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An Invariance Principle for Dependent Random Variables

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In this paper we give a weak invariance principle for a class of dependent random variables which contains martingale-like sequences and φ-mixing sequences. Stationarity is not required. Nevertheless, throughout the paper, various conditions are required which are a weak form of stationarity.

1. Introduction and Definitions

McLeish (1975) proves an invariance principle (cf. Theorem (2.6), [8]) under an assumption on the conditional expectations of variables with respect to the distant past. In Sect. 2 of this paper we give an invariance principle similar to that of McLeish for another class of random variables under an "asymptotic martingale" type condition. In Sect. 3, this result is used to extend the invariance principle obtained by McLeish (Theorem (5.1), [8]) for martingales to martingale-like sequences. We also prove an invariance principle for φ-mixing sequences under a variety of conditions for the φ-mixing rate and for the L_2 moments. One of the corollaries of this last theorem improves Theorem (3.8) of [8], showing that condition c) in this theorem, namely

$$\frac{E\left(\sum_{i=k+1}^{k+n} x_i\right)^2}{n} \to 0 \quad \text{as } \min(k,n)\to\infty$$

may be removed.

Proofs of the results of this paper are given in Sect. 4. Let $(\Omega, \mathscr{F}, P)$ be a probability space and $\{\mathscr{F}_n;\ n\geqq 1\}$ an increasing sequence of sub-σ-algebras of $\mathscr{F}$. $\{X_n, \mathscr{F}_n;\ n\geqq 1\}$ is said to be a stochastic sequence if X_n is $\mathscr{F}_n$-measurable for each n. We will denote the convergence in L_p and weak convergence by $\to_{L_p}$ and $\Rightarrow$ respectively. We will denote

$$E(X_i|\mathscr{F}_m) \text{ by } E_m X_i \quad \text{and} \quad E^{1/p}|U|^p \text{ by } \|U\|_p.$$

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Let $\{X_n, \mathscr{F}_n;\ n \geqq 1\}$ be a stochastic sequence of square integrable random variables and put $S_n = \sum_{i=1}^{n} X_i$. Throughout the paper we assume that

(1.1) $$E\frac{S_n^2}{n} \to \sigma^2 \quad \text{as } n \to \infty$$

where σ is a positive constant.

Consider the space $D[0,1]$, the set of all functions on the interval $[0,1]$ which have left hand limits and are continuous from the right at every point. We endow this space with the Skorohod topology. Let $\mathscr{B}$ be the Borel σ-algebra in D and define a random function by

(1.2) $$W_n(t) = \frac{S_{[nt]}}{n^{1/2}\sigma}, \qquad t \in [0,1]$$

where $[x]$ is the greatest integer contained in x. This is a measurable map from $(\Omega, \mathscr{F})$ into $(D, \mathscr{B})$, and we will establish the weak convergence of W_n to the standard Brownian motion process on D, denoted by W in the sequel.

(1.3) *Definition* [12]. A sequence W_n of random elements of a metric space is said to be Renyi-mixing (R-mixing) with limiting process W if: $P(W_n \in \cdot \mid E)$ converges weakly to the measure $P(W \in \cdot)$ for every $E \in \mathscr{F}$ such that $P(E) > 0$.

R-mixing is a useful concept when passing from non-random to random invariance principles (cf. Billingsley, Theorem 17.2, [1]).

2. The Invariance Principle

The following conditions are suggested by Gordin's condition (see [4]).

(2.1) For every fixed m the sequence $\left\{U_{m,n} = \sum_{i=m+1}^{n} E_m X_i;\ n > m\right\}$ converges to a function U_m in $L_2(\Omega)$ norm as $n \to \infty$.

(2.2) The sequence $\{U_m^2;\ m \geqq 1\}$ is uniformly integrable.

(2.3) **Theorem.** *Let $\{X_n, \mathscr{F}_n;\ n \geqq 1\}$ be a stochastic sequence of square integrable random variables which satisfies* (2.1) *and* (2.2). *If $\{X_i^2;\ i \geqq 1\}$ is uniformly integrable, then $\{W_n;\ n \geqq 1\}$ is tight in D and any limit process is a.s. continuous.*

(2.4) **Theorem.** *Suppose in addition to the conditions of Theorem* (2.3) *that*

(2.5) $$E_{k-m}\frac{(S_{k+n} - S_k)^2}{n} \xrightarrow[L^1]{} \sigma^2$$

as $(m,k,n) \to \infty$ such that $m \leqq k$.

Then W_n is R-mixing with limit W, a standard Brownian motion process on D.

3. Applications

One can apply Theorem (2.4) to martingale-like sequences.

(3.1) *Definition* [10]. The stochastic sequence $\{S_n, \mathscr{F}_n;\ n \geqq 1\}$ will be called a martingale in the L_2 limit if S_n is square integrable and $E_m(S_n - S_m) \xrightarrow[L_2]{} 0$ as $n \geqq m \to \infty$.

