A Uniform Ergodic Theorem for Certain Markov Operators on Lipschitz Functions on Bounded Metric Spaces★

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Summary. Some sufficient conditions are given for uniform convergence of the iterates of transition operators associated with random contractions of a bounded metric space.

1. Introduction

Let (T, d) be a metric space, $\mathcal{A}$ its Borel subsets. For any complex valued function f on T , let

$$|f| = \sup_{t \in T} |f(t)|,$$

$$m(f) = \sup_{s \neq t} \frac{|f(s) - f(t)|}{d(s, t)},$$

and

$$\|f\| = m(f) + |f|.$$

The class $L = \{f: \|f\| < \infty\}$ of bounded Lipschitz functions is a Banach space with respect to the norm $\|\cdot\|$. If g maps T into T , let

$$W(g) = \sup_{s \neq t} \frac{d(g(s), g(t))}{d(s, t)}.$$

Let (E, Γ) be a measurable space. For every $t \in T$ let $p(t, \cdot)$ be a probability on Γ such that, for some $M < \infty$ and all $G \in \Gamma$,

$$m(p(\cdot, G)) \leq M.$$

For any $x \in E$, let u_x map T into T , and for $x_1, \dots, x_n$ in E let $u_{x_1 \dots x_n} = u_{x_n} \circ \dots \circ u_{x_1}$. Assume that $\{(t, x): u_x(t) \in A\} \in \mathcal{A} \times \Gamma$ if $A \in \mathcal{A}$, from which it follows that

$$\{(t, x_1, \dots, x_n): u_{x_1 \dots x_n}(t) \in A\} \in \mathcal{A} \times \Gamma^n \quad \text{if } A \in \mathcal{A}.$$

Finally, suppose that there is a Γ measurable real valued function v and an $a < \infty$ such that

$$W(u_x) \leq v(x)$$

for all $x \in E$ and

$$\int_E p(t, dx) v(x) \leq a$$

for all $t \in T$.

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For every $f \in L$, Uf is the complex valued function

$$Uf(t) = \int_E p(t, dx) f(u_x(t)).$$

Clearly $Uf(t) = \int_T K(t, ds) f(s)$, where $K(t, A) = p(t, \{x: u_x(t) \in A\})$ is a Markov transition probability function in T . The Markov operator U is linear with $|Uf| \leq |f|$. The equality

$$\begin{aligned} Uf(t) - Uf(s) &= \int_E p(t, dx) (f(u_x(t)) - f(u_x(s))) + \int_E (p(t, dx) - p(s, dx)) f(u_x(s)) \end{aligned}$$

yields

$$m(Uf) \leq am(f) + R|f|, \quad (1)$$

where $R = 2M$, from which it follows that $Uf \in L$ and that U is bounded with $\|U\| \leq \max(a, R + 1)$.

A theorem of Ionescu Tulcea [1, Theorem 1] implies that, if (a) (T, d) is bounded (i.e. $b = \sup_{s, t \in T} d(s, t) < \infty$), (b) $W(u_x) \leq r$ for some $r < 1$ and all $x \in E$, and (c) $p(t, G) \geq \lambda \mu(G)$ for some $\lambda > 0$ and probability μ , and all $t \in T$ and $G \in \Gamma$, then there are constants $U^\infty f$, β and $\alpha < 1$ such that

$$|U^n f - U^\infty f| \leq \beta \alpha^{\sqrt{n}} \|f\| \quad (2)$$

for all $n \geq 1$. For no generality is lost in assuming that f is real valued with $\|f\| = 1$, and in that case (2) follows from Ionescu Tulcea's theorem on taking $a_n = M b r^n$, $c_n = m(f) b r^n$, and $s = [\sqrt{n}]$ (the largest integer that does not exceed $\sqrt{n}$), and noting that $\|f\|^+ = 1$. Ionescu Tulcea's $\|\cdot\|$ is my $|\cdot|$. Ionescu Tulcea [1, Section 8] further showed that

$$\|U^n - U^\infty\| \leq \gamma \zeta^n \quad (3)$$

for constants γ and $\zeta < 1$ under the additional assumption that (T, d) is compact.

