T, then w.p.* 1 $\lim X(T_n) = X(T)$ *a.e. on the set* $\{T < \infty, \lim X(T_n) \notin \mathscr{B} - \mathscr{S}\}$.

Proof. We prove the second assertion first; the first follows easily. We may assume $T < \infty$ everywhere. Let f be bounded and continuous on $I \cup \mathscr{A}$, and let $t > 0$. If $A \in \bigvee_n \mathscr{F}_{T_n}$,

$$E\{P_t f(\lim X(T_n)); A\} = E\{\lim P_t f(X(T_n)); A\} = E\{f(X(T+t)); A\}. \tag{8.1.1}$$

Let $t \to 0$; by right continuity, the right hand side goes to $E\{f(X(T)); A\}$. Now consider $A_x = \{T_n < T \ \forall n, \lim X(T_n) = x\}$ for $x \in I \cup \mathscr{B}$. Then we have:

$$\lim_{t \downarrow 0} P_t f(x) P\{A_x\} = E\{f(X(T)), A_x\}. \tag{8.1.2}$$

If $x \in I$, choose $f = I_x$, and if $x \in \mathscr{S}$, take $f = H_x$. In either case the left-hand side of (8.1.2) is $P\{A_x\}$ so we must have $f(X(T)) = 1$ a.e. on A_x, hence $X(T) = x$ on A_x. This verifies the second statement.

Now for $x \in I \cup \mathscr{B}$, (8.1.2) must hold for all bounded continuous f and predictable stopping times T, hence $\lim_{t \downarrow 0} P_t f(x) = f(x)$. It remains to show that all points of $\mathscr{B} - \mathscr{S}$ are branching points. Suppose $x \in \mathscr{B} - \mathscr{S}$. Take T to be the first hit of the boundary; then $P\{A_x\} > 0$. By Theorem 6.1, $X(T)$ can not be in $\mathscr{B} - \mathscr{S}$, hence $H_x(X(T)) < 1$ and (8.1.2) gives us:

$$\lim_{t \downarrow 0} P_t H_x(x) < 1 = H_x(x). \qquad \text{q.e.d.}$$

We now have a reasonably complete description of the boundary behavior of the paths of $X(t)$. Since X is gotten from the original process Z by killing at an exponential time independent of Z it follows that there is a version of Z which is right-continuous and has left limits in $I \cup \mathscr{A}$; this version is strongly Markov and has $\mathscr{B} - \mathscr{S}$ as its set of branching points.

We are left with the question of what happens if the entrance rather than the exit boundary is finite. This is easily answered by looking at the reversed process.

Theorem 8.2. *The process $\hat{\hat{X}}$ is a strong Markov process in $I \cup \mathscr{A}$. The set of branching points is Γ.*

Proof. It is enough to show that $s \to \tilde{P}_t f(\hat{\tilde{X}}_s)$ is right-continuous, or equivalently, that $\tilde{P}_t$ can be extended to $\mathscr{B} \cup \Gamma$ by fine and cofine continuity. We already know from section 3 that it can be extended to $\mathscr{B}_0^*$ and $\tilde{\mathscr{B}}_0^*$. But each point of $\mathscr{B}$ is either a single point or an equivalence class of such points. If $a \sim b$, we must have $\tilde{P}_t f(a) = \tilde{P}_t f(b)$ for otherwise $s \to \tilde{P}_t f(\hat{\tilde{X}}_s)$ would have an oscillatory I-discontinuity the first time it hit a (which equals the first time it hit b) contradicting Corollary 3.4.

As before, all points of I are nonbranching points, and points of Γ (excluding perhaps a negligible set) are branching points since, although $\hat{\tilde{X}}$ may have left limits in Γ, it takes its values in $I \cup \mathscr{B}$. Note that for $a \in \mathscr{B}$, if $\tau = \inf S_a^+$, that $\hat{\tilde{X}}(\tau) = a$: if $a \in \mathscr{B} - \mathscr{S}$ this is true since S_a^+ is discrete, and if $a \in \mathscr{S}$ this is true by right-continuity and the fact that τ is a limit from the right of S_a^+. Now for $t > 0$ and f bounded and continuous on $I \cup \mathscr{A}$, $\tilde{P}_t f(a) = \text{fine} \lim_{i \to a} \tilde{P}_t f(i)$; in either case above, it is easy to see that $\tilde{P}_t f(a) = E\{f(\hat{\tilde{X}}(\tau + t)) | \tau < \infty\}$, and this converges to $f(a)$ by continuity as $t \downarrow 0$. Thus $\mathscr{B}$ consists of nonbranching points and the theorem is proved. q.e.d.

