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# Risk-Aversion Concepts in Expected- and Non-Expected-Utility Models

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## Abstract

The non-expected-utility theories of decision under risk have favored the appearance of new notions of increasing risk like monotone increasing risk (based on the notion of comonotonic random variables) or new notions of risk aversion like aversion to monotone increasing risk, in better agreement with these new theories. After a survey of all the possible notions of increasing risk and of risk aversion and their intrinsic definitions, we show that contrary to expected-utility theory where all the notions of risk aversion have the same characterization ($u$ concave), in the framework of rank-dependent expected utility (one of the most well known of the non-expected-utility models), the characterizations of all these notions of risk aversion are different. Moreover, we show that, even in the expected-utility framework, the new notion of monotone increasing risk can give better answers to some problems of comparative statics such as in portfolio choice or in partial insurance. This new notion also can suggest more intuitive approaches to inequalities measurement.

**Key words:** increasing risk, risk aversion, non-expected utility

For most economists, risk aversion is exactly captured by the concavity of the utility function, and indeed, in the framework of expected-utility (EU) theory, which is their implicit model, risk averters—when intrinsically defined as those people who always prefer the expectation $E(X)$ to the random variable $X$—are characterizable by concave utilities.

For some authors, such as Allais [1952], diminishing marginal utility of wealth under certainty being meaningful is the natural interpretation of concavity of utility. Thus, for them, two independent psychological traits are necessarily and abusively linked in EU theory and a more flexible model is needed. During the last decade, several authors (Machina [1982a], Quiggin [1982], Yaari [1987], Segal [1989], and Allais [1987] himself), motivated by the poor quality of EU theory as a descriptive model,[1] have proposed various models, more general than EU theory. In these models, as we shall see, risk aversion no longer necessarily goes along with a concave utility function, unless, perhaps, the very definition of risk aversion is reconsidered.

The consideration of several models makes it necessary to look for *intrinsic*—that is, model-free—definitions of risk aversion.

In the literature on decision under risk, in addition to the already mentioned concept of *weak* risk aversion ($E(X)$ preferred to $X$), there is another classical concept—*strong* risk aversion: a decision maker is strongly risk averse if he prefers the random variable $X$ to any random variable $Y$ that is a "mean preserving spread" of $X$ (Hadar and Russell [1969], Rothschild and Stiglitz [1970]).

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In EU theory, these two notions coincide and are both characterized by the concavity of the utility function. For this reason, these two notions have often been considered as identical, whereas this identity is valid only in the framework of EU theory.

Corresponding to the emergence of new models of decision making under risk, new concepts of increasing risk and of risk aversion have appeared. In particular, Quiggin [1992] has introduced the notion of *monotone increasing risk* and the corresponding notion of *monotone risk aversion*. These two definitions involve comonotonic random variables[2] and are thus particularly fitted to Quiggin's rank-dependent expected-utility (RDEU) theory [1982], one of most well-known generalizations of EU theory, in which comonotonicity plays a fundamental part at the axiomatic level.

This by no means implies that these new notions do not have any interesting properties in the EU framework. On the contrary, as we shall see, they allow better answers to some problems of comparative statics.

The paper is organized as follows. In Section 1, I give the intrinsic definitions of the different notions of risk aversion found in the literature and their main properties. In Sections 2 and 3, I give the characterization of these different definitions in the framework of the EU model (Section 2) and in the framework of the rank-dependent expected-utility (RDEU) model (Section 3). I show, in Section 4, that the new notions of monotone increasing risk and monotone risk aversion will allow (1) better answers to some problems of comparative statics such as in portfolio theory, or in partial insurance, even in the framework of the EU model (2) a more intuitive approach to the measurement of inequalities. Some concluding remarks are provided in the last section.

## 1. Intrinsic definitions of risk aversion

### 1.1. Notations

Let $\mathcal{V} = \{X, Y, \ldots\}$ be the set of random variables from a set $\Omega = \{\omega\}$ of states of nature to a set $\mathcal{C}$ of outcomes. Here we assume that risk prevails and describe it through $\Omega = [0, 1]$ endowed with the uniform probability measure and that $\mathcal{C} = [-M, M] \subset \mathbb{R}$.[3] We denote by $\mathcal{V}^*$ the set $\{Y \in \mathcal{V} / \omega_1 \geq \omega_2 \Rightarrow Y(\omega_1) \geq Y(\omega_2)\}$. A decision maker has a preference relation $\succsim$ on $\mathcal{V}$ (the corresponding relations $\succ$ and $\sim$ are defined as usual).

Since any outcome $c$ of $\mathcal{C}$ can be identified with the degenerate random variable $I_c(\omega) = c$ for any $\omega$ in $\Omega$, preferences on $\mathcal{V}$ induces preferences on $\mathcal{C}$, which is also denoted by $\succsim$. For any $X$ in $\mathcal{V}$, we denote by $F_X$ (respectively $G_X$) the cumulative (resp. decumulative) distribution function of $X$ ($G_X = 1 - F_X$) and by $E(X)$, the expected value of $X$. Let $\mathcal{L}$ be the set of cumulative probability distribution functions on $\mathcal{C}$.

The first axiom required by most models under risk, though often implicitly, is the following:

**$A_0$:** *All random variables generating the same probability distribution over $\mathcal{C}$ are indifferent.*

Because of this assumption, throughout this paper, we can use the same symbol $\succsim$ to denote the preference relations on $\mathcal{V}$ and $\mathcal{L}$.