This concept generalizes the notion of martingale; conditions of this type have been considered in [2], [7].

The following theorem extends the Theorem (5.1) of [8].

(3.2) **Theorem.** *Let* $\{X_n, \mathscr{F}_n;\ n \geqq 1\}$ *be a sequence of differences of a martingale in the* L_2 *limit such that the set* $\{X_n^2;\ n \geqq 1\}$ *is uniformly integrable and*

$$\frac{1}{n} \sum_{i=1}^{n} E_{k-m} X_{k+i}^2 \xrightarrow[L_1]{} \sigma^2 \tag{3.3}$$

as $(m, k, n) \to \infty$ *such that* $m \leqq k$.

Then W_n *is R-mixing having as limit a standard Brownian motion process.*

Another result is an invariance principle under a φ-mixing condition.

Let $\{X_n;\ n \geqq 1\}$ be a sequence of random variables and put $\mathscr{R}_n^m = \sigma(S_m - S_n)$, $\mathscr{F}_n = \sigma(X_l;\ 1 \leqq l \leqq n)$, $\mathscr{F}_0 = \{0, \Omega\}$. For each $m \geqq 0$ define

$$\varphi_m = \sup_{(n,j)} \sup_{(A \in \mathscr{F}_n,\ P(A) \neq 0,\ B \in \mathscr{R}_{n+m}^{n+m+j})} |P(B \mid A) - P(B)|.$$

We say that $\{X_n;\ n \geqq 1\}$ is φ-mixing if $\varphi_m \to 0$. Obviously we can take φ_m nonincreasing.

(3.4) **Theorem.** *Let* $\{X_n;\ n \geqq 1\}$ *be a* φ*-mixing sequence of square integrable random variables centered at expectations and* f *a function* $f\colon R^+ \to R^+$ *such that* $f(\varphi_k) \geqq \varphi_k$ *for all sufficiently large* k. *If:*

$$\sum_k f(\varphi_k) < \infty \tag{3.5}$$

and

$$\lim_{c \to \infty} \int_{\{X_i^2 > c\}} X_i^2 \, dP (1 + F(i)) = 0 \qquad \textit{uniformly in } i, \tag{3.6}$$

where $F(i) = \sum_{m=1}^{i-2} \frac{\varphi_m - \varphi_{m+1}}{f(\varphi_m)}$, *then* W_n *is R-mixing with limiting process* W.

This theorem gives a variety of conditions for the mixing rate related to the L_2 moment conditions.

(3.7) *Remark.* If $F(i)$ converges, then the condition (3.6) in the above theorem may be replaced by

(3.8) the set $\{X_i^2;\ i \geqq 1\}$ is uniformly integrable.

If we take for example for $x>0$, $f(x)=x^\alpha$ with $\alpha<1$ or

$$f(x)=\begin{cases} x\,|\ln x|^\beta \text{ with } \beta>1 \text{ and } x\in\left(0,\frac{1}{e}\right) \\ x \text{ for } x\geqq\frac{1}{e} \end{cases},$$

then

$$\sup_i F(i)\leqq\int_0^1 \frac{dx}{f(x)}<\infty.$$

The following corollary improves Theorem (3.8) of [8] in the sense that condition c) in that theorem (requiring (1.1) for translations along the sequence) may be dropped.

(3.9) **Corollary.** *Let $\{X_i;\, i\geqq 1\}$ be a φ-mixing sequence of L_2-integrable random variables, such that the set $\{X_i^2;\, i\geqq 1\}$ is uniformly integrable and $\varphi_n=0\left(\frac{1}{nL_n^2}\right)$ where L_n is a sequence satisfying*

a) $\sum_n \frac{1}{nL_n}<\infty$,

b) $L_n-L_{n-1}=0\left(\frac{L_n}{n}\right)$,

c) *L_n is eventually non-decreasing.*

Then W_n is R-mixing with limiting process W.

Proof. We define $f(\varphi_n)=\frac{1}{nL_n}$, and note that by Remark (3.7) to verify the conditions in Theorem (3.4) amounts to show that $F(i)$ converges. Under conditions of this corollary we have

$$\begin{aligned} F(i)&=\sum_{m=1}^{i-2}\frac{\varphi_m-\varphi_{m+1}}{f(\varphi_m)}\leqq 1+\sum_{m=2}^{i-2}\varphi_m\left(\frac{1}{f(\varphi_m)}-\frac{1}{f(\varphi_{m-1})}\right) \\ &=0\left\{\sum_{m=1}^{i-2}\frac{1}{mL_m^2}[mL_m-(m-1)L_{m-1}]\right\}=0\left(\sum_{m=1}^{i-2}\frac{1}{mL_m}\right) \end{aligned}$$

and the result follows.