§ 9. The Finite Boundary Case

If we require that both the entrance and exit boundaries be finite, we get an esthetically more pleasing, if mathematically less useful case. The description of the boundary is certainly simpler. Let $\mathscr{B}_0 = \mathscr{B}_0^*/R$ and $\tilde{\mathscr{B}}_0 = \tilde{\mathscr{B}}_0^*/R$. Let us assume that both the backward and forward Kolmogorov differential equations are satisfied and that there are finite sets $\mathscr{B} \subset \mathscr{B}_0$ and $\tilde{\mathscr{B}} \subset \tilde{\mathscr{B}}_0$ of non-negligible points such that

$$P\{X(\alpha-) \in \mathscr{B} - \mathscr{B}_0\} = P\{\tilde{X}(\tilde{\alpha}-) \in \tilde{\mathscr{B}} - \tilde{\mathscr{B}}_0\} = 0.$$

If $\alpha \in \mathscr{B} - \mathscr{S}$, then $\Lambda(a)$ is empty. On the other hand, if $a \in \mathscr{S}$ $\Lambda(a)$ cannot be empty. An argument based on Lemma 5.1 shows that any atom in $\Lambda(a)$ must be reflecting; it follows that any two such atoms are equivalent, hence $\Lambda(a) = b$ for some $b \in \tilde{\mathscr{B}}$ and by symmetry, $\tilde{\Lambda}(b) = a$. We agree to identify a and b in this case, and with this in mind, we have:

Theorem 9.1. *A boundary atom a is reflecting iff it is in both the entrance and exit boundaries; if $a \in \mathscr{B}$ ($a \in \tilde{\mathscr{B}}$), a is nonreflecting iff there exists an $\mathscr{H}$ neighborhood of a and an $\mathscr{H}$ neighborhood of $\tilde{\mathscr{B}}$ ($\mathscr{B}$) which do not intersect.*

Proof. The non-intersecting neighborhoods of a and $\tilde{\mathscr{B}}$ can in fact be taken to be of the form $\{i: H_a(i) > 1 - \varepsilon\}$ and $\{i: H_a(i) < 1 - \varepsilon\}$ for some $\varepsilon > 0$, since if a is non-reflecting, $H_a(b) < 1$ for each of the finitely many $b \in \tilde{\mathscr{B}}$.

§ 10. Examples

Central to our discussion of reflecting boundary atoms is the notion of equivalence. Chung [2] has defined a related notion called indistinguishability; a and b are indistinguishable in $\mathscr{B}$ if $P_{aj}(t) = P_{bj}(t)$ for all $t > 0$ and $j \in I$. These two notions are the same for reflecting atoms, but it is possible for a reflecting atom and a nonreflecting atom to be indistinguishable and impossible for them to be equivalent. A more symmetric way of stating the relation is that a and b are equivalent iff both $P_{aj}(t) = P_{bj}(t)$ and $\tilde{P}_{aj}(t) = \tilde{P}_{bj}(t)$ for all $t > 0$, $j \in I$.

The question remains of the existence of equivalent but nonidentical atoms. That such atoms exist is a consequence of the construction theorems of Chung [2] and Dynkin [4]; however, explicit examples have not been given; here are two to indicate the possibilities.

Consider Brownian motion B_t on $(-\infty, \infty)-\{0\}$. There are 4 minimal regular functions for this space, corresponding to the points $+\infty$, $-\infty$, $0+$ and $0-$. The exit boundary for the process has two active points, $0+$ and $0-$. Since the first time the process hits 0 is a limit from the right of t for which $B(t-)=0-$ and of t for which $B(t-)=0+$, $0+$ and $0-$ are equivalent. This behavior can be translated into a Markov chain by using a device of Ito and McKean [8]. Let $\varphi(t, y)$ be Brownian local time at y, and set

$$\tau(t)=\sum_{-\infty}^{\infty} 2^{-|n|}\varphi\left(t, \frac{1}{n}\right).$$

Then the process $X(t)=1/B(\tau^{-1}(t))$ is a Markov chain with two equivalent boundary points, $+\infty$ and $-\infty$.

It is even possible to have a countable number of equivalent atoms. The process $|X(t)|$, X as above, is a Markov chain on $N^+=\{1, 2, \ldots\}$ with a single reflecting atom, $+\infty$. Let $P_{ij}(t)$ and $P_{ij}^m(t)$ be its transition probabilities and minimal transition probabilities, respectively. Define transition probabilities $\hat{P}$ on $N^+\times N^+$ by

$$\hat{P}_{(p,i)(n,j)}(t)=\delta_{np}\,P_{ij}^m(t)+\frac{1}{2^n}\,[P_{ij}(t)-P_{ij}^m(t)].$$

It is straightforward to verify that $\hat{P}$ is a semigroup, its minimal process has transition probabilities $\hat{P}_{(p,i)(n,j)}^m(t)=\delta_{np}\,P_{ij}^m(t)$ and has a Martin exit boundary $N^+\times\{\infty\}$. We claim these points are all equivalent. Let α be the first hit of the boundary; for any n, $t>0$, the process has probability $1/2^n$ of being in $\{n\}\times N^+$ at time $\alpha+t$. The only boundary point the process can hit from there is (n, ∞). On the other hand, the projection of the process on $1\times N^+$ is just $(1, |X(t)|)$, and the boundary of this process is reflecting. Our process must therefore hit the boundary infinitely often in any neighborhood of α. An application of Lemma 5.1 shows that the process hits (n, ∞) infinitely often in any neighborhood of α; this is true simultaneously for all n.

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