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On Brownian Motion and Certain Heat Equations

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Introduction

The connection of the standard Brownian motion process $B(t)$, $B(0)=0$, with various forms of the heat equation is well-known and well documented, going back (for example) to L. Bachelier (1912), A.N. Kolmogorov (1931), P. Lévy (1948), and M. Kac (1951). The present work is an effort at consolidation in one particular direction. On the purely analytical side, we obtain the fundamental solutions $p(t_1, z; t_2, x)$, $0<t_1<t_2$, of the two evolution equations

$$\frac{1}{2}\frac{\partial^2}{\partial x^2}Q-\frac{\partial}{\partial t}Q-\lambda t^{-2} I_{(0,\infty)}(x)\,Q=0, \tag{0.1}$$

$$\frac{1}{2}\frac{\partial^2}{\partial x^2}Q-\frac{\partial}{\partial t}Q-\lambda I_{(bt,\infty)}(x)\,Q=0, \tag{0.2}$$

where $t=t_2$, $Q=Q(x,t)$, $\lambda>0$, $b\geqq 0$, and I_B denotes the indicator function of B. For (0.1) with $t_1=0$ it is necessary to assume initial value $z<0$, or no fundamental solution exists. For $t_1>0$, however, the existence and uniqueness of the solutions (with continuous first order derivatives) is guaranteed by a theorem of M. Rosenblatt (1951). The outcome below is to give the explicit expressions of the solutions in terms of elementary functions. We largely leave it to the reader to experiment with changes of variable, etc., to obtain related solutions. Thus, for example, the restriction $b\geqq 0$ in (0.2) is only a matter of convenience.

Equation (0.2) can be interpreted as the heat equation for a material moving at constant velocity b past the juncture of two media having different linear heat transfer coefficients. It seems strange, therefore, that the solution is not well known, but we have found no hint of it in the standard references on the heat equation, such as [2, 11]. We mention at this point an equivalent form of (0.2) which is doubtless more familiar to probabilists. By making the substitution for $x+bt$ for x, it reduces to

$$\frac{1}{2}\frac{\partial^2}{\partial x^2}Q+b\frac{\partial}{\partial x}Q-\frac{\partial}{\partial t}Q-\lambda I_{(0,\infty)}(x)\,Q=0. \tag{0.3}$$

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Thus it corresponds to the sojourn times in $(0, \infty)$ of a Brownian motion with constant drift b. Moreover, the probabilistic method leads to a simple reduction of the general case to the case $b=0$, as expressed by (2.4) below.

On the probabilistic side, the main result of the work is to obtain the explicit distribution of the sojourn time of a standard B above a straight line bs, $0<s<t$, conditional on the pair $B(0)=z$, $B(t)=x$. This does not require any really new results or techniques. Rather, it is a combination of known transformations of B passage time distributions to lines, and the following basic result known as

P. Lévy's Law. Conditional on $B(0)=B(1)=0$, the sojourn time $\int_0^1 I_{(0,\infty)}(B(t))\,dt$ is uniformly distributed on $(0,1)$.

This result goes back to P. Lévy (1939), p. 323, who arrived at it by a remarkable train of probabilistic reasonings. Later it was derived rigorously by the analytical method of M. Kac (1951) (see Problem 1, Sect. 2.6, of [4]). Most of the other ingredients in our solutions are also implicit in Sect. 8, pp. 320–330, of Lévy (1939), but unlike Lévy's Law they do not seem to have received independent proofs. In fact, the extent of the present paper is simply to provide these proofs, and to show how the ingredients are used to solve the two equations. It should be noted here, perhaps, that the methods used below are relatively brief, and not along the lines of Lévy's argument (which is perhaps not rigorous by modern standards). However, the final expressions for the solutions of (0.1) and (0.2) depend on rather tedious algebraic computations. These may be omitted by the reader interested primarily in the probabilistic content.

