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$\mathscr{H}^p$ Stability of Solutions of Stochastic Differential Equations*

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Summary. Solutions of systems of stochastic differential equations are shown to be stable in $\mathscr{H}^p$ under $\mathscr{H}^p$ perturbations of semimartingale differentials. Analogous results are obtained in $\mathscr{S}^p$ when the solutions are not semimartingales but are only cadlag, adapted processes. Also, the solutions are shown to be stable under almost sure perturbations. These results are contrasted with the lack of stability under non-$\mathscr{H}^p$ perturbations, a result originally obtained by Wong and Zakai.

1. Introduction

Solutions of stochastic differential equations have long been assumed to be unstable under a small change in the (random) driving term. In 1965 Wong and Zakai [14] revealed instability by an a.s. approximation of a standard Brownian motion B by processes B^n which had piecewise continuously differentiable paths. If one restricts the approximations of Brownian motion to local martingales, however, a consequence of the results presented here is that the solutions are stable. Consider equations of the form

$$(1.1)\quad X_t = X_0 + \sum_{i=1,m} \int_0^t F_i X_{s-}\, dZ_s^i$$

where the driving terms Z^i are semimartingales. We show that if the driving terms of equations of the form (1.1) satisfy a technical uniformity condition and converge weak-locally in an $\mathscr{H}^p$ norm for semimartingales, then the solutions converge, also weak-locally in $\mathscr{H}^p$, to the solutions of the limiting equation (weak-local convergence and the $\mathscr{H}^p$ norm are defined in Section 2).

M. Emery [3,4] has extended to semimartingales the $\mathscr{H}^p$ norm ($1 \leqq p \leqq \infty$) for martingales. In Section 2 we recall some of the definitions and two lemmas of

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Emery. In Section 3 we consider equations of the form

$$(1.2)\quad X_t^n = X_0^n + \sum_{i=1,m} \int_0^t F_i X_{s-}^n \, dZ_s^{i,n}$$

$$(1.3)\quad X_t = X_0 + \sum_{i=1,m} \int_0^t F_i X_{s-} \, dZ_s^i.$$

We show that if (1) X_0^n converges to X_0 in L^p, (2) $Z^{i,n}$ are semimartingales satisfying a technical uniformity condition, (3) Z^n converges weak-locally in $\mathscr{H}^p$, then X^n converges to X weak-locally in $\mathscr{H}^p$. The main results are Theorem (3.3), Theorem (3.8) and Theorem (3.19). We show by example (3.12) that in general one cannot dispense with the uniformity condition (2) above. We also consider equations where X_0^n and X_0 in (1.2) and (1.3) are replaced with adapted, cadlag. processes $(J_t^n)_{t \geq 0}$ and $(J_t)_{t \geq 0}$ respectively, and we obtain analogous results.

In Section 4 we employ a technique due to Stricker [13] and the results of Section 3 to obtain almost sure convergence. This pertains particularly to the results of Wong and Zakai: if local martingales L^n converge a.s. to a local martingale L in the sup norm (i.e., if $\lim_{n\to\infty} (L^n - L)_t^* = 0$ for each t), then for some subsequence, solutions of Equations (1.4) below converge a.s. in the sup norm to the solution of (1.5):

$$(1.4)\quad X_t^n = X_0 + \int_0^t F X_{s-}^n \, dL_s^n + \int_0^t G X_{s-}^n \, ds,$$

$$(1.5)\quad X_t = X_0 + \int_0^t F X_{s-} \, dL_s + \int_0^t G X_{s-} \, ds.$$

The main result of Section 4 is Theorem (4.16). In Comment (4.17) we exhibit the relationship of this result to those of Wong and Zakai.

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2. Preliminaries

We assume the reader is familiar with the theory of stochastic integration as set forth in Meyer [7]. (In view of [12], however, all of our results hold as well for the stochastic integrals of E.J. McShane [6], provided the appropriate processes are "$K\Delta t$ after small amendments".) Our notation will be that of Meyer [7]; we recall in this section the recent definitions and results of M. Emery [3, 4] which are not contained in [7].

We assume throughout that $(\Omega, \mathscr{F}, P)$ is a fixed underlying complete probability space and that $(\mathscr{F}_t)$ is a right-continuous filtration, where $\mathscr{F}_0$ contains all the $\mathscr{F}$-null sets.

An $\mathscr{H}^p$ norm $(1 \leq p \leq \infty)$ has been proposed for semimartingales in [3, 4]. Meyer [8] has generalized the norm and further developed the properties of $\mathscr{H}^p$.

A process A is a *VF process* if $(A_t)_{t\geqq 0}$ is finite valued, adapted, and has right continuous paths which are of bounded variation on compact sets. For a local martingale N and a VF process A, we denote

$$(2.1)\quad j_p(N,A)=\left\|[N,N]_\infty^{1/2}+\int_{0-}^{\infty}|dA_s|\right\|_{L^p}$$

where $\int_{0-}^{\infty}|dA_s|$ is the random variable of the total variation of the paths of A, including the point mass at 0.

(2.2) *Definition.* Let X be a semimartingale. For $1\leqq p\leqq\infty$, define the norm

$$\|X\|_{\mathscr{H}^p}=\inf_{X=N+A} j_p(N,A)$$

where the infimum is over all possible decompositions $X=N+A$. The space $\mathscr{H}^p$ is the space of those X such that $\|X\|_{\mathscr{H}^p}<\infty$.

We refer the reader to Meyer [8] for properties of the semimartingale $\mathscr{H}^p$ norm. We observe that if X is a martingale then the usual $\mathscr{H}^p$ martingale norm is equal to the $\mathscr{H}^p$ semimartingale norm. If $X\in\mathscr{H}^p$, $1\leqq p\leqq\infty$, then X is a special semimartingale and so has a unique decomposition $X=\bar{N}+\bar{A}$, where $\bar{A}$ is a predictable VF process. Meyer shows that $j_p(\bar{N},\bar{A})$ gives a norm equivalent to $\|X\|_{\mathscr{H}^p}$ for $1\leqq p<\infty$. This then implies that $\left\|\bar{N}_\infty^*+\int_{0-}^{\infty}|d\bar{A}_s|\right\|_{L^p}$ also gives an equivalent norm ($1\leqq p<\infty$).

If a process X has paths which are right continuous with left limits it is said to have cadlag. paths. For X with cadlag. paths we denote

$$(2.3)\quad X_{t-}^*=\sup_{s<t}|X_s|,\qquad X_t^*=\lim_{\substack{s\to t\\ s>t}} X_{s-}^*$$

so that $t\to X_t^*$ is again cadlag.

