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Infinitely Divisible Representations of Lie Algebras

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In two previous papers [1, 2], we studied a special class of group representations, those called infinitely divisible. These are associated with infinitely divisible random variables, at least if the group is abelian [3]. In the non-abelian case, the theory was developed to provide representations of the current algebras of quantum field theory [4, 5]. More recently, interesting results have been obtained using the methods of probability theory [6].

In this paper we study cyclic representations of Lie algebras, with an analogous property, also called infinite divisibility. We obtain some results, which are the infinitesimal analogues of corresponding theorems for Lie groups, obtained in [2] and [5].

1. Cyclic Representations of Lie Algebras

Let $\mathscr{G}$ be a finite dimensional real Lie algebra; by a *cyclic representation* of $\mathscr{G}$ we shall mean a triple $(\mathscr{K}, \omega, \pi)$, where $\mathscr{K}$ is a Hilbert space with scalar product $\langle\ ,\ \rangle$, ω a unit vector in $\mathscr{K}$, and π a homomorphism from $\mathscr{G}$ to unbounded anti-symmetric operators on $\mathscr{K}$, such that ω is in the common domain D of each $\pi(A)$, $A\in\mathscr{G}$; D is invariant under each $\pi(A)$, and consists of the linear span of those vectors obtained from ω by applying $\pi(\mathscr{G})$.

The assumptions in this definition are not quite those obtained from a cyclic group representation by differentiation. First, our very strong domain assumptions would be true only in special cases; on the other hand, the Lie algebra definition is more general, in that it allows for local as well as global group representations.

If $(\mathscr{K}, \omega, \pi)$ is a cyclic representation of $\mathscr{G}$, the expressions $\langle\omega, \pi(A_1)\dots\pi(A_n)\,\omega\rangle$ for $A_j\in\mathscr{G}$, $j=1, 2, \dots, n$, are well defined, and will be called the *moments* of (ω, π) and denoted by $\langle A_1\dots A_n\rangle$. The problem, to find a representation of a Lie group G whose Lie algebra is $\mathscr{G}$, when the moments are given, is the non-commutative generalization of the moment problem.

The *cumulants* of (ω, π) are defined inductively as follows; they are the analogues of the truncated functions in a quantum field theory (see, for example, [7]), and will be denoted $\langle A_1\dots A_n\rangle_T$.

Let I be a partition of $(1, \dots, n)$ into p parts;

$$I=(i_1, \dots, i_{l_1})(i_{l_1+1}, \dots, i_{l_2})\dots(i_{l_{p-1}+1}, \dots, i_{l_p}).$$

In each part, the integers are written in their natural order. We define $\langle A\rangle_T=\langle A\rangle$ for all $A\in\mathscr{G}$, and

$$\langle A_1\dots A_n\rangle=\langle A_1\dots A_n\rangle_T+\sum_I \langle A_{i_1}\dots A_{i_{l_1}}\rangle_T\dots\langle A_{i_{l_{p-1}+1}}\dots A_{i_{l_p}}\rangle_T.$$

It is clear that the cumulants determine the moments, and conversely.

We shall say that two cyclic representations $(\mathscr{K}_1, \omega_1, \pi_1)$ and $(\mathscr{K}_2, \omega_2, \pi_2)$ of a Lie algebra $\mathscr{G}$ are *equivalent* if there is a unitary map V: $\mathscr{K}_1 \to \mathscr{K}_2$ such that $V\omega_1 = \omega_2$, and $V\pi_1 = \pi_2 V$. Equivalent representations will be treated as the same; in particular, the representations $(\mathscr{K}, \omega, \pi)$ and $(\mathscr{K}, e^{i\alpha}\omega, \pi)$ will be identified.

Let us define a *-operation on $\mathscr{G}$ by $A^* = -A$ for all $A \in \mathscr{G}$. Let $\mathscr{P} = \mathscr{P}(\mathscr{G})$ be the polynomial algebra over the vector space $\mathscr{G}$, with complex scalars, including polynomials of degree zero, that is, scalars. The involution * is defined in $\mathscr{P}$ by the requirement that $(AB)^* = B^* A^*$, $(\alpha A)^* = \bar{\alpha} A^*$ etc. Let $\mathscr{C}$ be the commutator ideal of $\mathscr{P}$, that is, the smallest 2 sided ideal containing all polynomials of the form $AB - BA - [A, B]$, where $[A, B]$ is Lie multiplication. We see that $\mathscr{C} = \mathscr{C}^*$, and so the quotient algebra $\mathscr{E} = \mathscr{P}/\mathscr{C}$ inherits the *-structure. $\mathscr{E}(\mathscr{G})$ is called the *enveloping algebra* of $\mathscr{G}$.

Since any cyclic representation $(\mathscr{K}, \omega, \pi)$ of $\mathscr{G}$ leads to a representation of $\mathscr{E}$, it also defines a representation of $\mathscr{P}$ in $\mathscr{K}$ (which vanishes on $\mathscr{C}$), with the same cyclic vector ω. Conversely, any *-representation of $\mathscr{P}$ with cyclic vector ω and common invariant domain D, that vanishes on $\mathscr{C}$, defines a cyclic representation of $\mathscr{E}$ (and thus of $\mathscr{G}$) by passing to the quotient.

An element p of $\mathscr{P}$ is said to be *positive* if $p = q^* q$ for some polynomial q. A linear map W: $\mathscr{P} \to \mathbb{C}$ is said to be *positive* if $W(p) \geqq 0$ for all positive p; it is said to be normalized if $W(1) = 1$, and *hermitian* if $\overline{W(p)} = W(p^*)$. Note that hermiticity follows from positivity. A linear functional will be called a *form*.

We can now state our preliminary theorem.

Theorem 1. i) *If $(\mathscr{K}, \omega, \pi)$ is a cyclic representation of a Lie algebra $\mathscr{G}$, then there exists a unique positive normalized form on $\mathscr{P}(\mathscr{G})$, $p \to W(p)$ say, such that*

$$\langle \omega, \pi(A_1) \dots \pi(A_n)\, \omega \rangle = W(A_1 \dots A_n) \qquad \textit{for } A_i \in \mathscr{G}.$$

This form vanishes on $\mathscr{C}$.

ii) *Two cyclic representations lead to the same form if and only if they are equivalent.*

iii) *Given a positive normalized form W on $\mathscr{P}$, vanishing on $\mathscr{C}$, there exists a cyclic representation $(\mathscr{K}, \omega, \pi)$ such that*

$$\langle \omega, \pi(A_1) \dots \pi(A_n)\, \omega \rangle = W(A_1 \dots A_n)$$

for all sets of n elements $A_1, \dots, A_n$ in $\mathscr{G}$, and all n.

The *proof* of this theorem follows exactly the same pattern as the Wightman reconstruction theorem in quantum field theory [8], and is omitted.

Let $\mathscr{P}_1$ denote the subalgebra of $\mathscr{P}$ of elements without constant term.

Theorem 2. *Given a cyclic representation $(\mathscr{K}, \omega, \pi)$ of $\mathscr{G}$, there exists a unique form W_T on $\mathscr{P}_1(\mathscr{G})$ such that $W_T(A_1 \dots A_n) = \langle A_1 \dots A_n \rangle_T$. Moreover, W_T vanishes on $\mathscr{C}$. Conversely, if a form W_T is hermitian, or vanishes on $\mathscr{C}$, the form W obtained from it has the same property.*

Proof. It is obvious from the definition that $A_1, \dots, A_n \to \langle A_1 \dots A_n \rangle_T$ is a multilinear map from $\mathscr{G} \times \dots \times \mathscr{G}$ to $\mathbb{C}$, and so possesses a unique extension, by linearity, to a form on $\mathscr{P}_1$.

