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A Limit Theorem for Renewal Sequences with an Application to Local Time*

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Summary. Let $0<h(\varepsilon)\uparrow$ and $\{S_n\}$ a renewal process. We find conditions under which as $t\to\infty$ $\sum_{n\geqq\pi(t)} h(S_n)\sim m^{-1}\int_t^\infty h(s)\,ds$ where $m=ES_1$, $\pi(t)=\min(n\colon S_n>t\}$. We apply these results to obtain sample path representation of local time at a point for a Markov process.

§1

Let $T_1, T_2, \ldots,$ be a sequence of i.i.d. nonnegative random variables and put $S_n=T_1+\ldots+T_n$, $S_0=0$, and assume $P[T_1>0]>0$. If $h>0$ is a decreasing integrable function on $[0,\infty)$ and if the T_i are nearly constant, say $T_1\equiv S_i-S_{i-1}\simeq m=ET_1$, then approximately

$$\sum_{k=n}^{\infty} h(S_k)\simeq 1/m \sum_{k=n}^{\infty} h(S_k)\,S_k-S_{k-1})\simeq 1/m \int_{S(n)}^{\infty} h(x)\,dx.$$

In other words, if we write $H(x)=\int_x^\infty h(s)\,ds$, then one should have

$$\lim_{n\to\infty} H(S_n)^{-1}\sum_{k=n}^{\infty} h(S_k)=1/m, \tag{1.1a}$$

a.s., or, equivalently (almost),

$$\lim_{t\to\infty} H(t)^{-1}\sum_{k=\pi(t)}^{\infty} h(S_k)=1/m \tag{1.1}$$

where

$$\pi(t)=\min\{n\colon S_n>t\}.$$

However, (1.1) need not hold even for nice h and nice T_i. See §3. One purpose of this paper is to find reasonably general conditions for (1.1). The motivation

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for studying this limit comes from a problem in the representation of local time at a point for a one-dimensional diffusion. See [5] pp. 212–222. We discuss this problem, its connection with (1.1) as well as some related matters in §§4–6. The main results are Theorems 1 and 2 in §2, Theorem 3 in §4, Theorem 4 in §5, and the example in §6.

Notation. i.o. = infinitely often; a.s. = almost surely; w.p.1. = with probability one: $0, o, \sim$ have the usual meanings; if A is a linear Borel set, $|A|$ denotes its Lebesgue measure. If $h \geqq 0$ and r is a Borel measure, $\int^{\infty} hr\{dx\} < \infty$ means $\int_a^{\infty} h(x)\, r\{dx\} < \infty$ for some $a > 0$, $\int^{\infty} hr\{dx\} = \infty$ means $\int_a^{\infty} h(x)\, r\{dx\} = \infty$ for all a. A similar meaning is assigned to $\int_{0+} hr\{dx\} < \infty$ or $= \infty$. If h is a function, $h(x)^{-1} = 1/h(x)$ but $h^{-1}(x)$ is the functional inverse of h, if it exists. lg = natural logarithm function. Integrable, unqualified, means integrable with respect to Lebesgue measure. The reader is warned that in §§1–4 m stands for the mean of a random variable, but in §§5 and 6 m stands for the speed measure of a diffusion. If x is a r.v., $I(x \in A) = 1$ or 0 according as $x \in A$ or $x \notin A$.

§2

In this section we state and prove the main results concerning (1.1). Throughout this section $[S_n, n \geqq 0]$ is as in §1, F is the common distribution of the T_i and U is the renewal measure:

$$U\{I\} = \sum_{n=0}^{\infty} P[S_n \in I] = \sum_{n=0}^{\infty} F^{n*}\{I\}$$

where F^{n*} has the usual meaning. We assume that F is not concentrated at 0. The following proposition gives a criterion for the finiteness of $\sum h(S_n)$. It will not be used explicitly in the sequel.

Proposition 1. *Let* $0 \leqq h \in \downarrow$, $h(0) < \infty$. *Then* $P\left[\sum_{n=0}^{\infty} h(S_n) < \infty\right] = 0$ *or* 1 *according as* $\int^{\infty} h(x)\, U\{dx\}$ *diverges or converges. If* $m = ET_1 = \int_0^{\infty} xF\{dx\} < \infty$, *then* $\int^{\infty} h(x)\, U\{dx\} < \infty$ *if and only if* $\int^{\infty} h(x)\, dx < \infty$.

Proof. Since $E\sum_0^{\infty} h(S_n) = \int_0^{\infty} h(s)\, U\{ds\}$, it is clear that $\int_0^{\infty} h(x)\, U\{dx\} < \infty$ assures that the random variable $\sum h(S_n)$ is integrable and hence finite w.p.1. So let us assume

$$EZ_n \uparrow \int_0^{\infty} h(x)\, U\{dx\} = \infty, \qquad n \to \infty, \tag{$*$}$$

where $Z_n = h(S_0) + \ldots + h(S_n)$. From $0 \leqq h \in \downarrow$ and $S_k \geqq 0$, we have for $j \geqq k$,

$$Eh(S_k)\,h(S_j) \leqq Eh(S_k)\,h(S_j - S_k) = Eh(S_k)\,Eh(S_{j-k}),$$

and then

$$\begin{aligned} EZ_n^2 &\leqq \sum_{k \leqq n} Eh(S_k)^2 + 2 \sum_{0 \leqq k \leqq j \leqq n} Eh(S_k)\,Eh(S_{j-k}) \\ &\leqq h(0)\,EZ_n + 2(EZ_n)^2 = (EZ_n)^2\,(2 + o(1)). \end{aligned}$$

In other words

$$\limsup_{n \to \infty} (EZ_n)^2 / EZ_n^2 \geqq \tfrac{1}{2} > 0,$$

and it follows from a generalization of the Borel-Cantelli Lemma, Kochen-Stone, [7], that $\limsup Z_n/EZ_n \geqq 1$ with a positive probability. But then $Z_n \to \infty$ w.p.1. by (*) and the $0-1$ law for symmetric events, Feller, [4], p. 124. The second assertion of the Proposition follows easily from the strong law of large numbers. There is a finite random variable N such that $\frac{1}{2}mk \leqq S_k \leqq 2mk$ for all $k \geqq N$. Hence

$$\sum_{k=N}^{\infty} h(2mk) \leqq \sum_{k=N}^{\infty} h(S_k) \leqq \sum_{k=N}^{\infty} h(\tfrac{1}{2}mk).$$

For monotone h, $Eh(ck) < \infty$ for a $c > 0$ if and only if $\int_0^\infty h(x)\,dx < \infty$.

Note 1. The second assertion, $P[\sum h(S_n) < \infty] = 1$ if and only if $\int_0^\infty h\,dx < \infty$, remains valid if the T's may take on negative values but $ET_1 = m > 0$. Note also that monotonicity may be replaced by a condition such as $h(x+y) \leqq \text{const.}\,h(x)$ for all $x, y \geqq 0$. The proof is the same.

A limit such as (1.1) seems to require either strong moment conditions on the T_i or else strong integrability conditions on h. See Theorems 1 and 2. There does not appear to be a single nice necessary and sufficient condition for (1.1), but the following proposition is a key ingredient. Before stating it we make the following assumptions regarding h and F to reamin in effect throughout this section.

h is a nonnegative, decreasing, (Lebesgue) integrable function on $[0, \infty)$. We write

$$H(x) = \int_x^\infty h(s)\,ds \tag{2.0}$$

and assume $H(x) > 0$ for all $x > 0$. Regarding F we assume $F\{0\} \equiv P[T_1 = 0] < 1$, F has at least 2 points of increase and

$$m = ET_1 = \int_0^\infty x F\{dx\} < \infty.$$

Proposition 2. *For each $\varepsilon > 0$ and $n = 0, 1, \ldots$, define the random variables.*

$$\beta_n = \max\{k \colon k \geqq 0,\ |S_{n+k} - S_n - km| \geqq k\varepsilon\}.$$

(i) *If as* $n \to \infty$

$$\beta_n = o(H(S_n)/h(S_n)) \quad a.s. \tag{2.1}$$

for every $\varepsilon > 0$, *then* (1.1) *holds a.s.*

(ii) *If for some* $r \geqq 1$

$$ET_1^{r+1} < \infty, \tag{2.2}$$

and

$$E\{h(S_{\pi(t)})/H(S_{\pi(t)})\}^r \to 0, \qquad t \to \infty \tag{2.3}$$

then (1.1) *takes place in mean of order r.*

Remark. Note that since $S_{\pi(t)} \to \infty$ as $t \to \infty$, (2.3) will hold whenever $\lim h(x)/H(x) = 0$.