The two sets $\mathcal{V}$ and $\mathcal{L}$ are *mixtures spaces* in the following sense: for any $\alpha$ of [0, 1], any $X$ and $Y$ of $\mathcal{V}$, the convex combination $\alpha X + (1 - \alpha) Y$ (mixing the outcomes) exists and belongs to $\mathcal{V}$, the convex combination $\alpha F_X + (1 - \alpha) F_Y$ (mixing the probabilities) exists and belongs to $\mathcal{L}$; however, mixtures on $\mathcal{V}$ and mixtures on $\mathcal{L}$ are completely different operations[4] as we can see in the following very simple example:

*Example 1.* Let $x$ and $y$ be two elements of $\mathcal{C}$, and define the degenerate random variables: $X = I_X$, $Y = I_y$. (1) The random variable $Z = 1/2\ X + 1/2\ Y$ takes the value $(x + y)/2$ with probability 1: $Z = I_{(x+y)/2}$. (2) The random variable $T$ having the probability distribution $1/2\ F_X + 1/2\ F_Y$ takes each value $x$ and $y$ with probability 1/2. The probability distributions generated by $Z$ and $T$ are obviously different. This remark will be important in Section 4, concerning comparative statics in EU theory.

Let us finally recall some classical definitions: (1) The *certainty equivalent* $c_X$ of a random variable $X$ is an element[5] of $\mathcal{C}$ such that $X \sim I_{c_X}$. (2) The *risk premium* $\pi_X$ of a random variable $X$ is defined by $\pi_X = E(X) - c_X$.

### 1.2. *Definition of weak risk aversion*

This first notion of weak risk aversion is based on the comparison between a random variable and its expected value.

**Definition 1:** (1) *A decision maker is weakly risk averse if he always prefers the expected value of any random variable with certainty to the random variable itself: for any $X$ of $\mathcal{V}$, $I_{E(X)} \succsim X$.* (2) *A decision maker is weakly risk seeking if he always prefers any random variable to its expected value with certainty: for any $X$ of $\mathcal{V}$, $X \succsim I_{E(X)}$.* (3) *A decision maker is weakly risk neutral if he is always indifferent between any random variable and its expected value with certainty: for any $X$ of $\mathcal{V}$, $X \sim I_{E(X)}$.*

*Remark.* There may, of course, exist decision makers who prefer $X$ to $I_{E(X)}$ for some elements of $\mathcal{V}$ and prefer $I_{E(Y)}$ to $Y$ for other elements of $\mathcal{V}$ and therefore do not belong to any of these three categories.

With these definitions, we get a first obvious result:

**Proposition 1:** *A decision maker is weakly risk averse if and only if the risk premium associated to any $X$ of $\mathcal{V}$ is always nonnegative.*

*Remark.* According to this result, the definition of weak risk aversion is the most significant for some agents like an insurance company.

With this result, we can also introduce a relation between decision makers:

**Definition 2.** *The decision maker $D_1$ is more risk averse than the decision maker $D_2$ if and only if for every $X$ of $\mathcal{V}$, the risk premium associated to $X$ is at least as great for $D_1$ than for $D_2$.*

*1.3. Definition of strong risk aversion*

This second notion of risk aversion is based on the definition of increasing risk (see Hadar and Russell [1969], Rothschild and Stiglitz [1970]). Let us recall the notion of a mean preserving spread.

**Definition 3:** *For two random variables X and Y, Y is a mean preserving spread of X if and only if* (1) $E(X) = E(Y)$ *and* (2) *X stochastically dominates Y in the second order, that is: For any T of* $[-M, M]$,

$$\int_{-M}^{T} F_X(t)\, dt \leq \int_{-M}^{T} F_Y(t)\, dt$$

*Remark.* When $\mathcal{C} = [-M, M]$, condition (1) is equivalent to

$$\int_{-M}^{M} F_X(t)\, dt = \int_{-M}^{M} F_Y(t)\, dt.$$

**Proposition 2:** *The relation mean preserving spread (MPS) has the following properties:* (1) *the relation MPS depends only on the probability distributions of the two random variables;* (2) *the relation MPS is only a partial order;* (3) *the relation MPS implies a nondecreasing variance but a nondecreasing variance does not imply a MPS.*[6]

We will need the definition of a simple mean preserving spread:

**Definition 4:** *Two cumulative distributions functions $F_X$ and $F_Y$ satisfy the simple crossing property if and only if*

$$\exists\, x^* \text{ of } \mathcal{C} \text{ s.t. } x \leq x^* \Leftrightarrow F_X(x) \leq F_Y(x),$$

*Y is a simple mean preserving spread of X if and only if* $E(X) = E(Y)$ *and their cumulative distributions functions $F_X$ and $F_Y$ satisfy the simple crossing property.*

A simple mean preserving spread is easily proved to be a mean preserving spread.

**Proposition 3** (Rothschild and Stiglitz [1979]: *For any two random variables X and Y, such that* $E(X) = E(Y)$, *the following statements are equivalent:* (1) *X stochastically dominates Y to the second order*; (2) *for any concave function u from* $\mathcal{C}$ *to* $\mathbb{R}$,

$$\int_{-M}^{M} u(x)\, dF_X(x) \geq \int_{-M}^{M} u(x)\, dF_Y(x),$$

*and* (3) *there exists a random variable θ such that Y has the same distribution as* $X + \theta$ *and* $E(\theta/X) = 0$.

*Remark.* For the moment, condition (2) is only a technical and intrinsic condition without any reference to a model.

The definition of strong risk aversion is based on this notion of increasing risk:

**Definition 5:** (1) *A decision maker is strongly risk averse, if and only if, for any X and Y of* $\mathcal{V}$ *such that Y is a mean preserving spread of X, he always prefers X to Y* ($X \succsim Y$); (2) *a decision maker is strongly risk seeking, if and only if, for any X and Y of* $\mathcal{V}$ *such that Y is a mean preserving spread of X, he always prefers Y to X* ($Y \succsim X$); (3) *a decision maker is strongly risk neutral, if and only if, for any X and Y of* $\mathcal{V}$ *such that Y is a mean preserving spread of X, he is always indifferent between Y and X.*

*Remarks.* (1) Also here, some decision makers may not belong to any of these three categories. (2) Whereas weak risk aversion can be viewed as aversion to risk, strong risk aversion can be viewed as aversion to any increase in risk.