(2.4) *Definition.* Let X be an (adapted) cadlag. process. For $1\leqq p\leqq\infty$, let

$$\|X\|_{\mathscr{S}^p}=\|X_\infty^*\|_{L^p}.$$

X is said to be in $\mathscr{S}^p$ if $\|X\|_{\mathscr{S}^p}<\infty$.

One easily checks that the $\mathscr{H}^p$ norm is a stronger norm than the $\mathscr{S}^p$ norm; i.e., $\|\cdot\|_{\mathscr{S}^p}\leqq c_p\|\cdot\|_{\mathscr{H}^p}$ for some universal constant c_p, $1\leqq p<\infty$.

The following elementary lemma is due to Emery [4], and is also proved (and extended) in Meyer [8].

(2.5) **Lemma.** *Given $1\leqq p\leqq\infty$, $1\leqq q\leqq\infty$, $\frac{1}{p}+\frac{1}{q}=\frac{1}{r}$, let H be predictable and X be a semimartingale. Suppose the stochastic integral $H\cdot X$ exists. Then*

$$\|H\cdot X\|_{\mathscr{H}^r}\leqq\|H\|_{\mathscr{S}^p}\|X\|_{\mathscr{H}^q},$$

$$\|H\cdot X\|_{\mathscr{S}^r}\leqq c_r\|H\|_{\mathscr{S}^p}\|X\|_{\mathscr{H}^q}.$$

In the theory of stochastic integration and differential equations the customary way theorems hold is *locally;* that is, on stochastic intervals $[\![0,T]\!]=\{(t,\omega): 0\leqq t\leqq T(\omega)\}$, where T is a stopping time. One *stops* a process at a stopping time T in the following way:

$$X_t^T=X_t 1_{[\![0,T[\![}+X_T 1_{[\![T,\infty[\![}.$$

Kazamaki [5], in defining weak martingales, pointed out the usefulness of stochastic intervals $[\![0,T[\![$. We will call a process X *weak-stopped* at a stopping time T if $(X_t)_{t\geqq 0}=(X_t^{T-})_{t\geqq 0}$, where X^{T-} is given by:

$$(2.6)\quad X_t^{T-}=X_t 1_{[\![0,T[\![}+X_{T-} 1_{[\![T,\infty[\![}.$$

Note that X^{T-} is continuous at T. If M is a local martingale then the weak-stopped process M^{T-} need not be a local martingale (unless T is a predictable stopping time). However, no such pathology occurs with semimartingales: if X is a semimartingale then so is the weak-stopped process X^{T-}.

(2.7) *Definition.* A result (R) is said to hold *weak-locally* for a cadlag. process X if there exists a sequence of stopping times $(T^n)_{n\geqq 1}$ increasing to ∞ a.s. such that (R) holds for the weak-stopped process X^{T^n-} for each n.

We caution the reader that results that hold weak-locally need not hold locally, where locally is used in the customary sense; that is, the result (R) is said to hold *locally* if it holds for the stopped process X^{T^n}, rather than for X^{T^n-}.

For simplicity of notation in stating results which hold weak-locally we introduce the notation (for $1\leqq p\leqq\infty$):

$$(2.8)\quad \begin{aligned}\|X\|_{\mathscr{H}^p(T)}&=\|X^{T-}\|_{\mathscr{H}^p}\\ \|X\|_{\mathscr{S}^p(T)}&=\|X^{T-}\|_{\mathscr{S}^p}.\end{aligned}$$

The following definition and Lemma (2.11) are due to Emery [4].

(2.9) *Definition.* Let $\alpha>0$, X be a semimartingale, and $(T_0,\ldots,T_k)$ be a *finite* sequence of increasing stopping times. This sequence is said *to carve X into slices smaller than α* if $X\in\mathscr{H}^\infty$, $X=X^{T_k-}$, and

$$(2.10)\quad \|\Delta_{]\!]T_i,T_{i+1}[\![}X\|_{\mathscr{H}^\infty}\leqq\alpha\qquad(1\leqq i\leqq k),$$

where $\Delta_{]\!]S,T[\![}X=(X-X^S)^{T-}$. For $\alpha>0$, we say that *X can be carved into slices smaller than α*, and we write

$$X\in D(\alpha)$$

if there exists a finite sequence $(T_0,\ldots,T_k)$ of increasing stopping times with $T_0=0$ that carve X into slices smaller than α.

(2.11) **Lemma.** *Let X be a semimartingale. For each $\alpha>0$ there exists a stopping time T arbitrarily large such that X^{T-} is bounded and in $D(\alpha)$.*

Let F be an operator mapping adapted cadlag. processes into adapted cadlag, processes and let Z be a semimartingale. Consider the following stochastic integral equation in which J is adapted and cadlag.:

$$(2.12) \quad X_t = J_t + \int_0^t F X_{s-}\, dZ_s.$$

Existence and uniqueness of solutions of (2.12) have been demonstrated in [1, 2, 10], and [11] under various additional hypotheses on F and/or J. Meyer has observed that the techniques used in the proofs essentially only use the fact that $F \in \mathrm{Lip}(K)$, as defined below. The Lip(K) definition is taken from Emery [4].

(2.13) *Defnition.* Let K be a constant and F be an operator that maps adapted, cadlag. processes into adapted, cadlag. processes. *F is said to be in* Lip(K) if the following two conditions are satisfied:

(2.14) for each stopping time T, $X^{T-} = Y^{T-}$ implies $(FX)^{T-} = (FY)^{T-}$.

(2.15) $(FX - FY)^* \leqq K(X-Y)^*$ as processes.

Note that if $f(\omega, t, x)$ is left continuous in t for fixed ω and x, is $\mathscr{F}_t$-measurable for fixed t and x, and is Lipschitz in the space variable, and if F is given by $(FX)_t(\omega) = f(\omega, t, X_t(\omega))_+$, then $F \in \mathrm{Lip}(K)$.

