Z. Wahrscheinlichkeitstheorie verw. Gebiete
44, 307–323 (1978)

Zeitschrift für
Wahrscheinlichkeitstheorie
und verwandte Gebiete

Study of a Filtration Expanded to Include an Honest Time

M.T. Barlow

Department of Pure Mathematics, University College of Swansea, Singleton Park, Swansea, SA2 8PP UK

Summary. Let $(\Omega, \mathscr{F}, P)$ be a measurable space, and $\{\mathscr{F}_t\}$ be a filtration on $(\Omega, \mathscr{F})$. Then, given a fixed honest time L a new filtration $\{\mathscr{G}_t\}$ is defined, the smallest containing $\{\mathscr{F}_t\}$ and for which L is a stopping time, and the martingales, semimartingales and stopping times of this new filtration are characterised.

0. Introduction

This paper presents a martingale approach to work on the decomposition of a process into its 'past' and 'future' relative to an honest random time. (See Millar [11], for a survey of the Markovian theory of such decompositions.)

Let $(\Omega, \mathscr{F}, P)$ be a complete probability triple, and $\{\mathscr{F}_t, t \geqq 0\}$ be a filtration consisting of sub-σ-fields of $\mathscr{F}$, satisfying the 'usual conditions': that is, the filtration is right-continuous and increasing, and $\mathscr{F}_0$ contains every P-null set in $\mathscr{F}$. A random time on $(\Omega, \mathscr{F})$ is any $\mathscr{F}$-measurable map $L: \Omega \to [0, \infty]$; a random time L is *honest* if for $s \leqq t$

$$\{L \leqq s\} = F_{st} \cap \{L \leqq t\} \qquad \text{for some } F_{st} \in \mathscr{F}_t.$$

This definition is equivalent (for a right-continuous filtration) to that given by Meyer, Smythe and Walsh in [10]. Most of the random times studied in connection with splitting-time theorems are honest: in particular optional, cooptional, and randomised coterminal times are all honest (see Millar [11]).

Let L be a fixed honest time, and for $t \in \mathbb{R}^+$ define

$$\mathscr{G}_t = \{\Lambda \in \mathscr{F}: \Lambda = (E \cap \{L \leqq t\}) \cup (F \cap \{L > t\}) \quad \text{for some } E, F \in \mathscr{F}_t\}.$$

Then $\mathscr{F}_t \subseteq \mathscr{G}_t$, L is a $\{\mathscr{G}_t\}$-stopping time, and $\{\mathscr{G}_t\}$ satisfies the usual conditions. We shall study the properties of the filtration $\{\mathscr{G}_t\}$, and in particular of its martingales.

Let $A_t = I\{t \geqq L\}$, and let A^o and $\hat{A}$ denote the optional and dual optional projections of A relative to $\{\mathscr{F}_t\}$. In Section 2 we shall establish a few basic results concerning these processes. Section 3 is devoted to a study of $\{\mathscr{F}_t\}$ and $\{\mathscr{G}_t\}$ martingales: here is the main results of the section.

0044-3719/78/0044/0307/$03.40

Theorem A. *Let M be a square integrable $\{\mathscr{F}_t\}$-martingale, and M' be defined by*

$$M'_t = M_t + \int_0^t [(1-A_{s-})(1-A^o_{s-})^{-1} - A_{s-}(A^o_{s-})^{-1}]\, d\langle M, A^o - \hat{A}\rangle_s.$$

Then M' is a square integrable $\{\mathscr{G}_t\}$-martingale.

As a corollary we show that every $\{\mathscr{F}_t\}$-semimartingale is a $\{\mathscr{G}_t\}$-semimartingale, providing a complement to a recent theorem of Stricker [12].

In Section 4 we investigate the 'measurable' structure of $\{\mathscr{G}_t\}$-progressive processes.

Theorem B. *Let T be a $\{\mathscr{G}_t\}$-stopping time. Then there exists a sequence (S_n) of disjoint $\{\mathscr{F}_t\}$-stopping times such that*

$$[\![T]\!] \subset [\![L]\!] \cup \bigcup_{i=1}^{\infty} [\![S_n]\!].$$

In Section 5 and 6 we prove a martingale representation theorem for $\{\mathscr{G}_t\}$-martingales.

Theorem C. *Suppose that $\{M^i: i\in I\}$ is a finite collection of continuous $\{\mathscr{F}_t\}$-local martingales, such that if Y is any continuous $\{\mathscr{F}_t\}$-local martingale then there exist $\{\mathscr{F}_t\}$-previsible processes C^i, $i\in I$ such that*

$$Y_t = \sum_{i\in I} \int_0^t C^i_s\, dM^i_s.$$

Then, if Z is any continuous $\{\mathscr{G}_t\}$-local martingale, there exist $\{\mathscr{G}_t\}$-previsible processes D^i, $i\in I$, such that

$$Z_t = \sum_{i\in I} \int_0^t D^i_s\, d(M^i)'_s.$$

To represent the jumps of $\{\mathscr{G}_t\}$-martingales we must use Jacod's theory of stochastic integrals relative to random measures.