Proof. Put

$$Z_n = \sum_{k=n}^{\infty} h(S_k) = \sum_{k=0}^{\infty} h(S_{n+k}).$$

Fix $0 < \varepsilon < m$. From monotonicity and the definition of β_n, we have

$$Z_n = \left(\sum_{k=0}^{\beta_n} + \sum_{k=\beta_n+1}^{\infty}\right) h(S_{n+k}) \leqq (\beta_n + 1)\, h(S_n) + \sum_{k=\beta_n+1}^{\infty} h(S_n + k(m+\varepsilon))$$

$$\leqq (\beta_n + 1)\, h(S_n) + (m-\varepsilon)^{-1} \int_{S_n}^{\infty} h(x)\, dx$$

or

$$H(S_n)^{-1} Z_n - 1/m \leqq (\beta_n + 1)\, q(S_n)^{-1} + \varepsilon',$$

where $\varepsilon' = \varepsilon/m(m-\varepsilon)$ and

$$q(x) = H(x)/h(x).$$

Similarly

$$Z_n \geqq \sum_{k=\beta_n+1}^{\infty} h(S_{n+k}) \geqq \sum_{k=\beta_n+1}^{\infty} h(S_n + k(m+\varepsilon))$$

$$\geqq (m+\varepsilon)^{-1}\{H(S_n + (\beta_n+1)(m+\varepsilon)) - H(S_n)\} + (m+\varepsilon)^{-1} H(S_n)$$

$$\geqq -(\beta_n+1)\, h(S_n) + (m+\varepsilon)^{-1} H(S_n),$$

or

$$H(S_n)^{-1} Z_n - 1/m \geqq -(\beta_n+1)\, q(S_n)^{-1} - \varepsilon'',$$

where $\varepsilon'' = \varepsilon/m(m+\varepsilon) \leqq \varepsilon'$. Combining the inequalities gives

$$|H(S_n)^{-1} Z_n - 1/m| \leqq (\beta_n+1)\, q(S_n)^{-1} + \varepsilon'. \tag{2.4}$$

Suppose that

$$\pi(t) \equiv \min\{k: S_k > t\} = n,$$

then $H(S_{n-1})^{-1} \leqq H(t)^{-1} \leqq H(S_n)^{-1}$, and, since $Z_{n-1} = Z_n + h(S_{n-1})$, it follows that

$$H(S_{n-1})^{-1} Z_{n-1} - q(S_{n-1})^{-1} \leqq H(t)^{-1} Z_n \leqq H(S_n)^{-1} Z_n. \tag{2.5}$$

Part (i) now follows immediately from (2.1), (2.4) and (2.5) as soon as we show that

$$q(S_n) \to \infty \quad \text{as } n \to \infty \quad \text{a.s.} \tag{2.6}$$

Suppose, to the contrary, that (2.6) fail. Then by the $0-1$ law for symmetric events, there exists a constant $a < \infty$ such that

$$P[q(S_n) \leqq a \text{ i.o.}] = 1.$$

Let $\sigma_0 = 0$ and

$$\sigma_k = \min\{n: n > \sigma_{k-1}, q(S_n) \leqq a\}.$$

Then $\sigma_k < \infty$ and $\sigma_k \uparrow \infty$ as $k \to \infty$ a.s.

Now notice the following more or less obvious facts: (a) β_r depends only on the random variables $T_{r+1}, T_{r+2}, \ldots$ and so must be independent of the event $[\sigma_k = r]$ which depends only on $S_0, S_1, \ldots, S_r$. (b) $\{\beta_r, r \geqq 0)$ is a stationary sequence. It follows from (a) and (b) that

$$P[\beta_{\sigma(k)} = j, \sigma_k = r] = P[\beta_r = j]\, P[\sigma_k = r] = P[\beta_0 = j]\, P[\sigma_k = r],$$

or, since $\sigma_k < \infty$ a.s.,

$$P[\beta_{\sigma(k)} = j] = \sum_{r=0}^{\infty} P[\beta_{\sigma(k)} = j, \sigma_k = r] = P[\beta_0 = j].$$

By assumption F has at least 2 points of increase, so $P[T_i = m = ET_i] < 1$ and for some $\varepsilon > 0$

$$P[\beta_0 = 0] = P[|S_k - km| \leqq k\varepsilon \text{ for all } k] < 1.$$

It follows that we can choose $j_0 > 0$ so that

$$P[\beta_0 = j_0] > 0,$$

and then

$$P[\beta_{\sigma(k)} = j_0 \text{ i.o. as } k \uparrow \infty] \geqq \lim_{k \to \infty} P[\beta_{\sigma(k)} = j_0] = P[\beta_0 = j_0] > 0.$$

We now have a contradiction, for, on the event $[\beta_{\sigma(k)} = j_0$ i.o.], we have on the one hand

$$\beta_{\sigma(k)}/q(S_{\sigma(k)}) \geqq j_0/a \quad \text{i.o.},$$

but on the other

$$\beta_{\sigma(k)}/q(S_{\sigma(k)}) \to 0 \quad \text{as } k \to \infty$$

by (2.1). This contradiction establishes (2.6).

It remains to prove (ii). First note that (2.2) implies $E\beta_0^r < \infty$. Indeed, one may show that (2.2) holds if and only if $E\beta_0^r < \infty$, See [3], pp. 361–371. Next, since the β_n are identically distributed and since $\beta_{\pi(t)}$ and $(\pi(t), S_0, \ldots, S_{\pi(t)})$ are independent, we get

$$E\beta_{\pi(t)}^r\, q(S_{\pi(t)})^{-r} = E\beta_0^r\, Eq(S_{\pi(t)})^{-r}.$$

We can now get L^r-convergence of $H(S_{\pi(t)})^{-1} Z_{\pi(t)}$ to $1/m$ from (2.3) and (2.4). L^r-convergence of $H(t)^{-1} Z_{\pi(t)}$ to $1/m$ then follows from (2.5).

Note 2. If we only assume that $q(S_n)\to\infty$ as $n\to\infty$ a.s. and $E\beta_0<\infty$ for every $\varepsilon>0$, then an average version of (1.1) holds: as $t\to\infty$

$$t^{-1}\int_0^t H(r)^{-1}\sum_{k\geqq\pi(r)} h(S_k)\,dr\to 1/m$$

a.s. Let us verify the average version of (1.1a). Put $W_n=H(S_n)^{-1}\sum_{k=n}^{\infty} h(S_k)$. Then by (2.4),

$$\left|n^{-1}\sum_{k=1}^{n} W_k-1/m\right|\leqq n^{-1}\sum_{k=1}^{n}\beta_k q(S_k)^{-1}+\varepsilon! \tag{$*$}$$

Now for fixed ε, $\{\beta_n, n\geqq 0\}$ is a stationary ergodic sequence fo random variables. The stationarity is clear from the definition of the β_n and stationarity of $\{T_n, n\geqq 1\}$, and the ergodicity follows from the fact that the tail σ-field of the β's is contained in the tail σ-field of the T's which is the trivial σ-field. Applying the ergodic theorem gives $n^{-1}(\beta_1+\ldots+\beta_n)\to E\beta_0<\infty$ a.s. as $n\to\infty$, and then, clearly, $n^{-1}\sum_{k=1}^{n}\beta_k q(S_k)^{-1}\to 0$. Since $\varepsilon'>0$ is arbitrary, we conclude from $(*)$ that $n^{-1}(W_1+\ldots+W_n)\to 1/m$ a.s. as $n\to\infty$ which is the average version of (1.1a).

Our first theorem involves strong conditions on the T_i but relatively weak conditions on h. It applies directly to the local time problem referred to in § 1, See § 4.

Theorem 1. *Suppose that for some* $b>0$

$$P[T_1>t]=O(e^{-bt}) \qquad t>0. \tag{2.7}$$

Suppose that h satisfies

$$\int^{\infty} e^{-aH(x)/h(x)}\,U\{dx\}<\infty \qquad \textit{for every } a>0, \tag{2.8}$$

and

$$h(x)=O(H(x)) \qquad x>0. \tag{2.9}$$

Then (1.1) *holds a.s. and in mean of order r for every* $r\geqq 1$.

Note 3. It will be seen from the proof that (2.9) is needed only to get the L^r-convergence.

For the next theorem we impose stronger conditions on h, weaker ones on the T_i.

Theorem 2. *If we replace* (2.7) *by*

$$ET_1^{r+1}<\infty \qquad \textit{for some } r>1 \tag{2.10}$$

and (2.8) *by*

$$\int^{\infty} [h(x)/H(x)]^r U\{dx\} < \infty, \qquad r \text{ as in } (2.10), \tag{2.11}$$

then (2.11) *holds a.s. and in mean of order r.* (*We do not need* (2.9) *here.*)

Note 4. One must have $r>1$ in (2.11); when $r=1$ the integral in (2.11) usually diverges. For example when F is spread out (i.e., when, for some n, F^{n*} has an absolutely continuous component with respect to Lebesgue measure), then we may write $U=U_0+U_1$ where U_0 is a finite measure and U_1 is an absolutely continuous measure with a bounded continuous density u_1 satisfying $u_1(x)\to 1/m$ as $x\to\infty$. (See Revuz, [8], pp. 153–154.) It follows that for B sufficiently large

$$\int_0^{\infty} h(x)/H(x)\, U\{dx\} \geqq (2m)^{-1} \int_B^{\infty} h(x)/H(x)\, dx = (2m)^{-1} \lim_{a\to\infty} \log H(B)/H(a) = \infty.$$

The above decomposition also shows that, in the spread out case, (2.8) is equivalent to

$$\int^{\infty} e^{-aH(x)/h(x)}\, dx < \infty \qquad \text{for every } a>0 \tag{2.8*}$$

and that (2.11) is equivalent to

$$\int^{\infty} [h(x)/H(x)]^r\, dx < \infty. \tag{2.11*}$$

(In proving the equivalence of (2.11) and (2.11*) one should note that (2.11*) actually implies that $h(x)/H(x)\equiv 1/q(x)\to 0$ as $x\to\infty$.) One can also show the equivalence of (2.8*), (2.11*) with (2.8), (2.11), respectively, in the case that $h(x)/H(x)$ is a decreasing function.