3. $\mathscr{H}^p$ Stability of Solutions

Fix p with $1 \leqq p < \infty$. Suppose that for $1 \leqq i \leqq m$, $(Z^{i,n})_{n \geqq 1}$ and Z^i are semimartingales, all locally in $\mathscr{H}^p$. Let F_i satisfy the Lip(K) conditions. Let $(X^n)_{n \geqq 1}$ and X be the unique solutions respectively of

$$(3.1) \quad X_t^n = X_0^n + \sum_{i=1,m} \int_0^t F_i X_{s-}^n\, dZ_s^{i,n}$$

$$(3.2) \quad X_t = X_0 + \sum_{i=1,m} \int_0^t F_i X_{s-}\, dZ_s^i.$$

That such unique solutions exist with each F_i satisfying the Lip(K) conditions was implicitly established in [1, 2] and [11], but it is first explicitly formulated by M. Emery in [4].

(3.3) **Theorem.** *Suppose for* $1 \leqq i \leqq m$, $(Z^{i,n})_{n \geqq 1}$, Z^i *are semimartingales and that* $\lim_{n\to\infty} Z^{i,n} = Z^i$ *weak-locally in* $\mathscr{H}^p$, $1 \leqq i \leqq m$. *Let* F_i *satisfy the* Lip(K) *conditions for* $1 \leqq i \leqq m$, *and let* $(X^n)_{n \geqq 1}$ *and* X *be given by* (3.1) *and* (3.2) *respectively. Let* $\lim_{n\to\infty} \|X_0^n - X_0\|_{L^p} = 0$. *Further, assume each* F_i *is bounded. Then* $\lim_{n\to\infty} X^n = X$ *weak-locally in* $\mathscr{H}^p$.

Proof. We prove the lemma for $m=1$ and for an arbitrary fixed p, $1 \leqq p < \infty$. The proof for arbitrary $m < \infty$ is analogous. Let $c = c_p$ be a constant such that $\|\cdot\|_{\mathscr{S}^p} \leqq c\|\cdot\|_{\mathscr{H}^p}$. Let K be the Lipschitz constant for F, and choose α such that

$0<\alpha<(1/c_p K)$. By Lemma (2.11) we know there exist stopping times $(T^k)_{k\geqq 1}$ increasing to ∞ a.s. such that $Z^{T-}\in D(\alpha)$ for each k. We fix k and implicitly stop $(Z^n)_{n\geqq 1}$ and Z at T^k; that is, we write Z instead of Z^{T^k-}. Thus $Z\in D(\alpha)$, and we let $0=R_0\leqq R_1\leqq\cdots\leqq R_l=T^k$ be the stopping times such that

$$\|\Delta_i Z\|_{\mathscr{H}^\infty}=\|(Z-Z^{R_{i-1}})^{R_i-}\|_{\mathscr{H}^\infty}\leqq\alpha.$$

Observe that $\|X^n-X\|_{\mathscr{S}^p}\leqq c\,\|X-X^n\|_{\mathscr{H}^p}$ and also

$$\begin{aligned}\|X-X^n\|_{\mathscr{H}^p}&\leqq\|X_0-X_0^n\|_{L^p}+\|F\|_\infty\,\|Z-Z^n\|_{\mathscr{H}^p}\\&\quad+K\,\|X-X^n\|_{\mathscr{S}^p}\,\|Z\|_{\mathscr{H}^\infty}.\end{aligned}$$

We conclude that $X^n\to X$ weak-locally in $\mathscr{H}^p$ if and only if $X^n\to X$ weak-locally in $\mathscr{S}^p$, when F is bounded.

Since our results are interpreted weak-locally, by weak-stopping if necessary we can assume without loss of generality that $(X-X^n)\in\mathscr{H}^p(R_i)$, $1\leqq i\leqq l$, for all $n\geqq 1$. Then

$$\begin{aligned}(3.4)\quad\|X-X^n\|_{\mathscr{S}^p(R_1)}&\leqq c_p\,\|X_0^n-X_0\|_{L^p}+c_p\,\|F\|_\infty\,\|Z-Z^n\|_{\mathscr{H}^p}\\&\quad+c_pK\,\|X^n-X\|_{\mathscr{S}^p(R_1)}\,\|\Delta_1 Z\|_{\mathscr{H}^\infty}\\&\leqq h_1(n,p)+r\,\|X^n-X\|_{\mathscr{S}^p(R_1)}\end{aligned}$$

where $0<r\leqq c_pK\alpha<1$, and $\lim_{n\to\infty}h_1(n,p)=0$. Since $r<1$ and $\|X^n-X\|_{\mathscr{S}^p(R_1)}<\infty$, iterating the inequality (3.4) yields

$$(3.5)\quad\|X-X^n\|_{\mathscr{S}^p(R_1)}\leqq h_1(n,p)(1/(1-r)).$$

Suppose now we have shown that $\lim_{n\to\infty}\|X-X^n\|_{\mathscr{S}^p(R_i)}=0$, for some i, $1\leqq i\leqq l$. We then have

$$\begin{aligned}(3.6)\quad\|X-X^n\|_{\mathscr{S}^p(R_{i+1})}&\leqq\|X-X^n\|_{\mathscr{S}^p(R_i)}+\|X_{R_i}^n-X_{R_i}\|_{L^p}\\&\quad+c_p\,\|F\|_\infty\,\|Z-Z^n\|_{\mathscr{H}^p}\\&\quad+c_pK\,\|X-X^n\|_{\mathscr{S}^p(R_{i+1})}\,\|\Delta_{i+1}Z\|_{\mathscr{H}^\infty}\\&\leqq h_i(n,p)+r\,\|X-X^n\|_{\mathscr{S}^p(R_{i+1})}.\end{aligned}$$

We wish to show $\lim_{n\to\infty}h_i(n,p)=0$. By assumption, $\lim_{n\to\infty}\|X-X^n\|_{\mathscr{S}^p(R_i)}=0$, and by hypothesis it suffices to show that $\lim_{n\to\infty}\|X_{R_i}^n-X_{R_i}\|_{L^p}=0$. For a process $(Y_t)_{t\geqq 0}$ and stopping time R, we denote $\delta Y_R=Y_R-Y_{R-}$, the jump at R. If Y is a semimartingale and $Y=M+A$ is a decomposition, we observe that

$$\begin{aligned}|\delta Y_s|&\leqq|\delta M_s|+|\delta A_s|\\&\leqq[M,M]_s^{1/2}+\int_0^s|dA_u|\\&\leqq j_\infty(M,A),\quad\text{a.s.}\end{aligned}$$