Acknowledgements. I wish to thank my supervisor, Professor D. Williams, for suggesting this problem, and for various improvements to the style of this paper. Lemma 3.1 short-circuits a rather involved argument, leading to essentially the same results.

Note. Some of the results of Section 2 appear in Azéma [1]. T. Jeulin and M. Yor, in [8] and [13], written at the same time as this paper, have obtained most of the results of Sections 3 and 4, and go further in certain respects. The representation results in Sections 5 and 6 have not appeared before.

1. Notation and Preliminaries

It is not possible to give here more than a very brief account of the general theory of processes and martingales on which this paper is based: see the books by Dellacherie and Meyer [2, 3], and [9], for details. Any unexplained notation and terminology will be found in Meyer [9], or Jacod and Yor [7].

If T is an optional time (that is, a stopping time) and X is any process we denote the stopped process by X^T, so that $X_t^T = X_{t \wedge T}$. If $\mathscr{C}$ is a class of processes we define the classes $\mathscr{C}^c$, $\mathscr{C}_{\text{loc}}$ by

$$\mathscr{C}^c = \{X \in \mathscr{C}: X.(\omega) \text{ is continuous for almost all } \omega\},$$

$$\mathscr{C}_{\text{loc}} = \{X: \text{there exists an increasing sequence } (T_n) \text{ of optional times, with } \lim_n T_n = +\infty, \text{ such that } X^{T_n} \in \mathscr{C} \text{ for each } n\}.$$

Let $\mathscr{M}$ be the class of uniformly integrable martingales M with $M_0 = 0$, and, for $p \geqq 1$, let $\mathscr{M}^p$ be the set of $M \in \mathscr{M}$ with $E|M_\infty|^p < \infty$; when we wish to discuss martingales relative to the filtrations $\{\mathscr{F}_t\}$ and $\{\mathscr{G}_t\}$, we shall write $\mathscr{M}(\mathscr{F})$, $\mathscr{M}(\mathscr{G})$, and so forth. Note that this is a slight departure from the usual notation, since we require every element of $\mathscr{M}$ to be null at the origin.

Let $\mathscr{A}^+$ be the collection of right-continuous, increasing, adapted processes A with $A_0 = 0$, and $A_t < \infty$ a.s. for each $t < \infty$. Let $\mathscr{A} = \mathscr{A}^+ - \mathscr{A}^+$. If $A - A_0 \in \mathscr{A}$, and $A_0 < \infty$ a.s., we shall say that A is a process of finite variation, or a VF process. Define $\mathscr{V}$ to be the subclass of $\mathscr{A}$ consisting of those processes A for which $E \int_0^\infty |dA_s| < \infty$. If $A - A_0 \in \mathscr{V}$, and $E|A_0| < \infty$, we shall say that A is of integrable variation, or a VI process; if $A - A_0 \in \mathscr{V}_{\text{loc}}$, and $E|A_0| < \infty$, that A is locally integrable, or an LI process.

A process X is a semimartingale [respectively: semimartingale (r)] if X has a decomposition of the form $X = X_0 + M + A$, where $M \in \mathscr{M}_{\text{loc}}$, $A \in \mathscr{A}$ [$M \in \mathscr{M}^2, A \in \mathscr{V}, E|X_0| < \infty$]. This decomposition is not unique, but M^c, the continuous martingale part of M, is unique, and is denoted X^c. If $M, N \in \mathscr{M}^c_{\text{loc}}$, we may define $\langle M, N\rangle$, the previsible variance process associated with M and N. For any pair X, Y of semimartingales we define $[X, Y]_t = \langle X^c, Y^c\rangle_t + \sum_{s \leqq t} \Delta X_s \Delta Y_s$. If $M, N \in \mathscr{M}_{\text{loc}}$ and $[M, N] \in \mathscr{V}_{\text{loc}}$, then we define $\langle M, N\rangle$ to be the dual previsible projection of $[M, N]$. In particular, if $M, N \in \mathscr{M}^2_{\text{loc}}$, then $MN - \langle M, N\rangle \in \mathscr{M}_{\text{loc}}$.

If X is a semimartingale and H a previsible process, we denote the stochastic integrals $\int H_s dX_s$, $\int H_{s-} dX_s$ by $H \cdot X$, $H_- \cdot X$, when they exist. If $M \in \mathscr{M}^2_{\text{loc}}$ [respectively: $M \in \mathscr{M}^2$] define $L^2_{\text{loc}}(M)$ [$L^2(M)$] to be the set of previsible processes H such that $H^2 \cdot \langle M, M\rangle \in \mathscr{V}_{\text{loc}}$ [$H^2 \cdot \langle M, M\rangle \in \mathscr{V}$]. If $M \in \mathscr{M}^2_{\text{loc}}$ and $H \in L^2_{\text{loc}}(M)$, then $H \cdot M \in \mathscr{M}^2_{\text{loc}}$; and if $M \in \mathscr{M}^2$ and $H \in L^2(M)$, then $H \cdot M \in \mathscr{M}^2$.