Finally we note that (2.11) is stronger than (2.8) since $e^{-aH/h} \leqq C(h/H)^r$ for some C depending only on a and r.

Proof of Theorem 1. The key step in the proof is the following estimate. Fix $0<\varepsilon<m$. Then there is a $z=z(\varepsilon)>0$ and $A<\infty$, both independent of n, so that

$$P[\beta_n(\varepsilon)>y] \leqq A e^{-zy}, \qquad y\geqq 0. \tag{2.12}$$

To prove this put $f(\theta)=Ee^{\theta(T_1-m)}=e^{-m}\int_0^{\infty} e^{\theta x} F\{dx\}$. Using (2.7) and an integration by parts we see that $f(\theta)<\infty$ for $|\theta|<b$, b as in (2.7). (Indeed, $f(\theta)<\infty$ for all $-\infty<\theta<b$.) Since $f(0)=1$ and $f'(0)=0$ the functions $\theta\mapsto f(\theta)\, e^{-\varepsilon\theta}$ and $\theta\mapsto f(-\theta)\, e^{\varepsilon\theta}$ are strictly decreasing near and have the value 1 at $\theta=0$. It follows that we can find $\theta_0>0$ and $z>0$ so that

$$f(\theta_0)\, e^{-\varepsilon\theta_0} < e^{-z} \quad \text{and} \quad f(-\theta_0)\, e^{-\varepsilon\theta_0} < e^{-z}.$$

Write $S'_k=(T_{n+1}-m)+\ldots+(T_{n+k}-m)$. From the definition of β_n, the preceding inequalities and Markov's inequality, we get

$$\begin{aligned}P[\beta_n=k]&\leqq P[S'_k>k\varepsilon]+P[-S'_k>k\varepsilon]\\&\leqq e^{-\theta_0\varepsilon k}[Ee^{\theta_0 S'_k}+Ee^{-\theta_0 S'_k}]\\&=e^{-\theta_0\varepsilon k}[f(\theta_0)^k+f(-\theta_0)^k]\leqq 2e^{-zk}.\end{aligned}$$

A summation gives (2.12) with $A\leqq 2(e^z-1)^{-1}$.

As in Proposition 2 write

$$q(x)=H(x)/h(x).$$

Since β_n and S_n are independent we have by (2.12)

$$\begin{aligned}\sum_{n=0}^{\infty}P[\beta_n>\delta q(S_n)]&=\sum_n E\{P[\beta_n>y]|_{y=\delta q(S_n)}\}\\&\leqq A\sum_n Ee^{-\delta z q(S_n)}=\int_0^{\infty}e^{-\delta z q(x)}\,U\{dx\}\end{aligned}$$

for any $\delta>0$. The last integral is finite for every $\delta>0$ by (2.8), and it follows from the Borel-Cantelli lemma that (2.1) holds.

To complete the proof we need to verify (2.3). But the random variables $\{q(S_{\pi(t)})^{-r},\ t>0\}$ are uniformly bounded by some constant C by (2.9), and, by (2.6), $q(S_{\pi(t)})^{-r}\to 0$ as $t\to\infty$ a.s. Dominated convergence now gives (2.3).

Proof of Theorem 2. Fix $0<\varepsilon<m$. From [3], pp. 365–370, it can be shown that (2.10) implies

$$E\beta_n^r=E\beta_0^r<\infty.$$

Computing as in the proof of Theorem 1 we get for any $\delta>0$

$$\sum_{n=0}^{\infty}P[\beta_n>\delta q(S_n)]\leqq\delta^{-r}E\beta_0^r\int_0^{\infty}q(x)^{-r}\,U\{dx\}<\infty$$

by (2.11). Hence $P[\beta_n>\delta q(S_n)\text{ i.o.}]=0$, i.e. (2.1) holds. To get (2.3) simply note that $q(S_{\pi(t)})^{-r}$ goes to 0 as $t\to\infty$ and is dominated by the integrable random variable $\sum_{n=0}^{\infty}q(S_n)^{-r}$.

§3

Examples. 1. Our first example shows that (1.1) need not hold even for nice h and F. Let $h(x)=e^{-x}$, $F(s)=P[T_i\leqq s]=1-e^{-s/m}$, $s\geqq 0$. Then $H(x)=e^{-x}$ and $ET_i=m$. Put

$$V_t=H(t)^{-1}\sum_{n\geqq\pi(t)}h(S_n)=\sum_{k=0}^{\infty}e^{-S_k}$$

where $S'_k = S_{\pi(t)+k} - t$. It is a well known property of the exponential distribution that the law of the sequence $\{S'_k, k \geqq 0\}$ does not depend on t (Indeed $\mathrm{Law}\{S'_k, k \geqq 0\} = \mathrm{Law}\{S_n, n \geqq 1\}$).

Consequently the distribution of V_t does not depend on t. Clearly this distribution has at least two distinct points in its support (i.e. V_0 is not a constant r.v.), and it follows that the limit (1.1) cannot exist. (In fact $\liminf V_t = 0$ and $\limsup V_t = \infty$ a.s. But note that $EV_t = 1/m$ for all t.)

2. If h is regularly varying, that is if as $t \to \infty$

$$h(tx)/h(t) \to x^c \quad \text{for all } x>0, \text{ for some constant } c \tag{3.1}$$

($c \leqq -1$ since h is supposed to be integrable), then (1.1) holds a.s. and in mean even if $m = \infty$ ($1/\infty \equiv 0$). To see this note first that (3.1) gives us the following. As $t \to \infty$

$$h(t+o(t)) = h(t)(1+o(1)), \tag{3.2}$$

$$h(t) = O(H(t)/t), \tag{3.3}$$

$$H(t+o(t)) = H(t)(1+o(1)). \tag{3.4}$$

The first of these is obvious from (3.1) and (3.3) follows from Feller [4], p. 281. Finally (3.4) is proved from (3.2) and (3.3) thus:

$$\begin{aligned}|H(t+o(t)) - H(t)| &= \left|\int_t^{t+o(t)} h(x)\,dx\right| \leqq (h(t) + h(t+o(t)))\,o(t)\\ &= o(th(t)) = o(H(t)).\end{aligned}$$

Consider the case $m < \infty$. (We continue to assume F is concentrated on $[0, \infty)$, $F\{0\} < 1$, but no other conditions on F need be imposed.) By the strong law of large numbers, we have as $n \to \infty$, $t \to \infty$ $S_n = nm(1+o(1))$, $\pi(t) = m^{-1} t(1+o(1))$ a.s. Hence

$$\begin{aligned}\sum_{n \geqq \pi(t)} h(S_n) &= \sum_{n \geqq \pi(t)} h(nm[1+o(1)])\\ &= (1+o(1))\, m^{-1} H(t+o(t)) = m^{-1} H(t)(1+o(1))\end{aligned}$$

as $t \to \infty$, and (1.1) follows. We leave it to the reader to establish (1.1) when $m = \infty$ (by a truncation argument) and to establish the mean convergence.

3. Here is a non-regularly varying example. Let $T_1, T_2, \ldots,$ be as in example 1 and

$$h(x) = e^{-x^d}$$

with $0 < d < 1$. Then $U\{dx\} = m^{-1} dx$, $x > 0$, $H(x) \sim d^{-1} x^{1-d} e^{-x^d}$, as $x \to \infty$, so (2.8) holds and Theorem 1 applies.

4. Here is a problem for the reader. Let $T_1, T_2, \ldots,$ be as in Example 1 and let $h(x) = e^{-x/\lg x}$, $x \geqq e$, $h(x) = \text{const.}$, $x < e$. Does (1.1) hold a.s.? Note that $H(x) \sim h(x) \lg x$ as $x \to \infty$ so (1.1) does hold in mean by Proposition 2. Theorems 1 and 2 do not apply.

