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Martin Boundaries for some Space-Time Markov Processes*

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1. Introduction and Summary

In this paper the general theory of the Martin boundary (or exit space) for Markov processes is applied to construct boundaries for several space-time processes. We shall consider the following processes in spacetime: Cauchy-, d-dimensional Bessel processes and Poisson processes composed with a symmetric binomial distribution. In [10] Wiener-, Poisson- and Gamma processes are also discussed. In all these cases the minimal part of the boundary will be determined using directly the definition of minimality.

Although the theory of the exit space of Markov processes is relatively old only a few boundaries for processes with continuous time and space are explicitly known. The aim of this note is to make this abstract construction more concrete, and perhaps convince readers of its great importance in the theory of stochastic processes.

2. Space Time Processes

Let $X=\{x(t)\}$, $t \geqq 0$, be a Markov process over some probability space $(\Omega, \mathscr{F}, \mathscr{P})$ taking values in a LCC space E; the σ-algebra generated by the open sets in E is denoted by $\mathscr{B}(E)$. If E is not compact, let Δ be a point not contained within E such that $E \cup \{\Delta\}$ is a one-point compactification of E; if E is compact, let Δ be an arbitrary point isolated from E. Further, let ζ be the lifetime of the process X, i.e., for $\omega \in \Omega$, $\zeta(\omega)=\inf\{t: x(t,\omega) \notin E\}$. We extend the process X to $E \cup \{\Delta\}$ by setting, in the case $\zeta(\omega)<\infty$,

$$x'(t,\omega):=\begin{cases} x(t,\omega) & \text{if } t \leqq \zeta(\omega), \\ \Delta & \text{if } t > \zeta(\omega) \end{cases}.$$

We maintain the notation X for this extended process.

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Let us construct a new process, $Y=\{y(t)\}$, $t \geqq 0$, over the same probability space by setting $y_t(\omega)=(t, x_t(\omega))$ for all $t \in T=[0, \infty)$ and $\omega \in \Omega$. It is clear that Y is also a Markov process and taking values in the product space $(T \times E, \mathscr{B}(T \times E))$. A stochastic process Y with this kind of a structure is called a *space-time process.*

Let $P(t, x, A)$, $t \in T$, $x \in E$, $A \in \mathscr{B}(E)$, and $P'(t, \mathbf{x}, \mathscr{A})$, $t \in T$, $\mathbf{x} \in T \times E$, $\mathscr{A} \in \mathscr{B}(T \times E)$, be the transition functions of the processes X and Y, respectively. From the construction of the process Y it follows that

$$P'(t, \mathbf{x}, \mathscr{A})=\begin{cases} P(t-s, x, A(t)), & \text{if } t \geqq s, \\ 0 \quad , & \text{otherwise,} \end{cases}$$

where $\mathbf{x}=(s, x)$ and $A(t)=\{x \in E:(t, x) \in \mathscr{A}\}$.

Let f be a non-negative, $\mathscr{B}(T \times E)$-measurable function and, for a fixed t, set $g^t := f(t, \cdot)$. Now, consider the function

$$\begin{aligned} P_t' f(\mathbf{x}) &= \int_{T \times E} P'(t, \mathbf{x}, du \times dy) f(u, y) \\ &= \int_E P(t-s, x, dy)\, g^t(y) := P_{t-s}\, g^t(x). \end{aligned}$$

f is called *space-time excessive* if for all $s \in T$ $P_{t-s}\, g^t \uparrow g^s$ pointwise as $t \downarrow s$, and *space-time invariant* if for all t, s, $t \geqq s$, $P_{t-s}\, g^t = g^s$.

3. The Exit Space of a Markov Process

The usual references for the Martin boundary theory of Markov processes are [2, 11, 6, 9, 3 and 4]. Using [4] we shall now list the basic assumptions under which the exit space can be constructed.

Let us consider the kernel

$$G(x, A) := \int_0^\infty P(t, x, A)\, dt, \qquad x \in E, \qquad A \in \mathscr{B}(E)$$

and assume that there exists a measure m on E which is finite in compact sets and such that

(i) $G(x, A)=\int_A g(x, y)\, m(dy)$,

where $g(x, y)$ is a non-negative, $\mathscr{B}(E) \times \mathscr{B}(E)$-measurable function such that

(ii) $\forall f \in C_c(E)$ (=continuous functions with compact support)

$$g(f, y)=\int_E m(dx) f(x)\, g(x, y)$$

is continuous and bounded by a constant which only depends on f.

