Z. Wahrscheinlichkeitstheorie verw. Gebiete
53, 157–173 (1980)

Zeitschrift für
Wahrscheinlichkeitstheorie
und verwandte Gebiete

Generalised Holley-Preston Inequalities on Measure Spaces and Their Products

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Summary. It is shown that if $(X, \mathscr{F}, \mu)$ is a product of totally ordered measure spaces and f_j $(j=1,2,3,4)$ are measurable non-negative functions on X satisfying

$$f_1(x) f_2(y) \leqq f_3(x \vee y) f_4(x \wedge y),$$

where $(\vee, \wedge)$ are the lattice operations on X, then

$$(\int f_1 \, d\mu)(\int f_2 \, d\mu) \leqq (\int f_3 \, d\mu)(\int f_4 \, d\mu).$$

This generalises results of Ahlswede and Daykin (for counting measure on finite sets) and Preston (for special choices of f_j).

1. Introduction

In recent years a number of inequalities have been discovered relating the cardinalities of subsets of a finite distributive lattice L and also the values of certain functions on L satisfying special conditions. One of the first of these was obtained by Kleitman [12] who showed that if S is any finite set, U an up-set and D a down-set in the lattice $L=2^S$ of all subsets of S, then

$$|L| \, |U \cap D| \leqq |U| \, |D| \tag{1.1}$$

where $|A|$ denotes the cardinality of A. This was strengthened by Daykin [5] who showed that for any subsets F and G of any finite distributive lattice L,

$$|F| \, |G| \leqq |F \vee G| \, |F \wedge G| \tag{1.2}$$

where

$$F \vee G = \{x \vee y : x \in F, y \in G\}, \qquad F \wedge G = \{x \wedge y : x \in F, y \in G\}.$$

0044-3719/80/0053/0157/$03.40

This result shows that (1.1) is valid in any finite distirbutive lattice, and also implies several other previously known inequalities. Daykin [6] later considered functions $A \to F_A$ and $A \to G_A$ of 2^S into 2^L, where S is any finite set and L is any finite distributive lattice. He showed that

$$\sum_{A \subset S} |F_A|\,|G_{S \setminus A}| \leqq \sum_{C \subset S} |H_C|\,|K_C| \tag{1.3}$$

where

$$H_C = \bigcup_{A \cup B = C} F_A \vee G_B, \qquad K_C = \bigcup_{A \cup B = C} F_{S \setminus A} \wedge G_{S \setminus B}.$$

When S is empty, (1.3) reduces to (1.2). If we put

$$f_1(A) = |F_A| \qquad f_2(A) = |G_A|$$
$$f_3(A) = |H_A| \qquad f_4(A) = |K_{S \setminus A}|$$

then (1.2) gives

$$f_1(A) f_2(B) \leqq f_3(A \cup B) f_4(A \cap B). \tag{1.4}$$

More general non-negative functions satisfying inequalities similar to (1.4) had already been considered by a number of authors. Fortuin, Kastelyn and Ginibre [9] showed that if f_1, f_2, f_3 and f_4 coincide on 2^S and satisfy (1.4), then

$$\Big(\sum_{A \subset S} u(A) f_1(A)\Big)\Big(\sum_{B \subset S} v(B) f_2(B)\Big) \leqq \Big(\sum_{C \subset S} u(C) v(C) f_3(C)\Big)\Big(\sum_{D \subset S} f_4(D)\Big) \tag{1.5}$$

for any increasing (non-negative) functions u and v on 2^S. Holley [10] extended this by showing that if $f_1 = f_3$, $f_2 = f_4$ and (f_1, f_2, f_3, f_4) satisfies (1.4), then

$$\Big(\sum_{A \subset S} f_1(A)\Big)\Big(\sum_{B \subset S} v(B) f_2(B)\Big) \leqq \Big(\sum_{C \subset S} v(C) f_3(C)\Big)\Big(\sum_{D \subset S} f_4(D)\Big). \tag{1.6}$$

(At first sight (1.6) may appear less general than (1.5) but this is only because of the stronger conditions imposed in [9]. To obtain (1.5) as a special case of (1.6) one simply replaces f_1 by uf_1 and f_3 by uf_3, noting that (1.4) is still valid.) A further extension was made by Preston [13] who showed that if $(X, \mathscr{F}, \mu)$ is a finite product of totally ordered measure spaces (so X is a lattice in the product ordering), f_1 and f_2 are non-negative integrable functions on X, v_1 is a bounded increasing measurable function on X, $f_3 = f_1$, $f_4 = f_2$, $v_2 = v_1$ and (f_1, f_2, f_3, f_4) satisfy the lattice-theoretic analogue of (1.4):

$$f_1(x) f_2(y) \leqq f_3(x \vee y) f_4(x \wedge y) \tag{1.7}$$

then

$$\Big(\int f_1\, d\mu\Big)\Big(\int v_1 f_2\, d\mu\Big) \leqq \Big(\int v_2 f_3\, d\mu\Big)\Big(\int f_4\, d\mu\Big). \tag{1.8}$$

The case of counting measure on a finite set gave (1.6). Preston's theorem was extended to countable products of probability spaces by the first author [3]. A simplified proof of (1.8) in finite products was found independently by Edwards [8] and Kemperman [11] (Edwards also considered countable products). However this method did not show the existence of certain interesting measures on

X^2 as established in [10, 13]. The original applications of (1.5) were to problems in statistical mechanics, but Seymour and Welsh [14, 15] have shown how (1.5), (1.6) and (1.8) are important in combinatorial and percolation theory.

Edwards and Kemperman in fact found that the inductive proof of (1.8) was made simpler if they replaced the assumption that v_1 and v_2 coincide and are increasing by the weaker condition:

$$v_1(x) \leqq v_2(y) \quad \text{whenever } x \leqq y \tag{1.9}$$

but they retained the conditions $f_3 = f_1$ and $f_4 = f_2$. Ahlswede and Daykin [1] dropped even these conditions, thereby making it unnecessary to introduce v_1 and v_2 at all, but they considered only finite sets. They showed that if f_1, f_2, f_3 and f_4 are non-negative functions satisfying (1.7) on a finite distributive lattice L then

$$\Big(\sum_{x \in L} f_1(x)\Big)\Big(\sum_{y \in L} f_2(y)\Big) \leqq \Big(\sum_{z \in L} f_3(z)\Big)\Big(\sum_{w \in L} f_4(w)\Big). \tag{1.10}$$

In [2] they considered more abstract situations in which the lattice L and operations $\vee$ and $\wedge$ are replaced by any finite set S and mappings ϕ and ψ of S^2 into S. Defining (ϕ, ψ) to be "$\mathfrak{M}$-expansive" if

$$\Big(\sum_{x \in S} f_1(x)\Big)\Big(\sum_{y \in S} f_2(y)\Big) \leqq \Big(\sum_{z \in S} f_3(z)\Big)\Big(\sum_{w \in S} f_4(w)\Big) \tag{1.11}$$

whenever

$$f_1(x) f_2(y) \leqq f_3(\phi(x,y)) f_4(\psi(x,y)), \tag{1.12}$$

they showed that the class of $\mathfrak{M}$-expansive pairs of mappings (ϕ, ψ) is closed under direct products. While studying direct products, they were led to introduce an apparently stronger notion of "$\mathfrak{M}$-explosiveness", which they showed to be equivalent to $\mathfrak{M}$-expansiveness.