If X is any bounded process (not necessarily adapted) we may take the optional and previsible projections of X relative to $\{\mathscr{F}_t\}$, and will denote them by X^o, X^p respectively. If X has increasing paths (or is the difference of two such processes) we may in addition define the dual optional and previsible projections, denoted by $\hat{X}$ and $\tilde{X}$ respectively. We shall write $\overset{c}{X} = X^o - \tilde{X}$, $\overset{\prime\prime}{X} = X^o - \hat{X}$. Note that if X has right-continuous paths, then $\overset{c}{X}$ and $\overset{\prime\prime}{X}$ are martingales. (This follows at once from the definition of these processes; see Dellacherie [2].) Set $A = I_{[\![L, \infty))}$. The processes A^o and $\hat{A}$ will prove to be of great importance.

Let us recall 'Dellacherie's formula', which we shall use frequently:

$$E\left(\int_s^t X_u^o \, dY_u | \mathscr{F}_s\right) = E\left(\int_s^t X_u \, d\hat{Y}_u | \mathscr{F}_s\right).$$

In Section 6 we shall make use of the theory of stochastic integrals with respect to random measures, which we require to discuss the representation of purely discontinuous martingales. We shall not use it elsewhere. For a full account of this theory see the papers by Jacod [4] and [5], and for a summary [6] or [7].

Let E be a Lusin space, $\mathscr{E}$ its Borel σ-field. Set $\tilde{\Omega} = \Omega \times [0, \infty) \times E$, $\tilde{\mathscr{P}} = \mathscr{P} \otimes \mathscr{E}$, $\tilde{\mathscr{O}} = \mathscr{O} \otimes \mathscr{E}$, where $\mathscr{P}$ and $\mathscr{O}$ are the previsible and optional σ-fields on $\Omega \times [0, \infty)$. A random measure $\mu(\omega; dt, dx)$ is a positive transition measure from $(\Omega, \mathscr{F})$ to $((0, \infty) \times E, \mathscr{B}((0, \infty)) \otimes \mathscr{E})$. For a function $U: \tilde{\Omega} \to \mathbb{R}^+$ let

$$(U * \mu)_t(\omega) = \int_{(0,t] \times E} U(\omega, t, x) \, \mu(\omega; dt, dx)$$

if this is finite, $+\infty$ otherwise. We say that μ is optional [respectively: previsible] if $U * \mu$ is optional [previsible] for all positive $\tilde{\mathscr{O}}$-measurable [$\tilde{\mathscr{P}}$-measurable] functions U. The random measure μ is said to be integer-valued if μ takes its values in $\mathbb{N} \cup \{+\infty\}$ and $\mu(\omega; \{t\} \times E) \leqq 1$ for each $t \in (0, \infty)$. Define the measure M_μ on $\tilde{\Omega}$ by setting $M_\mu(X) = E(X * \mu)_\infty$. From now on we shall take μ to be optional and integer-valued, with M_μ $\tilde{\mathscr{P}}$-σ-finite. Then μ is of the form

$$\mu(\omega; dt, dx) = \sum_{s > 0} I_D(s, \omega) \, \varepsilon_{(s, \alpha_s(\omega))}(dt, dx),$$

where α is an E-valued optional process, and D is an optional subset of $\Omega \times [0, \infty)$. Also, μ has a dual previsible projection ν. We may identify a space $\mathscr{G}^2_{\text{loc}}(\mu)$, of $\tilde{\mathscr{P}}$-measurable functions, and define, for $U \in \mathscr{G}^2_{\text{loc}}(\mu)$, the stochastic integral $U * (\mu - \nu)$, a purely discontinuous local martingale in $\mathscr{M}^2_{\text{loc}}$.

Suppose that $\{M^i : i \in I\}$ is a collection of continuous elements of $\mathscr{M}^2_{\text{loc}}(\mathscr{F})$. Then we shall say that $\{M^i : i \in I; \mu - \nu\}$ has the martingale representation property for $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}, P)$, and write $\{M^i : i \in I; \mu - \nu\} \in \mathscr{R}\mathscr{M}(\mathscr{F})$, if, whenever $Z \in \mathscr{M}^2_{\text{loc}}(\mathscr{F})$, $\langle Z, M^i \rangle = 0$ for every $i \in I$, and $\langle Z, U * (\mu - \nu) \rangle = 0$ for every $U \in \mathscr{G}^2_{\text{loc}}(\mu, \mathscr{F})$, then Z is null.

A more intuitive account of martingale representation is given by the following theorem.

Theorem 1.1 (Jacod, [6]). *The following are equivalent:*

(i) $\{M^i : i \in I; \mu - \nu\} \in \mathscr{R}\mathscr{M}(\mathscr{F})$.