A function g with these properties is called a *Green function* of the process X.

Further, we assume that there exists a finite measure γ such that

(iii) $q(y)=\int_E \gamma(dx)\, g(x, y)$

is positive and continuous,

(iv) there exists a positive, $\mathscr{B}(E)$-measurable function ψ such that

$$\gamma G\psi = \int_E \gamma(dx) \int_E g(x,y)\,\psi(y)\,m(dy) < \infty.$$

A measure γ with these properties is called *a standard (or reference) measure* of the process X. (Note that (iv) is a transience assumption.)

In general, the Green function need not be excessive. It is, however, possible to overcome this difficulty (see [4]) and, with this in mind, we assume that

(v) $\forall y \in E$ $g(\cdot, y)$ is excessive.

Set

$$K(x,y) = \begin{cases} \dfrac{g(x,y)}{q(y)} & \text{on } \{q < \infty\}, \\ 0 & \text{otherwise.} \end{cases}$$

The function $K(x,y)$ is called *the Martin function* of the process X.

Further, we make one more assumption which, usually, is a consequence of some other assumption:

(vi) $\forall f \in C_c(E)$ the function

$$K(f,y) = \int_E m(dx)\,f(x)\,K(x,y)$$

is bounded by a constant which only depends on f.

A Markov process X satisfying these assumptions together with certain regularity conditions is essentially Dynkin's special M-process (see [4]).

It can be proved that for a special M-process there exists a set $U \in \mathscr{B}(E^*)$ (E^* is a Martin compactification of E), called the *exit space* of the process X, which is in one-to-one correspondence with the set of all the *minimal excessive functions* of the process X. (An excessive function is called minimal if it cannot be represented in a non-trivial way as a linear combination of other excessive functions.) Given $y \in U$, the minimal excessive function, k_y, is either the Martin function $K(\cdot, y)$ or its extension.

Let h be any γ-integrable excessive function. Then there exists one and only one measure, σ^h, on U such that

$$h = \int_U k_y\,\sigma^h(dy).$$

The measure σ^h is called the *spectral measure* of the function h. Obviously, given $y \in U$, the spectral measure of the function k_y, denoted σ^y, must be Dirac's measure at y, denoted $\varepsilon_{\{y\}}$.

Further, let h be a γ-integrable excessive function and consider the transition function

$$P^h(t,x,A) = \frac{1}{h(x)} \int_A P(t,x,dy)\,h(y),$$

where $t \in T$, $x \in E^h = \{0 < h < \infty\}$, $A \in \mathscr{B}(E^h)$. The canonical realisation of a Markov process which has P^h as transition function is called the *h-process*. (Note that $h \equiv 1$ gives the original process X.)

Now, $\lim_{t\uparrow\zeta} x(t)$ exists in the *Martin topology* $\mathscr{P}^h_{\gamma^h}$ – a.s. where $\gamma^h = h\,d\gamma$ and $\mathscr{P}^h_{\cdot}$ are the probability measures induced by the h-process. Denote the limit variable by z_ζ, then it can be proved that

$$\sigma^h(\cdot) = \mathscr{P}^h_{\gamma^h}(z_\zeta \in \cdot).$$

Note that for $y \in U$ the k_y-process converges $\mathscr{P}^y_{\gamma y}$ – a.s. to the point y.

Before we start with the examples let us see how this framework looks for a space-time process. Consider the kernel

$$\begin{aligned} G'(\mathbf{x}, \mathscr{A}) &= \int_0^\infty P'(t, \mathbf{x}, \mathscr{A})\,dt \\ &= \int_s^\infty P(t-s, x, A(t))\,dt. \end{aligned}$$

Assume that there exists a measure m, finite in compact sets and such that $P(t, x, \cdot) \ll m$ for all t and x, and denote its density by $p(t, x, y)$. We have

$$\begin{aligned} &= \int_s^\infty \int_{A(t)} p(t-s, x, y)\,m(dy)\,dt \\ &= \int_{\mathscr{A}} p(t-s, x, y)\,m(dy)\,dt. \end{aligned}$$

Obviously a good candidate for a space-time Green function is:

$$g'(\mathbf{x}, \mathbf{y}) = \begin{cases} p(t-s, x, y) & \text{for } t \geqq s, \\ 0 & \text{otherwise,} \end{cases}$$

where $\mathbf{x} = (s, x)$, $\mathbf{y} = (t, y)$.