1. The First Equation

We begin with Eq. (0.1) because one can verify the solution directly by differentiations. In order to conform to the situation of [10] we first choose $b>0$ and replace $I_{(0,\infty)}$ by $I_{(b,\infty)}$ and t^{-2} by $(\max(\varepsilon,t))^{-2}$ for $\varepsilon>0$. Then letting $\varepsilon\to 0+$ it follows from Theorem 1 of [10] that the solution for $t_1=0$ approaching the unit mass at 0 as $t_2=t\to 0+$ can be expressed as

$$E\Big(\exp-\lambda\int_0^t I_{(b,\infty)}(B(s))\,s^{-2}\,ds\,|\,B(t)=x\Big)\,p(t,0,x), \tag{1.1}$$

where $p(t,x,y)=(2\pi t)^{-\frac{1}{2}}\exp(-(2t)^{-1}(x-y)^2)$, i.e. the fundamental solution of the heat equation.

We will use the notation $\equiv$ to denote equivalence in distribution, either for random variables or processes. Then we have

$$B(t)\equiv t\,B(t^{-1}), \tag{1.2}$$

also due to P. Lévy (see 1.4, Problem 3, of [4]). It follows that

$$\begin{aligned}\left(\int_0^t I_{(b,\infty)}(B(s))\,s^{-2}\,ds\,|\,B(t)=x\right) \\ =\left(\int_{t^{-1}}^{\infty} I_{(b,\infty)}(B(s^{-1}))\,ds\,|\,B(t)=x\right) \\ =\left(\int_{t^{-1}}^{\infty} I_{(bs,\infty)}(s\,B(s^{-1})\,ds\,|\,t^{-1}\,B(t)=t^{-1}\,x\right) \\ \equiv\left(\int_{t^{-1}}^{\infty} I_{(bs,\infty)}(B(s))\,ds\,|\,B(t^{-1})=t^{-1}\,x\right).\end{aligned} \tag{1.3}$$

Now the explicit distribution of the last expression in (1.3) will be obtained by using

Lemma 1.1. *For $b>0$, we have the equivalence of distribution*

$$\int_0^\infty I_{(bs,\infty)}(B(s))\,ds\equiv b^{-2}\,T_1^{-1}\,U,$$

where T_1 and U are independent random variables, U is uniformly distributed on $(0,1)$, and T_1 is the first passage time of B to the line $x=1$.

Proof. Let T_b denote the first passage time of B to $x=b$. Then, using the strong Markov property of B at T_b and (1.2), we have

$$\begin{aligned}\int_0^\infty I_{(bs,\infty)}(B(s))\,ds&\equiv\int_0^\infty I_{(bs,\infty)}\left(s\,B\left(\frac{1}{s}\right)\right)ds \\ &=\int_0^\infty I_{(b,\infty)}(B(t))\,t^{-2}\,dt\equiv\int_0^\infty I_{(0,\infty)}(B(t))(t+T_b)^{-2}\,dt \\ &=T_b^{-1}\int_0^\infty I_{(0,\infty)}(B(t))(1+t)^{-2}\,dt.\end{aligned} \tag{1.4}$$

On the other hand, by P. Lévy's Law,

$$\begin{aligned}U&\equiv\left(\int_0^1 I_{(0,\infty)}(B(t))\,dt\,\middle|\,B(1)=0\right) \\ &\equiv\left(\int_0^1 I_{(0,\infty)}(t\,B(t^{-1}))\,dt\,\middle|\,B(1)=0\right)=\left(\int_0^1 I_{(0,\infty)}(B(t^{-1}))\,dt\,\middle|\,B(1)=0\right) \\ &=\left(\int_1^\infty I_{(0,\infty)}(B(t))\,t^{-2}\,dt\,\middle|\,B(1)=0\right)\equiv\int_0^\infty I_{(0,\infty)}(B(t))(1+t)^{-2}\,dt.\end{aligned} \tag{1.5}$$

Finally, by the scaling property $B(t)\equiv k\,B(k^{-2}\,t)$ it is seen that $T_b\equiv b^2\,T_1$. Combining this with (1.4) and (1.5), we obtain Lemma 1.1.