§4

Let $\{X_t, t \geqq 0\}$ be a standard Markov process with state space Q. (With a slight change in notation our assumptions and definitions are as in [1], Chap. 5, §3, (except that we will assume $M_t \equiv 1$). See also [5]. Chap. 6.) Let 0 be a point in Q which satisfies

$$P_0[\eta_0 = 0] = 1, \tag{4.1}$$

where

$$\eta = \min[t: t > 0, X_t = 0\}.$$

As is well known, (4.1) is both necessary and sufficient for X to have a local time at 0, i.e., a continuous additive functional $L(t)$ which is non-negative and increase only at times $t \in Z$ where

$$Z = \overline{\{t: X_t = 0\}}.$$

L is unique up to a multiplicative constant. For proofs of the basic facts about L see the references cited above. When X is a canonical Brownian motion on the line, Levy gave a number of important sample path representations for $L(t)$. See [5] §2.3. One of these representations, (4.9) below, need not hold in general even if X is a nice diffusion, [5], §6.4. In this section after describing the representation we give a sufficient condition for its validity.

With very little loss of generality, we may suppose that 0 is regular for $\{0\}^c$ (as well as for $\{0\}$), that 0 is a recurrent point and that L has no linear part. Thus, in addition to (4.1)

$$P_0[\eta_{\{0\}^c} = 0] = 1, \tag{4.2}$$

$$P_0[X_t = 0 \text{ i.o. as } t \uparrow \infty] = 1, \tag{4.3}$$

$$P_0[\text{Lebesgue meas.} [t: X_t = 0] = 0] = 1. \tag{4.4}$$

Define the inverse local time L^{-1} by

$$L^{-1}(s) = \inf\{t: L(t) > s\}.$$

Note that $X(L^{-1}(s)) = 0$ a.s. (P_0) and that $L(L^{-1}(s)) = s$ by continuity but that $L^{-1}(L(t)) = t$ if and only if $t \in Z$ and t is not isolated from above in Z. The process L^{-1} is a subordinator, i.e., a non-decreasing right continuous process with stationary independent increments which increases only by jumps, moreover, under (4.1)–(4.4) every s-interval contains infinitely many discontinuities of L^{-1}. Let us denote by N the counting measure on $\mathscr{B}\{[0, \infty) \times (0, \infty)\}$ given by

$$N\{(s_1, s_2] \times (\varepsilon_1, \varepsilon_2]\} = \# \text{ jumps of } L^{-1} \text{ during } s_1 < s \leqq s_2$$
$$\text{of magnitude } L^{-1}(s) - L^{-1}(s-) \in (\varepsilon_1, \varepsilon_2],$$

then

$$L^{-1}(s) = \int_0^s \int_0^\infty x N\{du, dx\} = \int_0^\infty x N\{[0, s], dx\}.$$

The law of N, with respect to P_0, has a Poisson structure with mean measure

$$E_0 N\{(s_1, s_2] \times (\varepsilon_1, \varepsilon_2]\} = (s_2 - s_1)\, v\{(\varepsilon_1, \varepsilon_2]\}, \tag{4.5}$$

where v, the Levy measure of L^{-1}, satisfies

$$v\{(0, \infty)\} = \infty, \qquad \int_0^\infty \min(1, x)\, v\{dx\} < \infty. \tag{4.6}$$

In addition

$$-\lg E_0\, e^{-\alpha L^{-1}(s)} = \int_0^\infty (1 - e^{-\alpha x})\, v\{dx\} = s\, g_\alpha(0, 0)^{-1} \tag{4.7}$$

where $g_\alpha(0, 0)$ ($= u_\alpha(x_0)$ in the notation of [1]) is the α-potential of L. (More about this in § 5.)

Since Z is closed we may write

$$Z^c = \bigcup_{\beta=1}^{\infty} e_\beta$$

where the e_β are open pairwise disjoint intervals (the excursion intervals of X away from 0). Each e_β corresponds to a flat stretch in L and a jump in L^{-1}. Fix $t > 0$ and put

$$\begin{aligned} V(t, \varepsilon) &= \#\,\text{intervals } e_\beta \subset [0, t] \text{ with length } > \varepsilon \\ &= \sum_{\beta:\, e_\beta \subset [0, t]} I(|e_\beta| > \varepsilon). \end{aligned}$$

If $r = L^{-1}(L(t))$, then $L(t) = L(r)$ and

$$V(t, \varepsilon) \leqq V(r, \varepsilon) = N\{[0, L(r)] \times (\varepsilon, \infty)\} \leqq V(t, \varepsilon) + 1.$$

Now $\tilde{v}(\varepsilon) \equiv v\{(\varepsilon, \infty)\}$ (= the expected number of jumps $> \varepsilon$ per unit time in L^{-1}) $\to \infty$ as $\varepsilon \downarrow 0$, and this fact, the Poisson structure of N, and the strong law of large numbers gives

$$V(t, \varepsilon)/\tilde{v}(\varepsilon) \to L(t) \qquad \text{as } \varepsilon \downarrow 0 \text{ a.s. } (P_0). \tag{4.8}$$

In light of (4.8) it is plausible that one should also have for every t

$$W(t, \varepsilon) \Big/ \int_0^\varepsilon x\, v\{dx\} \to L(t) \qquad \text{as } \varepsilon \downarrow 0 \tag{4.9}$$

a.s. where $W(t, \varepsilon)$ is the total length of the intervals $e_\beta \subset [0, t]$ of length $\leqq \varepsilon$, i.e.

$$\begin{aligned} W(t, \varepsilon) &= \sum_{\beta:\, e_\beta \subset [0, t]} |e_\beta|\, I(|e_\beta| \leqq \varepsilon) \\ &= \int_0^\varepsilon x N\{[0, L(r)], dx\}, \qquad r = L^{-1}(L(t)). \end{aligned}$$

Note that $\int_0^\varepsilon x\, v\{dx\}$ may be roughly interpreted as the average length per unit time of a jump in the truncated process $s \mapsto \int_0^\varepsilon x N\{[0, s], dx\}$. As we noted earlier, however, Ito and McKean have discovered that, though (4.9) does hold in the Brownian notion case, it fails in general.

The following theorem provides a sufficient condition for (4.9) which seems to be very close to a necessary condition as well.

Theorem 3. *Suppose that* ν, *the Levy jump measure of* L^{-1}, *is non-atomic. If for every* $a>0$,

$$\int_{0+} \exp\left(-a\int_0^{\varepsilon} x\,\nu\{dx\}/\varepsilon\right)\nu\{d\varepsilon\}<\infty, \tag{4.10}$$

then for every $t>0$ (4.9) *holds a.s.* (P_0).

Note 1. The sure event on which (4.9) holds may be chosen independent of t and the convergence is uniform on bounded t-intervals.

Note 2. The measure ν will be non-atomic in the case that X is a diffusion on the line. Indeed in this case ν is absolutely continuous with a density ν' which is analytic! See Ito and McKean, [5], p. 217. For general conditions on X under which ν is either non-atomic or absolutely continuous, see [2], pp. 68–73.

Proof. Let us write

$$h(t)=\min\{\varepsilon\colon \tilde{\nu}(\varepsilon)\leqq t=\sup\{\varepsilon\colon \tilde{\nu}(\varepsilon)>t\}$$

where $\tilde{\nu}(\varepsilon)=\nu\{(\varepsilon,\infty)\}$ as above. Then h and $\tilde{\nu}$ are non-decreasing, $\tilde{\nu}$ is continuous since ν is non-atomic, h is right continuous and $h(0+)=\tilde{\nu}(0+)=\infty$, $h(\infty-)=\tilde{\nu}(\infty-)=0$. Moreover $\tilde{\nu}(h(t))=t$ for all $t\geqq 0$ but

$$\begin{aligned} &h(\tilde{\nu}(\varepsilon))=\varepsilon \quad \text{if and only if } \varepsilon\in l(\nu),\\ &l(\nu)=\{\varepsilon\colon \nu\{(\varepsilon-\delta,\varepsilon)\}>0 \text{ for all } \delta>0\}\subset \operatorname{supp}(\nu). \end{aligned} \tag{4.11}$$

Finally if $a=h(A)$, $b=h(B)$, $a<b$, then

$$\int_a^b f(\varepsilon)\,\nu(d\varepsilon)=\int_B^A f(h(t))\,dt \tag{4.12}$$

whenever either integral makes sense. Fix λ, $0<\lambda<\infty$, and let $Y_1, Y_2, \ldots$, be the atoms of the counting measure $A\mapsto N\{(0,\lambda]\times A\}\equiv N_\lambda(A)$. Since ν is non-atomic and L^{-1} on $(0,\lambda]$ has only finitely many jumps of magnitude $>\varepsilon$ for any fixed $\varepsilon>0$, we may suppose these Y's are listed in strict descending order: $Y_1>Y_2>\ldots$ (The Y's are random variables because $[Y_n>b]=[N_\lambda(b,\infty)\geqq n]$ for any $b\geqq 0$ and the latter event is measurable.)

Lemma 1.

$$P_0[h(\tilde{\nu}(Y_i))=Y_i \textit{ for all } i]=1.$$

Proof. For any Borel $A\subset(0,\infty)$, $P_0[Y_i\in A]\leqq P_0[N_\lambda A\geqq 1]\leqq E_0 N_\lambda A=\lambda\nu(A)$. Hence for every i the distribution of Y_i is absolutely continuous with respect to ν. Take $A=\operatorname{supp}(\nu)\setminus l(\nu)$. Then $P_0[Y_i\notin l(\nu)]\leqq\lambda\nu(A)=0$ since A is countable (A is the collection of right hand end points of the open intervals which make up $(0,\infty)\setminus\operatorname{supp}(\nu)$) and ν is non-atomic. Lemma 1 now follows form (4.11).