**Proposition 4:** (1) *Strong risk aversion implies weak risk aversion:* (2) *strong risk seeking implies weak risk seeking*; (3) *weak risk neutrality and strong risk neutrality are identical.*

Conditions (1) and (2) are true because, for any $X$ in $\mathcal{V}$, $X$ is always a mean preserving spread of $I_{E(X)}$. Condition (3) is true because weak risk neutrality implies that any $X$ is indifferent to $I_{E(X)}$ so that if $Y$ is a mean preserving spread of $X$, they are both indifferent to $I_{E(X)}$.

### *1.4. Definition of monotone risk aversion*

The notion of strong risk aversion can be considered as too strong by some decision makers. Let us look at the following example (from Landsberger and Meilijson, quoted in Quiggin [1991] of two distributions where one is a mean preserving spread of the other.

*Example 2.*

| | 1/1000 | 1/1000 | 498/1000 | 498/1000 | 1/1000 | 1/1000 |
|---|---|---|---|---|---|---|
| $X$ | $-2.10^6$ | 0 | $-10^3$ | $10^3$ | 0 | $2.10^6$ |
| $Y$ | $-2.10^6$ | $-2.10^6$ | 0 | 0 | $2.10^6$ | $2.10^6$ |

The random variables $X$ and $Y$ take the values indicated respectively in the second and the third lines with probabilities indicated in the first line.

It is clear that $Y$ is a mean preserving spread of $X$. Yet changing from $X$ to $Y$, more than 95 percent of the states move toward the mean. Having to choose between $X$ and $Y$, it seems that many decision makers although being weakly risk averse would choose $Y$. The definition of increasing risk by mean preserving spread includes perhaps too many random variables. This is one of the justifications given by Quiggin [1992] to define a new notion of increasing risk and as a consequence a new notion of risk aversion.

Let us first recall the definition of comonotonic functions (defined in Schmeidler [1989] and Yaari [1987]).

**Definition 6:** *Let f and g be two functions from* $\Omega$ *to* $\mathbb{R}$; *f and g are comonotonic*[7] *if and only if for any* $\omega$, $\omega'$ *of* $\Omega$, $(f(\omega) - f(\omega'))(g(\omega) - g(\omega')) \geq 0$.

Random variables of $\mathcal{V}^*$ are thus comonotonic. Let us just note that any constant random variable is comonotonic with every random variable.

Let us now give another definition of a mean preserving increasing risk: a mean preserving monotone spread.

The first concept has been found by Quiggin [1991] for comonotonic random variables of $\mathcal{V}^*$:

**Definition 7:** *For two random variables X and Y of* $\mathcal{V}^*$, *Y is a mean preserving monotone spread of X if* $E(X) = E(Y)$ *and* $Z = Y - X$ *belongs to* $\mathcal{V}^*$.

With this definition,[8] we are ensured that, since $X$ and $Z$ are comonotonic, there is no hedging whatsoever between $X$ and $Z$ and thus that $Y$ is "really" more risky than $X$.

The following Proposition 5 is used by Quiggin [1992] as the definition of mean preserving monotone spread because we find Definition 7 more intuitive, we take it as the definition while it is a characteristic property in Quiggin's paper.

**Proposition 5** (Quiggin [1992]): *Y is a mean preserving monotone spread of X, if and only if there is a smooth path* $\varphi : [0, 1] \rightarrow \mathcal{V}^*$ *such that* (1) $\varphi(\lambda) = (1 - \lambda)X + \lambda Y$; (2) $d(E(\varphi(\lambda)))/d\lambda = 0$ *for any* $\lambda$ *of* $[0, 1]$; (3) $\omega_1 \geq \omega_2 \Leftrightarrow d\varphi(\lambda)(\omega_1)/d\lambda \geq d\varphi(\lambda)(\omega_2)/d\lambda$ $\forall\ \omega_1, \omega_2 \in \Omega, \lambda \in [0, 1]$.

In fact, one can show that this concept could be extended to the distribution functions of two random variables of $\mathcal{V}$ in the following way:

**Definition 7':** *For two random variables X and Y of* $\mathcal{V}$, *Y is a mean preserving monotone spread of X if there exists a random variable* $\theta$ *such that Y has the same probability distribution as* $X + \theta$, *where* $E(\theta) = 0$ *and X and* $\theta$ *are comonotonic random variables.*

**Proposition 6** (Quiggin [1992]): *The mean preserving monotone spread (MS) relation possesses the following properties:* (1) *the MS relation is transitive;* (2) *any X of* $\mathcal{V}$ *is a MS of* $I_{E(X)}$; (3) *if Y is a MS of X, then Y is a simple mean preserving spread of X, and thus a mean preserving spread of X and the class of mean preserving monotone spreads is strictly smaller than the class of simple mean preserving spread*; (4) *when X and Y are comonotonic, and Y is a MS of X, then Y is a concave transform of X.*

*Remark.* Concerning the Property (3) of this proposition, one must understand that if there exists a random variable $\theta$ such that $Y$ has the same probability distribution as $X + \theta$, where $E(\theta) = 0$ and $X$ and $\theta$ are comonotonic random variables, there exists also *another* random variable $\theta'$ and such that $Y$ has the same distribution as $X + \theta'$ and $E(\theta'/X) = 0$ ($X$ and $\theta$ being comonotonic, $E(\theta/X) \neq 0$ except if $X$ is constant, and thus $\theta' \neq \theta$).

*Particular cases of mean preserving monotone spread.* (1) A mean preserving multiplicative spread $Y$ of $X$ is a mean preserving monotone spread of $X$, ($Y = X(1 + \epsilon)$ where $E(\epsilon) = 0$, and $X$ and $\epsilon$ are independent). (2) Let $X$ be a random variable whose probability distribution has been obtained by the truncature of the tail of distribution function[9] of a random variable $Y$: then $Y$ is a mean preserving monotone spread of $X$.

In a very interesting paper, Lansberger and Meilijson [1994] gave a characterization of the Bickel-Lehmann dispersion, which happens to be closely linked to Quiggin's notion of monotone spread (which they did not know).