To show that W_T vanishes on $\mathscr{C}$, it suffices to show that it vanishes on any element of $\mathscr{P}_1$ of the form $A_1 \dots A_j (AB - BA - C) B_1 \dots B_k$, where $C = [A, B]$; for, any element of $\mathscr{C}$ is a finite sum of these. Since $W_T(AB)=W(AB)-W(A)\,W(B)$, we see that $W_T(AB-BA-C)=W(AB-BA-C)=0$. Thus the above statement is certainly true if $j=k=0$. Suppose, then, as inductive hypothesis, that W_T vanishes on all elements of the form $A_1\ldots A_{j'}(AB-BA-C)\,B_1\ldots B_{k'}$, for $j'+k'<j+k$, $j'\leqq j$, $k'\leqq k$. Then, from the definition

$$W_T(A_1\ldots A_j(AB-BA-C)\,B_1\ldots B_k)=\langle A_1\ldots A_j(AB-BA-C)\,B_1\ldots B_k\rangle$$
$$-\{\textstyle\sum W_T(\ldots)\ldots W_T(\ldots)+\sum' W_T(\ldots)\ldots W_T(\ldots)+\sum'' W_T(\ldots)\ldots W_T(\ldots)\}$$

where $\sum$ is the sum over all partitions of $A_1\ldots A_j\,ABB_1\ldots B_k$, $\sum'$ the sum over all partitions of $A_1\ldots A_jBAB_1\ldots B_k$, and $\sum''$ is the sum over all partitions of $A_1\ldots A_jCB_1\ldots B_k$. Now, W vanishes on $\mathscr{C}$, so $\langle A_1\ldots A_j(AB-BA-C)\,B_1\ldots B_k\rangle=0$. If in a term in the sum $\sum$, A and B are in the same part of the partition, there is a corresponding partition in $\sum'$ and in $\sum''$, which may be combined to give a term of the form $W_T(\ldots)\ldots W_T(\ldots(AB-BA-C)\ldots)\ldots W_T(\ldots)$. By induction, this vanishes. This exhausts the partitions in $\sum''$. For the remaining partitions in $\sum$, those in which A and B are in different parts, there is a corresponding partition in $\sum'$, which cancels (there is no corresponding partition in $\sum''$ in this case). Therefore W_T vanishes on $A_1\ldots A_j(AB-BA-C)\,B_1\ldots B_k$, and the induction hypothesis is proved for $j+k+2$. The converse is proved in a similar way.

This proves Theorem 2.

Theorem 2 is a simple version of the well known fact in quantum field theory, that the truncated Wightman functions satisfy local commutativity.

2. Infinitely Divisible Representations

If $(\mathscr{K}_1,\omega_1,\pi_1)$ and $(\mathscr{K}_2,\omega_2,\pi_2)$ are cyclic representations of Lie algebras $\mathscr{G}_1$, and $\mathscr{G}_2$, we define the *tensor product* $\pi_1\otimes\pi_2$ to be the representation of $\mathscr{G}_1\oplus\mathscr{G}_2$, acting in $\mathscr{K}_1\otimes\mathscr{K}_2$; if $A_1\in\mathscr{G}_1$ and $A_2\in\mathscr{G}_2$, it is given by

$$\big((\pi_1\otimes\pi_2)(A_1\oplus A_2)\big)(\varphi_1\otimes\varphi_2)=\pi_1(A_1)\,\varphi_1\otimes\varphi_2+\varphi_1\otimes\pi_2(A_2)\,\varphi_2$$

on product vectors, and by linearity elsewhere. The cyclic vector of $\pi_1\otimes\pi_2$ is taken to be $\omega_1\otimes\omega_2$, and the domain, that generated from $\omega_1\otimes\omega_2$ by applying $\pi_1\otimes\pi_2$. With this definition, the tensor product is a cyclic representation of $\mathscr{G}_1\oplus\mathscr{G}_2$. Similarly, one defines the tensor product of any finite number of cyclic representations. It is well known that the associativity law

$$\big((\omega_1\otimes\omega_2)\otimes\omega_3,(\pi_1\otimes\pi_2)\otimes\pi_3\big)\cong\big(\omega_1\otimes(\omega_2\otimes\omega_3),\pi_1\otimes(\pi_2\otimes\pi_3)\big)$$

holds.

The algebra $\mathscr{G}\oplus\mathscr{G}$ possesses a subalgebra isomorphic to $\mathscr{G}$, namely the *diagonal subalgebra* $\tilde{\mathscr{G}}$ of elements of the form $A\oplus A$ with $A\in\mathscr{G}$. The restriction of a tensor product representation $\pi_1\otimes\pi_2$ of $\mathscr{G}\oplus\mathscr{G}$ to the subalgebra $\tilde{\mathscr{G}}$, acting on the Hilbert space generated from $\omega_1\otimes\omega_2$ by applying $\pi_1\otimes\pi_2(A,A)$, is a cyclic representation of $\mathscr{G}$. This is often again called the tensor product of π_1 and π_2, but to avoid confusion, we shall call it the *product* $\pi_1\times\pi_2$.

For abelian Lie algebras, the product of two representations corresponds to forming the convolution of the associated measures, at least when the moment problem has a unique solution.

Theorem 3. *The cumulants of $\pi_1 \otimes \pi_2 \otimes \cdots \otimes \pi_N$ are the sums of those of $\pi_1, \pi_2, \ldots, \pi_N$. In symbols, if $A^i_j \in \mathscr{G}_j$, $i=1, \ldots, n$ for each $j=1, \ldots, N$, if $\omega = \bigotimes_1^N \omega_j$ $\pi = \bigotimes_{j=1}^N \pi_j$, and $A^i = \bigoplus_{j=1}^N A^i_j$, $i=1, \ldots, n$, we have*

$$\langle \omega, \pi(A^1) \ldots \pi(A^n)\, \omega \rangle_T = \sum_{j=1}^N \langle \omega_j, \pi_j(A^1_j) \ldots \pi_j(A^n_j)\, \omega_j \rangle_T .$$

Proof. It is sufficient to prove it for $N=2$; the general case then follows immediately.

We proceed by induction on the order n of the cumulant. Since the first cumulant is the first moment, and these add, the inductive hypothesis is true for $n=1$. Suppose it is true for the cumulants of order $n-1$. Then, writing $A^i_1 = \pi(A^i_1, 0)$ and $A^i_2 = \pi(0, A^i_2)$, $i=1, \ldots, n$:

$$\begin{aligned}\langle \omega, \pi(A^1) \ldots \pi(A^n)\, \omega \rangle &= \langle \omega, (A^1_1 + A^1_2)(A^2_1 + A^2_2) \ldots (A^n_1 + A^n_2)\, \omega \rangle \\ &= \langle \omega, A^1_1 \ldots A^n_1\, \omega \rangle + \langle \omega, A^1_2 \ldots A^n_2\, \omega \rangle \\ &\quad + \sum_{\substack{J=(i_1 \ldots i_l) \\ K=(i_{l+1} \ldots i_n) \\ J \cup K = (1, \ldots, n)}} \langle \omega, A^{i_1}_1 \ldots A^{i_l}_1\, \omega \rangle \langle \omega, A^{i_{l+1}}_2 \ldots A^{i_n}_2\, \omega \rangle \\ &= \langle \omega_1, \pi_1(A^1_1) \ldots \pi_1(A^n_1)\, \omega_1 \rangle + \langle \omega_2, \pi_2(A^1_2) \ldots \pi_2(A^n_2)\, \omega_2 \rangle \\ &\quad + \sum_{J,K} \langle \omega_1, \pi_1(A^{i_1}_1) \ldots \pi_1(A^{i_l}_1)\, \omega_1 \rangle \langle \omega_2, \pi_2(A^{i_{l+1}}_2) \ldots \pi_2(A^{i_n}_2)\, \omega_2 \rangle .\end{aligned} \tag{1}$$

Here, the sum is over all partitions of $(1, \ldots, n)$ into two non-empty parts J, K; and the divisions (J, K) and (K, J) are regarded as distinct, and both enter.