Returning to the distribution of (1.3), let $T=\inf\{s>t^{-1}\colon B(s)=b\,s\}$, with $\inf\emptyset=\infty$. Then, conditional on $B(t^{-1})=t^{-1}\,x$, the distribution of $T-t^{-1}$ is the well-known passage time distribution to a line (see, for example, Karlin and Taylor (1975)). It has density on $0<y<\infty$ given by

$$|x-b|\,t^{-1}(2\pi y^3)^{-\frac{1}{2}}\exp\left(-(2y)^{-1}\left(\frac{|x-b|}{t}-by\right)^2\right), \tag{1.6}$$

with a residual mass at $+\infty$ if $x<b$. To compute the density of (1.3) we distinguish two cases as follows.

Case 1. $x>b$. Then we have a contribution $T-t^{-1}$, plus an independent contribution following time T and having the distribution given by Lemma 1.1.

Case 2. $x<b$. Then we have no contribution before T. After T, if $T<\infty$, we have a contribution as given by Lemma 1.1.

Now the Laplace transform of the density (1.6), also known from [6], is

$$\begin{aligned} &\exp(-t^{-1}(x-b)((b^2+2\lambda)^{\frac{1}{2}}-b)) \quad \text{for } x>b. \\ &\exp(-t^{-1}(b-x)((b^2+2\lambda)^{\frac{1}{2}}+b)) \quad \text{for } x<b. \end{aligned} \tag{1.7}$$

Hence we can express (1.1) directly as follows:

Case 1. $E\exp(-[t^{-1}(x-b)((b^2+2\lambda)^{\frac{1}{2}}-b)+\lambda b^{-2}T_1^{-1}U])\,p(t,0,x)$.

Case 2.

$$\begin{aligned}&[E(\exp(-\lambda b^{-2}T_1^{-1}U))\exp(-(2t^{-1}(b-x)b))\\ &\quad+(1-\exp(-2t^{-1}(b-x)b))]\cdot p(t,0,x)\end{aligned}$$

where we used the second transform of (1.7) at $\lambda=0$ to compute the probability of $\{T<\infty\}$ for Case 2.

Finally, to compute $E\exp-\lambda b^{-2}T_1^{-1}U$ we can use the fact that for $t>0$ the first derivatives with respect to x in the two cases must coincide at $x=b$. This leads directly to

$$E\exp(-\lambda b^{-2}T_1^{-1}U)=\lambda^{-1}b((b^2+2\lambda)^{\frac{1}{2}}-b). \tag{1.8}$$

Substituting and simplifying we obtain after translating by $-b(=z)$ and putting $x-z$ for x,

Theorem 1.1. *The fundamental solution* $p(t_1,z;t_2,x)$ *of* (0.1), *for* $t_1=0$ *and* $z<0$, *is*

$$p(0,z;t,x)=\begin{cases}(2\pi t)^{-\frac{1}{2}}|z|\,\lambda^{-1}((z^2+2\lambda)^{\frac{1}{2}}+z)\exp[(2t)^{-1}(2\lambda-((z^2+2\lambda)^{\frac{1}{2}}\\ \quad+x)^2)];\ x\geqq 0\\ (2\pi t)^{-\frac{1}{2}}\{(|z|\,\lambda^{-1}((z^2+2\lambda)^{\frac{1}{2}}+z)-1)\exp[(-2t)^{-1}(x+z)^2]\\ \quad+\exp[(-2t)^{-1}(x-z)^2]\};\ x\leqq 0.\end{cases}$$