Further it follows from the Chapman-Kolmogorov equation that the functions $g'(\cdot, \mathbf{y})$ are, for all $\mathbf{y}$, space-time invariant. Consequently, the assumption (v) is, in this case, always true. Because a space time process is very strongly transient the assumption (iv) should not cause any trouble when choosing the standard measure.

In order to construct the exit space we have to find measures m and γ, and show that the assumptions (ii) and (vi) are satisfied.

We shall now consider a space-time invariant function h and the corresponding h-process. Because h is invariant, the h-process never leaves the state space and, because the process moves deterministically in its first component, it follows that the spectral measure σ^h has all its mass concentrated to the *infinite part* of the exit space, denoted by M_∞. Therefore to prove the minimality of an invariant function k_y, we have to show that, if there exists a finite measure σ such that for all $\mathbf{x}$

$$k_y(\mathbf{x}) = \int_{M_\infty} k_z(\mathbf{x})\,\sigma(dz)$$

then $\sigma = \varepsilon_{\{y\}}$

4. Examples

4.1 Cauchy Processes

Cauchy processes belong to the family of stable processes. The expectation of a random variable distributed according to a Cauchy law does not exist.

A space-time Cauchy process has the following Green function

$$g(\mathbf{x},\mathbf{y})=\begin{cases}1 & \text{for } \mathbf{x}=\mathbf{y},\\ \dfrac{1}{\pi}\dfrac{t-s}{(t-s)^2+(y-x)^2} & \text{for } t\geqq s,\\ 0 & \text{otherwise,}\end{cases}$$

where $\mathbf{x}=(s,x)$ and $\mathbf{y}=(t,y)$. When showing that the assumptions (ii) and (vi) are satisfied it will turn out that the "right" state space for this process is $E=e\cup\{(0,\infty)\times(-\infty,\infty)\}$, where e is a point in the plane isolated from $(0,\infty)\times(-\infty,\infty)$ and playing the role of the origin. In this setting $\gamma_{\{e\}}$ can be used as a standard measure, and the Martin function takes the form

$$K(\mathbf{x},\mathbf{y})=\frac{t-s}{t}\,\frac{t^2+y^2}{(t-s)^2+(y-x)^2}.$$

It follows that no matter how we approach infinity $\lim K(\mathbf{x},\mathbf{y}_n)=1$ as $n\uparrow\infty$, where $\{\mathbf{y}_n\}$ is a point sequence in E such that $\lim\|\mathbf{y}_n\|=\infty$ ($\|\cdot\|$ is the usual Euclidean norm); so the Martin compactification is a one-point compactification, and the constants are the only space-time invariant functions.

4.2 d-dimensional Bessel Processes

Let $\{W(t)\}=\{(W_1(t),W_2(t),\ldots,W_d(t))\}$, $t\geqq 0$, be a standard d-dimensional Wiener process. Let us consider the one-dimensional process

$$\{B(t)\}=\{(W_1(t)^2+W_2(t)^2+\ldots+W_d(t)^2)^{1/2}\},\ t\geqq 0.$$

This process is called a d-dimensional Bessel process; when $d=1$ it is a reflected Brownian motion. A Bessel process has the following transition density (see [7] p. 60)

$$P(B(t)\in dy\,|\,B(s)=x)=p_d(t,y;s,x)\,dy$$
$$=\begin{cases}\dfrac{y^{d/2}\,x^{1-d/2}}{t-s}\exp\left(-\dfrac{y^2+x^2}{2(t-s)}\right)I_{d/2-1}\left(\dfrac{xy}{t-s}\right)dy & \text{for } x>0,\\ \dfrac{2^{1-d/2}\,y^{d-1}}{\Gamma(d/2)(t-s)^{d/2}}\exp\left(-\dfrac{y^2}{2(t-s)}\right)dy & \text{for } x=0,\end{cases}$$

where $I_u(x)$ is the modified Bessel function of order u. For $I_u(x)$ we have the approximate formulae:

(4.1) $I_u(t) \simeq (x/2)^u / \Gamma(u+1)$, when u is fixed and $\notin \{-1, -2, \ldots\}$ and $x>0$ is small (see [1] 9.6.7).