Lemma 1 below shows that (3) follows from (1) with $a = r < 1$ and (2), so that the compactness assumption is superfluous. Furthermore, Theorems 1 and 2 indicate that (b) and (c), respectively, can be considerably relaxed. One of the implications of Theorem 2 is that (3) obtains if, in addition to (a) and (b), (c) holds on a subset T' of T that is invariant under all u_x . Some applications of these results to learning models are presented in Section 4.

Since, in the case of compact T , there is a well developed theory of Markov operators satisfying (1) with $a < 1$ (see [2], [5, Parts II.B and II.C], and [6]) one naturally seeks an appropriate compactification of T . We note in this connection that boundedness of (T, d) does not insure that d can be extended to a metric on the Stone-Ćech compactification. Thus the reduction to the case of compact T in [3, Paragraph 1 of the proof of Lemma 1] is defective.

2. Uniform Convergence of U^n

Lemma 1 is slightly more general than our immediate needs require. For any bounded linear operator V on L , $\rho(V)$ is its spectral radius.

Lemma 1. Let U be a bounded linear operator on L with $|Uf| \leq |f|$, and for every $f \in L$ let $U^\infty f$ be a bounded complex valued function on T . If there are real numbers R and $r < 1$, a positive integer k , and a sequence $\{\delta_n\}$ with limit 0 such that

$$m(U^k f) \leq r m(f) + R |f| \quad (4)$$

and

$$|U^n f - U^\infty f| \leq \delta_n \|f\| \quad (5)$$

for all $n \geq 1$ and $f \in L$, then there is a bounded linear operator V on L such that

$$U^n = U^\infty + V^n \quad (6)$$

for all $n \geq 1$, and

$$\rho(V) < 1. \quad (7)$$

Proof. Iteration of (4) yields

$$m(U^{jk} f) \leq r^j m(f) + R \frac{1 - r^j}{1 - r} |f|. \quad (8)$$

Thus letting $j \rightarrow \infty$ on the right in

$$\begin{aligned} |U^\infty f(s) - U^\infty f(t)| &\leq m(U^{jk} f) d(s, t) \\ &\quad + |U^\infty f(s) - U^{jk} f(s)| + |U^{jk} f(t) - U^\infty f(t)| \end{aligned}$$

and using (5) we obtain

$$m(U^\infty f) \leq \frac{R}{1 - r} |f|. \quad (9)$$

Since (5) implies that $|U^\infty f| \leq |f|$, U^∞ is a bounded linear operator on L .

Since $U^{n+1} f$ converges uniformly to $U^\infty U f$, $U^\infty f$, and $U U^\infty f$, it follows that $U^\infty U = U^\infty = U U^\infty$. And since $U^n U^n f$ converges uniformly to $U^\infty U^\infty f$ and $U^\infty f$, $U^\infty{}^2 = U^\infty$. Let $V = U - U^\infty$. Then, by induction, $V^n = U^n - U^\infty$, and (6) is proved.

Replacing f by $f - U^\infty f$ in (8) and using (9) on the right hand side we obtain

$$m(V^{jk} f) \leq r^j m(f) + R' |f|, \quad (10)$$

where $R' = R(2+r)/(1-r)$. Eq. (5) yields

$$|V^{jk} f| \leq \delta_{jk} m(f) + \delta_{jk} |f|. \quad (11)$$

Changing f to $V^{jkn} f$ in (10) and (11) and letting

$$a_{j,n} = \begin{pmatrix} m(V^{jkn} f) \\ |V^{jkn} f| \end{pmatrix} \quad \text{and} \quad P_j = \begin{pmatrix} r^j & R' \\ \delta_{jk} & \delta_{jk} \end{pmatrix}$$

leads to the vector inequality $a_{j,n+1} \leq P_j a_{j,n}$, hence to $a_{j,n} \leq P_j^n a_{j,0}$. Adding co-ordinates on the right and left of this equation we obtain

$$\|V^{jkn}\| \leq |P_j^n|,$$

where $|P_j^n|$ is the maximum of the column sums of P_j^n . Now $\|V^{jkn}\|^{1/jkn} \rightarrow \rho(V)$ and $|P_j^n|^{1/n} \rightarrow \rho_j$ as $n \rightarrow \infty$, where ρ_j is the spectral radius of P_j . Thus

$$\rho(V) \leq \rho_j^{1/jk}.$$

Since $\rho_j < 1$ for j sufficiently large, the proof is complete.

