ON THE MEMBERSHIP OF INTEGRAL OPERATORS IN CLASSES $\mathbf{S}_p$ FOR $p \geq 2$

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1. In this paper we investigate, by the Lions–Peetre interpolation method of means [1], integral operators of the form

$$(\mathbf{M} f)(x) = \int_T M(x, y) f(y) \nu(dy). \tag{1}$$

Here M is a measurable function given on the product $(S \times T, \mu \times \nu)$ of two spaces with positive and σ-finite measures, (S, μ) and (T, ν). More precisely, we seek conditions in terms of the kernel M(x, y) which are sufficient for operator (1) to belong to the classes $\mathbf{S}_p(L^2_\mu, L^2_\nu) = \mathbf{S}_p$, $p \geq 2$ (see [2] for the definition and the properties of classes $\mathbf{S}_p$).

We find such conditions by interpolating two simple facts. Firstly, it is well known that

$$\|\mathbf{M}\|_2 = N(M(x, y)/L^2_{\mu\times\nu}), \tag{2}$$

where $\|\mathbf{M}\|_2$ is the absolute norm of operator **M**: everywhere in what follows the norm of an element a in a (quasi)normed space B is written as $N(a/B) = \|a\|_B$. Further, it is very well known that when $s_1, s_2 > 0$, $s_1 + s_2 = 2$,

$$\|\mathbf{M}\| \leqslant N(M(x, y)/L^\infty_y(L^{s_1}_x))^{s_1/2} N(M(x, y)/L^\infty_x(L^{s_2}_y))^{s_2/2}, \tag{3}$$

where $\|\mathbf{M}\|$ is the norm of the operator in the space $\mathbf{R} = \mathbf{R}(L^2_\mu, L^2_\nu)$ of bounded operators from L^2_μ into L^2_ν, and

$$N(M(x, y)/L^\infty_y(L^{s_1}_x)) = \nu\text{-}\sup_{y\in T} N(M(\cdot, y)/L^{s_1}_x) = \nu\text{-}\sup_{y\in T}\left[\int_S |M(x, y)|^{s_1} \mu(dx)\right]^{1/s_1}$$

(for example, see [3], where the proof is carried out for the case when $\mu = \nu$ is a Lebesgue measure, but it can be extended without difficulty to the case being considered here).

Let Φ be a mapping associating the integral operator **M** with the measurable function M(x, y) on $S \times T$ by means of formula (1). Obviously, Φ is linear, while (2) and (3) show that Φ acts continuously in the pairs of spaces

$$\Phi : L^2_{\mu\times\nu} \to \mathbf{S}_2, \tag{4}$$

$$\Phi : L^\infty_y(L^{s_1}_x) \cap L^\infty_x(L^{s_2}_y) \to \mathbf{R}, \tag{5}$$

where, as usual, the norm in the intersection of two (quasi)normed spaces A and B is introduced by the equality

$$\|\cdot\|_{A\cap B} = \max(\|\cdot\|_A, \|\cdot\|_B).$$

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Noting that the space L^s, $0 < s < 1$, is quasinormed,* we make use of the interpolation theorem in [4], wherein certain results of Lions–Peetre [1] are generalized to the quasinormed case.

Then Φ acts continuously in pairs of spaces of means:

$$\Phi : (L^2_{\mu\times\nu},\ L^\infty_y(L^{s_1}_x)\cap L^\infty_x(L^{s_2}_y))_{\theta,2,\infty} \to (S_2,\ R)_{\theta,2,\infty}. \tag{6}$$

Thus, it remains to characterize the spaces of means occurring in (6). For one of them this is simple: namely, by the standard methods (for example, see [1]) we obtain

$$(S_2,\ R)_{\theta,2,\infty} = S_{2/(1-\theta)},\ 0<\theta<1, \tag{7}$$

but for the second we encounter the question, nontrivial in general, of the commutativity of the intersection functor and the interpolation functor. However, in our case the spaces of type $L^\infty_y(L^{s_1}_x)$ being investigated possess an additional structure: namely, they are quasinormed lattices.