(4.2) $I_u(x) \simeq \exp(x)/\sqrt{2\pi x}$, when u is fixed and $x>0$ is large (see [1] 9.7.1).

Further, we have an upper bound

(4.3) $I_u(x) \leqq (x/2)^u \exp(x)/\Gamma(u+1)$, when $u > -1/2$ and $x>0$ (see [1] 9.1.62).

Remark 1. Note that we define Bessel processes, for all d, in the state space $[0, \infty)$, the usual one, for $d>1$, is $(0, \infty)$. We do this because it is convenient to be able to start the process from the origin. (See [7] p. 96.)

The case $d=1$ is very similar to the Wiener process case. Our state space is $e \cup \{(0, \infty) \times [0, \infty)\}$, where e is again an isolated point playing the role of the origin. $\gamma = \varepsilon_{\{e\}}$ can be used as a standard measure and the Martin boundary is homeomorphic to $[0, \infty) \cup \{\Delta\}$; the minimal part is $[0, \infty)$ and the minimal excessive functions are

$$k_z(\mathbf{x}) = \exp(-\tfrac{1}{2} s \cdot z^2) \cosh x z.$$

When $d>1$ the situation is more complicated. First we can consider our process only in the state space $E = e \cup \{(0, \infty) \times (0, \infty)\}$ (see [9]). However, this is not a serious problem because we know that the Bessel process, for $d>1$, does not touch $x=0$ at a positive time, i.e.

$$\mathscr{P}(B(t)>0 \quad \text{for all } t>0 \mid B(0)=0) = 1$$

(see [7] p. 61). We can therefore change our probability space without changing the finite dimensional distributions of the process, and consider a process taking values only in E.

Using (4.3) it is not difficult but tedious to prove that the assumptions (ii) and (vi) are satisfied (see [10]).

Now, consider the Martin function; this can be written in the form

$$K(\mathbf{x}, \mathbf{y}) = \Gamma\left(\frac{d}{2}\right) \left(\frac{2}{x}\right)^{d/2-1} \left(\frac{y}{t}\right)^{-d/2} \frac{y}{t-s} \cdot \exp\left(-\frac{x^2}{2(t-s)} - \frac{s y^2}{2t(t-s)}\right) I_{d/2-1}\left(\frac{x y}{t-s}\right).$$

Let $\mathbf{y}_n$ be a sequence such that $\|\mathbf{y}_n\| \uparrow \infty$ as $n \uparrow \infty$; it can be seen that $\lim K(\mathbf{x}, \mathbf{y}_n)$ exists if and only if $\lim y_n/t_n$ exists or "equals" ∞, and we have

$$\lim K(\mathbf{x}, \mathbf{y}_n) = \Gamma\left(\frac{d}{2}\right) \left(\frac{x z}{2}\right)^{1-d/2} \exp(-\tfrac{1}{2} s z^2) I_{d/2-1}(x z)$$
$$= \Gamma\left(\frac{d}{2}\right) \exp(-\tfrac{1}{2} s z^2) \sum_{k=0}^{\infty} \frac{\left(\frac{x z}{2}\right)^{2k}}{k!\, \Gamma(k+d/2)},$$

where

$$\begin{aligned} \lim y_n/t_n &= z\in(0,\infty); \text{ and} \\ &= 0 \quad \text{when } \lim y_n/t_n=\infty \text{ (use (4.2))}, \\ &= 1 \quad \text{when } \lim y_n/t_n=0 \text{ (use (4.1))}. \end{aligned}$$

Further

$$\lim K(\mathbf{x},\mathbf{y}_n)=\left(\frac{t_0}{t_0-s}\right)^{d/2}\exp\left(-\frac{1}{2}\frac{x^2}{t_0-s}\right),$$

when $y_n\downarrow 0$ and $t\to t_0>s$.

It follows that the Martin boundary of the d-dimensional space-time Bessel process is homeomorphic to $M=M_0\cup M_\infty\cup\{\Delta\}$, where $M_0=0\times(0,\infty)$ and $M_\infty=\infty\times[0,\infty)$. We will now prove that the minimal part is $M_0\cup M_\infty$.