We write the moments in terms of cumulants. Let $(J_\alpha) = (J_1, \ldots, J_j)$ be an arbitrary partition of J, and $(K_\beta) = (K_1, \ldots, K_k)$ an arbitrary partition of K. Let

$$\begin{aligned} W^1_T(J_\alpha) &= \langle \omega_1, \pi_1(A^{j_1}_1) \ldots \pi_1(A^{j_{p_\alpha}}_1)\, \omega_1 \rangle_T , \\ W^2_T(K_\beta) &= \langle \omega_2, \pi_2(A^{k_1}_2) \ldots \pi_2(A^{k_{q_\beta}}_2)\, \omega_2 \rangle_T , \end{aligned}$$

where $J_\alpha = (j_1, \ldots, j_{p_\alpha})$ and $K_\beta = (k_1, \ldots, k_{q_\beta})$; $\alpha = 1, \ldots, j$; $\beta = 1, \ldots, k$. Let $I = (I_1, \ldots, I_{j+k})$ be an arbitrary partition of $(1, \ldots, n)$ into $j+k$ parts. Then from (1)

$$\begin{aligned} &\langle \omega, \pi(A^1) \ldots \pi(A^n)\, \omega \rangle = \langle \omega_1, \pi_1(A^1_1) \ldots \pi_1(A^n_1)\, \omega_1 \rangle_T \\ &\quad + \sum_{\substack{I \\ j+k>1}} W^1_T(I_1) \ldots W^1_T(I_{j+k}) + \langle \omega_2, \pi_2(A^1_2) \ldots \pi_2(A^n_2)\, \omega_2 \rangle_T + \sum_{\substack{I \\ j+k>1}} W^2_T(I_1) \ldots W^2_T(I_{j+k}) \\ &\quad + \sum_{J,K} \langle \omega_1, \pi_1(A^{i_1}_1) \ldots \pi_1(A^{i_l}_1)\, \omega_1 \rangle_T \langle \omega_2, \pi_2(A^{i_{l+1}}_2) \ldots \pi_2(A^{i_n}_2)\, \omega_2 \rangle_T \\ &\quad + \sum_{J,K} \sum_{\substack{(J_\alpha) \\ (K_\beta)}} W^1_T(J_1) \ldots W^1_T(J_j)\, W^2_T(K_1) \ldots W^2_T(K_k). \end{aligned} \tag{2}$$

The 4 sums in (2) may be written

$$\sum_{j+k} \sum_{I=(I_1, \ldots, I_{j+k})} \big(W^1_T(I_1) + W^2_T(I_1)\big)\big(W^1_T(I_2) + W^2_T(I_2)\big) \ldots \big(W^1_T(I_{j+k}) + W^2_T(I_{j+k})\big),$$

where the sum is over all proper partitions. By the inductive hypothesis, $W_T^1(I_\gamma)+W_T^2(I_\gamma)$ can be replaced by $\langle\omega, \pi(A^{i_1})\dots\pi(A^{i_{p_\gamma}})\,\omega\rangle_T$ (where $I_\gamma=(i_1,\dots,i_{p_\gamma})$) $=W_T(I_\gamma)$ say. Thus (2) becomes

$$\begin{aligned}\langle\omega, \pi(A^1)\dots\pi(A^n)\,\omega\rangle &= \langle\omega_1, \pi_1(A_1^1)\dots\pi_1(A_1^n)\,\omega_1\rangle_T\\ &+\langle\omega_2, \pi_2(A_2^1)\dots\pi_2(A_2^n)\,\omega_2\rangle_T+\sum_{I=(I_1\dots I_p)} W_T(I_1)\dots W_T(I_p).\end{aligned}$$

Comparing with the definition of W_T, we see that the inductive hypothesis is true for n.

This proves Theorem 3.

Definition. If a Lie algebra $\mathscr{G}$ is isomorphic to a direct sum $\mathscr{G}_1\oplus\dots\oplus\mathscr{G}_n$, then we shall say a cyclic representation $(\mathscr{K}, \omega, \pi)$ of $\mathscr{G}$ is *factorizable relative to the direct sum* if there exist representations $(\mathscr{K}_i, \omega_i, \pi_i)$ of $\mathscr{G}_i$ such that

$$(\omega, \pi)\cong\bigotimes_{i=1}^{N}(\omega_i, \pi_i).$$

We shall say a cyclic representation of a Lie algebra is divisible if it is (equivalent to) the product of two non-trivial others.

Definition. A cyclic representation $(\mathscr{K}, \omega, \pi)$ of a Lie algebra $\mathscr{G}$ is *infinitely divisible* if, for any positive integer m, there exists a cyclic representation of $\mathscr{G}$, denoted π/m, such that (ω, π) is equivalent to $(\omega\otimes\dots\otimes\omega, \pi/m\times\pi/m\times\pi/m)$ (m factors).

Theorem 4. *A cyclic representation is ∞-divisible if and only if, for any integer m, its cumulants divided by m, are the cumulants of some cyclic representation.*

Proof (i). Suppose π is ∞-divisible; then, for a given m, there exists a cyclic representation π/m, such that $\pi\cong\pi/m\times\dots\times\pi/m$. Applying Theorem 3, restricted to the diagonal subalgebra of $\mathscr{G}\oplus\dots\oplus\mathscr{G}$, we see that the cumulants of π are m times those of π/m.

(ii) Conversely, if the cumulants of π are m times those of π_1, say, then $\pi_1\times\dots\times\pi_1$ (m factors) has the same cumulants as π. By Theorem 1, (ii), they are equivalent, showing π to be ∞-divisible.