The aim of this paper is to exhibit a single inequality which includes (1.1), (1.2), (1.5), (1.6), (1.8) and (1.10) as special cases. Thus it will be shown that if $(X, \mathscr{F}, \mu)$ is a finite product of totally ordered measure spaces and f_1, f_2, f_3 and f_4 are measurable functions of X into $[0, \infty]$ satisfying (1.7), then

$$\Big(\int f_1 \, d\mu\Big)\Big(\int f_2 \, d\mu\Big) \leqq \Big(\int f_3 \, d\mu\Big)\Big(\int f_4 \, d\mu\Big). \tag{1.13}$$

At this level of generality the inductive step in the proof becomes much simpler than that in [10] or [13]. Furthermore the extension to countably infinite products is now a straightforward application of the Fubini-Jessen theorem. The main difficulty in proving the infinite case in [3] and [8] was caused by uncertainty as to whether (1.8) is satisfied if (1.7) holds only μ^2-a.e. in X^2. We shall see here that in finite products this is the case, and indeed (1.13) is always valid if (1.7) holds μ^2-a.e.

The main results are presented for abstract pairs (ϕ, ψ) rather than $(\vee, \wedge)$, and they therefore include also (1.11) as a special case. Section 2 of the paper is devoted to a study of the inequality (1.12), Sect. 3 contains the main results showing that certain pairings (ϕ, ψ) satisfy the measure-theoretic analogue of $\mathfrak{M}$-expansiveness, and Sect. 4 contains a discussion of a measure-theoretic version of

$\mathfrak{M}$-explosiveness. The final section is devoted to finite distributive lattices, showing how inequalities such as (1.1), (1.2) and (1.3) are related to $\mathfrak{M}$-expansiveness and to each other.

It will be convenient to allow funcitons to take values in the extended non-negative reals $[0,\infty]$, and we shall adopt the usual conventions concerning the arithmetic of this system, except that we shall regard $0.\infty$ and $\infty.0$ as undefined. Furthermore an inequality will be considered to be satisfied if either side is undefined. We shall also need to consider n-tuples of functions taking values in $[0,\infty]$, and we shall denote these interchangably either componentwise as $(f_1,\dots,f_n)$ or as a single function $\mathbf{f}\colon X\to[0,\infty]^n$.

2. Compatibility

Our basic object of study will be a system $\mathscr{S}=(X,\mathscr{F},\mu,\phi,\psi)$ consisting of a σ-finite measure space $(X,\mathscr{F},\mu)$, whose measure-theoretic product with itself will be denoted by $(X^2, \mathscr{F}^2, \mu^2)$, together with a mapping (ϕ,ψ) of X^2 into itself. Such a system will be called a *paired measure space* with *pairing* (ϕ,ψ). If (ϕ,ψ) is $\mathscr{F}^2$-measurable, we shall say that $\mathscr{S}$ is *measurably paired.* As a matter of notational convenience, the component parts of a paired measure space denoted by $\mathscr{S}_\lambda$, where λ may be an index, will themselves always be denoted by $(X_\lambda, \mathscr{F}_\lambda, \mu_\lambda, \phi_\lambda, \psi_\lambda)$.

An *involution* on $\mathscr{S}$ is a bijection π of X such that for any x and y in X and E in $\mathscr{F}$, $\pi(\pi(x))=x$, $\phi(\pi(x),\pi(y))=\pi(\psi(y,x))$, $\pi(E)\in\mathscr{F}$ and $\mu(\pi(E))=\mu(E)$. Following [2], an $\mathscr{F}$-measurable function $\mathbf{f}$ of X into $[0,\infty]^4$ will be said to be *compatible* (resp. *μ-compatible*, resp. *diagonally μ-compatible*) with (ϕ,ψ) if

$$f_1(x)f_2(y)\leqq f_3(\phi(x,y))f_4(\psi(x,y)) \tag{2.1}$$

for all (resp. μ^2-almost all, resp. $\bar{\mu}$-almost all) pairs (x,y) in X^2, where $\bar{\mu}$ is the image of μ under the mapping $x\to(x,x)$ of X onto the diagonal Δ in X^2. Thus $\mathbf{f}$ is diagonally μ-compatible if

$$f_1(x)f_2(x)\leqq f_3(\phi(x,x))f_4(\psi(x,x)) \tag{2.2}$$

μ-a.e.(x). The sets of all compatible, μ-compatible and diagonally μ-compatible functions $\mathbf{f}$ will be denoted by $\mathfrak{K}(\phi,\psi)$, $\mathfrak{K}_\mu(\phi,\psi)$ and $\mathfrak{K}^d_\mu(\phi,\psi)$ respectively, or simply by $\mathfrak{K}$, $\mathfrak{K}_\mu$ and $\mathfrak{K}^d_\mu$ if no confusion is likely. A 4-tuple (E_1,E_2,E_3,E_4) of $\mathscr{F}$-measurable sets will be said to be *compatible* with (ϕ,ψ) if

$$\phi(E_1\times E_2)\subset E_3 \quad\text{and}\quad \psi(E_1\times E_2)\subset E_4 \tag{2.3}$$

or equivalently if their characteristic functions $(\chi_{E_1}, \chi_{E_2}, \chi_{E_3}, \chi_{E_4})$ are compatible. Thus the set of compatible 4-tuples of sets is in one-to-one correspondence with the set $\mathfrak{P}$ of functions in $\mathfrak{K}$ taking values in $\{0,1\}^4$.

For (x,y) in X^2, put $(x,y)^*=(y,x)$. For any function α on X^2, put $\alpha^*(z)=\alpha(z^*)$. The following properties of compatibility are clear:

$$\mathbf{f}\in\mathfrak{K}(\phi,\psi)\Leftrightarrow(f_2,f_1,f_3,f_4)\in\mathfrak{K}(\phi^*,\psi^*) \tag{2.4}$$

$$\mathbf{f}\in\mathfrak{K}(\phi,\psi)\Leftrightarrow(f_1,f_2,f_4,f_3)\in\mathfrak{K}(\psi,\phi) \tag{2.5}$$

$$\mathbf{f}, \mathbf{g} \in \mathfrak{K}(\phi, \psi) \Rightarrow (f_1 \cdot g_1, f_2 \cdot g_2, f_3 \cdot g_3, f_4 \cdot g_4) \in \mathfrak{K}(\phi, \psi) \tag{2.6}$$

$$\mathbf{f} \in \mathfrak{K}(\phi, \psi) \Leftrightarrow (f_1 \circ \pi, f_2 \circ \pi, f_3 \circ \pi, f_4 \circ \pi) \in \mathfrak{K}(\psi^*, \phi^*) \tag{2.7}$$

where π is an involution.

Suppose that Y is an $\mathscr{F}$-measurable subset of X such that $\phi(Y^2) \subset Y$ and $\psi(Y^2) \subset Y$. Let $\mathscr{S}|_Y = (Y, \mathscr{F}_Y, \mu_Y, \phi_Y, \psi_Y)$ be the paired measure space obtained from $\mathscr{S}$ by restriction to Y. Given $\mathbf{g}$: $Y \to [0, \infty]^4$, let $\mathbf{g}_X$ be the extension of $\mathbf{g}$ to X vanishing outside Y. Then

$$\mathbf{g} \in \mathfrak{K}(\phi_Y, \psi_Y) \Leftrightarrow \mathbf{g}_X \in \mathfrak{K}(\phi, \psi). \tag{2.8}$$

Both μ-compatibility and diagonal μ-compatibility have properties similar to (2.4), (2.5), (2.6), (2.7) and (2.8).

In general there is no reason to suppose that μ-compatibility implies diagonal μ-compatibility. However the technical lemma in [3] gave one particular circumstance in which this phenomenon does occur, and we shall be deeply involved with the property in Sect. 3. Thus we shall say that $\mathscr{S}$ is *diagonally settled* if $\mathfrak{K}_\mu$ is contained in $\mathfrak{K}_\mu^d$. If μ is purely atomic, then $\bar{\mu}$ is absolutely continuous with respect to μ^2, and $\mathscr{S}$ is diagonally settled. The following two propositions in this section give some other examples of diagonally settled spaces.