The following corollary shows that the mixing rate used by McLeish in Corollary (2.11) of [9] may be also obtained from Theorem (3.4).

(3.10) **Corollary.** *If $\{X_i;\, i\geqq 1\}$ is a φ-mixing sequence of centered, L_2 integrable random variables such that the set $\{X_i^2;\, i\geqq 1\}$ is uniformly integrable and*

$$\sum_{k=1}^{\infty}\left(\sum_{n=1}^{k}\frac{1}{\varphi_n}\right)^{-1/2}<\infty$$

then W_n is R-mixing with limiting process W.

Proof. We define $f(\varphi_k)=\left(\sum_{n=1}^{k}\frac{1}{\varphi_n}\right)^{-1/2}$. The condition $\sum_{k=1}^{\infty} f(\varphi_k)<\infty$, implies that $n\cdot f(\varphi_n)\to 0$ as $n\to\infty$. Therefore $\varphi_k\leqq \frac{k}{\sum_{i=1}^{k}\frac{1}{\varphi_i}}\leqq f(\varphi_k)$ for all sufficiently large k.

We have:

$$F(n)=\sum_{k=1}^{n-2}\frac{\varphi_k-\varphi_{k+1}}{f(\varphi_k)}=\sum_{k=1}^{n-2} f(\varphi_k)(\varphi_k-\varphi_{k+1})\sum_{i=1}^{k}\frac{1}{\varphi_i}.$$

We denote

$$S'_0=0,\qquad S'_k=\sum_{j=1}^{k}(\varphi_j-\varphi_{j+1})\sum_{i=1}^{j}\frac{1}{\varphi_i}$$

and we note that $S'_k\leqq k$. Consequently:

$$\begin{aligned}F(n)&=\sum_{k=1}^{n-2} f(\varphi_k)(S'_k-S'_{k-1})\leqq\sum_{k=1}^{n-3}(f(\varphi_k)-f(\varphi_{k-1}))\,k+(n-2)f(\varphi_{n-2})\\&\leqq\sum_{k=1}^{n-3} f(\varphi_k)+(n-2)f(\varphi_{n-2}).\end{aligned}$$

Therefore $F(n)$ converges, and the corollary follows by Remark (3.7). □

Taking now $f(x)=x$ for $x>0$, we observe that $F(i)\leqq\int_{\varphi_i}^{\varphi_1}\frac{dx}{x}\leqq|\ln\varphi_i|$ and by Theorem (3.4) follows:

(3.11) **Corollary.** *If $\{X_i;\, i\geqq 1\}$ is a φ-mixing sequence of L_2 integrable random variables centered at expectations, satisfying*

a) $\sum_{k=1}^{\infty}\varphi_k<\infty$,

b) $\lim_{c\to\infty}\int_{\{X_i^2>c\}} X_i^2\,dP\,|\ln(\varphi_i)|=0$ *uniformly in i, then W_n is* R-mixing with limiting process W.

4. Proofs

(4.1) **Lemma.** *Let $\{X_n,\mathscr{F}_n;\, n\geqq 1\}$ be a stochastic sequence satisfying* (2.1). *Then we have*

(4.2) $S_m=Z_m-U_m$ *for every* $m\geqq 1$
where U_m is defined by (2.1) *and $\{Z_m,\mathscr{F}_m;\, m\geqq 1\}$ is a martingale.*

Proof. Put $Z_m=S_m+U_m$. Clearly

$$E_{m-1}(S_m+U_{m,n})=S_{m-1}+U_{m-1,n}.$$

But, by (2.1) the left side of the above equality converges in L_2 to $E_{m-1}Z_m$ and the right side to Z_m.

(4.3) **Lemma.** *Let $\{X_n, \mathcal{F}_n;\ n \geqq 1\}$ be a stochastic sequence satisfying* (2.1) *and* (2.2). *If $\{X_i^2;\ i \geqq 1\}$ is uniformly integrable then the set*

$$\left\{\max_{j \leqq n} \frac{(S_{j+k}-S_k)^2}{n};\ k \geqq 1, n \geqq 1\right\} \tag{4.4}$$

is uniformly integrable.

Proof. Using the definition of Z_n,

$$\max_{j \leqq n} \frac{(S_{j+k}-S_k)^2}{n} \leqq 2\left(\max_{j \leqq n} \frac{(Z_{j+k}-Z_k)^2}{n} + \max_{j \leqq n} \frac{(U_{j+k}-U_k)^2}{n}\right).$$

But

$$\max_{j \leqq n} \frac{(U_{j+k}-U_k)^2}{n} \leqq 2\left(\sum_{j=k}^{k+n} \frac{U_j^2}{n} + U_k^2\right).$$

Therefore on account of Theorem 20, p. 36 of [6], the set

$$\left\{\max_{j \leqq n} \frac{(U_{j+k}-U_k)^2}{n};\ k \geqq 1,\ n \geqq 1\right\}$$

is uniformly integrable.