Corollary 1. *The product of a finite number of ∞-divisible representations is ∞-divisible.*

Corollary 2. *We may replace the integer m in the statement of Theorem* 4 *by any rational number. For, if $m=p/q$, the product of q copies of π/p gives the desired representation.*

Exponential Spaces and Representations

If $\mathscr{K}$ is a Hilbert space, the *symmetric tensor product* $\otimes_S^m\mathscr{K}$ is the subspace of $\mathscr{K}\otimes\dots\otimes\mathscr{K}$ spanned by vectors of the form

$$|\varphi_1, \varphi_2, \dots, \varphi_m\rangle=\frac{1}{\sqrt{m!}}\sum_{\gamma\in S_m}\varphi_{\gamma(1)}\otimes\dots\otimes\varphi_{\gamma(m)}$$

where S_m is the symmetric group on m indices. If $\mathscr{K}$ carries a representation π of $\mathscr{G}$, then $\otimes_S^m\mathscr{K}$ is invariant under $\otimes^m\pi$. The *symmetric Fock space* over $\mathscr{K}$ is the orthogonal direct sum $\exp \mathscr{K}$, given by

$$\exp \mathscr{K} = \mathbb{C} \oplus \mathscr{K} \oplus (\mathscr{K} \otimes \mathscr{K})_S \oplus \cdots \oplus \otimes_S^m \mathscr{K} \oplus \cdots .$$

It can be shown (see for example, [9]) that vectors of the form

$$e^{\varphi} = 1 \oplus \varphi \oplus |\varphi, \varphi\rangle \oplus \cdots \oplus |\varphi, \ldots, \varphi\rangle \oplus \cdots$$

with $\varphi \in \mathscr{K}$, span $e^{\mathscr{K}}$. Obviously

$$\langle e^{\varphi}, e^{\psi}\rangle = e^{\langle \varphi, \psi\rangle}.$$

If $(\mathscr{K}, \omega, \pi)$ is a cyclic representation of $\mathscr{G}$ with ω not necessarily normalized, we define $(e^{\mathscr{K}}, \Omega, e^{\pi})$ to the representation in $e^{\mathscr{K}}$, generated from the (normalized) cyclic vector $\Omega = \exp(-\frac{1}{2}\|\omega\|^2)\, e^{\omega}$ by applying

$$\exp \pi(A) = 0 \oplus \pi(A) \oplus \pi(A) \otimes \pi(A) \oplus \cdots$$

as A runs over $\mathscr{G}$.

Remark. This is the differential of an analogous concept defined for Lie groups [2]. Note that in this paper, exp is not related to the exponential map from a Lie algebra to a Lie group.

Theorem 5. *The cumulants of e^{π} are the moments of π.*

Proof. Let $\exp \pi(A) = \hat{\pi}(A)$. Then if $A_j \in \mathscr{G}$, $j = 1, \ldots, n$, we have

$$\begin{aligned} \hat{\pi}(A_1) \ldots \hat{\pi}(A_n)\, \Omega = \exp(-\tfrac{1}{2}\|\omega\|^2) \Big\{ &\pi(A_1) \ldots \pi(A_n)\, \omega \\ \oplus \sum_r{}^{\oplus} \frac{1}{\sqrt{r!}} \sum_{I_r} &\pi(A_{i_1}) \ldots \pi(A_{i_{j_1}})\, \omega \otimes \cdots \otimes \pi(A_{i_{j_{r-1}+1}}) \ldots \pi(A_{i_{j_r}})\, \omega \Big\}, \end{aligned} \tag{3}$$

where I_r is any partition of $(1, \ldots, n)$ into r parts (including cases where some of the parts are empty), and parts written in different orders are counted as distinct. The nth moment of $\hat{\pi}$, $(\Omega, \hat{\pi}(A_1) \ldots \hat{\pi}(A_n)\, \Omega)$, is then the sum over r, of the scalar product of (3) with the vector $\exp(-\frac{1}{2}\|\omega\|^2) \frac{1}{\sqrt{r!}}\, \omega \otimes \cdots \otimes \omega$ (r factors). Each of the different orders in which the parts of a partition I_r can be written, contributes the same term of this scalar product. The partition consisting of $r-1$ empty sets and the set $(1, \ldots, n)$, can be written in r different orders. The partition with $r-2$ empty sets and two non-empty sets, can be written in $r(r-1)$ different orders, and so on. Thus their respective contributions are multiplied by $r, r(r-1)$, and so on. If we write $W(I)$ for $\langle \omega, \pi(A_{i_1}) \ldots \pi(A_{i_j})\, \omega\rangle$, where $I = (i_1, \ldots, i_j)$, we get

$$\begin{aligned} &\left\langle e^{-\frac{1}{2}\|\omega\|^2} \frac{1}{\sqrt{r!}}\, \omega \otimes \cdots \otimes \omega, \hat{\pi}(A_1) \ldots \hat{\pi}(A_n)\, \Omega \right\rangle \\ &= \frac{1}{r!} e^{-\|\omega\|^2} \{ r \langle \omega, \pi(A_1) \ldots \pi(A_n)\, \omega\rangle \|\omega\|^{2(r-1)} \\ &\quad + r(r-1) \sum_{I_1, I_2} W(I_1)\, W(I_2) \|\omega\|^{2(r-2)} \\ &\quad + \cdots + \\ &\quad + r! \sum_{I_1, \ldots, I_r} W(I_1) \ldots W(I_r) \} \end{aligned}$$

(if $r > n$, series breaks off when a term is zero). Hence, by summing over r

$$\langle \Omega, \hat{\pi}(A_1) \dots \hat{\pi}(A_n)\, \Omega \rangle = \langle \omega, \pi(A_1) \dots \pi(A_n)\, \omega \rangle + \sum_{I=(I_1 \dots I_j)} W(I_1) \dots W(I_j).$$

Since $\langle \Omega, \hat{\pi}(A)\, \Omega \rangle_T = \langle \Omega, \hat{\pi}(A)\, \Omega \rangle = \langle \omega, \pi(A)\, \omega \rangle$, we have by induction

$$\langle \Omega, \hat{\pi}(A_1) \dots \hat{\pi}(A_n)\, \Omega \rangle_T = \langle \omega, \pi(A_1) \dots \pi(A_n)\, \omega \rangle.$$

This proves Theorem 5.

Corollary. *Any exponential representation of a Lie algebra $\mathscr{G}$ is ∞-divisible.*

For, if $(\Omega, \hat{\pi})$ is the exponential of (ω, π), and $\omega' = \omega / \sqrt{m}$, then the moments of (ω, π) are m times those of (ω', π). Hence the cumulants of $(\Omega, \hat{\pi})$ are m times those of $(e^{-\frac{1}{2}\|\omega'\|^2} e^{\omega'}, \exp \pi)$ by the theorem, and so $(\Omega, \hat{\pi})$ is ∞-divisible by Theorem 4, (ii).

Remark. By Gårdings theorem [10] there exist cyclic representations of any Lie algebra; so the corollary ensures that the set of ∞-divisible representations is non-empty. We shall see that, in general, not all ∞-divisible representations are of the form $\exp \pi$.

3. Current Algebras

Infinitely divisible representations of Lie groups and algebras arise in the study of the canonical and current commutation relations of quantum field theory [2]. If we choose a basis $A_1 \dots A_d$ in the Lie algebra $\mathscr{G}$ (where d is its dimension) an element $A = \sum_{j=1}^{d} \alpha_j A_j$ may be given coordinates $(\alpha_1, \dots, \alpha_d) \in \mathbb{R}^d$. The Euclidean norm on $\mathbb{R}^d$ defines a topology for $\mathscr{G}$ and a Borel structure independent of the choice of basis (A_j), so that $\mathscr{G}$ becomes a topological vector space. Let $(X, \mathscr{S}, \mu)$ be a measure space, with measurable sets $\mathscr{S}$ and measure μ. We may consider the vector space of bounded measurable maps $\underline{A}: X \to \mathscr{G}$, denoted $\mathscr{M}(X, \mathscr{G})$ or $\mathscr{M}$, and furnish it with the natural Lie bracket structure:

$$[\underline{A}, \underline{B}](x) = [\underline{A}(x), \underline{B}(x)].$$

With this definition, $\mathscr{M}(X, \mathscr{G})$ becomes an infinite dimensional Lie algebra, called a current algebra. We denote by $\mathscr{M}_0$ the subalgebra of maps $\underline{A}$ which take the value 0 except on a set of finite μ-measure. We note that $\mathscr{M}$ contains a subset of maps of the form $x \to f(x)\, A$ for $A \in \mathscr{G}$, where f is a suitable function, known as a test function. An element of $\mathscr{M}$ of this form is known as a "field (or current) smeared with test function f" and is written $A(f)$ or $\int A(x) f(x)\, d\mu$.