As remarked before, $p(0,z;t,x)$ does not exist for $z\geqq 0$, because of the singularity t^{-2} (this is easy to see probabilistically by reductio ad absurdum). However, for $t_1>0$ there is no theoretical diffuculty with either existence or uniqueness. On the other hand, the explicit expressions become still more complicated and involve integrals not easily evaluated. We will give the argument only for $t_1=1$, but the general case reduces to this by changes of scale. We again replace $I_{(0,\infty)}$ by $I_{(b,\infty)}$, and the solution is given by (1.1) except that in place of s^{-2} we have $(1+s)^{-2}$. Then instead of (1.3) we will use

$$\begin{aligned}
&\left(\int_0^t I_{(b,\infty)}(B(s))(1+s)^{-2}\,ds\,|\,B(t)=x\right)\\
&\equiv\left(\int_1^{1+t} I_{(b,\infty)}(B(s))\,s^{-2}\,ds\,|\,B(1)=0,\,B(1+t)=x\right)\\
&=\left(\int_{(1+t)^{-1}}^{1} I_{(bs,\infty)}(s\,B(s^{-1}))\,ds\,|\,B(1)=0,\,(1+t)^{-1}B(1+t)=(1+t)^{-1}x\right)\\
&\equiv\left(\int_{(1+t)^{-1}}^{1} I_{(bs,\infty)}(B(s))\,ds\,|\,B((1+t)^{-1})=(1+t)^{-1}x,\,B(1)=0\right).
\end{aligned}\tag{1.9}$$

Compared with (1.3) the effect here is to replace t by $1+t$, and more importantly to replace the upper limit ∞ by 1 together with a condition $B(1)=0$. Thus we require the same reasoning as before except that the passage times to the line bs, and the sojourn time above the line, must be computed conditionally on $B(1)=0$. To this effect, we have the following two lemmas which seem to have independent interest.

Lemma 1.2. *Let $b\geqq 0$ and β be constants, and let $T=\inf\{s\colon B(s)=bs+\beta\}$, with $B(0)=0$ and $\inf\emptyset=\infty$. Then for $t>0$ and any x, the density of T over $\{y<t\}$ conditional on $B(t)=x$ is given by*

$$f(y)=|\beta|\sqrt{t}\,(2\pi y^3(t-y))^{-\frac{1}{2}}\exp\left(-\frac{1}{2}\frac{(|\beta|(t-y)-(bt+\beta-x)\,y)}{t\,y(t-y)}\right).$$

Proof. The unconditional density of T is given by (1.6) in the form $|\beta|(2\pi y^3)^{-\frac{1}{2}}\exp(-(2y)^{-1}(|\beta|-by)^2)$, hence it is natural to expect that the conditional density is this multiplied by $p(t-y,\,by+\beta,x)\,p^{-1}(t,0,x)$. It turns out that this is correct, but to make the argument rigorous we reduce the problem to one involving the *Brownian bridge* $B(s)-s\,B(1)=B_0(s)$, $0<s<1$, and then use the well-known equivalence $B_0(s)\equiv(1-s)\,B(s(1-s)^{-1}))$, due to J.L. Doob (1949), and valid for any initial value $x=B_0(0)$. This is easily checked, in fact, using only the covariances, and it is similarly found that $B_0(s)$ and $B(1)$ are independent. Hence for $B(1)=0$ we have $B_0(s)\equiv(B(s)|B(1)=0)$ in a precise sense.

Translating $by-x$, and using the scaling property of B to see that

$$(B(s)|B(0)=-x,\,B(t)=0)\equiv\sqrt{t}\,(B(s\,t^{-1})|B(0)=-x\,t^{-\frac{1}{2}},\,B(1)=0)$$

we find that

$$\begin{aligned}
T&\equiv\inf\{s\colon B_0(s\,t^{-1})=b\,t^{-\frac{1}{2}}s+(\beta-x)\,t^{-\frac{1}{2}}\}\\
&=t\inf\{\tau\colon (1-\tau)\,B(\tau(1-\tau)^{-1})=b\,t^{\frac{1}{2}}\tau+(\beta-x)\,t^{-\frac{1}{2}}\}\\
&=t\,s_0(1+s_0)^{-1},
\end{aligned}\tag{1.10}$$

where $s_0=\inf\{s\colon B(s)=(b\,t^{\frac{1}{2}}+(\beta-x)\,t^{-\frac{1}{2}})\,s+(\beta-x)\,t^{-\frac{1}{2}}\}$. Now, as cited above, s_0 has the density

$$|\beta|\,t^{-\frac{1}{2}}(2\pi y^3)^{-\frac{1}{2}}\exp(-(2y)^{-1}(|\beta|\,t^{-\frac{1}{2}}-(b\,t^{\frac{1}{2}}+(\beta-x)\,t^{-\frac{1}{2}})\,y)^2),\qquad 0<y<\infty.$$