The fundamental example to be considered at this stage occurs when X is a lattice and ϕ and ψ are the lattice operations $\vee$ and $\wedge$ (cf. [1, 5, 13]). Then X is totally ordered if and only if $\{x \vee y, x \wedge y\} = \{x, y\}$ for all (x, y) in X^2. In general we shall say that the pairing (ϕ, ψ) is *selective*, and $\mathscr{S}$ is *selectively paired*, if $\{\phi(x, y), \psi(x, y)\} = \{x, y\}$ for all (x, y) in X^2. (In this case, ϕ is an arbitrary choice from each ordered pair (x, y), and ψ is the other choice.) We shall also say that $\mathscr{S}$ is *diagonally invariant* if $\phi(x, x) = \psi(x, x) = x$ for all x in X. Clearly any selectively paired space is diagonally invariant.

Proposition 2.1. *Any selectively paired measure space $\mathscr{S}$ is diagonally settled.*

Proof. Consider a μ-compatible function $\mathbf{f}$: $X \to [0, \infty]^4$, and let

$$X_0 = \{y \in X\colon (2.1) \text{ holds for } \mu\text{-almost all } x\}.$$

Then $X \backslash X_0$ is μ-null, so replacing $\mathscr{S}$ by $\mathscr{S}|_{X \backslash X_0}$, we may assume that $X = X_0$.

If $f_1(x) = 0$, $f_2(x) = 0$, $f_3(x) = \infty$ or $f_4(x) = \infty$, then (2.2) is automatically satisfied. Thus we can assume none of these possibilities occurs in X. Let $X_1 = \{x \in X\colon f_4(x) = 0\}$. Then $f_1(x) f_2(y) = 0$ μ^2-a.e. (x, y) in X_1^2. Hence either f_1 or f_2 vanishes μ-a.e. in X_1, so X_1 is null. Thus replacing $\mathscr{S}$ by $\mathscr{S}|_{X \backslash X_1}$, we can assume that $f_4(x) > 0$ and similarly that $f_3(x) > 0$, $f_1(x) < \infty$ and $f_2(x) < \infty$.

Put $g(x) = f_2(x) f_3(x)^{-1}$ and $h(x) = f_2(x) f_4(x)^{-1}$, so that $0 < g(x) < \infty$ and $0 < h(x) < \infty$. Then (2.1) shows that for fixed y,

$$f_1(x)\, g(y) \leqq f_4(x) \tag{2.9}$$

for μ-almost all x with $\phi(x, y) = y$, and

$$f_1(x)\, h(y) \leqq f_3(x) \tag{2.10}$$

for μ-almost all x with $\phi(x,y)=x$. For integers $k,n\geqq 1$, let $E_{kn}=g^{-1}((k-1)2^{-n}, k2^{-n}]$, and, assuming that E_{kn} is non-empty, choose a sequence y_{kn}^r $(r=1,2,\dots)$ in E_{kn} such that

$$\sup\{h(y_{kn}^r)\colon r=1,2,\dots\}=\sup\{h(x)\colon x\in E_{kn}\}. \tag{2.11}$$

Let $E'_{kn}=\{x\in E_{kn}\colon \phi(x,y_{kn}^r)=x \text{ for all } r\}$ and $E'=\bigcup_{k,n} E'_{kn}$. Applying (2.10), we see that for μ-almost all x in E'_{kn},

$$f_1(x)\,h(y_{kn}^r)\leqq f_3(x),$$

so it follows from (2.11) that $f_1(x)\,h(x)\leqq f_3(x)$, i.e. (2.2) holds μ-a.e. in E'.

Now consider x in $X\setminus E'$. For each n, there is a (unique) integer k such that x belongs to $E_{kn}\setminus E'_{kn}$. For some r, $\phi(x,y_{kn}^r)=y_{kn}^r$. It follows from (2.9) that for μ-almost all such x,

$$f_1(x)(g(x)-2^{-n})\leqq f_1(x)\,g(y_{kn}^r)\leqq f_4(x).$$

Letting $n\to\infty$, it now follows that $f_1(x)\,g(x)\leqq f_4(x)$, i.e. (2.2) holds. Thus $\mathbf{f}$ is diagonally μ-compatible.

We begin now our consideration of product spaces. The *direct product* of a finite family of paired measure spaces $\{\mathscr{S}_\lambda\colon \lambda\in\Lambda\}$, or an infinite family of paired probability spaces, is defined to be the paired measure space $\mathscr{S}=(X,\mathscr{F},\mu,\phi,\psi)$ where $(X,\mathscr{F},\mu)$ is the measure-theoretic product of $\{(X_\lambda,\mathscr{F}_\lambda,\mu_\lambda)\}$ and ϕ and ψ are defined by:

$$\phi(x,y)=(\phi_\lambda(x_\lambda,y_\lambda)),\qquad \psi(x,y)=(\psi_\lambda(x_\lambda,y_\lambda)).$$

In the case when $\Lambda=\{1,2\}$, we may denote $\mathscr{S}$, ϕ and ψ by $\mathscr{S}_1\times\mathscr{S}_2$, $\phi_1\times\phi_2$ and $\psi_1\times\psi_2$.

Proposition 2.2. *A finite direct product of diagonally settled measurably paired measure spaces is diagonally settled and measurably paired.*

Proof. It suffices by induction to consider the case when $\mathscr{S}=\mathscr{S}_1\times\mathscr{S}_2$. The measurability is clear. Take a function $\mathbf{f}$ which is μ-compatible with (ϕ,ψ). For μ_2^2-almost all (x_2,y_2) in X_2^2, the following inequality is valid μ_1^2-a.e. (x_1,y_1):

$$f_1(x_1,x_2)f_2(y_1,y_2)\leqq f_3(\phi_1(x_1,y_1),\phi_2(x_2,y_2))f_4(\psi_1(x_1,y_1),\psi_2(x_2,y_2)). \tag{2.12}$$

Thus if we put

$$\begin{aligned} h_1(x_1)&=f_1(x_1,x_2) & h_2(x_1)&=f_2(x_1,y_2)\\ h_3(x_1)&=f_3(x_1,\phi_2(x_2,y_2)) & h_4(x_1)&=f_4(x_1,\psi_2(x_2,y_2)) \end{aligned} \tag{2.13}$$

then $\mathbf{h}$ is μ_1-compatible and hence diagonally μ_1-compatible with (ϕ_1,ψ_1), so that the following inequality is valid for μ_1-almost all x_1 in X_1:

$$f_1(x_1,x_2)f_2(x_1,y_2)\leqq f_3(\phi_1(x_1,x_1),\ \phi_2(x_2,y_2))f_4(\psi_1(x_1,x_1),\psi_2(x_2,y_2)). \tag{2.14}$$

Hence for μ_1-almost all x_1, (2.14) holds μ_2^2-a.e. (x_2,y_2). (The measurability of (ϕ_j,ψ_j) is used here to allow Tonelli's form of Fubini's theorem to be applied.) A similar argument now shows that (2.14) holds $\bar{\mu}_2$-a.e. Hence $\mathbf{f}$ is diagonally μ-compatible with (ϕ,ψ).

Example 2.3. Let $\mathscr{S}$ be the direct product of a sequence of measurably paired probability spaces $\mathscr{S}_n$, and suppose that there exist $\mathscr{F}_n$-measurable sets E_n such that

$$\sum_n \mu_n(E_n) = \infty$$

$$\sum_n \mu_n^2(\phi_n^{-1}(E_n)) < \infty.$$

Let $E = \{x \in X\colon x_n \in E_n \text{ for infinitely many } n\}$. By the Borel-Cantelli lemmas, $\mu(E) = 1$ and $\mu^2(\phi^{-1}(E)) = 0$. Thus the function $(1, 1, 1-\chi_E, 1)$ is μ-compatible. However if each $\mathscr{S}_n$ and hence $\mathscr{S}$ is diagonally invariant, this function is not diagonally μ-compatible, so $\mathscr{S}$ is not diagonally settled.