Naturally, we identify elements of $\mathscr{M}$ or $\mathscr{M}_0$ that differ only on a set of μ-measure zero. We have in mind the special case where $X = \mathbb{R}^\nu$ and $d\mu = d^\nu x$, with $\nu = 3$ the most frequent case.

The problem of finding representations of $\mathscr{M}_0(X, \mathscr{G})$, where $(X, \mu) = (\mathbb{R}^\nu, d^\nu x)$ and $\mathscr{G} = s u_2$ or $s u_3$, is of some interest in physics [11]. The technique of continuous tensor products leads to representations of a special type, namely, those such that the cumulants are given by

$$\langle \underline{A}_1 \dots \underline{A}_n \rangle_T = \int_{\mathbb{R}^\nu} \langle \underline{A}_1(x) \dots \underline{A}_n(x) \rangle_T \, d^\nu x, \tag{4}$$

where $\langle A_1 \ldots A_n \rangle_T$ are the cumulants of some representation π of $\mathscr{G}$, so that the integrand is defined for each $x \in \mathbb{R}^\nu$. We note that the cumulants of π, being multilinear on $\mathscr{G} \times \cdots \times \mathscr{G}$ (n factors) are continuous and so measurable functions on $\mathscr{G} \times \cdots \times \mathscr{G}$. The integrand in (4) is therefore a bounded measurable function of compact support, and so the right hand side has a meaning. We shall now see that, in order for the left hand side of (4) to be, in fact, cumulants of some cyclic representation of $\mathscr{M}_0$, π cannot be chosen to be any old thing, but must be ∞-divisible. More exactly:

Theorem 6. *Let $\langle \ldots \rangle_T$ define a form on $\mathscr{P}_1(\mathscr{G})$, and define $\langle \underline{A}_1 \ldots \underline{A}_n \rangle_T$ by* (4). *Then the necessary and sufficient condition that $\langle \underline{A}_1 \ldots \underline{A}_n \rangle_T$ be the cumulants of some possible representation of $\mathscr{M}_0(\mathbb{R}^\nu, \mathscr{G})$, is that $\langle \ldots \rangle_T$ are the cumulants of some ∞-divisible representation of $\mathscr{G}$.*

Proof. (i) Suppose the moments $\langle \underline{A}_1 \ldots \underline{A}_n \rangle$ obtained from the cumulants (4) define a positive form on $\mathscr{P}(\mathscr{M}_0)$, vanishing on $\mathscr{C}(\mathscr{M}_0)$, the commutator ideal. Then in particular, these moments define a positive form on the subalgebra generated by $\underline{A} = A(f)$ as A runs over $\mathscr{G}$ and f is some fixed function. By choosing f to be a characteristic function of a set of finite measure, the sub-Lie algebra generated by $\{A(f), A \in \mathscr{G}\}$ is isomorphic to $\mathscr{G}$, so that $\langle A_1(f) \ldots A_n(f) \rangle_T = \int f^n(x)\, d^\nu x \langle A_1 \ldots A_n \rangle_T$ are the cumulants of some representation π_f of $\mathscr{G}$. If the support of f has volume 1, (4) leads to

$$\langle A_1(f) \ldots A_n(f) \rangle_T = \langle A_1 \ldots A_n \rangle_T .$$

Hence $\langle A_1 \ldots A_n \rangle_T$ are the cumulants of the representation π_f of $\mathscr{G}$. By choosing f^1 so that its support has volume $1/m$, (4) leads to

$$\langle A_1(f^1) \ldots A_n(f^1) \rangle_T = \frac{1}{m} \langle A_1 \ldots A_n \rangle_T .$$

Hence $\frac{1}{m} \langle A_1 \ldots A_n \rangle_T$ are the cumulants of π_{f^1}. Hence, by Theorem 4(ii), π_f is ∞-divisible.

(ii) Conversely, suppose π is ∞-divisible, and define a functional on $\mathscr{P}_1(\mathscr{M}_0)$ by (4).

Let us topologize $\mathscr{M}_0$ by means of a norm. We choose a basis in $\mathscr{G}$, and parametrize

$$A = \sum_{j=1}^{d} \alpha^j A_j \quad \text{by} \quad (\alpha^1, \ldots, \alpha^d) \in \mathbb{R}^d .$$

Then an element $\underline{A}$ of $\mathscr{M}_0$ becomes a bounded measurable function $\alpha\colon \mathbb{R}^\nu \to \mathbb{R}^d$, which is zero outside a set of finite measure. Define a scalar product on $\mathscr{M}_0$

$$\langle \underline{A}_1, \underline{A}_2 \rangle = \int d^\nu x \sum_j \alpha_1^j(x)\, \alpha_2^j(x) \tag{5}$$

and the corresponding norm. Although the scalar product depends on the choice of basis (A_j), the topology it defines clearly does not. With this definition, the cumulants defined by (4) are each separately continuous in variables of $\mathscr{M}_0 \times \cdots \times \mathscr{M}_0$. Let p be an arbitrary positive element of $\mathscr{P}(\mathscr{M}_0)$. Then p has a maximum degree N, and involves only a finite number of cumulants. It therefore suffices to prove positivity for the polynomials constructed out of some dense subalgebra $\mathcal{N}$ of $\mathcal{M}_0$.

Denote by $\mathcal{N}$ the subalgebra of $\mathcal{M}_0$ consisting of piecewise constant maps over a finite number of cubes K in $\mathbb{R}^\nu$ with rational numbers as boundaries:

$$K=\{x\in\mathbb{R}^\nu/r_\alpha \leqq x_\alpha \leqq s_\alpha,\ \alpha=1,\ldots,\nu;\ r_\alpha, s_\alpha \text{ rational}\}.$$

Then the completion of $\mathcal{N}$ in the norm given by (5) consists (in this coordinate system) of L_2 vector-valued functions $\mathbb{R}^\nu\to\mathbb{R}^d$, and this certainly includes $\mathcal{M}_0$. Any positive element $p\in\mathcal{P}(\mathcal{N})$ involves a finite number of elements from $\mathcal{N}$, and each element of $\mathcal{N}$ defines a finite number of rational boundaries $x_\alpha=r_\alpha$, $x_\alpha=s_\alpha$. The set of all these (r_α, s_α) coming from a given p divides the α-axis of $\mathbb{R}^\nu$ into a finite number of intervals, and doing this for all $\alpha=1,\ldots,\nu$ divides $\mathbb{R}^\nu$ up into a finite number of disjoint cubes $(K_\lambda)_{\lambda\leqq\Lambda}$ each with a rational volume V_λ, such that each of the elements of $\mathcal{N}$ involved in p is constant on each V_λ and zero outside $\bigcup_\lambda V_\lambda$. We note that the functions, $\mathbb{R}^\nu\to\mathcal{G}$, that are constant inside one V_λ, and zero outside, form a subalgebra $\mathcal{G}_\lambda$ of $\mathcal{M}_0$ isomorphic to $\mathcal{G}$. Moreover, $\mathcal{G}_\lambda$ commutes with $\mathcal{G}_{\lambda'}$, if $\lambda\neq\lambda'$. Therefore the set of maps, constant on each V_λ and zero outside $\bigcup_{\lambda=1}^{\Lambda} V_\lambda$, form a sub-Lie algebra $\mathcal{G}^\Lambda$ isomorphic to $\bigoplus_{\lambda=1}^{\lambda=\Lambda}\mathcal{G}_\lambda$. This isomorphism is given by $\underline{A}\leftrightarrow\underline{A}(x_1)\oplus\cdots\oplus\underline{A}(x_\Lambda)$ where x_λ is chosen in V_λ. In order to show that a form is positive on the given element $p\in\mathcal{P}(\mathcal{N})$, it is sufficient to show more, that it is positive on $\mathcal{P}(\mathcal{G}^\Lambda)$.