A change of variables from (1.10) then yields the stated density for T on $0<y<t$.

Our second lemma concerns the sojourn time above a line, conditional on $B(t)=x$. It is here that P. Lévy's Law provides the key ingredient.

Lemma 1.3. *For $b\geqq 0$, conditional upon $B(t)=0$, we have the equivalence in distribution*

$$\int_0^t I_{(bs,\infty)}(B(s))\,ds \equiv \frac{t\,U}{1+b^2\,t\,T_1},$$

where U and T_1 are independent random variables with U uniformly distributed on $(0,1)$ and T_1 the first passage time of B to the line $x=1$.

Proof. We again use the notations T_b for the unconditional first passage time of B to b, and $B_0(s)$ for the Brownian bridge (here, $B_0(0)=0$). Then the integral in the Lemma may be replaced by a sequence of equivalent expressions as follows.

$$\begin{aligned}
&\int_0^t I_{(bs,\infty)}(\sqrt{t}\,B_0(s\,t^{-1}))\,ds\\
&\quad = t\int_0^1 I_{(bt\tau,\infty)}(\sqrt{t}\,B_0(\tau))\,d\tau\\
&\quad \equiv t\int_0^1 I_{(b\sqrt{t}\tau,\infty)}((1-\tau)B(\tau(1-\tau)^{-1}))\,d\tau\\
&\quad = t\int_0^\infty I_{(b\sqrt{t}s,\infty)}(B(s))(1+s)^{-2}\,ds\\
&\quad \equiv t\int_0^\infty I_{(b\sqrt{t}s,\infty)}\left(s\,B\left(\frac{1}{s}\right)\right)(1+s)^{-2}\,ds\\
&\quad = t\int_0^\infty I_{(b\sqrt{t},\infty)}(B(t))(1+t)^{-2}\,dt\\
&\quad = t\int_{T_{b\sqrt{t}}}^\infty I_{(b\sqrt{t},\infty)}(B(t))(1+t)^{-2}\,dt\\
&\quad \equiv t\int_0^\infty I_{(0,\infty)}(B(t))(1+t+T_{b\sqrt{t}})^{-2}\,dt\\
&\quad = t(1+T_{b\sqrt{t}})^{-1}\int_0^\infty I_{(0,\infty)}B((1+T_{b\sqrt{t}})s)(1+s)^{-2}\,ds\\
&\quad \equiv t(1+T_{b\sqrt{t}})^{-1}\int_0^\infty I_{(0,\infty)}((1+T_{b\sqrt{t}})^{\frac{1}{2}}B(s))(1+s)^{-2}\,ds\\
&\quad \equiv t(1+T_{b\sqrt{t}})^{-1}\int_0^1 I_{(0,\infty)}(B_0(s))\,ds\\
&\quad \equiv t(1+T_{b\sqrt{t}})^{-1}\,U, \qquad (1.11)
\end{aligned}$$

where for the next-to-last identity we go back to second and third but with $b=0$ and $t=1$, and for the last identity we use P. Lévy's Law (of course, the strong Markov property of B at $T_{b\sqrt{t}}$ was used when needed, in the same way as in (1.4)). Replacing $T_{b\sqrt{t}}$ by $b^2 t T_1$ as after (1.5), Lemma 1.3 is proved.