As a specific example, we can take $\mathscr{S}_n$ to be the unit interval $[0, 1]$ equipped with Lebesgue measure and the lattice operations, and $E_n = [0, n^{-1}]$.

3. Expansiveness

We shall now introduce the concept of expansiveness as a generalisation of the set-theoretic ideas developed in [2]. A paired measure space $\mathscr{S}$ will be said to be *expansive* if

$$\mu(E_1)\,\mu(E_2) \leqq \mu(E_3)\,\mu(E_4) \tag{3.1}$$

for all 4-tuples (E_1, E_2, E_3, E_4) of sets compatible with (ϕ, ψ); $\mathfrak{M}$-*expansive* if

$$\mu(f_1)\,\mu(f_2) \leqq \mu(f_3)\,\mu(f_4) \tag{3.2}$$

for all compatible functions $\mathbf{f}$ (where $\mu(f_j)$ is written in place of $\int_X f_j\, d\mu$); *strongly* $\mathfrak{M}$-*expansive* if (3.2) holds for all μ-compatible $\mathbf{f}$.

We begin by making some observations concerning these definitions.

Remarks 3.1. (a) The inequality (3.2) is automatically satisfied if either f_3 or f_4 is not μ-integrable, while if they are both integrable, and neither f_1 nor f_2 is μ-null, (3.2) requires that f_1 and f_2 be integrable.

(b) It is clear from (2.3) that $\mathscr{S}$ is expansive if and only if

$$\mu(E_1)\,\mu(E_2) \leqq \mu^*(\phi(E_1 \times E_2))\,\mu^*(\psi(E_1 \times E_2))$$

for all E_1 and E_2 in $\mathscr{F}$, where μ^* denotes outer μ-measure. In particular, in an expansive probability space,

$$\mu^*(\phi(X^2)) = \mu^*(\psi(X^2)) = 1.$$

(c) If $\mathscr{S}$ is expansive (resp. $\mathfrak{M}$-expansive, strongly $\mathfrak{M}$-expansive), and Y is an $\mathscr{F}$-measurable subset of X with $\phi(Y^2) \subset Y$ and $\psi(Y^2) \subset Y$, then it follows from (2.8) that the restricted paired measure space $\mathscr{S}|_Y$ is expansive (resp. $\mathfrak{M}$-expansive, strongly $\mathfrak{M}$-expansive).

(d) It follows from (2.6) that for (E_1, E_2, E_3, E_4) and (f_1, f_2, f_3, f_4) compatible (or μ-compatible) with (ϕ, ψ), the function $(f_1\chi_{E_1}, f_2\chi_{E_2}, f_3\chi_{E_3}, f_4\chi_{E_4})$ is also compatible (or μ-compatible). Thus in an $\mathfrak{M}$-expansive (or strongly $\mathfrak{M}$-expansive) space,

$$(\int_{E_1} f_1\, d\mu)(\int_{E_2} f_2\, d\mu) \leqq (\int_{E_3} f_3\, d\mu)(\int_{E_4} f_4\, d\mu). \tag{3.3}$$

In particular if S is a finite set, μ_c is counting measure and (ϕ, ψ) is a mapping of S^2 into itself, then (ϕ, ψ) is expansive (resp. $\mathfrak{M}$-expansive) in the sense defined in [2, pp. 268–269] if and only if $(S, 2^S, \mu_c, \phi, \psi)$ is expansive (resp. $\mathfrak{M}$-expansive or equivalently strongly $\mathfrak{M}$-expansive) according to the above definitions. In this respect the following study of expansive measure spaces extends the detailed investigation carried out in [2]. The theorem and corollary in [1] now become the statements that if $S = 2^{\{1, \dots, n\}}$ then $(S, 2^S, \mu_c, \cup, \cap)$ is (strongly) $\mathfrak{M}$-expansive, and more generally that if S is a distributive lattice, then $(S, 2^S, \mu_c, \vee, \wedge)$ is (strongly) $\mathfrak{M}$-expansive. The reader is referred to [2] for some examples of expansive pairings on finite sets.

(e) Following [2], it would be possible to define $\mathfrak{C}$-expansiveness for an arbitrary subset $\mathfrak{C}$ of $\mathfrak{K}_\mu$ by requiring that (3.2) should hold for all $\mathbf{f}$ in $\mathfrak{C}$. Expansiveness, $\mathfrak{M}$-expansiveness and strong $\mathfrak{M}$-expansiveness correspond respectively to taking $\mathfrak{C} = \mathfrak{P}$, $\mathfrak{K}$ and $\mathfrak{K}_\mu$. Other interesting classes $\mathfrak{C}$ might include the bounded and the integrable functions in $\mathfrak{K}$ and in $\mathfrak{K}_\mu$ (note the comments on integrability in (a) above), and many of our results remain valid for such classes. However we shall not list these as it should be clear from the proofs under what circumstances these variations are valid.

Proposition 3.2. *Let π be an involution on a (strongly) $\mathfrak{M}$-expansive measure space $\mathscr{S}$. Then for any (μ-)compatible* $\mathbf{f}$,

$$\mu(f_1 \cdot (f_2 \circ \pi)) \leqq \mu(f_3 \cdot (f_4 \circ \pi)). \tag{3.4}$$

Proof. It follows from (2.4)–(2.7) that $(f_1 \cdot (f_2 \circ \pi), f_2 \cdot (f_1 \circ \pi), f_3 \cdot (f_4 \circ \pi), f_4 \cdot (f_3 \circ \pi))$ is (μ-)compatible. Since π is measure-preserving, (3.2) gives

$$\begin{aligned} \mu(f_1 \cdot (f_2 \circ \pi))^2 &= \mu(f_1 \cdot (f_2 \circ \pi))\, \mu(f_2 \cdot (f_1 \circ \pi)) \\ &\leqq \mu(f_3 \cdot (f_4 \circ \pi))\, \mu(f_4 \cdot (f_3 \circ \pi)) \\ &= \mu(f_3 \cdot (f_4 \circ \pi))^2. \end{aligned}$$

Example 3.3. Consider the case when $X = \{x, y\}$, $\mathscr{F} = 2^X$, $\mu = \mu_c$,

$$\phi(x,x) = \phi(x,y) = \phi(y,x) = x, \qquad \phi(y,y) = y$$

$$\psi(x,x) = x, \qquad \psi(x,y) = \psi(y,x) = \psi(y,y) = y$$

$$\pi(x) = y, \qquad \pi(y) = x.$$

Then π is an involution, and $\mathscr{S}$ is selectively paired and strongly $\mathfrak{M}$-expansive. The strong $\mathfrak{M}$-expansiveness follows as a very special case of [1]. For the ordering defined by taking $x > y$ is a distributive lattice-ordering in which ϕ and ψ are the lattice operations. Now taking $f_j(x) = a_j$, $f_j(y) = 1$, (3.4) becomes the following inequality observed by Preston [13, Lemma 2]:

$$0 \leqq a_1 \leqq a_3,\ 0 \leqq a_2 \leqq a_3,\ 0 \leqq a_4,\ a_1 a_2 \leqq a_3 a_4 \Rightarrow a_1 + a_2 \leqq a_3 + a_4. \tag{3.5}$$

Conversely putting $a_1=f_1(x)f_2(y)$, $a_2=f_1(y)f_2(x)$, $a_3=f_3(x)f_4(y)$, $a_4=f_3(y)f_4(x)$ in (3.5) gives:

$$\mathbf{f}\in\mathfrak{K} \Rightarrow f_1(x)f_2(y)+f_1(y)f_2(x)\leqq f_3(x)f_4(y)+f_3(y)f_4(x) \tag{3.6}$$

and adding in the additional inequalities $f_1(x)f_2(x)\leqq f_3(x)f_4(x)$ and $f_1(y)f_2(y) \leqq f_3(y)f_4(y)$ gives:

$$\mathbf{f}\in\mathfrak{K} \Rightarrow (f_1(x)+f_1(y))(f_2(x)+f_2(y))\leqq(f_3(x)+f_3(y))(f_4(x)+f_4(y)).$$

Thus the $\mathfrak{M}$-expansiveness of $\mathscr{S}$ is effectively equivalent to (3.5) and also to (3.6).