Looking at (4) for an element of $\mathcal{G}^\Lambda\times\cdots\times\mathcal{G}^\Lambda$, we get

$$\langle\underline{A}_1\ldots\underline{A}_n\rangle_T=\sum_\lambda\int_{V_\lambda}dx\langle\underline{A}_1(x_\lambda)\ldots\underline{A}_n(x_\lambda)\rangle_T=\sum_\lambda V_\lambda\langle\underline{A}_1(x_\lambda)\ldots\underline{A}_n(x_\lambda)\rangle_T$$

where $x_\lambda\in V_\lambda$ is any fixed point. Now, since $\langle\ldots\rangle_T$ on the right hand side are assumed to be possible cumulants of an ∞-divisible representation of $\mathcal{G}$, and each V_λ is rational, $V_\lambda\langle\ldots\rangle_T$ is also a possible cumulant for some representation π_λ say, of $\mathcal{G}$. Thus

$$\langle\underline{A}_1\ldots\underline{A}_n\rangle_T=\sum_\lambda\langle\omega_\lambda,\pi_\lambda(A_1^\lambda)\ldots\pi_\lambda(A_n^\lambda)\,\omega_\lambda\rangle_T \tag{6}$$

for $\underline{A}_j\in\bigoplus_\lambda\mathcal{G}_\lambda$. But the r.h.s. of (6) is the cumulant of the representation $\pi_1\otimes\cdots\otimes\pi_\Lambda$ of $\bigoplus_\lambda\mathcal{G}_\lambda$, by Theorem 3. Therefore the left hand side is the cumulant of a possible representation (the same one!) of $\bigoplus_\lambda\mathcal{G}_\lambda$, so the moments it defines are positive on $\mathcal{P}(\mathcal{G}^\Lambda)$, and in particular on p.

Since p was any element of $\mathcal{P}(\mathcal{N})$, (4) is positive on $\mathcal{P}(\mathcal{N})$ and by continuity, on $\mathcal{P}(\mathcal{M}_0)$.

This proves Theorem 6(ii), as the moments obviously vanish on the commutator ideal.

Corollary. *A representation $(\mathcal{K},\omega,\pi)$ is ∞-divisible only if its cumulants, divided by any positive number, are the cumulants of some cyclic representation.*

For, we need only replace m in the proof 6(i) by any positive number.

Remark. The representations encountered here have been called "ultra-local"; see [14].

Theorem 7. (i) *If $(\mathscr{K}, \omega, \pi)$ is ∞-divisible, then its cumulants define a positive form on $\mathscr{P}_1(\mathscr{G})$.*

(ii) *Given a positive form W_T on $\mathscr{P}_1(\mathscr{G})$ vanishing on $\mathscr{C}(\mathscr{G})$, then there exists a representation $(\mathscr{K}, \omega, \pi)$ of $\mathscr{G}$ such that the given W_T are its cumulants; moreover, $(\mathscr{K}, \omega, \pi)$ is ∞-divisible.*

Proof (i). $\frac{1}{m}\langle A_1 \dots A_k\rangle_T$ are the cumulants of a representation whose moments we shall denote by $\langle A_1 \dots A_k\rangle^{(m)}$. Then

$$\frac{1}{m}\langle A_1 \dots A_k\rangle_T + \sum_I \frac{1}{m^p}\langle A_{i_1} \dots A_{i_{l_1}}\rangle_T \dots \langle A_{i_{l_{p-1}+1}} \dots A_{i_{l_p}}\rangle_T = \langle A_1 \dots A_k\rangle^{(m)}.$$

Hence $\langle A_1 \dots A_k\rangle_T = \lim_{m\to\infty} m\langle A_1 \dots A_k\rangle^{(m)}$. The right hand side is a positive form on $\mathscr{P}_1(\mathscr{G})$.

Proof (ii). Let W_T be a positive form on $\mathscr{P}_1(\mathscr{G})$. This furnishes $\mathscr{P}_1$, regarded as a vector space, with a positive semi-definite sesqui-linear form. Let K be the Hilbert space obtained by separating and completing $\mathscr{P}_1(\mathscr{G})$ in the usual way. Let $p \to \psi(p)$ denote the canonical map from $\mathscr{P}_1$ into K.

Let now $A_1, \dots, A_n$ be any n elements of $\mathscr{G}$. We are going to define a corresponding vector $\Psi(A_1 \dots A_n) \in K$.

We use the symbol $|P_1, \dots, P_k\rangle$ to denote

$$\frac{1}{\sqrt{k!}} \sum_{\gamma \in S_k} \Psi(P_{\gamma(1)}) \otimes \cdots \otimes \Psi(P_{\gamma(k)}) \in (\otimes K^k)_S$$

where $P_1, \dots, P_k$ are elements of $\mathscr{P}_1$. If $I_1, \dots, I_k$ are k disjoint subsets of $1, 2, \dots, n$ then

$$|I_1, \dots, I_k\rangle \quad \text{denotes} \quad |(A_{i_1^1} \dots A_{i_{p_1}^1}), \dots, (A_{i_1^k} \dots A_{i_{p_k}^k})\rangle \in (\otimes K^k)_S$$

$$\text{where} \quad I_\alpha = (i_1^\alpha, \dots, i_{p_\alpha}^\alpha), \ \alpha = 1, 2, \dots, k.$$

Clearly, $|I_1, \dots, I_k\rangle$ is multi-linear in the variables $A_{i_1^1} \dots$ that enter.

Now let $k \leqq n$ be chosen, and let I be a partition of $(1, \dots, n)$ into p parts, $I_1, \dots, I_p$, where $p \geqq k$. Let $C_k = (c_1, \dots, c_k) \subset (1, \dots, p)$ be any selection of k indices from $(1, \dots, p)$. Then we define

$$\Psi_k = \sum_p \sum_{\substack{I \\ I=(I_1,\dots,I_p)}} \sum_{C_k} \prod_{\substack{\alpha \\ \alpha \notin C_k}} W_T(I_\alpha)\, |I_{c_1}, \dots, I_{c_k}\rangle \in (\otimes K^k)_S$$

and put $\Psi_k = 0$, $k > n$. Then

$$\Psi(A_1, \dots, A_n) = \bigoplus_{k=0}^{\infty} \Psi_k \in \exp K. \tag{7}$$

Again, Ψ is obviously multi-linear, and so depends only on the product $A_1 \dots A_n$ in $\mathscr{P}_1$. We define $\Psi(1)$ to be Ω, the Fock vacuum of $\exp K$; that is, a vector generating the one-dimensional space $\mathbb{C} = K^0$. Then Ψ has a unique extension by linearity to a map $\Psi: \mathscr{P} \to \exp K$.