Returning now to (1.9), it is necessary to distinguish the cases $x>b$ and $x<b$. Letting $y, (1+t)^{-1} \leqq y \leqq 1$, represent the arrival time after $(1+t)^{-1}$ of B at the line bs, we must integrate the density of y derived from Lemma 1.2, multiplying by $\exp(-\lambda(y-(1+t)^{-1}))$ for $x>b$, but not for $x<b$. Further, we must multiply by the conditional Laplace transform of the contribution following y, given y. In the second case, moreover, we must add $P\{$the line bs is not reached$\}$, which can be expressed by integrating the known density of y.

For brevity, we will write only the case $x>b$. To apply Lemma 1.2 we use the line $bs+(b-x)(1+t)^{-1}$, with the condition $B(1-(1+t)^{-1})=-x(1+t)^{-1}$ (as a figure makes clear). Then we replace y by $y-(1+t)^{-1}$ and $1-(1+t)^{-1}-y$ by $1-y$ to represent the arrival time after $(1+t)^{-1}$. This results in the expression

$$(x-b)\,t^{\frac{1}{2}}((2\pi)(1+t)^3)^{-\frac{1}{2}}((y-(1+t)^{-1})^3(1-y))^{-\frac{1}{2}}$$
$$\cdot \exp\left(-\frac{((1-y)(x-b)-b(1+t)(y-(1+t)^{-1}))^2}{2t(1+t)(y-(1+t)^{-1})(1-y)}\right)$$

for the density of the arrival time. Now if the arrival time is $y<1$, then $B(s)$ in $y<s<1$ becomes a Brownian motion starting and ending at 0 if we subtract the line segment from $B(y)=by$ to 0, in view of P. Lévy's observation that $B(t)-tB(1)$ is independent of $B(1)$. This means that in order to apply Lemma 1.3 to the contribution following y we must replace t by $1-y$ and also b by $b+by(1-y)^{-1}=b(1-y)^{-1}$. Hence the total contribution to (1.9), given y, is $y-(1+t)^{-1}+\frac{(1-y)^2 U}{(1-y)+b^2 T_1}$. Going back to the Laplace transform (1.1) multiplied by $p(t,0,x)$ (with $(1+s)^{-2}$ in place of s^{-2}, as noted before (1.9)), some algebraic simplification is possible in the integrand, and we finally are left with the expression

$$(2\pi)^{-1}\frac{x-b}{(1+t)^{3/2}}\exp\left(-\frac{(x-b)^2}{2(1+t)}\right)\int_{(1+t)^{-1}}^{1} ABC\,dy$$

where

$$A=(y-(1+t)^{-1})^3(1-y)^{-\frac{1}{2}},$$

$$B=\exp\left(-\frac{1}{2}\left[\left(\frac{x-b}{1+t}\right)^2\frac{1}{y-(1+t)^{-1}}+b^2 y(1-y)^{-1}\right]\right),$$

and

$$C=E\exp\left(-\lambda\left[y-(1+t)^{-1}+\frac{(1-y)^2 U}{(1-y)+b^2 T_1}\right]\right).$$

It is possible, but quite tedious, to check by differentiations that this does give the solution.

Finally, to obtain our elementary solution $p(1, z; t, x)$ with $z<0$ and $x>0$ we need only translate this result to $b=0$. Thus $z=-b<0$, and $x-b \to x>0$, $1+t \to t$, and we have

Theorem 1.2. *For $z<0<x$, $1<t$, the solution of* (0.1) *is*

$$p(1, z; t, x) = (2\pi)^{-1} x t^{-3/2} \left(\exp -\frac{x^2}{2t}\right) \int_{t^{-1}}^{1} (y-t^{-1})^3 (1-y)^{-\frac{1}{2}}$$

$$\cdot \exp\left(-\tfrac{1}{2}\left[(x t^{-1})^2 \frac{1}{y-t^{-1}} + z^2 y(1-y)^{-1}\right]\right) E\, dy$$

where

$$E = E \exp\left(-\lambda\left[y - t^{-1} + \frac{(1-y)^2 U}{(1-y) + z^2 T_1}\right]\right),$$

with U and T_1 as in Lemma 1.3.