Now consider the process $r \to N_\lambda(h(r), \infty) \equiv \tilde{N}(r)$, $r \geqq 0$. From the Poisson character of N we see that (i) $\tilde{N}$ is a right continuous process which increases only by unit jumps. (ii) $\tilde{N}(r)$ has a Poisson distribution with mean $E\tilde{N}(r) = \lambda \tilde{v}(h(r)) = \lambda r$ (iii) $\tilde{N}$ has independent increments. In other words $\tilde{N}$ is an ordinary Poisson process with rate λ. Let $S_1 < S_2 < \dots$ be the jump times of $\tilde{N}$ and put $S_0 = 0$. Then $\tilde{N}(r) = \max\{n: S_n \leqq r\}$ and the increments $T_i = S_i - S_{i-1}$, $i \geqq 1$, are mutually independent random variables with distribution

$$P_0[T_i > t] = e^{-\lambda t}, \qquad t > 0,$$

and mean $E_0 T_i = 1/\lambda$. From Lemma 1 we have that $h(S_i) = Y_i$ for all i and, if $\varepsilon = h(r)$, then

$$\begin{aligned}\int_0^\varepsilon x N\{[0,\lambda], dx\} &= \sum_i Y_i I[Y_i \leqq \varepsilon] \\ &= \sum_i h(S_i)\, I[h(S_i) \leqq \varepsilon] = \sum_{i=\pi(r)}^\infty h(S_i)\end{aligned}$$

where $\pi(r) = \min\{n: S_n > r\} = \tilde{N}(r) + 1$.

By (4.12) $\int_0^\varepsilon x\nu\{dx\} = \int_r^\infty h(s)\,ds = H(r)$ and

$$\int_{0+} \exp\left(-a \int_0^\varepsilon x\nu\{dx\}/\varepsilon\right) \nu\{d\varepsilon\} = \int^\infty e^{-aH(s)/h(s)}\,dr,$$

and from Theorem 1 in § 2 we get for each λ

$$\lim_{\varepsilon \to 0+} \left(\int_0^\varepsilon x\nu\{dx\}\right)^{-1} \int_0^\varepsilon x N\{[0,\lambda], dx\} = \lambda \tag{4.13}$$

a.s. (P_0) whenever (4.10) holds for each $a > 0$. (Note that the fact that $h(0+) = \infty$ does not matter; $\sum_{n \geqq \pi(r)} h(S_n) = \sum_{n \geqq \pi(r)} h_b(S_n)$ for all r sufficiently large where $h_b(x) = \min\{b, h(x)\}$.) Since the right hand side of (4.13) is a continuous function of λ and the left hand side is for each $\varepsilon > 0$, a non-decreasing function of λ we see that the sure event on which (4.13) holds may be chosen independent of λ. Setting $\lambda = L(t)$ we get

$$\lim_{\varepsilon \to 0+} \left(\int_0^\varepsilon x\nu\{dx\}\right)^{-1} \left(\int_0^\varepsilon y N\{[0, L(t)], dy\}\right) = L(t) \tag{4.14}$$

a.s. and again the sure event may be chosen independent of t. Now if $t \in Z^c$, then

$$\begin{aligned}W(t,\varepsilon) &\equiv \sum_{e_\beta \subset [0,t]} |e_\beta|\, I(|e_\beta| \leqq \varepsilon) \\ &= \int_0^\varepsilon y N\{[0, L(t)], dy\}\end{aligned}$$

as soon as $\varepsilon < |e_{\alpha(t)}|$ where $e_{\alpha(t)}$ is the unique excursion interval containing t. Thus (4.9) holds for $t \in Z^c$. But Z^c is dense in $(0, \infty)$ a.s. which implies that (4.9) holds for all $t > 0$ by continuity of L and monotonicity of $t \mapsto W(t, \varepsilon)$.

This completes the proof of Theorem 3.

§ 5. Diffusions

In this section we find a sufficient condition on the speed measure of a diffusion on the line in order that (4.9) hold. Since our condition is certainly not necessary, see § 6, we will confine our attention to the case of a natural scale diffusion on $[0, \infty]$ which is reflecting at 0. The reader should consult Ito and McKean, [5], Chaps. 3-6, for the proofs of various assertions we make about diffusion processes.

Let m be a measure on $B[0, \infty]$ such that

$$m\{I\} > 0 \qquad \text{for every open } I \subset [0, \infty). \tag{5.1}$$

Without loss of generality we may also suppose

$$m\{0\} = 0. \tag{5.2}$$

Let X be the diffusion process on $[0, \infty)$ with generator given by the differential operator

$$Af = \frac{d}{dm} f' \qquad \text{on } (0, \infty) \tag{5.3}$$

$\left(f' = \dfrac{df}{dx} = \text{ordinary derivative of } f\right)$ with the boundary condition

$$f^+(0) \equiv \lim_{\delta \downarrow 0} (f(\delta) - f(0)/\delta = 0, \tag{5.4}$$

provided the origin is both entrance and exit.

This will be the case if and only if

$$m(x) \equiv m\{[0, x]\} < \infty \qquad \text{for all } x > 0. \tag{5.5}$$

Proposition 1. *Under the above assumptions,* (4.1)–(4.4) *hold.*

Proof. Ito and McKean [5] Chap. 3 and 4.

Theorem 4. *Suppose, in addition to* (5.1)–(5.5), *that m is absolutely continuous on* $[0, \delta]$, *some $\delta > 0$, with a density $m'(x) = dm/dx$ which regularly varying at* 0 *with exponent $\beta \geqq -1$, i.e., for all $c > 0$*

$$\lim_{x \downarrow 0} m'(cx)/m'(x) = c^\beta, \tag{5.6}$$

then (4.9) *holds for any local time of X at* 0.

Proof. Let B be a standard one-dimensional canonical Brownian motion with local times $l_x(t) = l(t, x)$ jointly continuous in (t, x), normalized so that for $t > 0$, Borel F

$$\int_0^t 1_F(B(s)\,ds = 2\int_F l(t,x)\,dx \qquad \text{a.s.}$$

as customary. Define

$$\begin{aligned} \mu(t) &= \int_0^\infty l(t,x)\,m\{dx\}, \\ \mu^{-1}(t) &= \inf\{s\colon \mu(s)>t\}, \qquad t>0, \end{aligned} \tag{5.7}$$

(Note that $\mu(\infty-)=\mu^{-1}(\infty-)=\infty$ as one easily checks.) By [5], Chap. 5, the processes $B\circ\mu^{-1}$ and X are identical in law and we will write $X=B\circ\mu^{-1}$. A local time for X at 0 is then given by

$$L(t)=l_0(\mu^{-1}(t)), \qquad t\geqq 0, \tag{5.8}$$

(all other local times at 0 for X are, of course, multiples of this one) and its inverse by

$$L^{-1}(t)=\mu(l_0^{-1}(t))=\int_0^\infty l(l_0^{-1}(t),x)\,m\{dx\}. \tag{5.9}$$

(To prove (5.9) note that

$$\{s\colon L(s)>t\}=\{s\colon l_0(\mu^{-1}(s))>t=l_0(l_0^{-1}(t))=\{s\colon \mu^{-1}(s)>l_0^{-1}(t)\}$$

where the last equality is from $l_0^{-1}(t)=\inf\{r\colon l_0(r)>t\}=\max(r\colon l_0(r)=t\}$. But $\{k\colon \mu(k)>s\}=(\mu^{-1}(s),\infty)$, so $\{s\colon \mu^{-1}(s)>l_0^{-1}(t)\}=\{s\colon \mu(l_0^{-1}(t)<s\}$, and thus $L^{-1}(t)\equiv\inf\{s\colon L(s)>t\}=\mu(l_0^{-1}(t))$.) To prove Theorem 4 it clearly suffices to show that (4.9) holds for the L given in (5.8).