(ii) *For any $N \in \mathscr{M}^2_{\text{loc}}(\mathscr{F})$ there exist $U \in \mathscr{G}^2_{\text{loc}}(\mu, \mathscr{F})$, an increasing sequence (J_n) of finite subsets of I, and elements $u^i_n \in L^2_{\text{loc}}(M^i)$ such that for each $\{\mathscr{F}_t\}$-optional time T reducing N*

$$N_T = N_0 + U * (\mu - \nu)_T + \lim_n \sum_{i \in J_n} (u^i_n \cdot M^i)_T$$

(where the limit is taken in $L^2(\Omega, \mathscr{F}, P)$).

Finally, let us note a few results which will be of use later. As a simple consequence of Itô's Lemma for semimartingales (see [9, IV, 21]) we have

Lemma 1.2 (Meyer [9, IV, T23]). *If X, Y are semimartingales, then XY is a semimartingale, and* $X_t Y_t = \int_0^t X_{s-}\, dY_s + \int_0^t Y_{s-}\, dX_s + [X, Y]_t$. *If, further, X is a VF process, then* $X_t Y_t = \int_0^t X_{s-}\, dY_s + \int_0^t Y_s\, dX_s$.

Lemma 1.3. *If X is a continuous semimartingale, and $X^c = 0$, then X is in $\mathscr{V}_{\mathrm{loc}}$.*

Proof. By [9, IV, T32] X is a special semimartingale, and so has a decomposition $X = X_0 + M + A$ for which A is previsible, and locally integrable. Consequently M is previsible, and so, as M^c is null, it follows that M is null.

Lemma 1.4. *If $\{\mathscr{F}_t\}$, $\{\mathscr{G}_t\}$ are two filtrations on $(\Omega, \mathscr{F}, P)$, and X is a semimartingale relative to both, then $[X, X]$ is independent of the filtration.*

Proof. By [9, VI, T4], for each t, $[X, X]_t$ is the limit in probability of a sequence of random variables depending only on the path of X. Hence $[X, X](\mathscr{F})_t = [X, X](\mathscr{G})_t$ a.s., using an obvious notation. $[X, X]$ is right-continuous, increasing, so it follows that $[X, X](\mathscr{F}) = [X, X](\mathscr{G})$ a.s.

Lemma 1.5. *Suppose that $\mathscr{F}$ is the P-completion of a countably generated σ-field $\mathscr{F}^o$. Then, if $\mathscr{H}$ is any sub-σ-field of $\mathscr{F}$, there exists a countably generated σ-field $\mathscr{H}^o$ such that $\mathscr{H}$ is the P-completion of $\mathscr{H}^o$.* (This result is well known.)

Proof. A σ-field $\mathscr{F}$ is the P-completion of a countably generated σ-field $\mathscr{F}^o$ if and only if the function space $L^2(\Omega, \mathscr{F}, P)$ is separable. But if $(\xi_i)_{i=1}^{\infty}$ is a dense subset of $L^2(\Omega, \mathscr{F}, P)$, and $\eta_i = E(\xi_i | \mathscr{H})$ then $(\eta_i)_{i=1}^{\infty}$ is dense in $L^2(\Omega, \mathscr{H}, P)$.

Let us recall the Burkholder-Davis-Gundy inequalities: if $1 \leqq p < \infty$ there exist two constants, c_p and C_p with $0 < c_p < C_p < \infty$ such that, if $M \in \mathscr{M}_{\mathrm{loc}}$,

$$c_p E[M, M]_\infty^{p/2} \leqq E \sup_{(t)} |M_t|^p \leqq C_p E[M, M]_\infty^{p/2}.$$

Lemma 1.6. *If M is a local martingale, and for any $p \geqq 1$, $E[M, M]_\infty^{p/2} < \infty$, then M is a uniformly integrable martingale, and $M \in \mathscr{M}^p$.*

Proof. Let $\xi = \sup_{(t)} |M_t|^p$. If T is any optional time $|M_T|^p \leqq \xi$, and so $|M_T| \leqq \xi + 1$. Hence M is of class (D), and consequently a uniformly integrable martingale, – see Meyer [9, IV, 4c].

We see that $M \in \mathscr{M}^p$, since $E|M_\infty|^p < E\,\xi < \infty$.

2. The Projections of Λ

For each $t \in [0, \infty)$, define

$$\mathscr{G}_t = \{\Lambda \in \mathscr{F} : \Lambda = (E \cap \{L \leqq t\}) \cup (F \cap \{L > t\}) \quad \text{for some } E, F \in \mathscr{F}_t\}.$$

Lemma 2.1. (a) *For each* $t \geqq 0$, $\mathcal{G}_t$ *is a* σ*-field*, $\{L \leqq t\} \in \mathcal{G}_t$, *and* $\mathcal{F}_t \subseteq \mathcal{G}_t$.

(b) *The filtration* $\{\mathcal{G}_t, t \geqq 0\}$ *satisfies the usual conditions.*

The proof is not hard. Note, however, that the honesty of L is necessary if $\{\mathcal{G}_t\}$ is to be increasing.