Lemma 1 finishes the derivation of (3) from (a), (b), and (c). Theorem 1 reaches the same conclusion on the basis of assumptions more general than (b). Let $p_1 = p$, and, for $n > 1$, $t \in T$, and $A \in \Gamma^n$ let

$$p_n(t, A) = \int_E p(t, dx_1) \int_E \cdots \int_E p(u_{x_1 \dots x_{n-1}}(t), dx_n) I_A(x_1, \dots, x_n),$$

where I_A is the indicator function of A . Let μ^n be the n th Cartesian power of the probability μ on Γ .

Theorem 1. Suppose that, in addition to (a) and (c), the following conditions obtain: (b1) There is an $r < 1$, a positive integer k , and a Γ^k measurable real valued function h such that $W(u_{x_1 \dots x_k}) \leq h(x_1, \dots, x_k)$ for all $x_1, \dots, x_k \in E$, and

$$\int_E p_k(t, d(x_1, \dots, x_k)) h(x_1, \dots, x_k) \leq r$$

for all $t \in T$. (b2) There is a positive integer k' and a $\Gamma^{k'}$ measurable real valued function h' such that

$$W(u_{x_1 \dots x_{k'}}) \leq h'(x_1, \dots, x_{k'}) \quad \text{for all } x_1, \dots, x_{k'} \in E,$$

and

$$\int_{E^{k'}} \mu^{k'}(d(x_1, \dots, x_{k'})) h'(x_1, \dots, x_{k'}) < 1.$$

Then there are bounded linear operators U^∞ and V on L such that $U^\infty f$ is constant for any $f \in L$, $\rho(V) < 1$, and $U^n = U^\infty + V^n$ for all $n \geq 1$.

One proof of this theorem obtains (4) directly from (b1), and (5), with $U^\infty f$ constant, by a routine modification of the proof of Ionescu Tulcea's Theorem 1. We omit details.

3. u_x Invariant Subsets of T

Let T' be a Borel subset of T such that $u_x(t) \in T'$ for all $x \in E$ and $t \in T'$. The assumptions of the second paragraph of Section 1, as well as (a) and (b), carry over to the restrictions d' , p' , and u' of d , p , and u to T' . Hence if p' satisfies (c), (12) below holds, where U' is the operator on L' induced by p' . Thus Theorem 2 is applicable and (3) follows, as was claimed in Section 1.

Theorem 2. Under (a) and (b1), if there is a null sequence γ_n , and for every $f \in L'$ there is a constant $U'^\infty f$ such that

$$|U'^n f - U'^\infty f|' \leq \gamma_n \|f\|' \quad (12)$$

for all $n \geq 1$, then $U^n = U^\infty + V^n$, where $U^\infty f = U'^\infty f'$ and V is a bounded linear operator on L with $\rho(V) < 1$.

Proof. The function $\psi(t) = \inf_{s \in T'} d(t, s)$ is bounded and Lipschitz. For any $s \in T'$

$$\begin{aligned} \psi(u_{x_1 \dots x_k}(t)) &\leq d(u_{x_1 \dots x_k}(t), u_{x_1 \dots x_k}(s)), \\ \text{so} \quad \psi(u_{x_1 \dots x_k}(t)) &\leq h(x_1, \dots, x_k) \psi(t). \end{aligned}$$

Thus $U^k \psi(t) \leq r \psi(t)$. Iterating and using the positivity of U we obtain $U^{jk} \psi(t) \leq r^j \psi(t)$, so that

$$U^{jk} \psi(t) \leq b r^j \quad (13)$$

for all $j \geq 0$.