Lemma 1. *Let X_1, X_2 be two natural scale diffusions reflecting at 0 with speed measures m_1, m_2 satisfying (5.1), (5.2), (5.5). Let X_i be represented as $B\circ\mu_i^{-1}$, $\mu_i(\cdot)=\int_0^\infty l(\cdot,x)\,m_i\{dx\}$ with local time at 0 given by $L_i=l_0\circ\mu_i^{-1}$.*

Let ν_i be the Levy jump measure of L_i^{-1} as in §4. Suppose m_1 is absolutely continuous with respect to m_2 on $[0,\delta]$ for some $\delta>0$ and that

$$dm_1/dm_2\leqq K \qquad \text{on } [0,\delta], \tag{5.10}$$

for some constant K. Then

$$\limsup_{\varepsilon\downarrow 0}\frac{\tilde{\nu}_1(\varepsilon)}{\tilde{\nu}_2(\varepsilon/K)}\leqq 1. \tag{5.11}$$

Proof. Let $\eta=\min(t\colon B(t)=\delta\}$. Then for any $s\leqq l_0(\eta)$ we have $l(s,x)=l_x(s)=0$ for $x>\delta$ and

$$\begin{aligned}L_1^{-1}(s)-L_1^{-1}(s-)&=\int_0^{\delta}[l_x(l_0^{-1}(s))-l_x(l_0^{-1}(s-))]\,m_1\{dx\}\\&\leqq K\int_0^{\delta}[l_x(l_0^{-1}(s))-l_x(l_0^{-1}(s-))]\,m_2\{dx\}\\&=K[L_2^{-1}(s)-L_2^{-1}(s-)].\end{aligned}$$

It follows that for every $\varepsilon>0$,

$$N_1\{[0,s]\times(\varepsilon,\infty)\}\leqq N_2\{[0,s]\times(\varepsilon/K,\infty)\} \tag{5.12}$$

a.s. (P_0) on the event $[l_0(\eta)>s]$ where N_1, N_2 are the random Poisson counting measure for L_1^{-1}, L_2^{-1} as in §4. From the strong law of large numbers for N_i, see [5], §6.3, we have for $i=1,2$ and a.s. (P_0)

$$\lim_{\varepsilon\downarrow 0} N_i\{[0,s]\times(\varepsilon,\infty)\}/\tilde{v}_i(\varepsilon)=s \tag{5.13}$$

uniformly for s in bounded intervals (see §4). (5.11) is now clear from (5.12) and (5.13) and the fact that the event $[l_0(\eta)>s]$ has positive P_0 probability.

Corollary to Lemma 1. *If $(dm_1/dm_2)(x)\to K$ as $x\downarrow 0$, then $\tilde{v}_1(\varepsilon)/\tilde{v}_2(\varepsilon/K)\to 1$ as $\varepsilon\downarrow 0$.*

Lemma 2. *Let $q=q(x,\alpha)=q_\alpha(x)$ satisfy $q<\infty$, q is differentiable and*

$$Aq=\alpha q \quad \textit{on } (0,\infty) \tag{5.14}$$

where A is the differential operator (5.3). *Then for each constant $c>0$ the function*

$$x\mapsto q^c(x,\alpha)\equiv q(cx,\alpha/c) \tag{5.15}$$

satisfies

$$A^c q^c=\alpha q^c \tag{5.16}$$

where A^c is the operator (5.3) *with m replaced by m^c given by*

$$m^c\{dx\}\equiv m\{c\,dx\}. \tag{5.17}$$

Proof. With q^c as in (5.15) we have by definition of the operators A^c, A

$$\begin{aligned}\int_a^b A^c q_\alpha^c(y)\,m^c\{dy\}&=\frac{d}{dy}q_\alpha^c(y)\Big|_{y=a}^{y=b}=c\,\frac{d}{dx}q_{\alpha/c}^{(x)}\Big|_{x=ac}^{x=bc}\\&=c\int_{ac}^{bc}Aq_{\alpha/c}(x)\,m\{dx\}\end{aligned}$$

for every $0<a<b<\infty$. But $Aq_{\alpha/c}=\alpha/c\,q_{\alpha/c}$, so

$$\begin{aligned}\int_a^b A^c q_\alpha^c(y)\,m^c\{dy\}&=\alpha\int_{ac}^{bc}q(x,\alpha/c)\,m\{dx\}\\&=\alpha\int_a^b q(cy,\alpha/c)\,m^c\{dy\}.\end{aligned}$$

Since this holds for all $0<a<b<\infty$, (5.16) follows from (5.1) and continuity of q.

Let $G_\alpha(x,y)=G(\alpha;x,y)$ be the Green function for the operator A of (5.3). That is let g_1, g_2 satisfy $g_1\in\uparrow$, $g_2\in\downarrow$,

$$\begin{aligned} Ag_1(x)&=\alpha g_1(x), \quad x\geqq 0\\ Ag_2(x)&=\alpha g_2(x), \quad x>0 \end{aligned} \tag{5.18}$$

$g_1^+(0)=0$, and $g_1^+ g_2 - g_1 g_2^+ \equiv 1$, then $G_\alpha(x,y)=g_1(x)\,g_2(y)$, $x<y$, and $G_\alpha(x,y)=g_2(x)\,g_1(y)$, $x\geqq y$. From [5], § 6.2 we get

$$\int_0^\infty (1-\mathrm{e}^{-\alpha x})\,\nu\{\mathrm{d}x\}=\mathrm{G}_\alpha(0,0)^{-1}=-g_2^+(0,\alpha)/g_2(0,\alpha). \tag{5.19}$$

Corollary to Lemma 2. *Let $\tilde{\nu}^c$ correspond to m^c, then*

$$\tilde{\nu}^c(x)=c\,\tilde{\nu}(cx) \tag{5.20}$$

Proof. From (5.19), (4.6) and an integration by parts, we have

$$\alpha\int_0^\infty e^{-\alpha x}\,\tilde{\nu}(x)\,dx=-g_2^+(0;\alpha)/g_2(0;\alpha). \tag{5.21}$$

From Lemma 2 and the definitions of g_1^c, g_2^c (at (5.18)) we get,

$$g_1^c(x;\alpha)=g_1(cx;\alpha/c), \qquad g_2^c(x;\alpha)=c^{-1}\,g_2(cx,\alpha/c). \tag{5.22}$$

From these formulas and a change of variable, we get

$$\begin{aligned}\alpha\int_0^\infty e^{-\alpha x}\,\tilde{\nu}^c(x)\,dx&=-g_2^{c+}(0;\alpha)/g_2^c(0;\alpha)\\ &=-c\,g_2^+(0;\alpha/c)/g_2(0;\alpha/c)=\int_0^\infty e^{-\alpha x}\,\tilde{\nu}(cx)\,c\,dx.\end{aligned}$$

We get (5.20) on peeling away the Laplace transform and noting that $\tilde{\nu}$, $\tilde{\nu}^c$ are continuous, [5], p. 217.

Let us now finish the proof of Theorem 4. If m^c is as in (5.17) and if m satisfies (5.6), then for every $c>0$

$$\frac{dm^c}{dm}(x)=c\,m'(cx)/m'(x)\to c^{\beta+1}, \qquad x\downarrow 0.$$

Consequently

$$\tilde{\nu}^c(\varepsilon)/\tilde{\nu}(\varepsilon/c^{\beta+1})\to 1, \qquad \varepsilon\downarrow 0,$$

by the corollary to Lemma 1. But then

$$c\,\tilde{\nu}(c\varepsilon)/\tilde{\nu}(\varepsilon/c^{\beta+1})\to 1$$

by (5.20) and the above. It follows that as $t\downarrow 0$

$$\tilde{\nu}(tx)/\tilde{\nu}(t)\to x^{-r}, \qquad r=(\beta+2)^{-1},$$

for every $x>0$. In other words under (5.6), $\tilde{v}$ is regularly varying at 0 with exponent $-r$ where $0<r\leqq 1$. Now

$$\int_0^\varepsilon x\,v\{dx\} = -\varepsilon\tilde{v}(\varepsilon) + \int_0^\varepsilon \tilde{v}(x)\,dx$$

and all three terms go to 0 as $\varepsilon\downarrow 0$ by (4.6). It is a basic property of regular varying functions that

$$\varepsilon\tilde{v}(\varepsilon)\Big/\int_0^\varepsilon \tilde{v}(x)\,dx \to 1-r \quad \text{as } \varepsilon\downarrow 0$$

Hence for ε sufficiently small

$$\int_0^\varepsilon x\,v\{dx\} \geqq b\varepsilon\tilde{v}(\varepsilon)$$

for some constant $b>0$. From this and the elementary $e^{-z}<kz^{-q}$, $z>0$, k a constant depending on q, we see that

$$\int_{0+} \exp\left(-a\int_0^\varepsilon x\,v\{dx\}/\varepsilon\right) v\{d\varepsilon\} \leqq k_1 \int_{0+} \tilde{v}(\varepsilon)^{-q}\,v\{d\varepsilon\}$$

for some $k_1<\infty$. Again from standard facts about regular variation, the latter integral will be seen to be finite as soon as $q>1$.

Corollary to Theorem 4. *Let now X be any recurrent diffusion on an interval Q with scale s and speed measure m. Let $x_0\in Q$. If $u\to m\circ s^{-1}(u+x_0)$ has a regularly varying derivative at 0, then (4.9) holds for $L(t,x_0)$.*

Remark. We know from the proof that

$$\tilde{v}(\varepsilon)=\varepsilon^{-r}S(1/\varepsilon), \qquad r=(\beta+2)^{-1},$$

where for every x,

$$S(tx)/S(t)\to 1 \quad \text{as } t\to\infty.$$

By hypothesis we also have

$$m'(x)=x^\beta L(1/x)$$

where

$$L(tx)/L(t)\to 1.$$

The question is what is the asymptotic relation between S and L? Lemma 1 does not seem to be strong enough to answer this except in the case that L is known to be a constant, then S is also a constant.