The only other selective pairings on X are (ψ,ϕ), (ϕ_0,ψ_0) and (ψ_0,ϕ_0), where $\phi_0(z,w)=z$ and $\psi_0(z,w)=w$. It is now easy to verify that (3.6) remains valid when compatibility refers to any of these pairs.

We turn next to arbitrary selectively paired spaces.

Proposition 3.4. *Any selectively paired measure space $\mathscr{S}$ is strongly $\mathfrak{M}$-expansive.*

Proof. Consider a μ-compatible function $\mathbf{f}$, and assume without loss that f_3 and f_4 are μ-integrable (Remark 3.1(a)). For any (x,y) in $X^2\setminus\Delta$, let ϕ_{xy} and ψ_{xy} be the restrictions of ϕ and ψ to the two-point set $\{x,y\}$. These form a selective pairing on $\{x,y\}$, and we can ask whether the funciton $\mathbf{f}_{xy}$ obtained by restricting $\mathbf{f}$ to $\{x,y\}$ is compatible with (ϕ_{xy},ψ_{xy}). To check this, it is required to verify that

$$f_1(x')f_2(y')\leqq f_3(\phi(x',y'))f_4(\psi(x',y')) \tag{3.7}$$

whenever $(x',y')=(x,x)$, (x,y), (y,x) or (y,y). Since $\mathbf{f}$ is μ-compatible, (3.7) is valid when $(x',y')=(x,y)$ or (y,x) for μ^2-almost all pairs (x,y) in $X^2\setminus\Delta$. Also by Proposition 2.1, $\mathbf{f}$ is diagonally μ-compatible, so by (2.2), (3.7) holds for $(x',y')=(x,x)$ for μ-almost all x, and hence for μ^2-almost all pairs (x,y). Similarly (3.7) holds for $(x',y')=(y,y)$ for μ^2-almost all (x,y). Thus for μ^2-almost all (x,y) in $X^2\setminus\Delta$, $\mathbf{f}_{xy}$ is compatible with the selective pairing (ϕ_{xy},ψ_{xy}) on the two-point set $\{x,y\}$, so by (3.6)

$$f_1(x)f_2(y)+f_1(y)f_2(x)\leqq f_3(x)f_4(y)+f_3(y)f_4(x). \tag{3.8}$$

For (x,y) in Δ, (3.8) is equivalent to (2.1), so (3.8) holds μ^2-a.e. in Δ, and hence μ^2-a.e. in X^2. Integrating with respect to μ^2 over X^2 gives

$$2\mu(f_1)\,\mu(f_2)\leqq 2\mu(f_3)\,\mu(f_4).$$

There is a converse to Proposition 3.4.

Proposition 3.5. *Let X be any set, $\mathscr{F}_c$ be the σ-algebra of all countable and all cocountable subsets of X, (ϕ,ψ) be a mapping of X^2 into X^2, and suppose that $(X,\mathscr{F}_c,\mu,\phi,\psi)$ is expansive for every measure μ on $(X,\mathscr{F}_c)$. Then (ϕ,ψ) is selective.*

Proof. Take $E_1=E_2=\{x\}$ and $\mu=\delta_x$, where δ_x is the unit mass at x. Then Remark 3.1(b) gives

$$1\leqq\delta_x\{\phi(x,x)\}\,\delta_x\{\psi(x,x)\}.$$

Hence $\phi(x,x)=\psi(x,x)=x$.

Now for distinct x and y take $E_1=\{x\}$, $E_2=\{y\}$, and $\mu=2\delta_x+\delta_y$. Then Remark 3.1(b) gives

$$2\leqq(2\delta_x\{\phi(x,y)\}+\delta_y\{\phi(x,y)\})(2\delta_x\{\psi(x,y)\}+\delta_y\{\psi(x,y)\}).$$

Hence $\delta_x\{\phi(x,y)\}$ and $\delta_x\{\psi(x,y)\}$ are not both zero, i.e. $\phi(x,y)=x$ or $\psi(x,y)=x$. Similarly $\phi(x,y)=y$ or $\psi(x,y)=y$.

We turn now to our main consideration of direct products. The following lemma is an abstract version of [2, Lemma 1].

Lemma 3.6. *Let $\mathscr{S}$ be the direct product of two paired measure spaces $\mathscr{S}_1$ and $\mathscr{S}_2$, and suppose that $\mathscr{S}_1$ is (strongly) $\mathfrak{M}$-expansive. Let $\mathbf{f}: X\to[0,\infty]^4$ be (μ-)-compatible with (ϕ,ψ), and put*

$$g_j(x_2)=\int_{X_1} f_j(x_1,x_2)\,d\mu_1(x_1).$$

Then $\mathbf{g}$ *is (μ_2-)compatible with (ϕ_2,ψ_2).*

Proof. We consider the case when $\mathbf{f}$ is μ-compatible, the other being similar. For μ_2^2-almost all (x_2,y_2) in X_2^2, the function $\mathbf{h}$: $X_1\to[0,\infty]^4$ defined by (2.13) is μ_1-compatible with (ϕ_1,ψ_1), so, assuming that $\mathscr{S}_1$ is strongly $\mathfrak{M}$-expansive,

$$\mu_1(h_1)\,\mu_1(h_2)\leqq\mu_1(h_3)\,\mu_1(h_4).$$

This gives immediately

$$g_1(x_2)\,g_2(y_2)\leqq g_3(\phi_2(x_2,y_2))\,g_4(\psi_2(x_2,y_2))\qquad \mu_2^2\text{-a.e. }(x_2,y_2).$$

The next result is the main theorem concerning direct products, and is a measure-theoretic analogue of [2, Theorem 1].

Theorem 3.7. *The direct product of any finite family of (strongly) $\mathfrak{M}$-expansive paired measure spaces is (strongly) $\mathfrak{M}$-expansive.*

Proof. By induction it suffices to consider the product of two spaces. Using the notation of Lemma 3.6, for any $\mathbf{f}$ (μ-)compatible with (ϕ,ψ), $\mathbf{g}$ is (μ_2-)compatible with (ϕ_2,ψ_2). Since $\mathscr{S}_2$ is (strongly) $\mathfrak{M}$-expansive,

$$\mu_2(g_1)\,\mu_2(g_2)\leqq\mu_2(g_3)\,\mu_2(g_4).$$

But, by Fubini's theorem, $\mu_2(g_j)=\mu(f_j)$, and the theorem follows.

As in [3, 8], it is possible to extend Theorem 3.7 to cover infinite products. However at this level of abstraction, there is an additional technical complication.

Theorem 3.8. *Let $\mathscr{S}$ be the direct product of a family $\{\mathscr{S}_\lambda\colon \lambda\in\Lambda\}$ of $\mathfrak{M}$-expansive paired probability spaces. Suppose that there is no $\mathscr{F}$-measurable set E such that $\mu(E)=1$ and $(E\times E)\cap\phi^{-1}(E)\cap\psi^{-1}(E)$ is empty. Then $\mathscr{S}$ is $\mathfrak{M}$-expansive. In particular if each $\mathscr{S}_\lambda$ is diagonally invariant and $\mathfrak{M}$-expansive, then $\mathscr{S}$ is $\mathfrak{M}$-expansive.*

Proof. Take a function $\mathbf{f}: X \to [0, \infty]^4$ which is compatible with (ϕ, ψ). Since $\mathbf{f}$ is $\mathscr{F}$-measurable, there is a countable subset Λ_0 of Λ such that $f(x)=f(y)$ whenever $x_\lambda = y_\lambda$ for all λ in Λ_0. Thus replacing Λ by Λ_0, we may assume that Λ is countable, and hence that $\Lambda = \mathbb{N}$.