Now, again by (4.2) we have:

$$(Z_k - Z_{k-1})^2 \leqq 3(X_k^2 + U_k^2 + U_{k-1}^2)$$

for all $k \geqq 1$, whence by the hypothesis of this lemma it follows that the set $\{(Z_k - Z_{k-1})^2;\ k \geqq 1\}$ is uniformly integrable.

The proof of the fact that

$$\left\{\max_{j \leqq n} \frac{(S_{k+j}-S_k)^2}{n};\ k \geqq 1,\ n \geqq 1\right\}$$

is uniformly integrable is similar to that of Theorem 23.1 of Billingsley, [1], where instead of stationarity, we use the uniform integrability of the martingale differences $\{(Z_k - Z_{k-1})^2;\ k \geqq 1\}$.

Proof of Theorem (2.3). By [1], Theorem 8.4 adapted to D, the tightness condition will follow if we prove

$$\lim_{\lambda \to \infty} \lambda^2 P(\max_{j \leqq n} |S_{k+j} - S_k| > \lambda n^{1/2}) = 0$$

uniformly in (n, k). This follows from the uniform integrability of the set

$$\left\{\max_{j \leqq n} \frac{(S_{k+j}-S_k)^2}{n};\ k \geqq 1,\ n \geqq 1\right\}$$

which is shown in Lemma (4.3). Theorem 15.5 of [1] also shows that any weak limit process of W_n must be a.s. concentrated on the continuous functions.

Let d be the Skorohod's metric on D.

(4.5) **Lemma.** *If Z_n is R-mixing with limiting process W and $d(W_n, Z_n) \xrightarrow[P]{} 0$, then W_n is R-mixing with the same limiting process W.*

This is a minor extension of Theorem 4.1 of [1] and may be proved in a similar fashion.

Let Z_n be defined by (4.2) and put

$$V_n(t) = \frac{Z_{[nt]}}{n^{1/2}\sigma}, \qquad t \in [0,1]$$

V_n is a random function from $(\Omega, \mathscr{F})$ into $(D, \mathscr{B})$.

(4.6) **Lemma.** *Let $\{X_n, \mathscr{F}_n;\ n \geqq 1\}$ be a stochastic sequence satisfying* (2.1) *and* (2.2). *If V_n is R-mixing with limiting process W, then W_n is R-mixing with the same limiting process W.*

Proof. To use Lemma (4.5) it is enough to show that

$$P(d(W_n, V_n) \geqq \varepsilon) \to 0 \qquad \text{as } n \to \infty$$

for each positive ε.

Clearly

$$P(d(W_n, V_n) > \varepsilon) \leqq P\left(\sup_{1 \leqq k \leqq n} \frac{|U_k|}{n^{1/2}\sigma} > \varepsilon\right).$$

We have

$$P\left(\sup_{1 \leqq k \leqq n} \frac{|U_k|}{n^{1/2}\sigma} > \varepsilon\right)$$
$$\leqq \sum_{k=1}^{n} P\left(\frac{|U_k|}{n^{1/2}\sigma} > \varepsilon\right) \leqq \sum_{k=1}^{n} \frac{1}{\varepsilon^2 n \sigma^2} \int_{\{U_k^2 > n\sigma^2\varepsilon^2\}} U_k^2 \, dP.$$

Because the set $\{U_m^2;\ m \geqq 1\}$ is uniformly integrable we have that for every $\varepsilon > 0$:

$$P(d(W_n, V_n) > \varepsilon) < \varepsilon.$$

Proof of Theorem (2.4). We shall prove that the conditions of Theorem (5.1) of McLeish (1975) are satisfied for martingale differences $\{Z_n - Z_{n-1};\ n \geqq 1\}$.

The first condition is

a) The set $\{(Z_n - Z_{n-1})^2;\ n \geqq 1\}$ is uniformly integrable. This is already established in the proof of Lemma (4.3).

We verify now the second condition.

b) $\frac{1}{n} \sum_{i=1}^{n} E_{k-m}(Z_{k+i} - Z_{k+i-1})^2 \xrightarrow[L_1]{} \sigma^2$ as $(m, k, n) \to \infty$ such that $m \leqq k$.