Let us first give the definition of the Bickel-Lehmann dispersion in the sense of Bickel and Lehmann [1979]:

**Definition 8:** *For any two cumulative distributions $F$ and $G$ of $\mathcal{L}$, $F$ is Bickel-Lehmann less dispersed than $G$, if for every $0 < y < x < 1$, $F^{-1}(x) - F^{-1}(y) \leq G^{-1}(x) - G^{-1}(y)$.*

Their result is the following:

**Proposition 7** (Lansberger and Meilijson [1994]): *A distribution $F$ is Bickel-Lehmann less dispersed than a distribution $G$ if and only if there exist, on some probability space, two comonotonic random variables $X$ and $Z$ such that the distribution of $X$ is $F$ and the distribution of $X + Z$ is $G$.*

We get the following consequence (see Chateauneuf, Cohen, and Kast [1994]): $Y$ is a mean preserving monotone spread of $X$ if and only if $E(X) = E(Y)$ and $F_X$ is less dispersed than a distribution $F_Y$ in the sense of Bickel-Lehmann.

The definition of monotone risk aversion is based on the previous definition of mean preserving monotone spread:

**Definition 9:** (1) *A decision maker is monotone risk averse when for any $X$, $Y$ of $\mathcal{V}$, such that $Y$ is a mean preserving monotone spread of $X$, he always prefers $X$ to $Y$*; (2) *A decision maker is monotone risk seeking when for any $X$, $Y$ of $\mathcal{V}$, such that $Y$ is a mean preserving monotone spread of $X$, he always prefers $Y$ to $X$*; (3) *A decision maker is monotone risk neutral when for any $X$, $Y$ of $\mathcal{V}$, such that $Y$ is a mean preserving monotone spread of $X$, he is always indifferent between $Y$ and $X$.*

The notion of monotone risk aversion can be viewed as *aversion to monotone increasing risk.*

*Remarks.* (1) A decision maker may not belong to any of these three categories. (2) We will see, in Section 4, that this definition merges naturally with the rank-dependent expected utility theory (Quiggin [1982]).

**Proposition 8:** (1) *Strong risk aversion implies monotone risk aversion*; (2) *monotone risk aversion implies weak risk aversion*; (3) *weak risk neutrality, monotone risk neutrality, and strong risk neutrality are identical.*

Condition (1) is true because of Proposition 6, (3); Condition (2) is true because of Proposition 6, (2); Condition (3) is obvious.

*1.5. Definition of probabilistic risk aversion*

With the aim of characterizing risk aversion independently of marginal utility, Wakker [1994] defines *probabilistic risk aversion*, but we can already find this notion under *quasi-convexity in the probabilities* (see, for example, Machina [1982]). This purpose is obviously irrelevant to the EU model, where the two notions are captured by the same function. This notion is, thus, interesting only in a non-expected-utility model.

**Definition 10:** (1) *A decision maker is averse to probabilistic risk if and only if for $F_X$, $F_Y$ of $\mathcal{L}$,*

$$F_X \succsim F_Y \text{ implies } F_X \succsim \alpha F_X + (1 - \alpha) F_Y, \text{ for all } 0 < \alpha < 1;$$

(2) *A decision maker is prone to probabilistic risk if and only if*

$$F_X \succsim F_Y \text{ implies } \alpha F_X + (1 - \alpha) F_Y \succsim F_Y, \text{ for all } 0 < \alpha < 1;$$

(3) *A decision maker is neutral to probabilistic risk if and only if*

$$F_X \succsim F_Y \text{ implies } F_X \succsim \alpha F_X + (1 - \alpha) F_Y \succsim F_Y, \text{ for all } 0 < \alpha < 1$$

Part (1) of the definition means that the relation is quasi-convex with respect with probability mixtures. Part (3) is also called betweenness.

*Remark.* Let us mention that Safra and Zilcha [1988] have defined *$\alpha$-risk aversion* for any $\alpha$ in [0, 1]: For any $X$ of $\mathcal{V}$, they define a random variable $X_\alpha$ whose cumulative distribution function is defined by

$$F_\alpha(x) = F_X\left(\frac{x - \alpha E(X)}{1 - \alpha}\right) \text{ if } \alpha < 1 \text{ and } F_1(x) = F_{I_{E(X)}}.$$

A decision maker is *$\alpha$-risk averse* iff, for any $\beta$ of $[\alpha, 1[$, he always prefers $X_\beta$ to $X$ or equivalently iff for $\alpha \leq \beta \leq 1$, $(1 - \beta) X + \beta E(X) \succsim X$. 1-risk aversion is weak risk aversion, but for $\alpha \neq 1$, two different values of $\alpha$ give different concepts of risk aversion.

## 2. Results in the framework of the expected-utility model

Under expected-utility theory, a decision maker is characterized by his utility function $u$. Let us call him a *EU-decision maker. Throughout this section, we suppose that the decision maker has a preference relation $\succsim$ satisfying EU theory and characterized by his utility function u.*

**Proposition 9:** (1) *A EU-decision maker is weakly risk averse if and only if u is concave*; (2) *A EU-decision maker is strongly risk averse if and only if u is concave.*

These two results are well known. The proof of (1) is straightforward; the proof of (2) can be found in Rothschild and Stiglitz [1970].

**Corollary 10:** *In EU theory, the three notions of weak risk aversion, monotone risk aversion, and strong risk aversion are equivalent.*

This is an obvious consequence of Propositions 8 and 9. We can now speak of a risk-averse EU-decision maker without any need to specify (weak, monotone, or strong).

Let us now recall the Arrow-Pratt local measure of risk aversion:

**Definition 11:** *The absolute coefficient $R(x)$ of risk aversion of a EU decision maker is defined by for any $x$ of $\mathcal{C}$, $R(x) = (-u''(x))/(u'(x))$.*

We can now give the characterization of the relation more risk averse specific to the EU framework:

**Proposition 11:** *For two EU-decision makers $D_1$ and $D_2$ characterized respectively by (twice differentiable) utility functions $u_1$ and $u_2$, the following propositions are equivalent: (1) $D_1$ is more risk averse than $D_2$ (definition 2); (2) The absolute coefficient of risk aversion of $D_1$ is everywhere greater than the one of $D_2$: for any $x$ of $\mathcal{C}$, $-u_1''(x)/u_1'(x) \geq -u_2''(x)/u_2'(x)$; (3) $u_1$ is a concave transform of $u_2$.*

Let us now give an interpretation of Proposition 3 (characterization of mean preserving spread).