Thus a.s. $|\delta Y_R| \leqq \|Y\|_{\mathscr{H}^\infty}$ for any stopping time R. Using the above observation and notation we have

$$\begin{aligned}\|X^n_{R_i} - X_{R_i}\|_{L^p} &\leqq \|X^n_{R_i-} - X_{R_i-}\|_{L^p} + \|(FX^n_{R_i-})\,\delta Z^n_{R_i} - (FX_{R_i-})\,\delta Z^n_{R_i}\|_{L^p} \\ &\leqq \|X^n - X\|_{\mathscr{S}^p(R_i)} + c_p\,\|F\|_\infty\,\|\delta Z^n - \delta Z\|_{\mathscr{S}^p} \\ &\quad + c_p K\,\|X^n_{R_i-} - X_{R_i-}\|_{L^p}\,\|Z\|_{\mathscr{H}^\infty} \\ &\leqq \|X^n - X\|_{\mathscr{S}^p(R_i)} + 2c_p\,\|F\|_\infty\,\|Z^n - Z\|_{\mathscr{S}^p} \\ &\quad + c_p K\,\|Z\|_{\mathscr{H}^\infty}\,\|X^n - X\|_{\mathscr{S}^p(R_i)}\end{aligned}$$

which tends to 0 as n tends to ∞. Therefore $\lim_{n\to\infty} h_i(n,p)=0$, where $h_i(n,p)$ is given in (3.6). We conclude

$$(3.7)\quad \lim_{n\to\infty} \|X - X^n\|_{\mathscr{S}^p(R_i)} = 0, \quad 1 \leqq i \leqq l.$$

Since $R_l = T^k$ we deduce that (3.7) is equivalent to

$$\lim_{n\to\infty} \|X^n - X\|_{\mathscr{S}^p(T^k)} = 0$$

and since F is bounded and T^k tends to ∞ a.s. as k tends to ∞, we have the result for weak-local $\mathscr{H}^p$ convergence, and Theorem (3.3) is proved.

(3.8) **Theorem.** *Let the hypotheses of Theorem (3.3) be satisfied, except that the restriction that F_i be bounded ($1 \leqq i \leqq m$) is removed. Then there exists a subsequence $\{n_l\}$ such that $\lim_{n_l\to\infty} X^{n_l} = X$ weak-locally in $\mathscr{H}^p$.*

Proof. Once again, we only give the proof for $m=1$. The proof for arbitrary $m<\infty$ is analogous. Let $F^k = F \wedge k$. Then $F^k \in \mathrm{Lip}(K)$ when F is. Define $(X^{n(k)})_{n\geqq 1}$ and $X^{(k)}$ as solutions respectively of

$$(3.9)\quad X^{n(k)}_t = X^n_0 + \int_0^t F^k X^{n(k)}_{s-}\, dZ^n_s$$

$$X^{(k)}_t = X_0 + \int_0^t F^k X^{(k)}_{s-}\, dZ_s.$$

For each choice of k define

$$(3.10)\quad \begin{aligned} T^k &= \inf\{t: |X_t| > k\} \wedge k \\ T^{n,k} &= \inf\{t: |X^n_t| > k\}. \end{aligned}$$

Let $k>2$ be fixed. For notational simplicity, define

$$Y^n = X^{n(2k)}.$$

By the uniqueness of the solution of (3.9) it follows that $Y^n = X^{n(2k)} = X^{n(k+1)}$ on $[\![0, T^{n,k+1}]\!]$. By Theorem (3.3), for each k, $\lim_{n\to\infty} X^{n(2k)} = X^{(2k)}$ weak-locally in $\mathscr{H}^p$. But $\|X^{n(2k)} - X^{(2k)}\|_{\mathscr{H}^p(T^{k+2})} = \|Y^n - X\|_{\mathscr{H}^p(T^{k+2})}$, which implies that $\lim_{n\to\infty} Y^n$

$=X^{T^{k+2}-}$ weak-locally in $\mathscr{H}^p$. Hence there exist stopping times Q^m increasing to ∞ a.s. such that $1_{[\![0,Q^m]\!]}((Y^n-X)^{T^{k+2}-})^*$ tends to 0 in probability as n tends to ∞, for each fixed m. Since for any $\varepsilon>0$ we can choose m so large that $P(\{Q^m<T^{k+2}\})<\varepsilon$, we have that $\sup_t |Y_t^n-X_t|\,1_{[\![0,T^{k+2}[\![}$ tends to 0 in probability as n tends to ∞.

Let ε_k tend to 0 as k tends to ∞, and let δ be such that $0<\delta<1$. Then there exists an n_l such that

$$P(\{\sup_t |(Y_t^{n_l}-X_t)^{T^{k+2}-}|>\delta\})<(\varepsilon_k/2^l).$$

This implies that $P(\{T^{n_l,k+1}<T^k\})<\varepsilon_k/2^l$, where $T^{n_l,k+1}$ is as defined in (3.10). Let $\hat{S}^k=\inf_l T^{n_l,k+1}$ and define $S^k=\min(\hat{S}^k, T^k)$. Then

$$(3.11)\quad P\{S^k<T^k)\}<\varepsilon_k$$

and furthermore $X^{n_l}=Y^{n_l}$ and $X=X^{(2k)}$ on $[\![0,S^k[\![$. Thus since $\lim_{n_l\to\infty} Y^{n_l}=X^{(2k)}$ weak-locally in $\mathscr{H}^p(S^k)$, also $\lim_{n_l\to\infty} X^{n_l}=X$ weak-locally in $\mathscr{H}^p(S^k)$. Since T^k tends to ∞ a.s. as k tends to ∞, (3.11) implies that S^k does also. This completes the proof of Theorem (3.8).

We now give an example which shows that the hypotheses of Theorem (3.8) do not imply, in general, that the solutions converge weak-locally in $\mathscr{H}^p$. In Theorem (3.19) we impose an additional condition on the convergence of the semimartingales Z^n to Z which then guarantees the weak-local $\mathscr{H}^p$ converge of X^n to X.