First, consider the functions

$$k_t(\mathbf{x})=\begin{cases}\left(\frac{z}{z-s}\right)^{d/2}\exp\left(-\frac{1}{2}\frac{x^2}{z-s}\right) & \text{for } s<z,\\ 0 & \text{otherwise,}\end{cases}$$

where $z\in M_0$. It is not difficult to see that these functions are minimal and satisfy

$$P_t k_z(\mathbf{x})=\begin{cases}k_z(\mathbf{x}) & \text{for } t\geqq s, t<z,\\ 0 & \text{otherwise.}\end{cases}$$

Next, consider the functions

$$k_z(\mathbf{x})=\Gamma\left(\frac{d}{2}\right)\left(\frac{x\,z}{2}\right)^{1-d/2}\exp(-\tfrac{1}{2}s\,z^2)\,I_{d/2-1}(x\,z),$$

where $z\in M_\infty$. To prove the invariance of these functions, let us change the order of integration and summation; it follows that we have to verify the identity

$$(4.4)\qquad \begin{aligned}&(2(t-s))^{1-d/2}\exp\left(-\frac{1}{2}\left(\frac{x^2}{t-s}+(t-s)z^2\right)\right)\sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\frac{I}{F}\frac{x^{2k}z^{2m}}{(t-s)^{2k}}\\ &=\sum_{k=0}^{\infty}\frac{x^{2k}z^{2k}}{2^{2k}k!\,\Gamma(k+d/2)},\end{aligned}$$

where

$$F=2^{2k+2m}k!\,m!\,\Gamma(k+d/2)\,\Gamma(m+d/2)$$

$$I=\int_0^\infty\frac{y^{2m+2k+d-1}}{t-s}\exp\left(-\frac{1}{2}\frac{y^2}{t-s}\right)dy.$$

When integrated by parts,

$$I=\Gamma(m+k+d/2)(2(t-s))^{m+k+d/2-1} \quad \text{is obtained.}$$

After simple manipulations with the series in (4.4) it can be seen that it must be proved that, for all m, u,

$$\sum_{k=0}^{u} (-1)^{u-k} \binom{u}{k} \frac{\Gamma(m+k+d/2)}{\Gamma(k+d/2)\,\Gamma(m+d/2)} = \begin{cases} \dfrac{m!}{\Gamma(u+d/2)(m-u)!} & m \geqq u, \\ 0 & \text{otherwise.} \end{cases}$$

This is equivalent to

$$\sum_{k=0}^{u} (-1)^{u-k} \frac{\Gamma(u+d/2)}{(u-k)!\,\Gamma(k+d/2)} \frac{\Gamma(m+k+d/2)}{k!\,\Gamma(m+d/2)} = \begin{cases} \dbinom{m}{u} & m \geqq u, \\ 0 & \text{otherwise.} \end{cases}$$

But this identity is easily proved by comparing the coefficients of t^u in

$$(1-t)^{u+d/2-1}(1-t)^{-m-d/2} = (1-t)^{u-m-1}.$$

(Note that the coefficient of t^u is zero when $m<u$.)

Next, we shall prove the minimality of these functions: Let $z \in [0, \infty)$ and assume that there exists a finite measure μ and an interval $[z_1, z_2]$ such that $z \notin [z_1, z_2]$ and $\mu\{[z_1, z_2]\} = M > 0$ and

$$k_z(\mathbf{x}) = \int_0^\infty k_w(\mathbf{x})\,\mu(dw) \qquad \text{for all } \mathbf{x}.$$

Consequently for all $\mathbf{x}$ (assume $z_1 > 0$)

$$\begin{aligned} &k_z(\mathbf{x}) \geqq \int_{z_1}^{z_2} k_w(\mathbf{x})\,\mu(dw) \\ &\Leftrightarrow 1 \geqq M \left(\frac{z_2}{z}\right)^{1-d/2} \exp(-\tfrac{1}{2} s(z_2^2 - z^2)) \frac{I_{d/2-1}(z_1 x)}{I_{d/2-1}(z x)}. \end{aligned}$$

If $z < z_1$ consider the values of the right-hand side on the rays $x = c \cdot s$, where $c > \dfrac{1}{2}\dfrac{z_2^2 - z^2}{z_1 - z}$; and if $z > z_2$ we choose $c < \dfrac{1}{2}\dfrac{z^2 - z_2^2}{z - z_1}$. Letting $s \uparrow \infty$ and using (4.2) it can be seen that the right-hand side tends to infinity, which is a contradiction. If we cannot choose $z_1 > 0$ the only possibility is that $\mu = a \cdot \varepsilon_{\{z\}} + b \cdot \varepsilon_{\{0\}}$ and so for all

$$k_z(\mathbf{x}) = a\,k_z(\mathbf{x}) + b \Leftrightarrow k_z(\mathbf{x}) = \frac{b}{1-a},$$

which is absurd if $z > 0$.