**Corollary 12** (Hadar and Russell [1969]): *For $X$ and $Y$ of $\mathcal{V}$ such that $E(X) = E(Y)$, $Y$ is a mean preserving spread of $X$ if and only if all the risk-averse EU-decision makers prefer $X$ to $Y$.*

*Remark.* With the weak concept of risk aversion, this result no more holds if we omit *EU* in the statement of this corollary (see Proposition 16).

Finally, the characterization of probabilistic risk aversion is obvious (since the independence axiom of EU implies betweenness) but also completely irrelevant.

**Proposition 13:** *All the EU decision makers are neutral to probabilistic risk.*

Let us conclude this section. EU theory imposes restrictions to behavioral patterns. By Corollary 9, it is impossible to be weakly risk averse without being strongly risk averse. A EU decision maker who does not like risk (a weak risk-averse decision maker) and who is asked to choose between the two risky random variables $X$ and $Y$ of example 2, must choose $X$. We can put this result differently: a decision maker who is weakly risk averse and who prefers $Y$ to $X$ in example 2 cannot be an EU maximizer.

## 3. Results in the framework of the rank-dependent expected-utility model

As first noted by Machina [1982b, 1983], the EU equivalence between weak risk aversion and strong risk aversion does not carry over to generalized models. We focus here on RDEU (rank-dependent expected-utility) theory,[10] a generalization of EU theory, allowing in particular, the Allais paradox by weakening the independence axiom. RDEU theory was axiomatized first by Quiggin [1982] as *anticipated utility* then with some variants by Yaari [1987], Segal [1989], Allais [1987], Wakker [1994, 1st ed. 1990], and Chateauneuf [1990]. We will show that, in the RDEU framework, the different definitions of weak, monotone, and strong risk aversion are no more equivalent.

### *3.1. Definition of the rank-dependent expected-utility model*

In this section, for reasons of simplicity of the exposition, we assume that $\mathcal{C} = [0, 1]$. This can be done without loss of generality by changing the reference point and the units.

*Remark.* However, let us just mention that RDEU theory has been generalized to the case where $\mathcal{C}$ is any connected topological space (Wakker [1994], Chateauneuf [1990]).

**Definition 12:** *A decision maker satisfies RDEU theory if and only if his preference relation* $\succsim$ *can be represented by a real-valued function V such that for every X and Y of* $\mathcal{V}$,

$$X \succsim Y \text{ iff } V(X) \geq V(Y)$$

with

$$V(X) = -\int_0^1 u(x)\, df(G_X(x)) = +\int_0^1 f(Pr\{u(X) > t\})dt,$$

*where the function u is continuous, strictly increasing from* $[0, 1]$ *to* $\mathbb{R}$, *unique up to a positive affine transformation and the function f is continuous,*[11] *strictly increasing from* $[0, 1]$ *to* $[0, 1]$, *satisfying* $f(0) = 0$, $f(1) = 1$ *and unique.*

In all this section, we assume that the decision maker has a preference relation $\succsim$ satisfying RDEU theory and characterized by the transform function $f$ and the utility function $u$ as defined in Definition 12.

*Particular cases.* (1) When $f(p) = p$ for every $p$ of $[0, 1]$, this theory reduces to EU theory; (2) when $u(x) = x$, this theory is the dual theory of Yaari [1987]; (3) when $X$ has only a finite number of outcomes $(x_1, \leq x_2 \leq \ldots \leq x_n)$, $V$ can be written under the useful following form:

$$(1)\ V(X) = u(x_1) + \sum_{i=2}^{n} f\left(\sum_{j=i}^{n} p_j\right)(u(x_i) - u(x_{i-1})).$$

With this formulation, we can interpret the decision-maker behavior: obtaining the minimum satisfaction $u(x_1)$ with certainty, the decision maker then evaluates the successive additional utility differences as weighted by the associated transformed cumulative probabilities.

*Remarks.* (1) In some of the characterizations of the model, some authors use the transform $\phi$ of the cumulative function $F$ instead of the transform $f$ of the decumulative function $G$ (as we do here). In the sequel, our results show different properties of $f$ that we could express as dual properties of function $\phi$. (2) In this model, we can interpretate separately the functions $u$ and $f$ (Wakker [1994]): $u$ as the utility level under certainty ($u$ concave means a diminishing marginal utility for wealth) and $f$ as the perception of probabilities.

According to formula (1), the condition $f(p) \leq p$ means that the decision maker having, for sure, at least a satisfaction $u(x_1)$, systematically *underweights* the additional satisfactions $u(x_i) - u(x_{i-1})$ since

$$f\left[\sum_{j=i}^{n} p_j\right] \leq \sum_{j=i}^{n} p_j.$$

In this sense, we can give the following definition:

**Definition 13:** *A RDEU decision maker is weakly pessimistic under risk if and only if* $f(p) \leq p$, *for any* $p$ *of* [0, 1]; (2) *a RDEU decision maker is weakly optimistic under risk if and only if* $f(p) \geq p$, *for any* $p$ *of* [0, 1].

*Remark.* For a weakly pessimistic RDEU decision maker, for any random variable, the transformed expected value (the mathematical expectation of its transformed decumulative distribution function) is smaller than its expected value.

### *3.2. Characterization of strong risk aversion in the rank-dependent expected-utility model*

The following result is due to Chew, Karni, and Safra [1987] (see also Machina [1982]).

**Proposition 14** (Chew, Karni, and Safra [1987]): *A RDEU decision maker is strongly risk averse iff* $u$ *is concave and* $f$ *is convex.*[12]

Let us just note that a RDEU decision maker cannot be strongly risk averse without having a concave function $u$.