The other three case $z<0$, $x<0$; $z>0$, $x>0$; and $z>0$, $x<0$ are treated quite similarly. Thus for $z>0$ we need to use lines of slope $b<0$. However, by appealing to symmetry, the contribution to (1.9) with $x<b<0$ has the form $t(1+t)^{-1} - Z$ where Z is the same contribution using $-b$ and $-x$, hence reducing to the case just treated. We leave the details of these three cases to the reader.

2. The Second Equation

The methods developed in Sect. 1 (in particular, Lemmas 1.2 and 1.3) lead easily to the fundamental solution of (0.2). However, it is in the form of a double integral. A direct method based on the Green function of (0.3) leads to the same result by inverting a complicated Laplace transform, but this gives less insight.

According to Theorem 1 of [10], the fundamental solution of (0.2) with initial value $z=0$ is given (as in (1.1)) by

$$p_b(0, 0; t, x) = E\left(\exp -\lambda \int_0^t I_{(bs, \infty)}(B(s))\, ds \mid B(t) = x\right) p(t, 0, x). \tag{2.1}$$

By translating the problem to initial value z, it follows that

$$p_b(0, z; t, x) = E_z\left(\exp -\lambda \int_0^t I_{(bs, \infty)}(B(s))\, ds \mid B(t) = x\right) p(t, z, x), \tag{2.2}$$

where E_z denotes expectation for $B(s)$ with $B(0)=z$. Furthermore, it is easily checked by substitution (or seen from a picture) that

$$p_b(t_1, z; t_2, x) = p_b(0, z - b t_1; t_2 - t_1, x - b t_1), \tag{2.3}$$

hence only (2.2) need be computed. We now observe that it suffices to obtain the first factor on the right of (2.2) in the special case $b=0$. Here we use the fact that the process $B(s) - s t^{-1} B(t)$, $0 \leqq s \leqq t$, is independent of $B(t)$ (see the proof of

Lemma 1.2). Thus we can substract the line bs from the process $B(s)$ conditional upon $B(t)=x$ to get the process conditional upon $B(t)=x-bt$. The integral will be unchanged if we also replace $I_{(bs,\infty)}$ by $I_{(0,\infty)}$, hence we have

$$p_b(0,z;t,x)=p_0(0,z;t,x-bt)(p(t,z,x)/p(t,z,x-bt)).$$

Computing the last factor and combining with (2.3), we obtain

$$p_b(t_1,z;t_2,x)=p_0(0,z-bt_1;t_2-t_1,x-bt_2)\exp\left(\tfrac{1}{2}b^2(t_2-t_1)-b(x-z)\right). \quad (2.4)$$

It remains to find the explicit expression of $p_0(0,z;t,x)$ using (2.2). We treat first the case $x<0<z$. Letting T denote the passage time of $B(s)$ to 0, we need the conditional density of T given $B(0)=z$, $B(t)=x$. As justified by Lemma 1.2 (and some algebra) this is the density of T given just $B(0)=z$ multiplied by $p(t-y,0,x)/p(t,z,x)$, $0<y<t$. The last factor is cancelled by the last factor of (2.2), and the density given $B(0)=z$ only is $|z|(2\pi y^3)^{-1/2}\exp(-(2y)^{-1}z^2)$, as in (1.6). Finally, if $T=y<t$, the the contribution to the integral in (2.2) following time y is obtained from Lemma 1.3 by subtracting the line segment from $(y,0)$ to (t,x) from the Brownian path segment. This changes the slope b from 0 to $\frac{-x}{t-y}$, thus the factor in (2.2) is given by $E\exp\left(\frac{-\lambda(t-y)^2U}{(t-y)+x^2T_1}\right)$. Of course, for $z<0$ we must add the probability of not reaching 0, while for $x>0$ and $z>0$ we must add this times $\exp-\lambda t$. We can thus write the four cases as follows.