(3.12) *Example.* We wish to exhibit semimartingales $(Z^n)_{n\geqq 1}$ and Z such that $\lim_{n\to\infty} Z^n=Z$ weak-locally in $\mathscr{H}^p$, some $p\geqq 1$, but $\lim_{n\to\infty} X^n\neq X$ weak-locally in $\mathscr{H}^p$. We will do this for $p=1$. Here $F\in\mathrm{Lip}(K)$, and X^n and X satisfy (3.1) and (3.2) respectively, with $m=1$. Let $\Omega=[0,1]$, P be Lebesgue measure on $[0,1]$, and $\mathscr{F}_t$ be the Lebesgue sets, for $0\leqq t\leqq\infty$. Let $\varphi(t)=\min(t,1)$ for $t\geqq 0$. Let $f_n\geqq 0$, and let $A_t^n(\omega)=\varphi(t)f_n(\omega)$, $\omega\in[0,1]$. We let $F\in\mathrm{Lip}(K)$ be given by $FX=X$, and for simplicity we choose $X_0^n=X_0=1$. Thus X^n satisfies

$$(3.13)\quad X_t^n=1+\int_0^t X_s^n\,dA_s^n$$

and hence $X_t^n=\exp(A_t^n)$. Suppose that $\lim_{n\to\infty} E[f_n]=0$, but $\lim_{n\to\infty} E[f_n^p]\neq 0$. Then for each t, $\lim_{n\to\infty} E[e^{A_t^n}]\neq 1$. However, a priori there may exist stopping times T^k tending to ∞ such that $\lim_{n\to\infty} E[e^{A_{T^k}^n}]=1$. We will need the following lemma.

(3.14) **Lemma.** *There exist nonnegative functions on* $[0,1]$ *such that*

$$(3.15)\quad \lim_{n\to\infty}\int_0^1 f_n(\omega)\,d\omega=0,$$

(3.16) $\limsup_{n\to\infty} \int_\Lambda f_n(\omega)^p\, d\omega = +\infty$

for all $p>1$, and all Lebesgue sets Λ such that $P(\Lambda)>0$.

Let us for the moment assume the truth of Lemma (3.14). Let $A_t^n(\omega) = \varphi(t) f_n(\omega)$ be such that the functions $\{f_n\}_{n\geqq 1}$ satisfy (3.15) and (3.16). Let T be any stopping time (which in this framework is merely a nonnegative random variable) which is not a.s. equal to 0. Then there exists a constant $\zeta \leqq 1$ such that if $\Lambda = \{T > \zeta\}$, then $P(\Lambda) = \eta > 0$. We then have, for $p>1$,

$$\begin{aligned}(3.17)\quad E[e^{A_T^n}] &\geqq E[(A_T^n)^p] \\ &\geqq E[1_\Lambda \varphi(T)^p (f_n)^p] \\ &\geqq \zeta^p E[1_\Lambda (f_n)^p].\end{aligned}$$

The inequalities (3.17) shows that $\limsup_{n\to\infty} E[e^{A_T^n}] = \infty$, by property (3.16). But property (3.15) shows that $\lim_{n\to\infty} \|A^n\|_{\mathscr{H}^1} = 0$, and so we conclude that $\lim_{n\to\infty} A^n = 0$ in $\mathscr{H}^1$, but X^n as given in (3.13) does not converge to 1 weak-locally in $\mathscr{H}^1$.

Proof of Lemma (3.14). Let $\varepsilon_m = (1/\log(m+1))$, for $m \in \mathbb{N}$. For each m; $k = 1, 2, \ldots, m$; and $\omega \in [0,1]$, define

$$f_{m,k}(\omega) = \begin{cases} m\,\varepsilon_m & \text{if } k - 1/m < \omega < k/m \\ 0 & \text{otherwise} \end{cases}.$$

Put the $f_{m,k}$ into a sequence: $f_{1,0}, f_{1,1}, f_{2,0}, \ldots$. Then $E[f_{m,k}] = \varepsilon_m$ which implies $\limsup_{m\to\infty} E[f_{m,k}] = 0$. Now define, relative to a given Lebesgue set Λ with $P(\Lambda) = \eta > 0$,

$$\Lambda_{m,k} = \Lambda \cap \,]k - 1/m, k/m[.$$

For each m there exists at least one $k = k(m)$ such that $P(\Lambda_{m,k(m)}) \geqq \eta/m$. Taking $p>1$ we have

$$\begin{aligned} E[1_\Lambda (f_{m,k(m)})^p] &= (m\,\varepsilon_m)^p P(\Lambda_{m,k(m)}) \\ &\geqq m^{p-1} (\varepsilon_m)^p \eta \end{aligned}$$

and $\limsup_{m\to\infty} m^{p-1} \varepsilon_m^p \eta = \infty$ since $\varepsilon_m = (1/\log(m+1))$. This completes the proof of Lemma (3.14).

(3.18) *Definition.* A family of semimartingales $\{M^\beta\}_{\beta\in B}$ *is said to be uniformly in* $D(\alpha)$ if there exist increasing stopping times $\{T^i\}_{i\in\mathbb{N}}$, a stopping time T with $T^i \leqq T$ for all i, and a constant $C = C(\alpha)$, such that for all $\beta \in B$: (1) $(M^\beta)^{T-} = M^\beta$; (2) $\|M^\beta\|_{\mathscr{H}^\infty} \leqq C$; and (3) $\|\Delta_{]\!]T^i, T^{i+1}[\![} M^\beta\|_{\mathscr{H}^\infty} \leqq \alpha$ for each $i \in \mathbb{N}$.

(3.19) **Theorem.** *Suppose for $1 \leqq i \leqq m$, $(Z^{i,n})_{n\geqq 1}$ are weak-locally uniformly in $D(\alpha)$ with $\alpha < (1/c_p K)$. Let F_i satisfy the* Lip(K) *conditions, and let $(X^n)_{n\geqq 1}$ and X be given by* (3.1) *and* (3.2). *Suppose further that $\lim_{n\to\infty} Z^n = Z$ weak-locally in $\mathscr{H}^p$ $(p \geqq 1)$. Then $\lim_{n\to\infty} X^n = X$ weak-locally in $\mathscr{H}^p$.*

Proof. By weak-stopping at T for a large stopping time T, we can assume without loss of generality that: (1) $(Z^{i,n})_{n\geqq 1}$ are uniformly in $D(\alpha)$, (2) $\lim_{n\to\infty} Z^n = Z$ in $\mathscr{H}^p$, (3) $\|FX\|_{\mathscr{S}^\infty} \leqq C_1$, for some constant C_1. We only give the proof for $m = 1$.