So we have proved that the functions k_z, $z \in M_\infty$, are spacetime invariant and minimal. A by-product is that

$$\begin{aligned} \lim_{t \uparrow \infty} \frac{B(t)}{t} &= 0 \qquad \mathscr{P}_e - \text{a.s.} \\ &= z \qquad \mathscr{P}_e^z - \text{a.s.} \end{aligned}$$

The Kolmogorov backward equation for an ordinary Bessel process is given by

$$\frac{\partial f}{\partial t} = \frac{1}{2}\frac{\partial^2 f}{\partial x^2} + \frac{d-1}{2x}\frac{\partial f}{\partial x}.$$

It follows that, in smooth functions with compact support, the infinitesimal operator of the space-time Bessel process coincides with the differential operator

$$Af=\frac{1}{2}\frac{\partial^2 f}{\partial x^2}+\frac{d-1}{2x}\frac{\partial f}{\partial x}+\frac{\partial f}{\partial t}.$$

It can easily be seen that, as the functions k_z are spacetime invariant, the infinitesimal operator of the k_z-process coincides with the differential operator

$$\begin{aligned}A^z f&=\frac{1}{2}\frac{\partial^2 f}{\partial x^2}+\frac{d-1}{2x}\frac{\partial f}{\partial x}+\frac{1}{k_z}\frac{\partial k_z}{\partial x}\frac{\partial f}{\partial x}+\frac{\partial f}{\partial t}\\ &=\frac{1}{2}\frac{\partial^2 f}{\partial x^2}+\left(\frac{d-1}{2x}-\frac{d-2}{2x}+z\frac{I'_{d/2-1}(xz)}{I_{d/2-1}(xz)}\right)\frac{\partial f}{\partial x}+\frac{\partial f}{\partial t}.\end{aligned}$$

Using the formula $I'_u(x)=I_{u+1}(x)+\frac{u}{x}I_u(x)$ (see [1] 9.6.26) this can be written in the form

$$=\frac{1}{2}\frac{\partial^2 f}{\partial x^2}+\left(\frac{d-1}{2x}+z\frac{I_{d/2}(zx)}{I_{d/2-1}(zx)}\right)\frac{\partial f}{\partial x}+\frac{\partial f}{\partial t}.$$

It follows that the k_z-process on the real line is a diffusion process with a drift rate

$$a(t,x)=\frac{d-1}{2x}+z\frac{I_{d/2}(zx)}{I_{d/2-1}(zx)}.$$

Note that for large values of x the d-dimensional Bessel process behaves like a Brownian motion and, as is natural, the k_z-process like a Brownian motion with a drift z (use 4.2).

Remark 2. Let $d=3$ and consider the k_z-process in the interval $(0,+\infty)$. It follows from the above that, in smooth functions with compact support, the infinitesimal operator of this process coincides with the differential operator

$$A^z=\frac{1}{2}\frac{d^2}{dx^2}+\mu_z(x)\frac{d}{dx},$$

where

$$\mu_z(x)=\begin{cases} z\coth zx & \text{if } z>0,\\ \lim_{z\downarrow 0}\mu_z(x)=1/x & \text{if } z=0.\end{cases}$$

This diffusion figures in [12] and [8], and could be interpreted as an absorbing Brownian motion with a drift $-z$ conditioned to converge to the Martin boundary point $+\infty$. As shown above these diffusions could be called Bessel processes with a drift $z(\geqq 0)$, and so their appearance in the papers [12] and [8] becomes more natural and intuitive.

Further, we have the following fact: Consider a space-time Brownian motion with a drift $-z$, $z>0$ and absorption at $-\varepsilon$, $\varepsilon>0$. The minimal part of the

Martin boundary of this process is $\{(+\infty, -\varepsilon)\} \cup \{+\infty \times [0, +\infty)\}$. It can easily be proved that, as $\varepsilon \downarrow 0$, the finite dimensional distributions of the $k_{(\infty,0)}$- and $k_{(\infty,z)}$-processes (when considered on the real line) converge to the distributions of a usual 3-dimensional Bessel process and one with a drift z, respectively.