2. In this section we briefly recall certain definitions and interpolation constructions needed subsequently.

We say that the quasinormed spaces F_0 and F_1 form an interpolation pair if they are continuously imbedded in a certain linear topological space $\mathbf{F}$. If (F_0, F_1) is an interpolation pair, we construct the space $F_0 + F_1$ consisting of all elements of $\mathbf{F}$ representable in the form $f_0 + f_1$, $f_i \in F_i$, with the quasinorm

$$\|f\|_{F_0+F_1} = \inf(\|f_0\|_{F_0} + \|f_1\|_{F_1}),$$

where the inf is taken over all representations

$$f = f_0 + f_1,\quad f_i \in F_i,\ i = 0,\ 1.$$

If (F_0, F_1) is an interpolation pair, we construct the space of means

$$(F_0,\ F_1)_{\theta,p_0,p_1},\ \theta\in(0,\ 1),\ p_i\in(0,\ \infty],\ i=0,\ 1,$$

by setting

$$(F_0,\ F_1)_{\theta,p_0,p_1} = \{f\in F_0+F_1,\ f=f_0(n)+f_1(n),$$
$$n=0,\ \pm 1,\ \ldots,\ e^{(i-\theta)n}\|f_i(n)\|_{F_i}\in l^{p_i},\ i=0,1\}$$

(see [1, 4] and the survey article [5]). The space of means is once again quasinormed with the quasinorm

$$\|f\|_{\theta,p_0,p_1} = \inf\max_{i=0,1}\left(N(e^{(i-\theta)n}\|f_i(n)\|_{F_i}\,|\,l^{p_i})\right),$$

where the inf is taken over all representations of f in the form

$$f = f_0(n) + f_1(n).$$

It is easy to see that if $p_i \le q_i$, $i = 0, 1$, then $(F_0, F_1)_{\theta,p_0,p_1} \subset (F_0, F_1)_{\theta,q_0,q_1}$ and the imbedding is continuous (the monotonicity theorem).

Further, the Riesz theorem is true. Let F be a quasinormed space and let (S, μ) be a measurable space with a σ-finite positive measure μ. By $L^p(F) = L^p_\mu(S; F)$, $p > 0$, we shall denote the space of F-valued measurable functions for which $\int_S \|F(x)\|^p_F\,d\mu < \infty$. This is a quasinormed space with the quasinorm $\|f\|^p = \int_S \|f(x)\|^p_F\,d\mu$.

*We recall that a linear topological space F is quasinormed if on F there is given a real function $\|\cdot\|_F$ with the properties: 1) $\|\cdot\|_F \geqslant 0$, $\|f\|_F = 0$ only if $f = 0$; 2) $\|\lambda f\|_F = |\lambda|\,\|f\|_F$; and 3) $\|f+g\|_F \leqslant C(F)(\|f\|_F + \|g\|_F)$.

It is easy to see that if F is a complete quasinormed space with a continuous quasinorm, then $L^p(F)$ and $l^p(F)$ are complete. However, if (F_0, F_1) is an interpolation pair of complete quasinormed spaces, then $(F_0, F_1)_{\theta,p_0,p_1}$ also is complete.

By arguing mainly as in [1], we can prove the Riesz theorem.

Let (F_0, F_1) be an interpolation pair of complete quasinormed spaces with continuous quasinorms. Then

$$(L^{p_0}(F_0),\ L^{p_1}(F_1))_{\theta,\, p_0,\, p_1} = L^p((F_0,\ F_1)_{\theta,\, p_0,\, p_1}),$$

with equivalent quasinorms, where

$$\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1},\ p_i \in (0,\ \infty],\ i=0,\ 1,\ p \in (0,\ \infty).$$

We recall also that a quasinormed space A of complex-valued functions given on some set S is called a quasinormed lattice on S if the quasinorm of space A is monotonic, i.e., from $|f(x)| \leq |g(x)|$, $x \in S$, it follows that $N(f\,|\,A) \leq N(g|A)$.