For the rest of this section let p, q, r denote generic rationals. Since L is honest, we can choose sets F_{pq}, for $0 \leqq p \leqq q$, such that

(i) $F_{pq} \in \mathcal{F}_q$;

(ii) $\{L \leqq p\} = F_{pq} \cap \{L \leqq q\}$;

(iii) $F_{pp} = \Omega$.

By setting $F'_{pq} = \bigcap_{p \leqq r \leqq q} F_{rq}$ we see that the F_{pq} may be chosen so that they are increasing in the first argument and decreasing in the second.

Now define $C_p(\omega) = \inf\{q \leqq p : \omega \in F_{qp}\}$. It is readily seen that C_p is $\mathcal{F}_p$ measurable, that $C_p(\omega) \leqq p \wedge L(\omega)$, and that (C_p) is increasing. Define $C_t = \inf_{p > t} C_p$: C is then a right-continuous, increasing, $\{\mathcal{F}_t\}$-adapted process. Furthermore we have

$$L = \sup\{t : C_t = t\}, \tag{2.1}$$

$$\{L \leqq s\} = \{C_u < u \text{ for all } u \in (s, t]\} \cap \{L \leqq t\}, \quad \text{for } s \leqq t. \tag{2.2}$$

Remark. If $L(\omega) \leqq t$ then $C_t(\omega) = L(\omega)$, so that C_t is the value L must have if L is less than t.

The following lemma is a consequence of (2.2):

Lemma 2.2. *Suppose that* T *is* $\{\mathcal{F}_t\}$*-optional. Then*

(i) $\mathcal{G}_T = \{\Lambda \in \mathcal{F} : \Lambda = (E \cap \{L \leqq T\}) \cup (F \cap \{L > T\})$ *for some* $E, F \in \mathcal{F}_T\}$,

(ii) $\mathcal{G}_{T-} = \{\Lambda \in \mathcal{F} : \Lambda = (E \cap \{L < T\}) \cup (F \cap \{L \geqq T\})$ *for some* $E, F \in \mathcal{F}_{T-}\}$.

The projections A^o and $\hat{A}$ contain most of the probabilistic information about L which we will require. The next few results clarify the behaviour of these processes.

Lemma 2.3. *If* T *is* $\{\mathcal{F}_t\}$*-previsible, then* $A^o_{T-} = E(A_{T-} | \mathcal{F}_{T-})$.

Proof. Since T is previsible there exists a sequence (T_n) of $\{\mathcal{F}_t\}$-optional times increasing to T, and $\mathcal{F}_{T-} = \bigvee_{n \geqq 1} \mathcal{F}_{T_n}$. Therefore it is enough to show that, given $\varepsilon > 0$, we can find an $n_0(\varepsilon)$ such that $|E A_{T-} I_F - E A^o_{T-} I_F| < \varepsilon$ for all $F \in \mathcal{F}_{T_n}$ with $n > n_0(\varepsilon)$. But this holds if we choose $n_0(\varepsilon)$ such that, whenever $n > n_0(\varepsilon)$,

$$E|A^o_{T_n} - A^o_{T-}| < \tfrac{1}{2}\varepsilon, \quad \text{and} \quad E|A_{T-} - A_{T_n}| < \tfrac{1}{2}\varepsilon.$$

By Dellacherie [2, V, T14, T15], it is immediate that A^o_- is the previsible projection of A_-.

Lemma 2.4. *For* P*-almost all* ω, *we have* $A^o_t > 0$, $A^o_{t-} > 0$ *whenever* $t > L(\omega)$.

Proof. Let $u, v \in \mathbb{R}^+$, with $u < v$, and put $T = \inf\{s \in (u, v] : A^o_s = 0\}$, $S = \inf\{s \in (u, v] : A^o_{s-} = 0\}$. Note that $A^o_T = 0$ on $\{T < \infty\}$, that S is previsible, and that $A^o_{S-} = 0$ on $\{S < \infty\}$.

But

$$P(L<u, T<\infty) \leqq \int_{\{T<\infty\}} A_T \, dP = \int_{\{T<\infty\}} A_T^o \, dP = 0.$$

and

$$P(L<u, S<\infty\} \leqq \int_{\{S<\infty\}} A_{S-} \, dP = \int_{\{S<\infty\}} A_{S-}^o \, dP = 0.$$

Lemma 2.5. *If* $T = \inf\{t \geqq 0: A_t^o = 1$ or $A_{t-}^o = 1\}$ *then the processes* A^o *and* $(A^o)^T$ *are indistinguishable.*

The proof is essentially the same as that of Lemma 2.4. From now on we shall take $A^o = (A^o)^T$.

Lemma 2.6. *Let* T *be any* $\{\mathscr{F}_t\}$*-optional time.*

(i) *If* $T \geqq L$ a.s. *on* $\{T<\infty\}$ *then* $A_T^o = 1$ a.s. *on* $\{T<\infty\}$.