By (4.2) and the Cauchy-Schwarz inequality, we have

$$\frac{1}{n} E\,|(Z_{k+n} - Z_k)^2 - (S_{k+n} - S_k)^2|$$
$$\leqq \frac{1}{n} E(U_{k+n} - U_k)^2 + \frac{2}{n} \|U_{k+n} - U_k\|_2 \cdot \|S_{k+n} - S_k\|_2.$$

On account of the fact that the set $\{U_k^2;\ k \geqq 1\}$ is uniformly integrable $\frac{1}{n} E(U_{k+n}-U_k)^2 \to 0$ as $n \to \infty$, and by Lemma (4.3) the sequence $\left\{\frac{1}{n} E(S_{k+n}-S_k)^2,\ n \geqq 1\right\}$ is bounded. Then

$$\frac{1}{n} E|(Z_{k+n}-Z_k)^2-(S_{k+n}-S_k)^2| \to 0, \qquad \text{as } n \to \infty \tag{4.7}$$

and b) follows from (2.5).

The conditions of Theorem (5.1) of [8] are satisfied. Therefore V_n is R-mixing with limit W, where W is a standard Brownian motion process on D.

Proof of Theorem (3.2). If $\{X_n;\ n \geqq 1\}$ is a sequence of differences of a martingale in the L_2 limit then $\|U_{m,n}\|_2 \to 0$ as $n > m \to \infty$. For fixed m and $n \leqq n'$, we have

$$\|U_{m,n}-U_{m,n'}\|_2 \leqq \|U_{n,n'}\|_2 \to 0 \qquad \text{as } n' \geqq n \to \infty.$$

Therefore $\{U_{m,n};\ n \geqq m\}$ converges in L_2 for m fixed, to a stochastic sequence $\{U_m,\ \mathscr{F}_m;\ m \geqq 1\}$ and so the condition (2.1) is satisfied. Because $\|U_{m,n}\|_2 \to 0$ as $n > m \to \infty$, it follows that $\{U_m;\ m \geqq 1\}$ also converges in L_2 to 0, whence (2.2) follows.

It remains to verify the condition (2.5). By (2.1) and the Cauchy-Schwarz inequality we have

$$E|(Z_i-Z_{i-1})^2-X_i^2| \leqq E(U_{i+1}-U_i)^2+2\|X_i\|_2\ \|U_{i+1}-U_i\|_2.$$

Therefore $(Z_i-Z_{i-1})^2-X_i^2 \xrightarrow[L_1]{} 0$ and by (3.3) and (4.7), (2.5) holds.

From the proof of Theorem 2.2 of [13], we deduce the following:

(4.8) **Lemma.** *If* $\{X_n;\ n \geqq 1\}$ *is a sequence of random variables centered at expectations such that for some* C, $|X_i| \leqq C$ *a.s., then for* $m \leqq i$

$$\|I_A E(X_i|\mathscr{F}_m)\|_2 \leqq 2CP(A)^{1/2} \varphi_{i-m} \tag{4.9}$$

for every $A \in \mathscr{F}$.

(4.10) **Lemma.** *If the sequence* $\{X_i;\ i \geqq 1\}$ *satisfies the conditions of Theorem* (3.4) *then the set* $\left\{\max_{i \leqq n} \frac{1}{n}(S_{k+i}-S_k)^2;\ k \geqq 1,\ n \geqq 1\right\}$ *is uniformly integrable.*

Proof. For positive C put

$$X_i^c = X_i\, I_{\{|X_i| \leqq C\}},$$

$$Y_i = X_i^c - EX_i^c$$

and

$$V_i = X_i - X_i^c + E(X_i^c - X_i).$$

Note that $X_i = Y_i + V_i$.

Let us use the notation $E^{(y)} U = \int_{\{U > y\}} U\, dP$,

$$\bar{Y}_k(j) = \sum_{i=k+1}^{k+j} Y_i, \quad \bar{V}_k(j) = \sum_{i=k+1}^{k+j} V_i, \quad \text{and} \quad \bar{S}_k(j) = S_{k+j} - S_k.$$

Then

$$\bar{S}_k^2(j) \leqq 2(\bar{Y}_k^2(j) + \bar{V}_k^2(j))$$

and hence

$$E^{(y)}\left(\max_{j\leqq n} \frac{\bar{S}_k^2(j)}{n}\right) \leqq \frac{4}{n} E^{(y/4)}\left(\max_{j\leqq n} \bar{Y}_k^2(j)\right) + \frac{4}{n} E\left(\max_{j\leqq n} \bar{V}_k^2(j)\right).$$

On account of the fact that

$$V_i = \sum_{m=0}^{i-1} (E_{i-m} V_i - E_{i-m-1} V_i)$$

we have

$$\bar{V}_k(j) = \sum_{m=0}^{k} Y_{k,m}(j) + \sum_{m=k+1}^{k+j-1} Z_{k,m}(j)$$

where

$$(4.11)\qquad \begin{cases} Y_{k,m}(j) = \sum_{i=k+1}^{k+j} (E_{i-m} V_i - E_{i-m-1} V_i) \\ Z_{k,m}(j) = \sum_{i=m+1}^{k+j} (E_{i-m} V_i - E_{i-m-1} V_i). \end{cases}$$

Obviously for fixed (k,m), $m \leqq k$, $\{Y_{k,m}(j),\ \mathscr{F}_{k+j-m},\ j \geqq 1\}$ and $\{Z_{k,m}(j),\ \mathscr{F}_{k+j-m},\ j \geqq 1\}$ are martingales.