Theorem 2.1. *The fundamental solution of* (0.2) *is derived from the case* $b=0$ *by the substitution* (2.4). *For the latter, we have*

Case 1. $z>0$, $x<0$.

$$p(0,z;t,x)=\frac{|z|}{2\pi}\int_0^t\frac{\exp(-\lambda y-1/2(z^2/y+x^2/(t-y)))}{\sqrt{y^3(t-y)}}E\exp\left(\frac{-\lambda(t-y)^2U}{(t-y)+x^2T_1}\right)dy.$$

Case 2. $z<0$, $x<0$.

$$p(0,z;t,x)$$
$$=\frac{-z}{2\pi}\int_0^t\frac{\exp-1/2(z^2/y+x^2/(t-y))}{\sqrt{y^3(t-y)}}\left(E\exp\left(\frac{-\lambda(t-y)^2U}{(t-y)+x^2T_1}\right)-1\right)dy+p(t,z,x).$$

Case 3. $z<0$, $x>0$.

$$p(0,z;t,x)=p(0,x;t,z)$$

Case 4. $z>0$, $x>0$.

$$p(0,z;t,x)=(\exp-\lambda t)\,p^*(0,-z;t,-x)$$

where p^* denotes p with $-\lambda$ in place of λ. In the above, U and T_1 are independent random variables with U having density 1 on $(0,1)$ and T_1 having density $(2\pi)^{-1/2}y^{-3/2}\exp-(2y)^{-1}$ on $(0,\infty)$.

Remarks. The symmetry of p in (z,x) is a consequence of the self-adjointness of $\frac{1}{2}\frac{d^2}{dx^2}-\lambda I_{(0,\infty)}(x)$, and holds for all (z,x). It can be made evident by routine changes of variable in Case 1, which yield the symmetric expression

$$p(0,z;t,x)$$
$$=|xz|\,\lambda^{-1}(2\pi)^{-\frac{3}{2}}\int_0^t(1-e^{-\lambda s})s^{-\frac{3}{2}}\int_0^{t-s}\frac{\exp(-\lambda v-1/2(z^2/v+x^2/t-s-v))}{(v(t-s-v))^{\frac{3}{2}}}\,dv\,ds.$$

The reduction given in Case 4 is also valid in Case 3. Thus for $x>0$ the slope $-x/(t-y)$ used in Case 1 and 2 is negative, which necessitates that the expectation be replaced by $E\exp-\lambda\left(t-y-\frac{(t-y)^2 U}{(t-y)+x^2 T_1}\right)$. This leads easily to the asserted reductions.

References

1. Bachelier, L.: Calcul des Probabilités. Paris: Gauthier-Villars 1912
2. Carslaw, H.S., Jaeger, J.C.: Conduction of Heat in Solids. Oxford: Clarendon 1959
3. Doob, J.L.: Heuristic approach to the Kolmogorov-Smirnov theorems. Ann. Math. Statist. **20**, 393–403 (1949)
4. Ito, K., McKean, H.P. Jr.: Diffusion Processes and their Sample Paths. New York: Springer 1965
5. Kac, M.: On some connections between probability theory and differential and integral equations. Proc. Second Berkeley Symposium on Math. Statist. and Probab. U. of Calif. Press, **1**, 189–215 (1951)
6. Karlin, S., Taylor, H.: A First Course in Stochastic Processes, 2nd ed. New York: Academic Press (1975)
7. Kolmogorov, A.N.: Über die analytischen Methoden in der Wahrscheinlichkeitsrechnung. Math. Ann. **104**, 415–458 (1931)
8. Lévy, P.: Sur certains processus stochastiques homogenes. Compositio Math. **7**, 283–339 (1939)
9. Lévy, P.: Processus Stochastiques et Mouvement Brownien. Paris: Gauthier-Villars 1948
10. Rosenblatt, M.: On a class of Markov processes. Trans. Amer. Math. Soc. **71**, 120–135 (1951)
11. Widder, D.V.: The Heat Equation. New York: Academic Press 1975

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