Let $\mathscr{T}_m = (Z_m, \mathscr{G}_m, v_m, \rho_m, \sigma_m)$ and $\mathscr{T}_m' = (Z_m', \mathscr{G}_m', v_m', \rho_m', \sigma_m')$ be the respective direct products of $\{\mathscr{S}_n: n > m\}$ and $\{\mathscr{S}_n: 1 \leqq n \leqq m\}$, and identify $\mathscr{S}$ with $\mathscr{T}_m' \times \mathscr{T}_m$. By Theorem 3.7, $\mathscr{T}_m'$ is $\mathfrak{M}$-expansive. Thus if

$$\tilde{f}_{mj}(z_m', z_m) = f_{mj}(z_m) = \int_{Z_m'} f_j(w_m', z_m)\, dv_m'(w_m') \quad (z_m' \in Z_m', z_m \in Z_m)$$

Lemma 3.6 shows that $\mathbf{f}_m$ is compatible with (ρ_m, σ_m), i.e.

$$f_{m1}(z_m) f_{m2}(w_m) \leqq f_{m3}(\rho_m(z_m, w_m)) f_{m4}(\sigma_m(z_m, w_m)) \qquad (z_m, w_m \in Z_m)$$

or equivalently,

$$\tilde{f}_{m1}(x)\tilde{f}_{m2}(y) \leqq \tilde{f}_{m3}(\phi(x,y))\tilde{f}_{m4}(\psi(x,y)) \qquad (x, y \in X). \tag{3.9}$$

By the reverse version of the Fubini-Jessen theorem [7, Theorem III.11.27], $\tilde{f}_{mj}(x)$ converges to the constant $\mu(f_j)$ as $m \to \infty$ for all x in some set E with $\mu(E) = 1$. Taking (x, y) in the non-empty set $(E \times E) \cap \phi^{-1}(E) \cap \psi^{-1}(E)$ and letting $m \to \infty$ in (3.9) gives (3.2).

If each $\mathscr{S}_\lambda$ and hence $\mathscr{S}$ is diagonally invariant, choosing x in E and putting $y = x$ in (3.9) gives the result.

Corollary 3.9. *The direct product of any family of selectively paired probability spaces is $\mathfrak{M}$-expansive.*

Proof. This is immediate from Proposition 3.4, Theorem 3.8 and the fact that a selective pairing is diagonally invariant.

In the notation of Theorem 3.8, Remark 3.1(b) shows that $\mu_\lambda^*(\phi_\lambda(X_\lambda^2)) = \mu_\lambda^*(\psi_\lambda(X_\lambda^2)) = 1$, so $\mu^*(\phi(X^2)) = \mu^*(\psi(X^2)) = 1$. Hence if $\mu(E) > 0$, then $\phi^{-1}(E)$ and $\psi^{-1}(E)$ are non-empty. However it is not clear that $(E \times E) \cap \phi^{-1}(E) \cap \psi^{-1}(E)$ is non-empty, even if $\mu(E) = 1$, nor indeed whether Theorem 3.8 remains valid if this condition is dropped. Note however that if $(E \times E) \cap \phi^{-1}(E) \cap \psi^{-1}(E)$ is empty, then $(2\chi_E, \chi_E, 2 - \chi_E, 2 - \chi_E)$ is compatible. Hence if $\mathscr{S}$ is $\mathfrak{M}$-expansive, then $\mu(E) < 1$.

Strong $\mathfrak{M}$-expansiveness is rarely preserved under infinite direct products. For instance the μ-compatible function $(1, 1, 1 - \chi_E, 1)$ as constructed in Example 2.3 does not satisfy (3.2). Indeed using Proposition 2.2 and the forward version of the Fubini-Jessen theorem, it is possible to show that a product $\mathscr{S}$ of diagonally settled, diagonally invariant, strongly $\mathfrak{M}$-expansive, measurably paired spaces is strongly $\mathfrak{M}$-expansive if and only if $\mathscr{S}$ is diagonally settled. A simple modification of the proof of Theorem 3.8 shows that this does occur if (and only if) there is no $\mathscr{F}$-measurable set E with $\mu(E) = 1$ such that $\phi^{-1}(E) \cap \psi^{-1}(E)$ is μ^2-null.

Example 3.10. Suppose that X is a lattice and v_1 and v_2 are functions of X into $[0, \infty]$ satisfying (1.9), viz. $v_1(x) \leqq v_2(y)$ whenever $x \leqq y$. Then $(1, v_1, v_2, 1)$ is compatible with $(\vee, \wedge)$. Furthermore if $\mathbf{f}$ is compatible with $(\vee, \wedge)$, then by (2.6), so is $(f_1, v_1 f_2, v_2 f_3, f_4)$.

Now if $(X, \mathscr{F}, \mu)$ is a product of totally ordered measure spaces and X is given the product ordering, then X is a lattice, and $\mathscr{S}=(X, \mathscr{F}, \mu, \vee, \wedge)$ is $\mathfrak{M}$-expansive by Corollary 3.9. Thus for v_1, v_2 and $\mathbf{f}$ as above we recover the inequality (1.8), viz.

$$\mu(f_1)\,\mu(v_1 f_2) \leqq \mu(v_2 f_3)\,\mu(f_4).$$

Thus in the terminology of [11], $\mathscr{S}$ is an "FKG-space". The inequality (1.8) was first proved in [13] for finite products and in [3] for infinite products under the additional assumptions that $v_1 = v_2$, $f_1 = f_3$ and $f_2 = f_4$. Edwards [8] and Kemperman [11] gave simplified proofs and allowed v_1 and v_2 to differ, but they still required that $f_1 = f_3$ and $f_2 = f_4$. If the proof of Corollary 3.9 is followed through, it seems that our proof of the more general result is simpler than those in [8, 11] both in finite cases and in the infinite extension. Furthermore for a finite product, Proposition 3.4 and Theorem 3.7 show that $\mathscr{S}$ is strongly $\mathfrak{M}$-expansive, so our result also improves that of [13] in that for (1.8) to be valid it is sufficient that (1.7) holds μ^2-a.e. (and that $v_1(x) \leqq v_2(x \vee y)\,\mu^2$-a.e.). However to establish this required considerably more technical detail in the proof of Proposition 3.4.

Allowing for example f_1 and f_3 to differ makes it pointless to seek measures on $\{(x,y) \in X^2 : x \leqq y\}$ with certain marginals as was successfully done in [13].

4. Explosiveness

Again following [2], we now introduce the concept of explosiveness for a paired measure space $\mathscr{S}$. An $\mathscr{F}$-measurable function $\mathbf{g}: X \to [0, \infty]^8$ is *bicompatible* (resp. *μ-bicompatible*) with (ϕ, ψ) if:

$$g_1(x)\,g_2(x')\,g_3(y)\,g_4(y') \leqq g_5(\phi(x,y'))\,g_6(\psi(y,x'))\,g_7(\phi(y,x'))\,g_8(\psi(x,y')) \quad (4.1)$$

for all (resp. μ^4-almost all) (x, x', y, y') in X^4. For subsets F_1 and F_2 of X^2, put

$$\phi\,\psi(F_1, F_2) = \{(\phi(x,y'), \psi(y,x')) : (x,x') \in F_1, (y,y') \in F_2\}. \quad (4.2)$$

A 4-tuple (F_1, F_2, F_3, F_4) of $\mathscr{F}^2$-measurable sets is *bicompatible* with (ϕ, ψ) if

$$\phi\,\psi(F_1, F_2) \subset F_3 \quad \text{and} \quad \phi\,\psi(F_2, F_1) \subset F_4. \quad (4.3)$$

The space $\mathscr{S}$ is *explosive* if

$$\mu^2(F_1)\,\mu^2(F_2) \leqq \mu^2(F_3)\,\mu^2(F_4)$$

for all (F_1, F_2, F_3, F_4) bicompatible with (ϕ, ψ); *(strongly) $\mathfrak{M}$-explosive* if

$$\Big(\int_{F_1} g_1 \otimes g_2\, d\mu^2\Big)\Big(\int_{F_2} g_3 \otimes g_4\, d\mu^2\Big) \leqq \Big(\int_{F_3} g_5 \otimes g_6\, d\mu^2\Big)\Big(\int_{F_4} g_7 \otimes g_8\, d\mu^2\Big) \quad (4.4)$$

for all (μ-)bicompatible $\mathbf{g}$ and bicompatible (F_1, F_2, F_3, F_4), where

$$(h \otimes h')(x, x') = h(x)\,h'(x') \quad (x, x' \in X).$$

For counting measure on a finite set, these definitions of explosiveness and $\mathfrak{M}$-explosiveness reduce to those of [2].