If D is a quasinormed space, then by A(D) we shall denote the class of D-valued functions $f(x)$ such that $\| f(x) \|_D \in A$ with the quasinorm $\| f \|_{A(D)} = N(\| f(x) \|_D \,|\, A)$.

3. LEMMA. Let $\{A_i\}_{i=1}^n$, $\{B_j\}_{j=1}^m$ be quasinormed lattices of positive functions on S and let D be a quasinormed space such that the spaces $A_i(D)$ and $B_j(D)$ are continuously imbedded in some linear topological space. Then

$$\left(\bigcap_{i=1}^n A_i(D),\ \bigcap_{j=1}^m B_j(D)\right)_{\theta,\, p_0,\, p_1} = \bigcap_{i=1}^n \bigcap_{j=1}^m (A_i(D),\ B_j(D))_{\theta,\, p_0,\, p_1}. \tag{8}$$

Proof. On one side the imbedding is obvious, since $\bigcap_{i=1}^n A_i(D)$ and $\bigcap_{j=1}^m B_j(D)$ are continuously imbedded in $A_i(D)$ and $B_j(D)$, respectively. Let us verify the inverse for n = 1, m = 2 (it is not difficult to obtain the general assertion from this by induction). Let

$$f \in (A_1(D),\ B_1(D))_{\theta,\, p_0,\, p_1} \cap (A_1(D),\ B_2(D))_{\theta,\, p_0,\, p_1}.$$

Then, by definition of a space of means (Paragraph 2), f can be represented in the form

$$f = f_0(n) + f_1(n) = g_0(n) + g_1(n) \tag{9}$$

(we emphasize that for each n, $f_i(n)$ and $g_i(n)$, i = 0, 1, are functions on S with values in D, which will sometimes be noted by writing them as

$$f_i(n;\ x),\ g_i(n;\ x),\ x \in S);$$

moreover,

$$e^{-\theta n} N(f_0(n)/A_1(D)),\ \ e^{-\theta n} N(g_0(n)/A_1(D)) \in l^{p_0} \tag{10}$$

and

$$e^{(1-\theta)n} N(f_1(n)/B_1(D)),\ \ e^{(1-\theta)n} N(g_1(n)/B_2(D)) \in l^{p_1}. \tag{11}$$

We examine a set E_n by setting

$$E_n = \{x \in S;\ \|f_1(n;\ x)\|_D \leqslant \|g_1(n;\ x)\|_D\},$$

and for any n we define the functions $h_0(n)$, $h_1(n)$ by the equalities

$$h_i(n) = h_i(n;\ x) = \begin{cases} f_i(n;\ x), & x \in E_n, \\ g_i(n;\ x), & x \in S/E_n. \end{cases}$$

It is obvious that here $f = h_0(n) + h_1(n)$,

$$\|h_1(n;\ x)\|_D \leqslant \min(\|f_1(n;\ x)\|_D,\ \|g_1(n;\ x)\|_D),$$
$$\|h_0(n;\ x)\|_D \leqslant \|f_0(n;\ x)\|_D + \|g_0(n;\ x)\|_D). \tag{12}$$

From the properties of a quasinormed lattice it now follows that

$$N(h_1(n)/B_1(D)) \leqslant N(f_1(n)/B_1(D)),$$
$$N(h_1(n)/B_2(D)) \leqslant N(g_1(n)/B_2(D)),$$

and, therefore, by the definition of a norm in an intersection of spaces (Section 1),

$$N(h_1)n)/B_1(D) \cap B_2(D)) \leqslant \max(N(f_1(n)/B_1(D)),$$
$$N(g_1(n)/B_2(D))).$$

From (11) we then get that

$$e^{(1-\theta)n} N(h_1(n)/B_1(D) \cap B_2(D)) \in l^{p_1}. \tag{13}$$

On the other hand, from (12) and from property 3 of a quasinorm it follows that

$$N(h_0(n)/A_1(D)) \leqslant 2c\max(N(f_0(n)/A_1(D)),\ N(g_0(n)/A_1(D))),$$

whence, using (10) and once again property 3 of a quasinorm, we get that

$$e^{-\theta n} N(h_0(n)/A_1(D)) \in l^{p_1}. \tag{14}$$

By the definition of a space of means, (9), (13), and (14) show that

$$(A_1(D),\ B_1(D) \cap B_2(D))_{\theta,\ p_0,\ p_1}$$
$$\supset (A_1(D),\ B_1(D))_{\theta,\ p_0,\ p_1} \cap (A_1(D),\ B_2(D))_{\theta,\ p_0,\ p_1},$$

moreover, as is easy to note, the continuity of the imbedding follows automatically from the preceding arguments. The lemma is proved.