We have

$$E\left(\max_{j\leqq n} \frac{\bar{V}_k^2(j)}{n}\right) \leqq 2\left[E \frac{\left(\sum_{m=0}^{k} \max_{j\leqq n} |Y_{k,m}(j)|\right)^2}{n} + E \frac{\left(\sum_{m=k+1}^{k+n-1} \max_{j\leqq n} |Z_{k,m}(j)|\right)^2}{n}\right].$$

Whence by Cauchy-Schwartz inequality,

$$E\left(\max_{j\leqq n} \frac{\bar{V}_k^2(j)}{n}\right) \leqq \frac{2}{n}\left[\sum_{m=0}^{k} f(\varphi_m) \sum_{m=0}^{k} \frac{1}{f(\varphi_m)} E\left(\max_{j\leqq n} Y_{k,m}^2(j)\right) + \sum_{m=k+1}^{k+n-1} f(\varphi_m) \sum_{m=k+1}^{k+n-1} \frac{1}{f(\varphi_m)} E\left(\max_{j\leqq n} Z_{k,m}^2(j)\right)\right].$$

Using Doob's inequality (p. 317 of [3]), (3.5) and (4.11), we obtain for some positive constant K,

$$\begin{aligned} E\left(\max_{j\leqq n} \frac{\bar{V}_k^2(j)}{n}\right) &\leqq \frac{K}{n}\left\{\sum_{m=0}^{k} \frac{1}{f(\varphi_m)} \sum_{i=k+1}^{k+n} [E(E_{i-m} V_i)^2 - E(E_{i-m-1} V_i)^2] \right. \\ &\quad \left. + \sum_{m=k+1}^{k+n-1} \frac{1}{f(\varphi_m)} \sum_{i=m+1}^{k+n} [E(E_{i-m} V_i)^2 - E(E_{i-m-1} V_i)^2]\right\} \\ &= \frac{K}{n} \sum_{i=k+1}^{k+n} \sum_{m=0}^{i-1} \frac{1}{f(\varphi_m)} [E(E_{i-m} V_i)^2 - E(E_{i-m-1} V_i)^2] \end{aligned}$$

This inequality yields.

$$E\left(\max_{j\leqq n}\frac{\bar{V}_k^2(j)}{n}\right)$$
$$\leqq\frac{K}{n}\sum_{i=k+1}^{k+n}\left[\frac{EV_i^2}{f(\varphi_0)}+\sum_{m=1}^{i-1}\left(\frac{1}{f(\varphi_m)}-\frac{1}{f(\varphi_{m-1})}\right)E(E_{i-m}V_i)^2\right].$$

By Theorem (2.2) of [13]

$$E(E_{i-m}V_i)^2\leqq 4\varphi_m EV_i^2.$$

Therefore, for some constant K_1

$$E\left(\max_{j\leqq n}\frac{\bar{V}_k^2(j)}{n}\right)$$
$$\leqq\frac{K_1}{n}\sum_{i=k+1}^{k+n}E(X_i-X_i^c)^2\left[\frac{1}{f(\varphi_0)}+\sum_{m=1}^{i-1}\varphi_m\left(\frac{1}{f(\varphi_m)}-\frac{1}{f(\varphi_{m-1})}\right)\right]$$
$$\leqq\frac{K_1}{n}\sum_{i=k}^{k+n}E(X_i-X_i^c)^2\left[\frac{1}{f(\varphi_0)}+\frac{\varphi_{m-1}}{f(\varphi_{m-1})}+\sum_{m=1}^{i-2}\frac{1}{f(\varphi_m)}(\varphi_m-\varphi_{m+1})\right].$$

The convergence in (3.6) being uniformly in i, we may choose and fix C, such that

$$E\left(\max_{j\leqq n}\frac{\bar{V}_k^2(j)}{n}\right)\leqq\frac{\varepsilon}{8}\qquad\text{for every } n\geqq 1 \text{ and } k\geqq 1.$$

For this fixed value of C, we apply Lemma (4.3) to the sequence Y_i. By Lemma (4.8) for $n'\geqq n\geqq m$, we have

$$\left\|\sum_{i=n}^{n'}E(Y_i|\mathscr{F}_m)\right\|_2\leqq 2C\sum_{i=n}^{n'}\varphi_{i-m}.$$

Using (3.5) and the fact that $f(\varphi_k)\geqq\varphi_k$ for all sufficiently large k it follows that $\left\{\sum_{i=m+1}^{n}E(Y_i|\mathscr{F}_m), n\geqq m\right\}$ is Cauchy in L_2 for m fixed, and therefore (2.1) is verified. By the same argument it follows that for all $m\leqq n$

$$\left\|\sum_{i=m+1}^{n}E(Y_i|\mathscr{F}_m)\right\|_2\leqq 2C\sum_{k=1}^{\infty}\varphi_k$$

which implies that the limit sequence $\{U_m^2;\ m\geqq 1\}$ is uniformly bounded in L_1. This and (4.9) shows that the sequence $\{U_m^2;\ m\geqq 1\}$ is uniformly integrable.