As always, $\|X^n - X\|_{\mathscr{S}^p} \leqq c_p \|X^n - X\|_{\mathscr{H}^p}$. In the above situation we also have

$$\begin{aligned}\|X^n - X\|_{\mathscr{H}^p} &\leqq \|X_0 - X_0^n\|_{L^p} + \|FX\|_{\mathscr{S}^\infty} \|Z - Z^n\|_{\mathscr{H}^p} \\ &\quad + K\,\|X - X^n\|_{\mathscr{S}^p} \|Z^n\|_{\mathscr{H}^\infty} \\ &\leqq \|X_0 - X_0^n\|_{L^p} + C_1 \|Z - Z^n\|_{\mathscr{H}^p} + K C(\alpha)\,\|X - X^n\|_{\mathscr{S}^p}\end{aligned}$$

and so *we conclude that* $\lim_{n\to\infty} X^n = X$ *weak-locally in* $\mathscr{H}^p$ *if and only if* $\lim X^n = X$ *weak-locally in* $\mathscr{S}^p$, *under the assumptions of Theorem* (3.19). Since $\|FX\|_{\mathscr{S}^\infty} \leqq C_1$ and $\|Z^n\|_{\mathscr{H}^\infty} \leqq C$ independently of n, the rest of the proof is almost exactly the same as the proof of Theorem (3.3), so we omit it.

The proofs of Theorem (3.3) and Theorem (3.19) carry over exactly (except for obvious modifications) to include what has become known as "the equations of C. Doléans-Dade". We state here, without proof, the analogue of Theorem (3.3). The interested reader will easily do the same for the analogue of Theorem (3.19).

(3.20) **Theorem.** *Let* $(J^n)_{n\geqq 1}$ *and* J *be adapted cadlag. processes. For* $1 \leqq i \leqq m$ *let* $(Z^{i,n})_{n\geqq 1}$, Z^i *be semimartingales such that* $\lim_{n\to\infty} Z^{i,n} = Z^i$ *weak-locally in* $\mathscr{H}^p$, *for* $1 \leqq p < \infty$. *Let* $(X^n)_{n\geqq 1}$, X *be respectively the unique solutions of* (3.21) *and* (3.22) *below:*

$$(3.21)\quad X_t^n = J_t^n + \sum_{i=1,m} \int_0^t F_i X_{s-}^n \, dZ_s^{i,n}$$

$$(3.22)\quad X_t = J_t + \sum_{i=1,m} \int_0^t F_i X_{s-} \, dZ_s^i$$

where $F_i \in \mathrm{Lip}(K)$ *and is bounded* $1 \leqq i \leqq m$. *Assume either*

(a) $(J^n)_{n\geqq 1}$, J *are semimartingales and* $\lim_{n\to\infty} J^n = J$ *weak-locally in* $\mathscr{H}^p$; *or*

(b) $(J^n)_{n\geqq 1}$, J *are cadlag., adapted processes and* $\lim_{n\to\infty} J^n = J$ *weak-locally in* $\mathscr{S}^p$.

If (a) *holds then* $\lim_{n\to\infty} X^n = X$ *weak-locally in* $\mathscr{H}^p$; *if* (b) *holds then* $\lim_{n\to\infty} X^n = X$ *weak-locally in* $\mathscr{S}^p$.

We remark that the results of this section and also those of Section 4 hold true for systems as well. If $X = (X^1, \dots, X^k)$ is a vector of semimartingales, define

$$\|X\|_{\mathscr{H}^p} = \sum_{i=1,k} \|X^i\|_{\mathscr{H}^p}$$

and one can define $\|X\|_{\mathscr{S}^p}$ analogously. If one then has a system of equations of the form:

$$X_t^j = J_t^j + \sum_{i=1,m} \int_0^t F_i^j X_{s-}\, dZ_s^i$$

for $1 \leqq j \leqq q$, the analogous theorems to Theorems (3.3), (3.19) and (4.16) can be proved in exactly the same fashion.

4. Almost Sure Stability of Solutions

In Section 3 we saw that if the semimartingale differentials converge in an $\mathscr{H}^p$ norm then the solutions converge along a subsequence in an $\mathscr{H}^p$ norm. One might ask if one can get similar stability results if one has only almost sure convergence of the differentials. It was almost sure convergence that Wong and Zakai considered in [14] when they revealed a lack of stability. In this section *we show that one does have almost sure stability, provided the semimartingales converge* a.s. *in an appropriate fashion.* Our proof relies on an idea due to Stricker [13], the importance of which was emphasized by Meyer [9]. By a change to an equivalent probability we obtain $\mathscr{H}^1$ convergence. We then invoke Theorem (3.8) and get $\mathscr{H}^1$ and hence a.s. convergence for a subsequence of the solutions. Since the probabilities are equivalent, the solutions must also converge a.s. for the original probability law. Let Y be a semimartingale and let $Y = N + A$ be any decomposition of Y, where N is a local martingale and A is an adapted process whose paths are right continuous and of bounded variation on compact sets. Let $\int_{0-}^t |dA_s|$ denote the random variable giving the total variation of the path up to time t. Let N^* be as defined in (2.3). For any stopping time T, define

$$(4.1)\quad \nu_T(N, A) = N_T^* + \int_{0-}^T |dA_s|.$$

We now prove a lemma that is an adaptation of Stricker's theorem ([13] or [9]).

(4.2) **Lemma.** *Let* $(Z^{i,n})_{n \geqq 1}$, Z^i *be semimartingales* $(1 \leqq i \leqq m)$, *and let*

$$(4.3)\quad Z^i - Z^{i,n} = N^{i,n} + A^{i,n}\,(P)$$

be some decomposition. Let T^k *be stopping times increasing to* ∞ a.s. *such that*

$$(4.4)\quad \lim_{n \to \infty} \nu_{T^k}(N^{i,n}, A^{i,n}) = 0 \quad \text{a.s.} \quad (1 \leqq i \leqq m)$$

for each k. *Then for each* k *there exists a subsequence* $\{n_l\}$ *and a probability* Q^k *equivalent to* P *such that under* Q^k

$$\lim_{n_l \to \infty} \|Z^{i,n_l} - Z^i\|_{\mathscr{H}^1(T^k)} = 0.$$

Proof. We point out that (4.4) need not hold for all decompositions; we merely require it to hold for some sequence of decompositions. We give the proof here for m

$=1$. An obvious modification gives the proof for arbitrary $m<\infty$. We separate the proof into three steps.

Step 1. We construct a probability R equivalent to P such that for each n, $Y^n=Z-Z^n$ is special under R. We let $Y^n=M^n+B^n$ be its canonical decomposition, and we show that $[Y^n, Y^n]$, $[M^n, M^n]$ are in $L^1(dR)$.