*Particular case of the dual theory of Yaari* [1987]. A dual theory decision maker is strongly risk averse if and only if $f$ is convex.

*Remark.* This last result has been directly proved for dual theory by Yaari [1987] and Roell [1985].

### *3.3. Study of monotone risk aversion in the rank-dependent expected-utility model*

The following result is due to Quiggin [1992]:

**Proposition 15** (Quiggin [1992]): (1) *A RDEU decision maker who is monotone risk averse and whose u is concave,*[13] *has a f satisfying:* $f(p) \leq p$ *for every p of* [0, 1]; (2) *a RDEU-decision maker for whom* $f(p) \leq p$ *for every p of* [0, 1] *and u concave, is monotone risk averse.*

*Particular case of the dual theory of Yaari* [1987]. A dual-theory decision maker is monotone risk averse if and only if his function $f$ satisfies $f(p) \leq p$ for every $p$ of [0, 1]. This last result has been proved by Quiggin [1991].

*Characterization of monotone increasing risk.* The following result of Quiggin [1992] shows that, as for mean preserving spreads, there is a characterization of mean preserving monotone spreads:

**Proposition 16** (Quiggin [1992]): *For X and Y of* $\mathcal{V}$ *such that* $E(X) = E(Y)$, *Y is a mean preserving monotone spread of X if and only if all the RDEU monotone risk-averse decision makers prefer X to Y.*

*Remarks.* (1) As already noticed, Corollary 10 is only true for risk-averse EU decision makers. (2) Getting rid of reference to any model, one could also state the following:

**Proposition 16′:** *For any two random variables X and Y, such that* $E(X) = E(Y)$, *the following statements re equivalent:* (1) *Y is a mean preserving monotone spread of X*; (2) *for any concave function u from* $\mathcal{C}$ *to* $\mathbb{R}$ *and for any continuous function f s.t.* $f(0) = 0, f(1) = 1$, *and* $f(p) \leq p$ *for any p of* [0, 1],

$$-\int_0^1 u(x)\, df(G_X(x)) \geq -\int_0^1 u(x)\, df(G_Y(x)).$$

### *3.4. Study of weak risk aversion in the rank-dependent expected-utility model*

The results presented next are based on Chateauneuf and Cohen [1994], who have obtained necessary conditions on one side and sufficient conditions in the other side. The sufficient conditions are the following:

**Proposition 17** (Chateauneuf and Cohen [1994]): *Consider a RDEU decision maker with differentiable functions u and f*: (1) *if u satisfies* ($C$): $\exists k \geq 1$ *s.t.* $u'(x) \leq k(u(x) - u(y)/(x - y))$, $0 \leq y < x \leq 1$ *and if f satisfies* : $f(p) \leq p^k$ $\forall p \in [0, 1]$, *he is weakly risk averse*; (2) *if u satisfies* ($D$): $\exists h \geq 1$ *s.t.* $u'(y) \leq h(u(x) - u(y)/(x - y))$, $0 \leq y < x \leq 1$ *and if f satisfies*: $f(p) \geq 1 - (1 - p)^h$, $\forall p \in [0, 1]$, *he is weakly risk seeking.*

*Particular cases.* In the following particular cases, it can be proved (Chateauneuf and Cohen [1994]) that the conditions are necessary and sufficient:

1. A RDEU decision maker whose function $u$ is concave and differentiable is weakly risk averse if and only if $f$ satisfies $f(p) \leq p$ for every $p$ of [0, 1] (since condition ($C$) with $k = 1$ implies $u$ concave).

2. A RDEU decision maker whose $u$ is convex and satisfies (for $x \in [0, 1]$), $u(x) = x^n$, $n \geq 1$ is weakly risk averse if and only if $f$ satisfy: $f(p) \leq p^n$ for $p \in [0, 1]$ (since $u$ satisfies condition $(C)$ with $k = n$). One can thus compensate a convex function $u$ by a very pessimistic function $f$.

The most interesting part of Proposition 17 is obtained for weakly risk-seeking decision makers:

3. A RDEU decision maker whose $u$ is a concave function satisfying (for $x \in [0, 1]$, $u(x) = 1 - (1 - x)^n$, $n \geq 1$, is weakly risk seeking if and only if $f$ satisfy $f(p) \geq 1 - (1 - p)^n$, $\forall p \in [0, 1]$ (since $u$ satisfies condition $(D)$ with $h = n$).

*A RDEU decision maker can be weakly risk seeking despite a diminishing marginal utility of wealth*[14] *if he is sufficiently optimistic.* Such a behavior could not be explained by EU theory.

*Particular case of the dual theory of Yaari* [1987]. A dual-theory decision maker is weakly risk averse if and only if $f$ satisfies $f(p) \leq p$ for every $p$ of [0, 1]. This result has been proved directly by Yaari [1987], Roell [1985], and Quiggin [1991].

*Remark.* Contrary to the case of EU theory, in the dual-theory framework weak risk aversion and strong risk aversion do not lead to the same characterization. In this sense, we can say that, concerning behavior under risk, dual theory is more flexible than EU theory.[15] The different results can be summarized in the following proposition:

**Proposition 18** (Chateauneuf and Cohen [1994]): *Let a decision maker comply with RDEU theory with a differentiable u satisfying condition* $(C)$ *and condition* $(D)$ *(defined in Proposition 17):* (1) *if f satisfies* $f(p) \leq p^k$ $\forall p \in [0, 0]$, *he is weakly risk averse*; (2) *if f satisfies* $f(p) \geq 1 - (1 - p)^h$ $\forall p \in [0, 1]$, *he is weakly risk seeking.*

Many functions $u$ simultaneously satisfy conditions $(C)$ and $(D)$. For instance, it is the case of any continuously differentiable function $u$ such that $u'(x) > 0$ for any $x$ of [0, 1] (see Chateauneuf and Cohen [1994].

In Figure 1, the RDEU decision maker characterized by functions $u$ and $f_1$ is weakly risk averse while the RDEU decision maker characterized by the *same* function $u$ and function $f_2$ is weakly risk seeking.