(ii) *If* $T \leqq L$ a.s. *on* $\{T<\infty\}$ *then* $A_T^o = \Delta \hat{A}_T$ a.s. *on* $\{T<\infty\}$.

Proof. (i) is an immediate consequence of Dellacherie [2, V, T15].

(ii) Set $B_t = I\{t = T\}$. Then if $\xi \in \mathscr{F}_T$, by Dellacherie's formula,

$$E \int_0^\infty \xi B_s \, dA_s = E \int_0^\infty \xi B_s \, d\hat{A}_s,$$

so that $E(I\{L=T\} | \mathscr{F}_T) = \Delta \hat{A}_T$, proving (ii).

Lemma 2.7. $\hat{A}$ is constant on $((L, \infty))$.

Proof. The process $t - C_t$ is $\{\mathscr{F}_t\}$-optional, so, by Dellacherie's formula,

$$E \int_0^\infty (t - C_t) \, d\hat{A}_t = E \int_0^\infty (t - C_t) \, dA_t = E(L - C_L) = 0.$$

Since $t - C_t$ is nonnegative, $\hat{A}$ is constant whenever $t - C_t > 0$, and in particular on $((L, \infty))$.

3. $\{\mathscr{F}_t\}$ and $\{\mathscr{G}_t\}$ martingales

Lemma 3.1 (Williams). *Let* M *be a* $\{\mathscr{G}_t\}$*-adapted process, whose paths are right-continuous with left limits. Suppose that, for each* t, $E|M_t| < \infty$. *Then* M *is a* $\{\mathscr{G}_t\}$*-martingale if and only if the following conditions are satisfied:*

(i) $E(M_t | \mathscr{F}_s) = E(M_s | \mathscr{F}_s)$ *for* $s \leqq t$. (3.1)

(ii) $E(A_s M_t | \mathscr{F}_s) = E(A_s M_s | \mathscr{F}_s)$ *for* $s \leqq t$. (3.2)

Proof. This is immediate from the definition of $\mathscr{G}_s$.

Lemma 3.2. *Suppose that* M *is* $\{\mathscr{G}_t\}$*-adapted, that* $E|M_t| < \infty$ *for each* $t \in \mathbb{R}^+$ *and that the paths of* M *are right-continuous with left limits. If* M *is zero on* $[\![0, L]\!]$, *or* *constant on* $((L,\infty))$, *then* M *is a* $\{\mathscr{G}_t\}$*-martingale if and only if*

$$E(M_t|\mathscr{I}_s)=E(M_s|\mathscr{I}_s) \quad \textit{for } s\leqq t. \tag{3.3}$$

Proof. The necessity of (3.3) is immediate from Lemma 3.1.

Suppose first that M is constant on $((L,\infty))$. Then $A_s M_t = A_s M_L = A_s M_s$ for every ω, so that M satisfies (3.2).

Now let M be zero on $[\![0,L]\!]$, so that $A_t M_t = M_t$. Fix $s\leqq t$, and set $T=\inf\{u>s\colon C_u=u\}$. Then, since $T\leqq L$ on $\{T<\infty\}$ we have $M_T I\{T<\infty\}=0$. Note that since $|M_{t\wedge T}|=|M_t I\{t<T\}|\leqq|M_t|$ we have that $M_{t\wedge T}$ is integrable. Set M^o to be the optional projection of M with respect to $\{\mathscr{I}_t\}$: then since M satisfies (3.3) M^o is an $\{\mathscr{I}_t\}$-martingale. To show that M is a $\{\mathscr{G}_t\}$-martingale it is enough to show that M satisfies (3.2). However, $A_s=A_t I\{t<T\}$, so we have

$$\begin{aligned} E(A_s M_t|\mathscr{I}_s) &= E(A_t I\{t<T\}\, M_t|\mathscr{I}_s)=E(A_t M_{t\wedge T}|\mathscr{I}_s)=E(M_{t\wedge T}|\mathscr{I}_s)\\ &=E(M^o_{t\wedge T}|\mathscr{I}_s)=E(M^o_s|\mathscr{I}_s)=E(M_s|\mathscr{I}_s)=E(A_s M_s|\mathscr{I}_s), \end{aligned}$$

completing the proof.

Corollary 3.3. $A-\hat{A}$ *is a* $\{\mathscr{G}_t\}$*-martingale.*

Proof. The process $A-\hat{A}$ is constant on $((L,\infty))$, so it is enough to show that $A-\hat{A}$ satisfies (3.3). But, by Dellacherie's formula $E(A_t-A_s|\mathscr{I}_s)=E(\hat{A}_t-\hat{A}_s|\mathscr{I}_s)$.

Lemma 3.4. *Let* J *be a previsible element of* $\mathscr{V}(\mathscr{I})$.