4.3 *Poisson Processes Composed with a Symmetric Binomial Distribution*

Let α be a random variable with

$$\mathscr{P}(\alpha = +1) = \mathscr{P}(\alpha = -1) = \tfrac{1}{2},$$

and $\{C_t\}$ a Poisson process composed with this distribution. Simple calculations show that

$$\mathscr{P}(C_t = k \mid C_s = n) = \exp(-\lambda(t-s))\, I_{|k-n|}(\lambda(t-s)),$$

where λ is the intensity of our Poisson process, $t \geqq s$, k, n are integers and $I_u(x)$ is the modified Bessel function of order u.

Consider this process in space-time; the Martin function is then given by

$$K(\mathbf{x}, \mathbf{y}) = \exp(\lambda s) \frac{I_{|k-n|}(\lambda(t-s))}{I_{|k|}(\lambda t)},$$

where $\mathbf{x} = (s, n)$, $\mathbf{y} = (t, k)$ and $t > s$. $\gamma = \varepsilon_{\{(0,0)\}}$ is used as a standard measure.

We shall first determine $\lim K(\mathbf{x}, \mathbf{y}_m)$ as $m \uparrow \infty$, where $\mathbf{y}_m$ is a point sequence in the state space such that $\|\mathbf{y}_m\| \uparrow \infty$. The following expansion holds for modified Bessel functions

$$I_u(x) = \frac{1}{\sqrt{2\pi}} (u^2 + x^2)^{-1/4} \cdot \exp\left((u^2 + x^2)^{1/2} - u \sinh^{-1} \frac{u}{x}\right) (1 + O(x^{-1})),$$

where $u, x > 0$ (see [5] 7.13.(8) p. 86). With the aid of this, the Martin function can be written in the form

$$K(\mathbf{x}, \mathbf{y}) = T_1(\mathbf{x}, \mathbf{y}) \exp(\lambda s + T_2(\mathbf{x}, \mathbf{y}) + T_3(\mathbf{x}, \mathbf{y})) \frac{1 + O((\lambda(t-s)^{-1})}{1 + O((\lambda t)^{-1})},$$

where

$$T_1(\mathbf{x}, \mathbf{y}) = \left(\frac{k^2 + \lambda^2 t^2}{(k-n)^2 + \lambda^2 (t-s)^2}\right)^{1/4},$$

$$T_2(\mathbf{x}, \mathbf{y}) = ((k-n)^2 + \lambda^2 (t-s)^2)^{1/2} - (k^2 + \lambda^2 t^2)^{1/2},$$

$$T_3(\mathbf{x}, \mathbf{y}) = |k| \sinh^{-1} \frac{|k|}{\lambda t} - |k-n| \sinh^{-1} \frac{|k-n|}{\lambda(t-s)}.$$

It is obvious that $\lim T_1(\mathbf{x},\mathbf{y}_m)=1$ as $m\uparrow\infty$. Let us now consider T_2, and assume that $\lim k_m/t_m=z$, for example, exists. Then it can easily be seen that

$$\lim T_2(\mathbf{x},\mathbf{y}_m)=-\frac{z\cdot n+\lambda^2\cdot s}{\sqrt{z^2+\lambda^2}}.$$

If $\lim k_m/t_m=\pm\infty$, then $\lim T_2=-n$ and $+n$, respectively. T_3 can be written in the form

$$T_3(\mathbf{x},\mathbf{y})=\left(\frac{|k|}{\lambda t}+\sqrt{1+\left(\frac{k}{\lambda t}\right)^2}\right)^{|k|}\Big/\left(\frac{|k-n|}{\lambda(t-s)}+\sqrt{1+\left(\frac{k-n}{\lambda(t-s)}\right)^2}\right)^{|k-n|}.$$

We can now assume that $\lim k_m/t_m=z$, for example, exists. Using l'Hospital's rule, it can be seen that

$$\lim T_3(\mathbf{x},\mathbf{y}_m)=\left(\frac{|z|}{\lambda}+\sqrt{1+\left(\frac{z}{\lambda}\right)^2}\right)^{\operatorname{sgn}(z)n}$$
$$\cdot\exp\left(-|z|s+\frac{zn}{\sqrt{\lambda^2+z^2}}+\frac{|z|\lambda^2 s}{\sqrt{\lambda^2+z^2}(|z|+\sqrt{\lambda^2+z^2})}\right).$$

Combining these results, we get

$$\lim K(\mathbf{x},\mathbf{y}_m)=\left(\frac{|z|}{\lambda}+\sqrt{1+\left(\frac{z}{\lambda}\right)^2}\right)^{\operatorname{sgn}(z)n}$$
$$\cdot\exp\left(\left(\lambda-|z|-\frac{\lambda^2}{|z|+\sqrt{\lambda^2+z^2}}\right)s\right),$$

when $\lim k_m/t_m=z$ exists. If $\lim k_m/t_m=\pm\infty$, then $\lim K(\mathbf{x},\mathbf{y}_m)=0$.