To establish step 1 fix k and without loss of generality assume $(Z^n)_{n\geqq 1}$ and Z are implicitly stopped at T^k. Then $\lim_{n\to\infty} v_\infty(N^n, A^n)=0$ a.s. Thus $\sum_s |\Delta A_s^n| \leqq \int_{0-}^{\infty} |dA_s^n|$ and so $\lim_{n\to\infty} \sum_s (\Delta A_s^n)^2=0$. Let

$$(4.5)\quad G^1=\sup_n \{v_\infty(N^n, A^n)^2+\sum_s (\Delta A_s^n)^2\}$$

$$G^2=1/(1+G^1)$$

$$G=G^2/E(G^2).$$

Define the equivalent probability R by

$$(4.6)\quad dR=G\,dP.$$

Then $E_R\{(N^{n*})^2\}<\infty$ and so $E\{[N^n, N^n]_\infty\}<\infty$ for all n. Since $[Z-Z^n, Z-Z^n]_\infty \leqq [N^n, N^n]_\infty+\sum(\Delta A_s^n)^2$, we have

$$E_R\{[Y^n, Y^n]_\infty\}<\infty,$$

where $Y^n=Z-Z^n$. Thus Y^n is an R-special semimartingale. Let

$$(4.7)\quad Y^n=M^n+B^n$$

be the canonical decomposition. For any predictable stopping time S and all n, since Y^n is special, $E_R\{\Delta Y_S^n|\mathscr{F}_{S-}\}=\Delta B_S^n$. Then Jensen's inequality implies $E_R\{(\Delta B_S^n)^2\}\leqq E_R\{(\Delta Y_S^n)^2\}$. Since B^n is predictable a countable number of predictable stopping times can be found which exhaust its jumps. Thus

$$E_R\{[B^n, B^n]_\infty\}\leqq E\{[Y^n, Y^n]_\infty\}<\infty.$$

Since $[M^n, M^n]\leqq 2([Y^n, Y^n]+[B^n, B^n])$, also $[M^n, M^n]_\infty \in L^1(dR)$.

Step 2. We show that $\int_{0-}^{\infty} |dB_s^n|$ tends to 0 in $L^1(dR)$, where B^n is as defined by (4.7), and R is as given in (4.6).

By Girsanov's theorem [7, p. 377] we know one decomposition of the semimartingale Y^n relative to R is

$$(4.8)\quad Y_t^n=\left(N_t^n-\int_0^t \frac{1}{G_s}\,d[N^n, G]_s\right)+\left(A_t^n+\int_0^t \frac{1}{G_s}\,d[N^n, G]_s\right)$$

where $G_t=E_P[G|\mathscr{F}_t]$, and N^n and A^n are as given in (4.3). Rewrite (4.8) as

$$(4.9)\quad Y^n=\hat{M}^n+\hat{B}^n.$$

Then B^n is the dual predictable projection of $\hat{B}^n$, and if we show that $\lim_{n\to\infty} E_R\left\{\int_0^\infty |d\hat{B}_s^n|\right\}=0$, then also $\lim_{n\to\infty} E_R\left\{\int_0^\infty |dB_s^n|\right\}=0$ (cf. [7, p. 257]). We know $\lim_{n\to\infty} E_R\left\{\int_0^\infty |dA_s^n|\right\}=0$ by construction of R. Also $\int_0^t \frac{1}{G_s} d[N^n, G]_s$ tends to 0 in $L^1(dR)$ by an application of the Kunita-Watanabe inequality and the construction of G.

Step 3. We construct a probability Q equivalent to R such that Y^n is special for Q and if $Y^n=L^n+C^n$ is its canonical decomposition, then $L^n\in\mathscr{H}^1$, $\|L^n\|_{\mathscr{H}^1}$ tends to 0 (under Q), and the total variation of C^n tends to 0 under Q.

In step 2 we saw that $\int_{0-}^\infty |dB_s^n|$ tends to 0 in $L^1(dR)$. Let $\{n_l\}$ be a subsequence such that $\int_{0-}^\infty |dB_s^{n_l}|$ tends to 0 a.s. Let

$$D^1=\sup_{n_l}\int_{0-}^\infty |dB_s^{n_l}|,$$

$$D^2=1/(1+D^1),$$

$$D=D^2/E_R(D^2).$$

Define the equivalent probability Q by:

$$dQ=D\,dR.$$

By Girsanov's theorem the canonical decomposition for Y^n under Q is

$$(4.10)\quad Y_t^{n_l}=\left(M_t^{n_l}-\int_{0-}^t \frac{1}{D_{s-}} d\langle M^{n_l}, D\rangle_s\right)+\left(B_t^{n_l}+\int_{0-}^t \frac{1}{D_{s-}} d\langle M^{n_l}, D\rangle_s\right)$$
$$=L_t^{n_l}+C_t^{n_l}$$

where M^n and B^n are given in (4.7), and $D_t=E_R\{D|\mathscr{F}_t\}$.

We wish to show $\lim_{n_l\to\infty}\|Y^{n_l}\|_{\mathscr{H}^1}=0$, under Q. As Meyer [8] has observed, it is equivalent to show that $\lim_{n_l\to\infty} E\{v_\infty(L^{n_l}, C^{n_l})\}=0$, since $Y^{n_l}=L^{n_l}+C^{n_l}$ is the canonical decomposition for each n_l.

We first show that $\lim_{n_l\to\infty} E_Q\left\{\int_{0-}^\infty |dC_s^{n_l}|\right\}=0$. Since $\int_{0-}^\infty |dB_s^{n_l}|\leq D^1\in L^1(dQ)$ and $\lim_{n_l\to\infty}\int_{0-}^\infty |dB_s^{n_l}|=0$ a.s., we have that $\int_{0-}^\infty |dB_s^{n_l}|$ tends to 0 in $L^1(dQ)$. On the other hand,

$$(4.11)\quad E_Q\left\{\int \frac{1}{D_{s-}}|d\langle M^{n_l}, D\rangle_s|\right\}=E_R\left\{D\int \frac{1}{D_{s-}}|d\langle M^{n_l}, D\rangle_s|\right\}$$
$$=E_R\left\{\int |d\langle M^{n_l}, D\rangle_s|\right\}.$$