(i) *If* $\int_0^t I\{A^o_{s-}=1\}\,|dJ_s|=0$ a.s., *then*

$$E\int_0^t (1-A_{s-})(1-A^o_{s-})^{-1}\,dJ_s=EJ_t.$$

(ii) *If* $\int_0^t I\{A^o_{s-}=0\}\,|dJ_s|=0$ a.s., *then*

$$E\int_0^t A_{s-}(A^o_{s-})^{-1}\,dJ_s=EJ_t.$$

Proof. Recall from Lemma 2.3 that the previsible projection of A_- is A^o_-. The process J is the difference of two right-continuous, previsible, increasing processes: we may therefore take J to be increasing.

Now set $U^n=(1-A^o_{s-})^{-1}(1-A_{s-})\,I\left\{A^o_{s-}<1-\frac{1}{n}\right\}$. The process U^n is bounded, so has a previsible projection, and it is clear that $(U^n)^p_s=I\left\{A^o_{s-}<1-\frac{1}{n}\right\}$. Hence, by Dellacherie's formula,

$$E\int_0^t U^n_s\,dJ_s=E\int_0^t (U^n)^p_s\,dJ_s=E\int_0^t I\left\{A^o_{s-}<\frac{1}{n}\right\}dJ_s.$$

A monotone convergence argument now completes the proof of (i), and (ii) is proved in a similar fashion.

Define $\mathscr{C}(\hat{A}\_)$ to be the class of $M \in \mathscr{M}^2(\mathscr{I})$ such that

(i) $\hat{A}\_ \in L^2(M)$: i.e. $E \int_0^\infty \hat{A}_{s-}^2 \, d\langle M, M\rangle_s < \infty$,

(ii) $E|\hat{A}_t M_t| < \infty$ for each $t \geqq 0$.

The following is an immediate consequence of Lemma 1.2 and Dellacherie's formula.

Lemma 3.5. *Suppose that $M \in \mathscr{C}(\hat{A}\_)$. Then if $s \leqq t$,*

$$E(M_t \hat{A}_t - M_s \hat{A}_s | \mathscr{I}_s) = E\left(\int_s^t M_u \, d\hat{A}_u | \mathscr{I}_s\right) = E\left(\int_s^t M_u \, dA_u | \mathscr{I}_s\right)$$
$$= E((A_t - A_s) M_L | \mathscr{I}_s). \tag{3.4}$$

Corollary 3.6. *Let $M \in \mathscr{C}(\hat{A}\_)$. Then, if M is a $\{\mathscr{G}_t\}$-martingale, $\langle M, \overset{\prime\prime}{A}\rangle = 0$.*

Proof. It is enough to show that $M\overset{\prime\prime}{A}$ is an $\{\mathscr{I}_t\}$-martingale. But

$$E(\overset{\prime\prime}{A}_t M_t - \overset{\prime\prime}{A}_s M_s | \mathscr{I}_s) = E(A_t M_t - A_s M_s | \mathscr{I}_s) - E(\hat{A}_t M_t - \hat{A}_s M_s | \mathscr{I}_s)$$
$$= E(A_t(M_t - M_L) - A_s(M_s - M_L) | \mathscr{I}_s),$$

by Lemma 3.5. The last term is $E((M_t - M_t^L) - (M_s - M_s^L) | \mathscr{I}_s)$, which is zero since M and M^L are $\{\mathscr{G}_t\}$-martingales.

If $M \in \mathscr{M}_{\text{loc}}^2(\mathscr{I})$, so that $\langle M, \overset{\prime\prime}{A}\rangle$ exists, define the $\{\mathscr{G}_t\}$-optional process M' by setting

$$M'_t = M_t + \int_0^t \left(\frac{1 - A_{s-}}{1 - A_{s-}^o} - \frac{A_{s-}}{A_{s-}^o}\right) d\langle M, \overset{\prime\prime}{A}\rangle_s. \tag{3.5}$$

Proposition 3.7. *If $M \in \mathscr{C}(\hat{A}\_)$ then M' is a $\{\mathscr{G}_t\}$-martingale.*

Proof. Set

$$X_t = M'_{t \wedge L} = M_{t \wedge L} + \int_0^{t \wedge L} (1 - A_{u-}^o)^{-1} \, d\langle M, \overset{\prime\prime}{A}\rangle_u,$$

$$Y_t = A_t(M'_t - M'_L) = A_t(M_t - M_L) - \int_L^t (A_{u-}^o)^{-1} \, d\langle M, \overset{\prime\prime}{A}\rangle_u.$$

We will use Lemma 3.2 to prove that X and Y are $\{\mathscr{G}_t\}$-martingales.