### *3.5. Characterization of probabilistic risk aversion in the rank-dependent expected-utility model*

P. Wakker [1994] proved the following result:

**Proposition 19** (Wakker [1994]): *Under RDEU, a decision maker is* (1) *averse to probabilistic risk if and only if f is convex*; (2) *prone to probabilistic risk, if and only if f is concave*; (3) *neutral to probabilistic risk, if and only if f is the identity.*

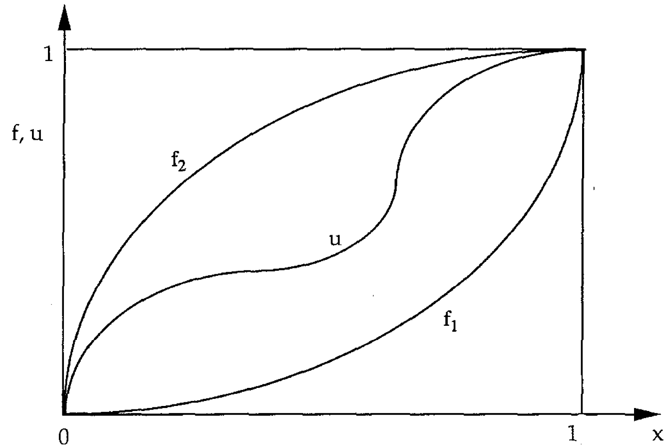


*Figure 1.* A weakly risk-averse RDEU decision maker ($u$ and $f_1$) and a weakly risk-seeking RDEU ($u$ and $f_2$).

This result confirms the fact that the notion of probabilistic risk aversion captured in this model by function $f$ is completely independent of the notion of cardinal utility under certainty captured by function $u$. In the dual-theory framework, this notion is called *pessism* by Yaari. Since, in the framework of RDEU theory, the convexity of function $f$ implies $f(p) \leq p$, we prefer to call Yaari's definition *strong pessimism under risk.*

## 4. New results in the expected-utility model

### *4.1. Comparative statics in the expected-utility model*

This section discusses how new notions such as monotone increasing risk provide better answers to some problems of comparative statics even in the framework of EU model. "In EU theory, the analysis of comparative statics under uncertainty shows many counterintuitive results" (Meyer [1989]). Here is such an example:

*Counterintuitive result 1.* In the standard problem of one-safe-asset, one-risky-asset portfolio, a natural prediction is that, if the return of the risky asset become more risky, then EU risk-averse investors will want less of it. Rothschild and Stiglitz [1971] have shown that this prediction is not always true when the new risky asset is a mean preserving spread of the previous one.

In fact, there are a lot of such paradoxes not only in portfolio choice but in many other economic problems—in the selection of the optimal level of coverage in insurance, in the theory of the competitive firm under price uncertainty, in the theory of hiring under uncertainty. Let us give a general explanation of all these apparent paradoxes. In EU theory, we get the following result: for any $F_1$, $F_2$, and $F$ of $\mathcal{L}$, and $\alpha$ in [0, 1], if $F_1$ is a mean preserving spread of $F_2$, then for any EU risk-averse decision maker $F_1 \precsim F_2$, and then, because of the independence axiom, $\alpha F_1 + (1 - \alpha) F \precsim \alpha F_2 + (1 - \alpha)F$. Now most of the economic problems quoted can be seen as selecting the value of a choice variable $\alpha$ that maximizes $\alpha X + (1 - \alpha)Z$ where $X$ and $Z$ are known random variables ($Z$ may also be a constant). Let $X_1$ and $X_2$ be such that $X_2$ is a mean preserving spread of $X_1$ and let $\alpha_1^*$ and $\alpha_2^*$ be the associated optimal values in this problem. For all risk-averse EU decision makers $X_1 \succsim X_2$ but, as noticed in Section 1.1, $\alpha X_1 + (1 - \alpha) Y$ differing from the correspondant mixture on $\mathcal{L}$, there is no reason that the independence axiom applies and that $\alpha X_1 + (1 - \alpha)Y \succsim \alpha X_2 + (1 - \alpha)Y$. This fact can explain some apparent counterintuitive examples of comparative statics in EU.

*Remark.* In Yaari's theory, the independence axiom is replaced by a *dual independence axiom* where he precisely assumes the stability of the relation $\succsim$ by convex combination of comonotonic random variables. We can then anticipate that, in dual theory, when $X_1$ and $Z$ are comonotonic (in particular when $Z$ is constant), there is no such paradox.

In several problems of comparative statics, many authors have studied stronger conditions under which one can suppress the counterintuitive results on increasing risk: Sandmo [1971], Rothschild and Stiglitz [1971], Diamond and Stiglitz [1974], Eeckoudt and Hansen [1980], Ross [1981], Machina [1982b], Meyer and Ormiston [1983, 1985, 1989],[16] Gollier [1991]. According to the different authors, these conditions may be the restriction of the possible concave-utility functions or the restriction of the notion of increasing risk.

In the classical one-safe-asset, one-risky-asset portfolio problem,[17] Rothschild and Stiglitz [1971] have shown that a sufficient condition for the allocation to the risky asset to be reduced by a mean preserving spread is (1) decreasing absolute risk aversion (DARA) and (2) relative risk aversion that is increasing and less than or equal to unity.

According to the following result of Quiggin [1991], the notion of mean preserving monotone spread seems more adapted to this portfolio problem.

**Proposition 20** (Quiggin [1991]):[18] *In the problem of one-safe-asset, one-risky-asset portfolio problem, a sufficient condition for the allocation to the risky asset to be reduced by a mean preserving monotone spread is decreasing absolute risk aversion (DARA).*

Only the first and more intuitive part of the conditions (DARA) remains. The second very restrictive part is not needed anymore. Here is another counterintuitive result in comparative statics to which the notion of monotone increasing risk is more appropriate.