§6

In this section we construct an example of a diffusion which shows that (5.6) is not necessary for (4.9). The example is of independent interest inasmuch as we compute the asymptotic behavior of both the speed measure density $m'(x)$, $x\downarrow 0$, and of the tailsum $\tilde{\nu}(\varepsilon)$, $\varepsilon\downarrow 0$, of the Levy jump measure of the inverse local time. This example is patterned after an example in Ito and McKean, [5], § 6.4. Whenever possible we will use the details of their computations without comment.

In what follows C's are finite positive constants independent of any variables within the formulas in which they occur.

Fix $k>0$ and let

$$r(\theta)=r_k(\theta)\equiv\int_0^1 \theta^s s^{k-1}\Gamma(s+1)^{-1}\,ds, \qquad 0\leqq\theta<\infty. \tag{6.1}$$

Then

$$0<r'(\theta)=(1/\theta)\int_0^1 \theta^s s^{k-1}\Gamma(s)^{-1}\,ds\uparrow\infty \qquad \text{as } \theta\downarrow 0. \tag{6.2}$$

Let us write $\theta(x)=r^{-1}(x)$ (functional inverse of r). Then

$$\theta(0)\equiv\theta(0+)=\theta'(0+)=0$$

and, since $\lambda\to r(e^{-\lambda})$ has a positive second derivative we also have

$$\theta''(x)\,\theta'(x)^{-2}<\theta(x)^{-1}, \qquad x>0. \tag{6.4}$$

Lemma 1. *As $\theta\downarrow 0$, $x\downarrow 0$ we have*

$$r(\theta)=\Gamma(k)\,|\lg\theta|^{-k}(1+O(|\lg\theta|^{-1})), \tag{6.5}$$

$$r'(\theta)=(1/\theta)\,\Gamma(k+1)\,|\lg\theta|^{-k-1}(1+O(|\lg\theta|^{-1})), \tag{6.6}$$

$$\theta(x)=\exp\{-c_0 x^{-1/k}(1+O(x^{1/k}))\}, \tag{6.7}$$

$$\theta'(x)=c_1 x^{-(1/k)-1}\exp\{-c_0 x^{-1/k}(1+O(x^{1/k}))\}\,(1+O(x^{1/k})). \tag{6.8}$$

Proof. By straightforward calculus methods, we obtain as $\theta\downarrow 0$

$$J_k(\theta)\equiv\int_0^1 \theta^s s^{k-1}\,ds=\Gamma(k)\,|\lg\theta|^{-k}(1+O(\theta|\lg\theta|^{k-1})). \tag{6.9}$$

By Taylor's formula $\Gamma(s+1)^{-1}=1+O(s)$, hence $r_k(\theta)=J_k(\theta)+O(J_{k+1}(\theta))$. Apply (6.9) to get (6.5) then (6.6) follows from (6.5) and $r_k'(\theta)=(1/\theta)\,r_{k+1}(\theta)$. (6.7) and (6.8) are easily checked by substituting these expressions into $r(\theta(x))=x$ and $r'(\theta(x))\,\theta'(x)=1$ and applying (6.5)-(6.6).

Now let X be the recurrent natural scale diffusion on $[0,\infty)$ reflecting at 0 with speed measure

$$m\{dx\}=\theta'(x)^2\,dx, \qquad x>0,\ m\{0\}=0. \tag{6.10}$$

From (6.7)-(6.8) we have for x near 0

$$m'(x)=c_1^2 x^{-(2/k)-2}\exp\{-2c_0 x^{-1/k}(1+O(x^{1/k}))\}\,(1+O(x^{1/k})). \tag{6.11}$$

The exponential term forces (5.5) but destroys (5.6), nevertheless we will show that (4.9) still holds provided $k>1$. One should note that the Ito-McKean example is the special case $k=1$ for which (4.9) fails.

To continue, let g_1, g_2 be the increasing and decreasing solutions to $Ag \equiv (\theta')^{-2} g = \alpha g$ exactly as in (5.18) but with the additional initial condition $g_1(0)=1$. Then

$$g_2(x) = g_1(x)^{-1} \int_x^\infty g_1(s)^{-2}\, ds, \qquad x \geqq 0. \tag{6.12}$$

Define g_3 on the range of θ by $g_3(\theta)=g_1(x)$, $\theta=\theta(x)$. Then

$$g_3''(\theta) + \theta''(x)\,\theta'(x)^{-2}\, g_3'(\theta) = \alpha g_3(\theta), \tag{6.13}$$

$$g_3(0)=1, \qquad g_3^+(0)=0 \quad \text{and} \tag{6.14}$$

$$g_3''(\theta) < \alpha g_3(\theta) < g_3''(\theta) + (1/\theta)\, g_3'(\theta), \qquad \theta>0, \tag{6.15}$$

by virtue of (6.4).

Lemma 2. *Let Y be a diffusion on $[0, \infty)$ with 0 an entrance point. Let D be the generator and D_0 the differential operator coinciding with D on the domain of D. Suppose $u \in C[0, \infty)$ is locally in the domain of D and satisfies* i) $u(0)=u(0+)=0$, ii) $D_0 u = \lambda u + f$ *on* $(0, \infty)$ *where* iii) $f>0$, $f \in C(0, \infty)$. *Then $u(\theta)>0$ for all $\theta>0$.*

Proof. Let $\hat{Y}$ be Y killed at time σ where σ is $\operatorname{Exp}(\lambda)$ independent of Y. Then $\hat{D}v = Dv - \lambda v (= D_0 v - \lambda v$ on $(0, \infty))$, $v \in \operatorname{domain}(\hat{D}) = \operatorname{domain}(D)$. Fix a and $x \in (0, a)$ and let $u=v$ on $(0, a)$ where $v \in \operatorname{domain}(D)$. Put $T = \min\{\eta_x, \sigma\}$ where $\eta_x = \min\{t: Y_t = x\}$. Applying Dynkin's formula and recalling $v(\hat{Y}_\sigma) = v(\Delta) = v(0) = u(0) = 0$ gives us

$$\hat{E}_0 \int_0^T \hat{D} v(\hat{Y}_t)\, dt = \hat{E}_0\, v(\hat{Y}_T) - v(0) = E_0[u(x), \eta_x < \sigma] = u(x)\, P_0(\eta_x < \sigma).$$

But $\hat{E}_0 \int_0^T \hat{D} v(\hat{Y}_t)\, dt = E_0 \int_0^T [Du(Y_t) - \lambda u(Y_t)]\, dt > 0$ by iii), the fact that $Y_t \in [0, x]$ for $t \leqq T$, and $P_0(T>0)=1$. Since $P_0(\eta_x < \sigma) > 0$, the conclusion follows.

We apply the lemma first with $D_0 = \frac{1}{2} d^2/dx^2$, $u(\theta) = \exp(\alpha^{\frac{1}{2}} \theta) - g_3(\theta)$, then with $D_0 = \frac{1}{2}(d^2/dx^2 + (1/\theta)\, d/dx$, $u(\theta) = g_3(\theta) - I_0(\alpha^{\frac{1}{2}} \theta)$ where I_0 is the usual modified Bessel function, $I_0(0)=1$. In both cases $\gamma = \frac{1}{2}\alpha$, and the fact that $f \equiv D_0 u - \frac{1}{2} \alpha u > 0$ on $(0, \infty)$ is from (6.15). The result is

$$I_0(\alpha^{\frac{1}{2}} \theta(x)) \leqq g_1(x) \leqq \exp(\alpha^{\frac{1}{2}} \theta(x)), \qquad x \geqq 0, \tag{6.16}$$

and thus, see (6.12) and [5] p. 221,

$$\int_0^\infty \exp(-\beta^{\frac{1}{2}} \theta(x))\, dx \leqq g_2(0) \leqq \beta^{-\frac{1}{2}} \lg \beta + \int_0^\infty \exp(-(\gamma\beta)^{\frac{1}{2}} \theta(x))\, dx \tag{6.17}$$

where $\beta = 4\alpha$ and $\gamma \uparrow 1$ as $\alpha \uparrow \infty$, γ independent of x. But, see (6.2), (6.9),

$$\int_0^\infty \exp(-\beta^{\frac{1}{2}}\theta(x))\,dx = \int_0^\infty \exp(-\beta^{\frac{1}{2}}\theta)\,r'(\theta)\,d\theta$$
$$= \int_0^1 \beta^{-\frac{1}{2}s} s^{k-1}\,ds = J_k(\beta^{-\frac{1}{2}})$$
$$= 2^k \Gamma(k)(\lg\beta)^{-k}(1+O(\beta^{-\frac{1}{2}}(\lg\beta)^{k-1})),$$

as $\beta\uparrow\infty$, and it follows that

$$c_2^{-1}(\lg\beta)^{-k}(1-c_3\beta^{-\frac{1}{2}}(\lg\beta)^{k-1}) \leqq g_2(0) \leqq c_2^{-1}(\lg\gamma\beta)^{-k}(1+c_4\beta^{-\frac{1}{2}}(\lg\gamma\beta)^{k-1}), \quad (6.18)$$