For any $f \in L$, $t \in T$ and $s \in T'$,

$$\begin{aligned} |f(t)| &\leq |f(t) - f(s)| + |f(s)|, \\ \text{so that} \quad |f(t)| &\leq m(f) \psi(t) + |f'|, \end{aligned}$$

where f' is the restriction of f to T' . Therefore

$$|U^{jk} f(t)| \leq m(f) U^{jk} \psi(t) + |f'|,$$

or, using (13),

$$|U^{jk} f| \leq b r^j m(f) + |f'|. \quad (14)$$

Replacing f by $U^i f - U^\infty f$ in (14) we obtain

$$|U^{jk+i} f - U^\infty f| \leq b r^j m(U^i f) + |U^i f' - U^\infty f'|. \quad (15)$$

It has already been observed that (b1) implies (4), and the latter implies, in turn, that $m(U^i f) \leq J \|f\|$ for some constant J and all i . This inequality along with (12) and (15) yields

$$|U^{jk+i} f - U^\infty f| \leq (b J r^j + \gamma_i) \|f\|. \quad (16)$$

Letting $j = [n/2k]$ and $i = n - jk$ we obtain (5). An application of Lemma 1 completes the proof.

4. Applications

In many psychological learning experiments, a subject makes a sequence of choices from a measurable space $(Y, \mathcal{A})$, and each of these is followed by an outcome from a measurable space (Z, Σ) . Let $(E, \Gamma) = (Y, \mathcal{A}) \times (Z, \Sigma)$. The experimenter prescribes the probability distribution $\xi(y, \cdot)$ over Σ after a choice y . We assume that $\xi(\cdot, B)$ is $\mathcal{A}$ measurable for each $B \in \Sigma$. In many learning models, the subject is characterized at the time of a choice by a probability distribution t over Y . T is the set of choice distributions considered (e.g. the probabilities absolutely continuous with respect to some fixed measure on $\mathcal{A}$), and d is the total variation metric. Obviously (T, d) is bounded. The joint distribution of response and outcome is

$$p(t, G) = \int_Y t(dy) \int_Z \xi(y, dz) I_G(y, z).$$

Clearly $m(p(\cdot, G)) \leq 1$ for all $G \in \Gamma$. Occurrence of the response-outcome pair $x = (y, z)$ effects the transformation $t \rightarrow u_x(t)$ of the choice distribution. In linear models $u_x(t) = (1 - \theta_x) t + \theta_x v_x$, for measurable mappings θ and v of (E, Γ) into the Borel subsets of $[0, 1]$ and T , respectively. We assume convexity of T to insure that $u_x(T) \subseteq T$, and separability of T to insure joint measurability of u . Clearly $W(u_x) = 1 - \theta_x \leq 1$, and (b) is satisfied if $\theta_x \geq \delta > 0$ for all $x \in E$.

We now consider two sets of additional assumptions that permit us to conclude that $U^n = U^\infty + V^n$, where $U^\infty f$ is constant for all $f \in L$ and V is bounded on L with $\rho(V) < 1$. Case A is applicable to Suppes' [7] model, case B to Norman's [4].

A. The functions θ and v depend only on z , and $\xi(y, B) \geq \lambda \mu(B)$ for some $\lambda > 0$ and probability μ on Σ . Since u doesn't depend on y , U is the same as the operator U^* corresponding to $E^* = Z$, $p^*(t, B) = p(t, Y \times B)$, and $u_z^* = u_{(y, z)}$. But $p^*(t, B) \geq \lambda \mu(B)$, so p^* satisfies (c).

B. $v_x(B) \geq \lambda \omega(B)$ for some $\lambda > 0$ and probability ω on $\mathcal{A}$. Clearly the convex hull T' of the range of v is u_x invariant for all $x \in E$, and $t(B) \geq \lambda \omega(B)$ for all $t \in T'$. Hence $p(t, G) \geq \lambda \mu(G)$ for all $t \in T'$ and $G \in \Gamma$, where μ is the probability

$$\mu(G) = \int_Y \omega(dy) \int_Z \xi(y, dz) I_G(y, z)$$

on Γ .

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