From interpolation results of the type of the monotonicity theorem and the Riesz theorem (Paragraph 2) and from the lemma just proved, it is now not difficult to show that

$$L_{\theta,\ s} = L_y^{2/(1-\theta)}(L_x^{p_\theta}) \cap L_x^{2/(1-\theta)}(L_y^{q_\theta}) \subset (L^2_{\mu\times\nu},\ L_y^\infty(L_x^{s_1}) \cap L_x^\infty(L_y^{s_2}))_{\theta,\ 2,\ \infty}, \tag{15}$$

where

$$\frac{1}{p_\theta} = \frac{1-\theta}{2} + \frac{\theta}{s_1},\ \frac{1}{q_\theta} = \frac{1-\theta}{2} + \frac{\theta}{s_2},\ 0 < \theta < 1,$$

and that the imbedding is continuous. Now from (6), (7), and (15) we see that Φ acts continuously in the pairs of spaces:

$$\Phi : L_{\theta,\ s} \to \mathbf{S}_{2/(1-\theta)}$$

and, by the same token, the following result holds.*

<u>THEOREM.</u> If a measurable function $M(x, y)$ belongs to the space $L_{r;p,q} = L_y^r(L_x^p) \cap L_x^r(L_y^q)$, where $r > 2$, $0 < p, q < 2$, $p + q > 2$, $(2^{-1} - r^{-1})^{-1} = (p^{-1} - r^{-1})^{-1} + (q^{-1} - r^{-1})^{-1}$, then the integral operator $\mathbf{M}$ with kernel $M(x, y)$ belongs to class $\mathbf{S}_r$ and

$$N(\mathbf{M}/\mathbf{S}_r) \leqslant \text{const}\, N(M(x,\ y)/L_{r;\ p,\ q}). \tag{16}$$

*A result analogous in character, but less general, was obtained in [6] with the aid of complex interpolation.

Let us try to improve bound (16). To do this we introduce into consideration a new measure $\nu_\varepsilon = \varepsilon\nu$, $\varepsilon > 0$, and by $\mathbf{M}_\varepsilon$ we denote an integral operator with kernel M(x, y) acting from L^2_μ into $L^2_{\nu_\varepsilon}$. It is easily seen that

$$N(\mathbf{M}/\mathbf{S}_r(L^2_\mu,\ L^2_\nu)) \leqslant \varepsilon^{-1/2} N(\mathbf{M}_\varepsilon/\mathbf{S}_r(L^2_\mu,\ L^2_{\nu_\varepsilon})). \tag{17}$$

From (16) and (17) we find without difficulty that

$$N(\mathbf{M}/\mathbf{S}_r) \leqslant \text{const}\max(\varepsilon^{r^{-1}-2^{-1}} N(M/L^r_y(L^p_x)),\ \varepsilon^{q^{-1}-2^{-1}} N(M/L^r_x(L^q_y))),$$

and, by minimizing with respect to $\varepsilon > 0$, we finally obtain

$$N(\mathbf{M}/\mathbf{S}_r) \leqslant \text{const}\, N^{s_1/2}(M/L^r_y(L^p_x))\, N^{s_2/2}(M/L^r_x(L^q_y)),$$

where

$$\frac{s_1}{2} = \left(\frac{1}{2} - \frac{1}{r}\right)\left(\frac{1}{p} - \frac{1}{r}\right)^{-1},\quad \frac{s_2}{2} = \left(\frac{1}{2} - \frac{1}{r}\right)\left(\frac{1}{q} - \frac{1}{r}\right)^{-1}.$$

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