*Counterintuitive result 2.* Under partial insurance, the Arrow-Pratt measure of risk aversion does not ensure that more risk-averse EU-decision makers are willing to pay more for the reduction of some risk (Ross [1981]). Ross's idea was to adopt a restrictive concept of more risk averse in order to eliminate this counterintuitive result. By the following result, the mean preserving monotone spread concept offers an appealing alternative to Ross's suggestion:

**Proposition 21** (Landsberger and Meilijson [1993], Quiggin [1991]): *In the class of all decision makers with nondecreasing utility functions, more risk-averse decision makers pay a higher risk premium for a reduction in risk if and only if the less risky distribution is Bickel-Lehmann less dispersed.*

By Proposition 7, "$X$ is Bickel-Lehmann less dispersed than $Y$" means exactly that $Y$ is a mean preserving monotone spread of $X$.

We have shown with these examples that the concept of mean preserving monotone spread is more fitted to some problems of comparative statics in the framework of EU theory.[19]

*Remarks.* A generalization of these comparative statics has been very elegantly done by Quiggin [1991], Ormiston and Quiggin [1994] to the RDEU model, and by Machina [1989] for his model [1982a] generalizing EU theory.

### 4.2. *Measurement of inequalities*

As for comparative statics in EU theory, the new concept of monotone increasing in risk may be of help. The following example is due to Chateauneuf [1994]:

*Example 3.* Let us look at the three following distributions of income $X$, $Y$, and $Z$ for four agents:

| | *1* | *2* | *3* | *4* |
|---|---|---|---|---|
| $X$ | 800 | 900 | 1,000 | 1,100 |
| $Y$ | 800 | 950 | 950 | 1,100 |
| $Z$ | 850 | 925 | 975 | 1,050 |
| $X$–$Y$ | 0 | −50 | +50 | 0 |
| $X$–$Z$ | −50 | −25 | +25 | +50 |

Let us compare first $X$ and $Y$: $X$ is obviously a mean preserving spread of $Y$ but not a mean preserving monotone spread of $Y$ since $\theta = X - Y$ is not comonotonic with $Y$. In terms of mean preserving spread, one can say that $Y$ reduces inequalities. Yet the first agent may feel frustrated since he gets nothing more, whereas the second one, who had already more than him, get moreover an additional payment. Similarly, the third one loses 50, whereas the fourth, who had already more than him, loses nothing. One can think then that a decrease in distribution by mean preserving spread is not enough but that a monotone decrease in risk should be a good criterion for reduction of inequalities. If we compare now $X$ and $Z$, one can say that $Z$ really reduces inequalities since $X - Z = \theta'$ and $Z$ and $\theta'$ are comonotonic.

It is on the base of this remark that authors as Chateauneuf [1994], have tried to give a stronger characterization of decreasing inequalities. Similarly, Yaari [1988a] and Ben-Porath and Gilboa [1994] have used the dual-theory value function to axiomatize new measurement of inequalities.

## 5. Conclusion

We have shown that several different concepts of increasing risk and of risk aversion could be introduced and that their definitions did not need make references to any particular model.

In RDEU theory, all these notions of risk aversion receive different characterizations and thus enlarge the typologies of possible behaviors. In particular, the weak concept of risk aversion (risk seeking) allows for the following behavior: a decision maker having a utility function with convex and concave parts can be weakly risk averse or weakly risk seeking depending on his degree of pessimism or optimism under risk.

As for EU theory where all the definitions of risk aversion are equivalent and amount to the concavity of the utility function, we have shown that the introduction of the new notion of monotone increasing risk, particularly adapted to the RDEU model, was still of interest in the EU framework by providing better answers to some problems of comparative statics as to the standard one-safe-asset, one-risky-asset portfolio problem or to partial insurance.

There is hope that this notion of monotone increasing risk will prove to be equally helpful in many other problems of comparative statics both in EU and non-EU theories.

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## Notes

1. See, for instance, Machina [1983] for a survey on this area.
2. I define the two notions of monotone risk aversion and comonotonic random variables and explain the link between them later in the paper.
3. This work is an overview more pedagogical and intuitive than technical. We take the simplest hypothesis on $\mathcal{V}$ and $\mathcal{C}$ but extensions to more general $\mathcal{V}$ and $\mathcal{C}$ are possible.
4. Yaari [1987] calls them respectively horizontal and vertical mixtures (see also Roell [1985]).
5. Such an element will always exist and be unique in all the models presented in this paper.
6. The fact that the relation mean preserving spread is only a partial order shows the difference with the relation increasing variance, which is obviously complete.
7. See Chateauneuf, Cohen, and Kast [1994] for a survey on this subject.
8. The reason why the initial definition is given only for elements of $\mathcal{V}^*$ is that, if $X$ and $Y - X$ are comonotonic random variables, so must be $X$ and $Y$.
9. Eeckould and Hansen [1980] and Meyer and Ormiston [1983] have given an economic application of this kind of reduction of risk to price-band stabilization for a producer.
10. This theory also is called *expected utility with rank-dependent probabilities* (EURDP).
11. Peter Wakker has given an axiomatization of RDEU where function $f$ can be discontinuous.
12. For Chew, Karni, and Safra [1987] who use the transform $\phi$ of the cumulative distribution function, $f$ convex becomes $\phi$ concave.
13. Concavity of $u$ is not needed since Chateauneuf and Cohen [1994] have proved that weakly risk-averse decision makers had necessarily $f(p) \leq p$.
14. Yaari [1987] and Jaffray [1988], each one in the framework of his model, have already shown that a decision maker could be weakly risk averse without a decreasing marginal utility for wealth, whereas Yaari [1987] and Cohen [1992] have shown that a decision maker could be weakly risk seeking without an increasing marginal utility for wealth.
15. On the other hand, EU is naturally more flexible concerning cardinal utility of wealth.

16. Meyer and Ormiston's [1985] *strong increase in risk* seems to be a particular case of mean preserving monotone spread.
17. See Gollier [1991] for an overview on sufficient conditions to solve the portfolio problem.
18. See also Ormiston and Quiggin [1994].
19. More such examples can be found in Quiggin [1991].

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