for all β sufficiently large. But, see (5.19),

$$g_2(0)^{-1} = (g_1(0)\,g_2(0))^{-1} = \alpha\int_0^\infty e^{-\alpha x}\,\tilde{v}(x)\,dx,$$

so

$$c_2(\lg\gamma\beta)^k(1-c_4\beta^{-\frac{1}{2}}(\lg\gamma\beta)^{k-1}) \leqq \alpha\int_0^\infty e^{-\alpha x}\,\tilde{v}(x)\,dx \leqq c_2(\lg\beta)^k(1+c_5\beta^{-\frac{1}{2}}(\lg\beta)^{k-1}), \quad (6.19)$$

where $\beta=4\alpha$. (If the reader is also following the example in [5], he should note that γ in 7), p. 221, must be adjusted to account for the term $\alpha^{-\frac{1}{2}}\lg\alpha$ in 5).) Since $\tilde{v}$ is convex, $\alpha\int_0^\infty e^{-\alpha x}\,\tilde{v}(x)\,dx > \tilde{v}(1/\alpha)$, and then

$$\tilde{v}(x) \leqq c_2\,|\lg\tfrac{1}{4}x|^k + c_6\,\tfrac{1}{2}x^{\frac{1}{2}}\,|\lg\tfrac{1}{4}x|^{2k-1} \equiv q(\tfrac{1}{4}x) \quad (6.20)$$

for all x sufficiently small. By choosing c_6 sufficiently large (but don't touch c_2!) we may suppose (6.20) valid for all $x>0$. Also q and $\tilde{v}$ are nonincreasing near 0, so

$$q(1/\alpha)-\tilde{v}(2/\alpha) < \tfrac{1}{2}\alpha\int_{1/2\alpha}^{1/\alpha}[q(x)-\tilde{v}(4x)]\,dx \leqq \tfrac{1}{2}e\alpha\int_0^\infty e^{-\alpha x}[q(x)-\tilde{v}(4x)]\,dx.$$

But by (6.19)

$$\alpha\int_0^\infty e^{-\alpha x}\,\tilde{v}(4x)\,dx = \tfrac{1}{4}\alpha\int_0^\infty e^{-\frac{1}{4}\alpha x}\,\tilde{v}(x)\,dx \geqq c_2(\lg\gamma\alpha)^k - c_7\,\alpha^{-\frac{1}{2}}(\lg\gamma\alpha)^{2k-1}$$

where $\gamma\uparrow 1$ as $\alpha\uparrow\infty$, and

$$\alpha\int_0^\infty e^{-\alpha x}\,q(x)\,dx = c_2\int_0^\infty e^{-x}|\lg x-\lg\alpha|^k\,dx + c_6\,\alpha^{-\frac{1}{2}}\int_0^\infty e^{-x}|\lg x-\lg\alpha|^{2k-1}\,dx.$$

Combining these inequalities and noting the elementary

$$|A|^k - |B|^k \leqq k|A \pm B|\,(|A|^{k-1} + |B|^{k-1}),$$

$$\int_0^\infty e^{-x}|\lg x|^a\,|\lg x - \lg \alpha|^b\,dx = O(\lg^b \alpha), \qquad \alpha\uparrow\infty,\ a, b \geqq 0,$$

we get

$$\begin{aligned}
q(1/\alpha) - \tilde{v}(2/\alpha) &= O\left(\int_0^\infty e^{-x}(\lg x - \lg\alpha|^k - |\lg\alpha + \lg\gamma|^k)\,\mathrm{dx}\right.\\
&\quad + O\left(\alpha^{-\frac{1}{2}}\int_0^\infty e^{-x}|\lg x - \lg\alpha|^{2k-1}\,dx\right)\\
&\quad + O(\alpha^{-\frac{1}{2}}|\lg\gamma\alpha|^{2k-1})\\
&= O\left(\int_0^\infty e^{-x}|\lg x\gamma|\,(|\lg x - \lg\alpha|^{k-1} + |\lg\gamma\alpha|^{k-1})\,dx\right.\\
&\quad + O(\alpha^{-\frac{1}{2}}|\lg\alpha|^{2k-1})\\
&= O(\lg\alpha)^{k-1}, \qquad \alpha\uparrow\infty.
\end{aligned}$$

In other words for x near 0

$$\begin{aligned}
\tilde{v}(x) &\geqq c_2|\lg\tfrac{1}{2}x|^k - c_8|\lg x|^{k-1}\\
&\geqq c_2|\lg\tfrac{1}{4}x|^k - c_9|\lg x|^{k-1},
\end{aligned} \tag{6.21}$$

or, if we combine (6.20) and (6.21)

$$|\tilde{v}(x) - c_2|\lg\tfrac{1}{4}x|^k| = O(|\lg\tfrac{1}{4}x|^{k-1}) \tag{6.22}$$

as $x\downarrow 0$. On setting $h(t) = \tilde{v}^{-1}(t)$, as before, we get that for t sufficiently large

$$c_{10}\exp(-(t/c_2)^{1/k}) \leqq h(t) \leqq c_{11}\exp(-(t/c_2)^{1/k}), \tag{6.23}$$

$$c_{12}\,t^{1-(1/k)}\,h(t) \leqq H(t) \equiv \int_t^\infty h(s)\,ds \leqq c_{13}\,t^{1-(1/k)}\,h(t). \tag{6.24}$$

Therefore, if $a_1 = c_{12}\,a$, then

$$\begin{aligned}
\int_{0+} \exp\left(-a\int_0^\varepsilon x\,v\{dx\}/\varepsilon\right)d\varepsilon &= \int^\infty \exp(-aH(t)/h(t)\,dt\\
&\leqq \int^\infty \exp(-a_1 t^{1-(1/k)})\,dt < \infty
\end{aligned} \tag{6.25}$$

for all $a > 0$ as soon as $k > 1$ and then (4.9) holds by Theorem 3.

Remark 1. A general class of diffusions for which the asymptotic behavior of both m' and $\tilde{v}$ can be computed may be constructed by replacing (6.1) with

$$r(\theta) = \int_0^1 \theta^s s^{k-1}\,L(s)\,ds \tag{6.26}$$

where L is slowly varying 0; for all $x>0$,

$$L(tx)/L(t)\to 1 \quad \text{as } t\downarrow 0 \tag{6.27}$$

Defining $\theta(x)$, $m'(x)=\theta'(x)^2$ as before and applying a standard Abelian Theorem, [4], Theorem 3, p. 445 (put $\theta=e^{\lambda}$), we obtain first that

$$m'(x)=b_1 L_*(x)^{-2/k} x^{-(2/k)-2} \exp\{-b_2 x^{-1/k}(1+O(1))\} \tag{6.28}$$

where b_1, b_2 are positive constants and L_* is a slowly varying function such that

$$L_*(x)\, L(x^{1/k} L_*(x)^{1/k})\to 1, \qquad x\downarrow 0, \tag{6.29}$$

(Such an L_* may be shown to exist and to be asymptotically unique.) Applying Abelian and Tauberian theorems to first estimate $g_2(\alpha;0)$, as $\alpha\uparrow\infty$, and then $\tilde{v}(\varepsilon)$, $\varepsilon\downarrow 0$, we end up with

$$\tilde{v}(\varepsilon)=b_3|\lg\varepsilon|^k L(|\lg\varepsilon|^{-1})^{-1}(1+O(1)). \tag{6.30}$$

Note that (6.28) and (6.30) are not as precise as (6.11) and (6.23) and, indeed, one cannot determine if the first two integrals in (6.25) converge or not. (From (6.30) one can only get $b_4\exp(-b_5 t^{1/k})\leqq h(t)\leqq b_6\exp(-b_7 t^{1/k})$ for some unknown constants b_4, b_5, b_6, b_7, and one must have $b_5=b_7$ to get anywhere.) If one could differentiate (6.30), which seems reasonable since $\tilde{v}'$ is monotone, one could in fact determine the convergence or divergence in (6.25). Unfortunately the right hand side is a slowly varying function which is a borderline case, i.e., if $\tilde{v}(\varepsilon)\sim\varepsilon^l S(\varepsilon)$ where S is slowly varying and $l>0$, and if $\tilde{v}'$ is monotone, then $\tilde{v}'(\varepsilon)\sim l\varepsilon^{l-1}S(\varepsilon)$. (See [4], p. 446, for the case $\varepsilon\to\infty$.) This result only allows $\tilde{v}'(\varepsilon)=o(S(\varepsilon))$ in the case $l=0$ which is not strong enough. One should be able to compare with the case when $L\equiv 1$ to conclude that (4.9) holds when $k>1$ in (6.26).

§7

It has recently come to my attention that B.E. Fristedt and S.J. Taylor have also obtained, by entirely different methods, conditions under which there will be some normalization $b(\varepsilon)$, not necessarily $b(\varepsilon)=\int_0^{\varepsilon} x\nu\{dx\}$, for which $W(t,\varepsilon)/b(\varepsilon)\to L(t)$ a.s. They did not consider the relationship of speed measures to such representations.

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