Define a pairing $(\overline{\phi\,\psi}, \underline{\phi\,\psi})$ on X^2 by:

$$\overline{\phi\,\psi}((x,x'),(y,y')) = (\phi(x,y'), \psi(y,x'))$$
$$\underline{\phi\,\psi}((x,x'),(y,y')) = (\phi(y,x'), \psi(x,y'))$$

so that $\underline{\phi\,\psi}(z) = \overline{\phi\,\psi}(z^*) = \overline{\psi\,\phi}(z)^*$. (The asterisks here refer to the operation changing the order of pairs in $X^2 \times X^2$. In [2], $\overline{\phi\,\psi}$ and $\underline{\phi\,\psi}$ were denoted by ϕ^2 and ψ^2 respectively, but we feel such notation might be in conflict with that adopted elsewhere in this paper.) Then it is clear from (2.1), (2.3), (4.1), (4.2) and (4.3) that

$$\mathbf{g} \text{ is } (\mu\text{-})\text{bicompatible with } (\phi,\psi) \Leftrightarrow$$
$$(g_1 \otimes g_2, g_3 \otimes g_4, g_5 \otimes g_6, g_7 \otimes g_8) \text{ is } (\mu\text{-})\text{compatible with } (\overline{\phi\,\psi}, \underline{\phi\,\psi}) \tag{4.5}$$
$$\phi\,\psi(F_1,F_2) = \overline{\phi\,\psi}(F_1 \times F_2), \qquad \phi\,\psi(F_2,F_1) = \underline{\phi\,\psi}(F_1 \times F_2)$$
$$(F_1,F_2,F_3,F_4) \text{ is bicompatible with } (\phi,\psi) \Leftrightarrow$$
$$(F_1,F_2,F_3,F_4) \text{ is compatible with } (\overline{\phi\,\psi}, \underline{\phi\,\psi}). \tag{4.6}$$

The following is a measure-theoretic extension of [2, Theorem 6].

Theorem 4.1. *The following are equivalent for a paired measure space* $\mathscr{S}$:

(i) $(X^2, \mathscr{F}^2, \mu^2, \phi \times \psi^*, \psi \times \phi^*)$ is *(strongly)* $\mathfrak{M}$*-expansive*
(ii) $(X^2, \mathscr{F}^2, \mu^2, \overline{\phi\,\psi}, \underline{\phi\,\psi})$ *is (strongly)* $\mathfrak{M}$*-expansive*
(iii) $\mathscr{S}$ *is (strongly)* $\mathfrak{M}$*-explosive*
(iii)′ *For any* $\mathbf{g}$*(*μ*-)bicompatible with* (ϕ,ψ),

$$\mu(g_1)\,\mu(g_2)\,\mu(g_3)\,\mu(g_4) \leqq \mu(g_5)\,\mu(g_6)\,\mu(g_7)\,\mu(g_8)$$

(iv) $\mathscr{S}$ *is (strongly)* $\mathfrak{M}$*-expansive*
(v) $(X, \mathscr{F}, \mu, \psi^*, \phi^*)$ *is (strongly)* $\mathfrak{M}$*-expansive.*

Proof. We shall consider the weaker properties, the equivalence of the stronger versions being analogous.

(i)$\Leftrightarrow$(ii). It is readily verified (cf. [2, p. 285]) that (f_1, f_2, f_3, f_4) is compatible with $(\overline{\phi\,\psi}, \underline{\phi\,\psi})$ if and only if (f_1, f_2^*, f_3, f_4^*) is compatible with $(\phi \times \psi^*, \psi \times \phi^*)$. Furthermore

$$\mu^2(f_1)\,\mu^2(f_2) = \mu^2(f_1)\,\mu^2(f_2^*), \qquad \mu^2(f_3)\,\mu^2(f_4) = \mu^2(f_3)\,\mu^2(f_4^*).$$

(ii)$\Rightarrow$(iii). The inequality (4.4) follows from (ii) via (4.5), (4.6) and (3.3).

(iii)$\Rightarrow$(iii)′. This is immediate on restricting (4.4) to the case $F_j = X^2$.

(iii)′$\Rightarrow$(iv). For $\mathbf{f}$ compatible with (ϕ,ψ), put

$$g_1 = g_3 = f_1, \qquad g_2 = g_4 = f_2, \qquad g_5 = g_7 = f_3, \qquad g_6 = g_8 = f_4.$$

Then $\mathbf{g}$ is bicompatible with (ϕ,ψ), so

$$\mu(f_1)^2\,\mu(f_2)^2 = \mu(g_1)\,\mu(g_2)\,\mu(g_3)\,\mu(g_4) \leqq \mu(g_5)\,\mu(g_6)\,\mu(g_7)\,\mu(g_8) = \mu(f_3)^2\,\mu(f_4)^2.$$

(iv)$\Leftrightarrow$(v). By (2.4) and (2.5), (f_1, f_2, f_3, f_4) is compatible with (ϕ, ψ) if and only if (f_2, f_1, f_4, f_3) is compatible with (ψ^*, ϕ^*).

(iv)$\Rightarrow$(i). This follows immediately from the already proved equivalence of (iv) with (v) together with Theorem 3.7.

Proposition 4.2. *Consider the following properties of a paired measure space* $\mathscr{S}$:

(i) $(X^2, \mathscr{F}^2, \mu^2, \phi\times\psi^*, \psi\times\phi^*)$ *is expansive*
(ii) $(X^2, \mathscr{F}^2, \mu^2, \overline{\phi\,\psi}, \underline{\phi\,\psi})$ *is expansive*
(iii) $\mathscr{S}$ *is explosive*
(iv) $\mathscr{S}$ *is expansive*
(v) $(X, \mathscr{F}, \mu, \psi^*, \phi^*)$ *is expansive.*

The following implications are valid:

$$(\mathrm{i})\Leftrightarrow(\mathrm{ii})\Leftrightarrow(\mathrm{iii})\Rightarrow(\mathrm{iv})\Leftrightarrow(\mathrm{v}).$$

Proof. Specialising the proof of Theorem 4.1 to the case of characteristic functions leads to a proof of all implications except (iii)$\Rightarrow$(ii), which is immediate from (4.6).

5. Finite Distributive Lattices

Example 3.3 shows that $(\{0,1\}, 2^{\{0,1\}}, \mu_c, \vee, \wedge)$ is an $\mathfrak{M}$-expansive space. Applying Theorem 3.7 to a finite family of copies of this space shows that for any finite set Λ, $\mathscr{S}(\Lambda)=(2^\Lambda, 2^{2^\Lambda}, \mu_c, \cup, \cap)$ is also $\mathfrak{M}$-expansive. Any finite distributive lattice L can be embedded in 2^Λ for some finite set Λ [4, Corollary, p. 59], so it follows from Remark 3.1(c) that $\mathscr{S}_L=(L, 2^L, \mu_c, \vee, \wedge)$ is $\mathfrak{M}$-expansive, i.e. inequality (1.10) holds. (An extension to infinite lattices can then be made by a simple limiting argument.) This result was first obtained in [1], and itself has many important consequences as listed in [2, Sect. 9]. The expansiveness of $\mathscr{S}_L$ gives inequality (1.2).