By Lemma (4.3) it follows that the sequence

$$\left\{\max_{j\leqq n}\frac{\bar{Y}_k^2(j)}{n};\ n\geqq 1;\ k\geqq 1\right\}$$

is uniformly integrable, and therefore we may choose y sufficiently large such that

$$E^{(y/4)}\left(\max_{j\leqq n}\frac{\bar{Y}_k^2(j)}{n}\right)\leqq\varepsilon/8\qquad\text{for every } n\geqq 1 \text{ and } k\geqq 1.$$

Then, for this value of y

$$E^{(y)}\left(\max_{j\leqq n}\frac{\bar{S}_k^2(j)}{n}\right)\leqq\varepsilon \quad \text{for every } n\geqq 1 \text{ and } k\geqq 1.$$

So:

$$\left\{\max_{j\leqq n}\frac{1}{n}(S_{k+j}-S_k)^2;\ n\geqq 1;\ k\geqq 1\right\} \tag{4.12}$$

is uniformly integrable.

For the proof of Theorem (3.4) we need the following theorem (Theorem 19.2, [1]).

(4.13) **Theorem.** *Let $\{X_n,\ n\geqq 1\}$ be a sequence of random functions in D with asymptotically independent increments such that $\{X_n^2(t);\ n\geqq 1\}$ is uniformly integrable for each t, $E(X_n(t))\to 0$ and $E(X_n^2(t))\to t$ as $n\to\infty$. Suppose that for each positive ε and η, there exists a positive δ such that for all sufficiently large n*

$$P(w(X_n,\delta)\geqq\varepsilon)\leqq\eta \tag{4.14}$$

where $w(x,\delta)$ is the modulus of continuity of x. Then $X_n\Rightarrow W$.

Proof of Theorem (3.4). Since $\bigcup_{n=1}^{\infty}\mathscr{F}_n$ is an algebra generating $\sigma(W_1,W_2,\ldots)$ we need only to verify that for arbitrary m and $F\in\mathscr{F}_m$ with $P(F)>0$ it follows $P(W_n\in\cdot\,|F)\Rightarrow W$. We shall apply Theorem (4.13) with $P(.)$ replaced by $\hat{P}(.)=P(.\,|F)$, $\hat{E}(.)$ by $E(.|F)$. We have first to show that $0\leqq s_1\leqq t_1<s_2\leqq t_2\ldots<s_r\leqq t_r\leqq 1$ implies, for all linear Borel sets $H_1,\ldots,H_r$, that the difference

$$\hat{P}\{W_n(t_i)-W_n(s_i)\in H_i\ i=1,2,\ldots,r\}-\prod_{i=1}^{r}\hat{P}\{W_n(t_i)-W_n(s_i)\in H_i\}$$

converges to 0 as $n\to\infty$. This follows from the definition of φ-mixing sequences by induction on r, as in the proof of the Theorem 20.1 of [1], taking into account that for sufficiently large n, $F\in\mathscr{F}_{[nt_1]}$. Tightness of the measures $P(W_n\in.|F)$ and the uniform integrability of $\{W_n^2(t),\ n\geqq 1\}$ for each $t\in[0,1]$, follow directly from (4.12). To verify that $\hat{E}W_n(t)\to 0$ for every t, we note that for n sufficiently large $F\in\mathscr{F}_{[n^{1/2}t]}$.

Therefore

$$\begin{aligned}|\hat{E}\,W_n(t)|&=\frac{1}{P(F)}\left|\int_F\frac{S_{[nt]}}{n^{1/2}}\,dP\right|\\&\leqq\frac{1}{P(F)}\left|\int_F E_{[n^{1/2}t]}\frac{(S_{[nt]}-S_{2[n^{1/2}t]})\,dP}{n^{1/2}}\right|+\frac{1}{P(F)}E\frac{|S_{2[n^{1/2}t]}|}{n^{1/2}}\\&\leqq\frac{1}{P(F)}\left[2\varphi_{[n^{1/2}t]}^{1/2}\frac{\|S_{[nt]}-S_{2[n^{1/2}t]}\|_2}{n^{1/2}}+\frac{\|S_{2[n^{1/2}t]}\|_2}{n^{1/2}}\right.\end{aligned}$$

and using (1.1) it follows that $\hat{E}W_n(t)\to 0$ as $n\to\infty$.