Now define a positive semi-definite sesqui-linear form on $\mathscr{P}$ by

$$W(q', q) = \langle \Psi(q'), \Psi(q) \rangle \qquad q', q \in \mathscr{P}.$$

We now prove that $W(q', q) = W(q'^* q)$, where $W(q)$ is the moment function defined from the given W_T. Clearly, it suffices to prove this for the special case where q and q' are simple products of elements of $\mathscr{G}$. If $q' = A'_1 \ldots A'_{n'}$ and $q = A_1 \ldots A_n$, we find

$$\langle \Psi(q'), \Psi(q) \rangle = \sum_k \sum_{p, p'} \sum_{\substack{I = (I_1, \ldots, I_p) \\ I' = (I'_1, \ldots, I'_{p'})}} \sum_{C'_k, C_k} \prod_{\substack{\alpha' \notin C'_k \\ \alpha' \in (1, \ldots, p')}} \prod_{\substack{\alpha \notin C_k \\ \alpha \in (1, \ldots, p)}} W_T(I'_{\alpha'}) \, W_T(I_\alpha)$$

$$\cdot \sum_{\gamma \in S_k} W_T(I'_{c'_{\gamma(1)}} I_{c_1}) \ldots W_T(I'_{c'_{\gamma(k)}} I_{c_k}).$$

The right-hand side is a sum of products of cumulants corresponding to partitions of $A^{*\prime}_{n'}, \ldots, A^{*\prime}_1, A_1, \ldots, A_n$. Here, k is the number of parts containing both dashed and undashed indices, and runs from 1 to n; $p-k$ and $p'-k$ are the numbers of parts with respectively undashed and dashed indices only; I_p and $I'_{p'}$ are any partitions with p and p' parts; C'_k and C_k are any 2 choices of k indices; α and α' are the remaining indices; and γ is any permutation of the remaining indices. This is a precise enumeration of the possible partitions of $A^{*\prime}_{n'}, \ldots, A'^*_1, A_1, \ldots, A_n$. Therefore,

$$\langle \Psi(q'), \Psi(q) \rangle = W(q'^* q).$$

It follows that the moments are positive semi-definite on $\mathscr{P}$.

If now the cumulants vanish on $\mathscr{C}$, then so do the moments (Theorem 2). Hence, by Theorem 1, W defines a cyclic representation π of $\mathscr{G}$; the given functions W_T are its cumulants. π is ∞-divisible by theorem (4); for, $\frac{1}{m} W_T$ is positive on $\mathscr{P}_1$, and so is the cumulant function of some cyclic representation, by what has just been proved.

This proves Theorem 7.

Remark. One can give the following explicit heuristic expression for π: let $A \in \mathscr{G}$; then

$$\begin{aligned} \pi(A) |P_1, \ldots, P_n\rangle &= \sum_j W_T(AP_j) |P_1, \ldots, \hat{P}_j, \ldots, P_n\rangle \\ &\quad + W_T(A) |P_1, \ldots, P_n\rangle + \sum_j |P_1, \ldots, (AP_j), \ldots, P_n\rangle + |A_1 P_1, \ldots, P_n\rangle \end{aligned} \tag{8}$$

and

$$\pi(A)\,\Omega = W_T(A)\,\Omega + |A\rangle.$$

Indeed, applying (8) successively to Ω, one obtains (by induction)

$$\pi(A_1) \ldots \pi(A_n)\,\Omega = \Psi(A_1 \ldots A_n)$$

confirming that W are the moments of (Ω, π). The formula (8) for π is implicit in [9], and with a change of notation may be found in [5].

4. Coboundaries and Cocycles

Most of this section is the differential form of results in [5].

Let $\mathscr{E}$ be an associative algebra, D a linear space, and π a representation of $\mathscr{E}$ on D. Then $C^p(\mathscr{E}, D, \pi)$ is the linear space of *p-cochains* ψ; that is, ψ is a multilinear map $\mathscr{E}\times\mathscr{E}\times\cdots\times\mathscr{E}\to D$. The co-boundary operator is a map $\delta: C^p\to C^{p+1}$ given by

$$(\delta\psi)(X_1,\ldots,X_{p+1})=\pi(X_1)\psi(X_2,\ldots,X_{p+1})+\sum_1^p(-1)^j\psi(X_1,\ldots,X_jX_{j+1},\ldots,X_{p+1}).$$

This is multilinear, and satisfies $\delta^2=0$. The cohomology theory defined in this way is a special case of the usual one [12], in that the right action is taken to be trivial.

Defining as usual the p-cocycle group $\mathscr{Z}^p$ as the kernel of $\delta: C^p\to C^{p+1}$, and the p-coboundary group $\mathscr{B}^p$ as the image of $\delta: C^{p-1}\to C^p$, we see that a 1-cocycle is a map $\psi: \mathscr{E}\to D$ that satisfies $\psi(XY)=\pi(X)\psi(Y)$, and a 1 co-boundary is a map $\psi:\mathscr{E}\to D$ such that $\psi(X)=\pi(X)\Omega$ for some $\Omega\in D$.

If $D=\mathbb{C}$ and $\pi=0$, a 2 co-boundary is a bilinear form $\psi(X, Y)$ of the form $\psi(X,Y)=-\varphi(XY)$ for some linear form φ on $\mathscr{E}$. A 2 co-boundary ψ in $\mathscr{B}^2(\mathscr{E},\mathbb{C},0)$ is said to be *positive* if $\psi(X^*,X)\geqq 0$ for all $X\in\mathscr{E}$.

If $\mathscr{E}=\mathscr{E}_1(\mathscr{G})$, the enveloping algebra of a Lie algebra without identity, and D is a dense invariant common domain of π in a Hilbert space $\mathscr{K}$, then we may restate Theorem 7 as follows:

There is a 1:1 correspondence between ∞-divisible representations (ω,π) of $\mathscr{G}$, and positive elements of $\mathscr{B}^2(\mathscr{E}_1(\mathscr{G}),\mathbb{C},0)$. This is the analogue of the lemma, [2], p. 254, and Theorem 5.1 of [5].

Let $\mathscr{E}_1'$ be the space of forms on $\mathscr{E}_1$, and $\mathscr{E}_1'^{\,+}$ the subset of positive forms. Let us say $W_T\in\mathscr{E}_1'^{\,+}$ is *pure* if it is not the sum $W_T=W_T'+W_T''$, with W_T' and $W_T''\in\mathscr{E}_1'^{\,+}$, with neither proportional to W_T. The decomposition of an impure form gives rise to the factorization of an ∞-divisible representation into two others, neither of which is a fractional power of the given one. It also corresponds to the decomposition used by Johansen [13].

There is a relationship between positive elements of $\mathscr{B}^2(\mathscr{E}_1,\mathbb{C},0)$, and the elements of $\mathscr{Z}^1(\mathscr{E}_1,\mathscr{K},\pi')$ for some representation π'. (This is the analogue of [5], or [2], Theorem 11.)