Conversely let L be a finite lattice such that $\mathscr{S}_L$ is $\mathfrak{M}$-expansive. Then $\mathscr{S}_L$ is an FKG-space (see Example 3.10) and hence L is distributive [11, Theorem 7]. Thus $\mathscr{S}_L$ is $\mathfrak{M}$-expansive if and only if $\mathscr{S}_L$ is an FKG-space, or, equivalently, L is distributive.

For any finite set S, there is an involution π on $\mathscr{S}(S)$ given by $\pi(A)=S\setminus A$. Let F_1 and F_2 be functions of 2^S into 2^L, and put

$$\begin{aligned} F_3(C)&=\bigcup_{A\cup B=C} F_1(A)\vee F_2(B)\\ F_4(C)&=\bigcup_{A\cap B=C} F_1(A)\wedge F_2(B) \qquad (5.1)\\ f_j(A)&=|F_j(A)| \qquad (j=1,2,3,4). \end{aligned}$$

Then (1.2) shows that $\mathbf{f}$ is compatible with $(\cup, \cap)$ on 2^S. Now applying Proposition 3.2 gives the inequality

$$\sum_{A\subset S}|F_1(A)|\,|F_2(S\setminus A)| \leqq \sum_{C\subset S}|F_3(C)|\,|F_4(S\setminus C)| \qquad (5.2)$$

which (apart from notational changes) is the same as (1.3). Thus [6, Theorem 1] can be deduced from (1.2) and $\mathfrak{M}$-expansiveness. Since (1.2) is itself a consequence of expansiveness, (5.2) could be obtained directly from a single application of expansiveness. Rather than giving explicit details of this, we prefer to show how (5.2) can be obtained more fundamentally from an application of (1.2) in a different lattice.

Let $\mathscr{L}$ be the lattice $2^S \times L \times \tilde{L}$ with the product ordering, where $\tilde{L}$ is the lattice obtained from L by reversing the ordering in L. Put

$$\mathscr{A} = \{(A, x_1, x_2) \in \mathscr{L}\colon x_1 \in F_1(A), x_2 \in F_2(S \setminus A)\}$$
$$\mathscr{B} = \{(B, y_1, y_2) \in \mathscr{L}\colon y_1 \in F_2(B), y_2 \in F_1(S \setminus B)\}.$$

Then

$$\mathscr{A} \vee \mathscr{B} \subset \{(C, z_1, z_2) \in \mathscr{L}\colon z_1 \in F_3(C), z_2 \in F_4(S \setminus C)\}$$
$$\mathscr{A} \wedge \mathscr{B} \subset \{(D, w_1, w_2) \in \mathscr{L}\colon w_1 \in F_4(D), w_2 \in F_3(S \setminus D)\}$$
$$|\mathscr{A}| = |\mathscr{B}| = \sum_{A \subset S} |F_1(A)|\,|F_2(S \setminus A)|$$
$$|\mathscr{A} \vee \mathscr{B}|, \quad |\mathscr{A} \wedge \mathscr{B}| \leqq \sum_{C \subset S} |F_3(C)|\,|F_4(S \setminus C)|.$$

Thus (1.2) applied to $\mathscr{A}$ and $\mathscr{B}$ gives (the square of) (5.1) immediately.

Recall that a subset U of L is an *up-set* if

$$x \in U, \quad y \in L, \quad x \leqq y \Rightarrow y \in U.$$

Down-sets are defined similarly, and a down-set of the form $\{y \in L\colon y \leqq x\}$ is an *ideal* in L.

Daykin [6] has obtained a number of other inequalities concerning up-sets, down-sets and ideals in 2^A, but it was not clear from his methods that the results could be transferred to arbitrary finite distributive lattices. (An up-set in one lattice may fail to be an up-set in a larger lattice.) We shall now show that they do remain valid by deducing them from (5.2).

Theorem 5.1. *Let I and J be ideals in a finite distributive lattice L, S be a finite set, U, V, W and X be increasing functions from 2^S into the set of up-sets in L (ordered by inclusion), and D and E be decreasing functions of 2^S into the set of down-sets in L. Suppose that $I \subset J$, $U(A) \subset V(A)$, $W(A) \subset X(A)$ and $D(A) \subset E(A)$ for all A in 2^S. Then*

$$|I| \sum_{A \subset S} |J \cap V(A) \cap D(A)| + |J| \sum_{A \subset S} |I \cap U(A) \cap E(A)|$$
$$\leqq \sum_{C \subset S} |I \cap U(C)|\,|J \cap D(S \setminus C)| + \sum_{C \subset S} |I \cap E(C)|\,|J \cap V(S \setminus C)| \tag{5.3}$$
$$\sum_{A \subset S} |I \cap U(A)|\,|J \cap X(S \setminus A)| + \sum_{A \subset S} |I \cap W(A)|\,|J \cap V(S \setminus A)|$$
$$\leqq |I| \sum_{C \subset S} |J \cap X(C) \cap V(C)| + |J| \sum_{C \subset S} |I \cap U(C) \cap W(C)|. \tag{5.4}$$

Proof. Let S' be the set obtained by adjoining one additional point ω to S, and for A contained in S, let $A' = A \cup \{\omega\}$. Put

$$F_1(A) = I \cap U(A) \cap E(A) \qquad F_1(A') = J \cap V(A) \cap D(A)$$
$$F_2(A) = I \qquad\qquad F_2(A') = J.$$

Then F_1 and F_2 are functions from $2^{S'}$ into 2^L, and if F_3 and F_4 are defined by (5.1) with S' replacing S,

$$\begin{aligned} F_3(C) &= \bigcup_{A \cup B = C} (I \cap U(A) \cap E(A)) \vee I \\ &\subset I \cap U(C) \end{aligned}$$

$$\begin{aligned} F_3(C') = \bigcup_{A \cup B = C} \{&[(I \cap U(A) \cap E(A)) \vee J] \cup [(J \cap V(A) \cap D(A)) \vee I] \\ &\cup [(J \cap V(A) \cap D(A)) \vee J]\} \\ \subset J \cap V(C).& \end{aligned}$$

Similarly

$$F_4(C) \subset I \cap E(C), \qquad F_4(C') \subset J \cap D(C).$$

Now applying (5.2) with S replaced by S' gives

$$\begin{aligned} |I| \sum_{A \subset S} &|J \cap V(A) \cap D(A)| + |J| \sum_{A \subset S} |I \cap U(A) \cap E(A)| \\ &= \sum_{A \subset S} |F_1(A)|\,|F_2(S' \setminus A')| + \sum_{A \subset S} |F_1(A)|\,|F_2(S' \setminus A)| \\ &= \sum_{B \subset S'} |F_1(B)|\,|F_2(S' \setminus B)| \\ &\leqq \sum_{B \subset S'} |F_3(B)|\,|F_4(S' \setminus B)| \\ &\leqq \sum_{C \subset S} |I \cap U(C)|\,|J \cap D(S \setminus C)| + \sum_{C \subset S} |I \cap E(C)|\,|J \cap V(S \setminus C)|. \end{aligned}$$

A similar argument, the details of which are left to the reader, leads to (5.4).

The inequalities (5.3) and (5.4) were obtained in [6, Theorems 2, 3] in the case when $L = 2^A$ for some finite set A. Combining the methods of this section, they could each have been deduced from a single application of expansiveness in $\mathcal{S}_{\mathcal{L}'}$, where $\mathcal{L}' = 2^{S'} \times L \times \tilde{L}$. Putting $S = \emptyset$, $I = J = L$, $U(S) = \emptyset$, $V(S) = U$, $D(S) = E(S) = D$ gives (1.1) as a special case of (5.3).

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Received August 1, 1979