Before we proceed, let us consider a Poisson process composed with the following distribution

$$\mathscr{P}(\alpha=+1)=p,\qquad \mathscr{P}(\alpha=-1)=1-p=q.$$

It is clear that, in this case,

$$\mathscr{P}(C_t=k\,|\,C_0=0)=\left(\frac{p}{q}\right)^{k/2}\exp(-\lambda t)\,I_k(2\lambda\sqrt{p\,q}\,t)$$

Set $\lambda_1=2\lambda p$, $\lambda_2=2\lambda q$; then this can be written

$$\left(\frac{\lambda_1}{\lambda_2}\right)^{k/2}\exp(-\tfrac{1}{2}(\lambda_1+\lambda_2)t)\,I_k(\sqrt{\lambda_1\lambda_2}\cdot t).$$

Let $k_z(\mathbf{x})=\lim K(\mathbf{x},\mathbf{y}_m)$ when $\lim k_m/t_m=z$; and consider the k_z-transformation of the transition function of our Poisson process. It can be seen that

$$\mathscr{P}^z(C_t=k\,|\,C_0=0)$$
$$=\left(\frac{|z|+\sqrt{\lambda^2+z^2}}{\lambda}\right)^{\operatorname{sgn}(z)k}\exp(-\sqrt{\lambda^2+z^2}\,t)\,I_k(\lambda t)$$

Set $\lambda_1=\sqrt{\lambda^2+z^2}+|z|$, $\lambda_2=\sqrt{\lambda^2+z^2}-|z|$; then this takes the form

$$\left(\frac{\lambda_1}{\lambda_2}\right)^{\operatorname{sgn}(z)k/2}\exp(-\tfrac{1}{2}(\lambda_1+\lambda_2)t)\,I_k(\sqrt{\lambda_1\lambda_2}\cdot t).$$

It follows that the k_z-process is a compound Poisson process with parameter $\lambda'=\sqrt{\lambda^2+z^2}$ and the distribution

$$\begin{aligned}\mathscr{P}(\alpha=+1)&=\frac{1}{2}\left(1+\frac{|z|}{\sqrt{z^2+\lambda^2}}\right)\quad \text{for } z>0,\\ &=\frac{1}{2}\left(1-\frac{|z|}{\sqrt{z^2+\lambda^2}}\right)\quad \text{for } z<0.\end{aligned}$$

The invariance of the functions k_z, $z\in(-\infty,+\infty)$, is an immediate consequence of these considerations, and the minimality can be proved as in the Bessel process case.

5. Space Time Invariant Functions as Radon-Nikodym Derivatives

Let X be a Markov process on the real line starting from 0, and let us assume that the corresponding process in space time is a special M-process with a standard measure $\gamma=\varepsilon_{\{(0,0)\}}$ (or perhaps $\varepsilon_{\{e\}}$, where e is an isolated point). Set $\mathscr{F}_0=\{\emptyset,\Omega\}$, $\mathscr{F}_t^x=\sigma(x_s, s<t)$, $\mathscr{F}^x=\sigma(\bigcup_{t>0}\mathscr{F}_t^x)$. Denote by $\mathscr{P}^0$ and $\mathscr{P}^z$ probability measures on $(\Omega,\mathscr{F}^x)$ induced by the process X and the k_z-process on the real line, respectively, where k_z is a minimal space-time invariant function. Denote by $\mathscr{P}^0|_t$ and $\mathscr{P}^z|_t$ the restrictions of the measures $\mathscr{P}^0$ and $\mathscr{P}^z$, respectively, on the σ-algebra $\mathscr{F}_t^x$. It can now easily be proved that $\mathscr{P}^z|_t$ is absolutely continuous with respect to $\mathscr{P}^0|_t$ with a Radon-Nikodym derivative $k_z(t,x_t)$.

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