By (1.1) $EW_n^2(t)\to t$ as $n\to\infty$. Using the fact that

$$S_{[nt]}^2-(S_{[nt]}-S_{2[n^{1/2}t]})^2 = S_{2[n^{1/2}t]}^2+2S_{2[n^{1/2}t]}(S_{[nt]}-S_{2[n^{1/2}t]})$$

we obtain by (1.1).

$$E\left|\frac{S_{[nt]}^2}{n}-\frac{(S_{[nt]}-S_{2[n^{1/2}t]})^2}{n}\right|\to 0 \quad \text{as } n\to\infty$$

and

$$\hat{E}\left|\frac{S_{[nt]}^2}{n}-\frac{(S_{[nt]}-S_{2[n^{1/2}t]})^2}{n}\right|\to 0 \quad \text{as } n\to\infty.$$

Therefore

$$\left||\hat{E}W_n^2(t)-EW_n^2(t)|-\left|\frac{\hat{E}(S_{[nt]}-S_{2[n^{1/2}t]})^2}{n}-\frac{E(S_{[nt]}-S_{2[n^{1/2}]})^2}{n}\right|\right|\to 0 \quad \text{as } n\to\infty. \tag{4.15}$$

If, for some positive C, we denote $A_C=\{|S_{[nt]}-S_{2[n^{1/2}t]}|\leqq C\}$ and $\bar{A}_C=\{|S_{[nt]}-S_{2[n^{1/2}t]}|>C\}$ we obtain for n sufficiently large

$$\begin{aligned}
&\left|\frac{\hat{E}(S_{[nt]}-S_{2[n^{1/2}t]})^2}{n}-\frac{E(S_{[nt]}-S_{2[n^{1/2}t]})^2}{n}\right| \\
&\leqq \frac{1}{P(F)}\int_F\left|E_{[n^{1/2}t]}\frac{(S_{[nt]}-S_{2[n^{1/2}t]})^2}{n}I_{A_C}-E\frac{(S_{[nt]}-S_{2[n^{1/2}t]})^2}{n}I_{A_C}\right| \\
&\quad+\hat{E}\frac{(S_{[nt]}-S_{2[n^{1/2}t]})^2}{n}I_{\bar{A}_C}+E\frac{(S_{[nt]}-S_{2[n^{1/2}t]})^2}{n}I_{\bar{A}_C} \\
&\leqq 2\varphi_{[n^{1/2}t]}C+\frac{2}{P(F)}E\frac{(S_{[nt]}-S_{2[n^{1/2}t]})^2}{n}I_{\bar{A}_C}.
\end{aligned}$$

By Lemma (4.10) we choose and fix C sufficiently large that the second term is less than $\frac{\varepsilon}{2}$. The first term can be made $<\frac{\varepsilon}{2}$ for n sufficiently large. Therefore by (4.15) $|\hat{E}W_n^2(t)-EW_n^2(t)|\to 0$ as $n\to\infty$ hence $\hat{E}W_n^2(t)\to t$ as $n\to\infty$. □

References

1. Billingsley, P.: Convergence of probability measures. New York: Wiley 1968
2. Blake, L.: A generalization of martingale and two consequent convergence theorems. Pacific J. Math. **35**, 2, 279–284 (1970)
3. Doob, I.L.: Stochastic processes. New York: Wiley 1953
4. Gordin, I.M.: The central limit theorem for stationary processes. Soviet. Math. Dokl. **10**, 1174–1176 (1969)

5. Iosifescu, M., Theodorescu, R.: Random processes and learning. Berlin Heidelberg New York: Springer 1969
6. Meyer, P.A.: Probabilités et potentiel. Paris: Hermann 1966
7. Mucci, A.G.: Another martingale convergence theorem. Pacific J. Math., **64**, 2, 539–542 (1976)
8. McLeish, D.L.: Invariance principles for dependent variables. Z. Wahrscheinlichkeitstheorie verw. Gebiete **32**, 165–178 (1975)
9. McLeish, D.L.: On the invariance principle for nonstationary mixingales, Ann. Probability, **5**, 616–621 (1977)
10. Peligrad, M.: Limit theorems and law of the large numbers for martingale-like sequences. Math. Nachr. **99**, (1981) [To appear]
11. Philipp, W., Webb, G.R.: An invariance principle for mixing sequences of random variables. Z. Wahrscheinlichkeitstheorie und Verw. Gebiete, **25**, 223–237 (1973)
12. Renyi, A.: On mixing sequences of random variables. Acta Math. Acad. Sci. Hungar. **9**, 389–393 (1958)
13. Serfling, R.J.: Contributions to central limit theory for dependent variables. Ann. Math. Statist. **39**, 1158–1175 (1968)

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