Let W_T on $\mathscr{E}_1$ be positive. Then, as in the proof of Theorem 7, (ii) we construct a Hilbert space K with invariant domain D, and a representation π' of $\mathscr{E}_1$ in K, and a map $\psi:\mathscr{E}_1\to K$. K is the separated, completed space $\mathscr{E}_1$ furnished with the scalar product $\langle X,Y\rangle=W_T(X^*Y)$ and ψ is the canonical embedding. The representation π' is defined by $\pi'(X)\psi(Y)=\psi(XY)$. That is, ψ is a 1-cocycle relative to π'. However, in general, not all 1-cocycles of a given representation π' can occur from this construction. For, if ψ is a 1-cocycle of π' arising from some W_T, then $W_T(X^*Y)=\langle\psi(X),\psi(Y)\rangle$, that is, $\langle\psi(X),\psi(Y)\rangle$ is a 2 boundary. This means, it is a functional depending only on X^*Y and not X and Y separately; and moreover, it can be extended to elements of degree 1 (i.e. elements of $\mathscr{G}$) in a consistent manner, so that it vanishes on the commutator ideal of $\mathscr{P}(\mathscr{G})$. We have the following partial result.

Let $\mathscr{P}_2(\mathscr{G})=\mathscr{P}_2$ be the subalgebra of $\mathscr{P}(\mathscr{G})$ generated by monomials of degree $\geqq 2$.

Lemma. *If* $\psi\in\mathscr{Z}^1(\mathscr{P}_1(\mathscr{G}),\mathscr{K},\pi)$ *then* $W_T: X, Y\to\langle\psi(X),\psi(Y)\rangle$ *lies in* $\mathscr{B}^2(\mathscr{P}_2,\mathbb{C},0)$. *If* ψ *vanishes on* $\mathscr{C}$, *then* Re $W_T\in\mathscr{B}_2(\mathscr{E}_1(\mathscr{G}),\mathbb{R},0)$.

Proof. For any monomial $X=A_1\dots A_n$, $A_j\in\mathscr{G}$, we may define

$$W_T(A_1\dots A_n)=\langle\psi(A_j^*\dots A_1^*),\psi(A_{j+1}\dots A_n)\rangle.$$

This is independent of the choice of j, since

$$\begin{aligned}W_T(A_1\dots A_n)&=\langle\psi(A_j^*\dots A_1^*),\pi(A_{j+1})\,\psi(A_{j+2}\dots A_n)\rangle\\&=\langle\pi(A_{j+1}^*)\,\psi(A_j^*\dots A_1^*),\psi(A_{j+2}\dots A_n)\rangle\\&=\langle\psi(A_{j+1}^*\dots A_1^*),\psi(A_{j+2}\dots A_n)\rangle.\end{aligned}$$

W_T can be extended to the whole of $\mathscr{P}_2$, so it lies in $\mathscr{B}^2(\mathscr{P}_2,\mathbb{C},0)$.

If ψ vanishes on $\mathscr{C}$, then Re W_T vanishes on $\mathscr{C}_3$ (elements of $\mathscr{C}$ of degree 3 at least). For elements of degree 2, $C=AB-BA\in\mathscr{C}$, we see

$$\operatorname{Re}W_T(AB)-\operatorname{Re}W_T(BA)=\operatorname{Re}\langle\psi(A^*),\psi(B)\rangle-\operatorname{Re}\langle\psi(B^*),\psi(A)\rangle.$$

Since $A^*=-A$, $B^*=-B$ for $A,B\in\mathscr{G}$, we get $\operatorname{Re}W_T(A,B)=0$. Hence Re W_T vanishes on $\mathscr{C}$, and so defines a functional on $\mathscr{P}_2(\mathscr{G})$, which vanishes on $[\mathscr{G},\mathscr{G}]$. We can extend it to $\mathscr{P}_1(\mathscr{G})$ by requiring that on $\mathscr{G}$ it is some character χ of $\mathscr{G}$ (a character of $\mathscr{G}$ is a linear functional vanishing on $[\mathscr{G},\mathscr{G}]$). This means that Re $W_T\in\mathscr{B}^2(\mathscr{E}_1,\mathbb{R})$. This proves the lemma.

We may therefore again rephrase Theorem 7. There is a 1:1 correspondence between ∞-divisible representations $(\mathscr{K},\omega,\pi)$ of $\mathscr{G}$, and triplets (π',ψ,χ), where π' is a representation of $\mathscr{G}$ in some prehilbert space K, $\psi\in\mathscr{Z}^1(\mathscr{E}_1,K,\pi')$ is some cocycle of π' with the property that

$$\operatorname{Im}\langle\psi(X),\psi(Y)\rangle\in\mathscr{B}^2(\mathscr{E}_1,\mathbb{R},0)\tag{9}$$

and χ is a real character of $\mathscr{G}$.

In this correspondence

$$\begin{aligned}&W_T(X^*Y)=\langle\psi(X),\psi(Y)\rangle\qquad X,Y\in\mathscr{P}_1,\\&W_T(A)=\langle\psi(A_1^*),\psi(A_2)\rangle-\langle\psi(A_2^*),\psi(A_1)\rangle\quad\text{if}\quad A=[A_1,A_2]\in\mathscr{G},\\&W_T(A)=\chi(A)\quad\text{if}\quad A\notin[\mathscr{G},\mathscr{G}]\end{aligned}$$

(cf. [5], Theorem 5.1).

It is possible to show that if (9) fails to hold, then the triple (π',ψ,χ) corresponds to a projective rather than a true representation of $\mathscr{G}$; and that this representation say $(\mathscr{K},\omega,\pi)$ is ∞-divisible in a suitable sense. This is the differential analogue of recent theorems of Parthasarathy and Schmidt [15].

It is quite possible for π' to be zero. This happens when all the cumulants beyond the second vanish, as for the Gaussian. The cocycles of $\pi'=0$ are the positive semi-definite bilinear forms in $\mathscr{G}$.

The following is somewhat analogous to Lemma 7.2 of [5].

Let $\mathscr{E}$ be an associative *-algebra. Let $\mathscr{E}'^{+}$ be the set of positive forms on $\mathscr{E}$. Let $D(\mathscr{E}'^{+})$ be the linear space generated by $\mathscr{E}'^{+}$, and $X\to\pi'(X)$ the following representation of $\mathscr{E}$ in D: let $\Omega\in D$; then define for each $X\in\mathscr{E}$ the following form on $\mathscr{E}$, denoted $\pi'(X)\,\Omega$:

$$\big(\pi'(X)\,\Omega\big)(Y)=\Omega(X^*YX).$$

If $\Omega\geqq 0$ then $\pi'(X)\,\Omega\geqq 0$; therefore π' maps D into D.

Suppose now K is a prehilbert space carrying a representation π'' of $\mathscr{E}$, and $\psi \in \mathscr{Z}^1(\mathscr{E}, K, \pi'')$ is a cocycle satisfying the extra condition (9). If $\mathscr{E}$ contains the identity then $\psi(X) = \pi(X)\psi(1)$, and so the cocycle is a coboundary. We are interested in the case where $\mathscr{E} = \mathscr{E}_1(\mathscr{G})$, without identity. Condition (9) together with the lemma ensures that there exists $\Omega \in \mathscr{E}'^+$ such that $\langle \psi(X), \psi(Y) \rangle = \Omega(X^* Y)$. We may embed K in $\mathscr{E}'^+$ (assuming K consists of $\psi(\mathscr{E})$) by the map $i: K \to \mathscr{E}'^+$ given by:

$$i(\psi(X))(A) = \langle \psi(X), \pi''(A)\psi(X) \rangle = \Omega(X^* A X) = (\pi'(X))\,\Omega(A)$$

by definition.

Clearly, $\pi'|_K = \pi''$, by the cocycle condition on ψ and π''. Therefore $\psi(X)$ may be identified with $\pi'(X)\,\Omega$, that is, ψ is a coboundary in the cohomology theory $(\mathscr{E}, D, \pi')$.

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