But $\langle M^{n_l}, D\rangle$ is the dual predictable projection of $[M^{n_l}, D]$ and therefore

$$(4.12)\quad E_R\{\int |d\langle M^{n_l}, D\rangle_s|\} \leqq E_R\left\{\int_{0-}^{\infty} |d[M^{n_l}, D]_s|\right\}$$

and by Fefferman's inequality

$$(4.13)\quad E_R\left\{\int_{0-}^{\infty} |d[M^{n_l}, D]_s|\right\} \leqq c E_R\{(M_\infty^{n_l})^*\} \, \|D\|_{\mathscr{BMO}}$$

$$\leqq c\sqrt{5}\, \|D_\infty\|_{L^\infty} E_R\left\{(Y^{n_l})_\infty^* + \int_{0-}^{\infty} |dB_s^{n_l}|\right\}$$

$$\leqq c\sqrt{5}\, \|D_\infty\|_{L^\infty} E_R\left\{v_\infty(N^{n_l}, A^{n_l}) + \int_{0-}^{\infty} |dB_s^{n_l}|\right\}.$$

Combining (4.11), (4.12) and (4.13) yields

$$(4.14)\quad E_Q\left\{\int \frac{1}{D_{s-}} |d\langle M^{n_l}, D\rangle_s|\right\} \leqq C E_R\left\{v_\infty(N^{n_l}, A^{n_l}) + \int_0^\infty |dB_s^{n_l}|\right\}$$

where C is a constant. By (4.4) and the construction of R we know that $\lim_{n_l\to\infty} E_R\{v_\infty(N^{n_l}, A^{n_l})\} = 0$, and we saw in step 2 that $\lim_{n\to\infty} E_R\left\{\int_{0-}^{\infty} |dB_s^n|\right\} = 0$. So the inequality (4.14) implies that $\lim_{n_l\to\infty} E_Q\left\{\int \frac{1}{D_{s-}} |d\langle M^{n_l}, D\rangle_s|\right\} = 0$, and thus $\lim_{n_l\to\infty} E_Q\left\{\int_{0-}^{\infty} |dC_s^{n_l}|\right\} = 0$.

We next show $\lim_{n_l\to\infty} E_Q\{(L^{n_l})^*\} = 0$. Observe that

$$(4.15)\quad E_Q\{(L^{n_l})^*\} \leqq E_Q\{(M^{n_l})^*\} + E_Q\left\{\int \frac{1}{D_{s-}} |d\langle M^{n_l}, D\rangle_s|\right\}.$$

The second term on the right of (4.15) tends to 0 by (4.14). As for the first term on the right,

$$E_Q\{(M^{n_l})^*\} = E_R\{D(M^{n_l})^*\}$$
$$\leqq \|D\|_{L^\infty} E_R\{(M^{n_l})^*\}$$
$$\leqq \|D\|_{L^\infty} E_R\left\{v_\infty(N^{n_l}, A^{n_l}) + \int_{0-}^{\infty} |dB_s^{n_l}|\right\}$$

which we have seen tends to 0 as n_l tends to ∞. This completes the proof of Lemma (4.2).

(4.16) **Theorem.** *Let* $(Z^n)_{n\geqq 1}$, Z *be semimartingales. Let* $Z - Z^n = N^n + A^n$ *be decompositions such that there exists a sequence of stopping times* $(T^k)_{k\geqq 1}$ *increasing to* ∞ a.s. *and* $\lim_{n\to\infty} v_{T^k}(N^n, A^n) = 0$ a.s. *Let* $(J^n)_{n\geqq 1}$, J *be cadlag. adapted processes such that* $\lim_{n\to\infty} ((J^n - J)^*)^{T^k -} = 0$, *for each k. Let* $F \in \mathrm{Lip}(K)$ *and* $(X^n)_{n\geqq 1}$, X *be solutions* *respectively of*

(4.17) $X_t^n = J_t^n + \int_0^t F X_{s-}^n \, dZ_s^n,$

(4.18) $X_t = J_t + \int_0^t F X_{s-} \, dZ_s.$

Then there exists a subsequence $\{n_l\}$ *such that* $\lim_{n_l \to \infty} ((X^{n_l} - X)^*)^{T^k-} = 0$ a.s., *for each k.*

Proof. We fix a k. By changing the choice of G^1 in (4.5) in the proof of Lemma (4.2) to

$$G^1 = \sup_n \{(J^n - J)^* + v_\infty(N^n, A^n)^2 + \sum_s (\Delta A_s^n)^2\}$$

we can conclude that there exists a subsequence $\{n_l\}$ and a probability Q^k equivalent to P such that $\lim_{n_l \to \infty} \|J^{n_l} - J\|_{\mathscr{S}^1} = 0$ and $\lim_{n_l \to \infty} \|Z^{n_l} - Z\|_{\mathscr{H}^1} = 0$, under Q^k. One can prove an analogous theorem to Theorem (3.8) for equations of the form given in (4.17) and (4.18), rather than (3.1) and (3.2) respectively. One then has the conclusion that there is a subsequence of $\{n_l\}$ such that $\lim X^n = X$ weak-locally in $\mathscr{S}^1$, where the convergence is along the subsequence. Hence there exists yet a further subsequence along which $((X^n - X)^*)^{T^k-}$ tends to 0 a.s. (Q^k). Since Q^k and P are equivalent, the convergence is also a.s. (P). This completes the proof.

(4.19) *Comment.* One might hope to circumvent the instability of Wong and Zakai by approximating Brownian motion with *VF* local martingales and then use the results of Theorems (3.3), (3.19), or (4.16). Unfortunately one cannot do so, as we show here.

Suppose the filtration $(\mathscr{F}_t)_{t \geq 0}$ is large enough to admit a standard Brownian motion B and a sequence $(M^n)_{n \geq 1}$ of *VF* local martingales. Then for a stopping time T and $1 \leq p < \infty$

$$\begin{aligned} \|B - M^n\|_{\mathscr{H}^p(T)} &= \|[B - M^n, B - M^n]_{T-}^{1/2}\|_{L^p} \\ &= \|([B, B]_T + [M^n, M^n]_{T-})^{1/2}\|_{L^p} \end{aligned}$$

since B and M^n are orthogonal for each n. Since $[B, B]_T = T$, the sequence $(M^n)_{n \geq 1}$ cannot approximate B in $\mathscr{H}^p$.

By using Lemma (4.2) and Girsanov's theorem one can even show that there cannot exist a sequence of *VF* local martingales $(M^n)_{n \geq 1}$ such that $\lim_{n \to \infty} (B - M^n)_T^* = 0$ a.s., where T is any stopping time not equal to 0 a.s.

The preceding argument shows more generally that any semimartingale X with a non-zero continuous martingale part cannot be approximated in $\mathscr{H}^p$ by *VF* semimartingales. Indeed, X cannot even be approximated almost surely if the convergence is required to be of the form described in Lemma (4.2).

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