Note that $EA_t|M_L| \leqq E \sup_{s \leqq t} M_s < \infty$, by Doob's inequality, since $M \in \mathscr{M}^2(\mathscr{I})$. Now

$$E(M_{t \wedge L} - M_{s \wedge L} | \mathscr{I}_s) = -E(A_t(M_t - M_L) - A_s(M_s - M_L) | \mathscr{I}_s)$$
$$= -E(A_t^o M_t - A_s^o M_s | \mathscr{I}_s) + E(A_t M_L - A_s M_L | \mathscr{I}_s)$$
$$= -E(\overset{\prime\prime}{A}_t M_t - \overset{\prime\prime}{A}_s M_s | \mathscr{I}_s) \quad \text{by Lemma 3.5}$$
$$= -E(\langle M, \overset{\prime\prime}{A}\rangle_t - \langle M, \overset{\prime\prime}{A}\rangle_s | \mathscr{I}_s).$$

Consequently,

$$E(X_t - X_s | \mathscr{I}_s) =$$

$$E\left(\int_s^t (1-A_{u-})(1-A^o_{u-})^{-1}\, d\langle M, \ddot{A}\rangle_u | \mathscr{I}_s\right) - E(\langle M, \ddot{A}\rangle_t - \langle M, \ddot{A}\rangle_s | \mathscr{I}_s),$$

and a similar equation holds for Y. To complete the proof it is therefore enough to show that $\langle M, \ddot{A}\rangle$ satisfies the conditions of Lemma 3.4: that is, if $U_s = I\{A^o_{s-} = 0\}$, and $V_s = I\{A^o_{s-} = 1\}$, that $\int_0^t U_s |d\langle M, \ddot{A}\rangle_s| = 0$ a.s., and that $\int_0^t V_s |d\langle M, \ddot{A}\rangle_s| = 0$ a.s.

By Lemma 2.4, the previsible process U is zero on $((L, \infty))$, so that the martingale $U \cdot M$ is constant on $((L, \infty))$. Thus by Lemma 3.2, $U \cdot M$ is a $\{\mathscr{G}_t\}$-martingale, and by Corollary 3.6, $\langle U \cdot M, \ddot{A}\rangle = 0$. However, $U \cdot \langle M, \ddot{A}\rangle_s = \langle U \cdot M, \ddot{A}\rangle_s = 0$ for all s, and hence $\int_0^t U_s |d\langle M, \ddot{A}\rangle_s| = 0$.

Let $T = \inf\{t \geq 0 : A^o_t = 1\}$. Then, by Lemma 2.5, $A^o_t = 1$ if $t \geq T$, and $\hat{A}$ is constant on $((T, \infty))$, since $T \geq L$. The martingale $V \cdot \ddot{A}$ is zero on $((0, T))$, and constant on $((T, \infty))$. But if $V_T = 1$ then $A^o_{T-} = 1$, so that $\Delta \ddot{A}_T = 0$; therefore $V \cdot A = 0$. It follows, as in the case of U, that $\int_0^t V_s |d\langle M, \ddot{A}\rangle_s| = 0$.

Theorem 3.8. *If $M \in \mathscr{M}^2(\mathscr{I})$ then $M' \in \mathscr{M}^2(\mathscr{G})$.*

Proof. The process $\hat{A}$ has jumps bounded by 1, so, if we set $S_n = \inf\{t \geq 0 : \hat{A}_t \geq n\}$, then $M^{S_n} \in \mathscr{C}(\hat{A}_-)$. Thus, by 3.7, $(M^{S_n})' = (M')^{S_n}$ is a $\{\mathscr{G}_t\}$-martingale, and M' is a $\{\mathscr{G}_t\}$-local martingale.

We may decompose M as follows: $M = U + V$, where $U, V \in \mathscr{M}^2(\mathscr{I})$ V is a purely discontinuous martingale and jumps only at previsible times, and U jumps only at totally inaccessible times. Then $[U, V] = 0$, since U and V have no common jumps, and $V^c = 0$. See [9, II, 8–11] for details.

For notational convenience set

$$H_u = (1-A_{u-})(1-A^o_{u-})^{-1} I\{A^o_{u-} < 1\} - A_{u-}(A^o_{u-})^{-1} I\{A^o_{u-} > 0\}.$$

Then $H \cdot \langle U, \ddot{A}\rangle$, $H \cdot \langle V, \ddot{A}\rangle$ are VF processes, so that U and V are $\{\mathscr{G}_t\}$-semimartingales, since U' and V' are $\{\mathscr{G}_t\}$-local martingales.

Now U jumps only at totally inaccessible times, so $\langle U, \ddot{A}\rangle$ is continuous, and hence $H \cdot \langle U, \ddot{A}\rangle$ is continuous. Thus $[U, U] = [U', U']$.

Let T be $\{\mathscr{I}_t\}$-previsible. Then T is $\{\mathscr{G}_t\}$-previsible, and $E(\Delta V'_T | \mathscr{G}_{T-}) = 0$. But $\Delta V'_T = \Delta V_T + H_T \Delta\langle V, \ddot{A}\rangle_T$, and therefore $E((\Delta V'_T)^2 | \mathscr{G}_{T-}) = E((\Delta V_T)^2 | \mathscr{G}_{T-}) - (H_T \Delta\langle V, \ddot{A}\rangle_T)^2$, so that $E(\Delta V'_T)^2 \leq E(\Delta V_T)^2$.

However, the jumps of V' are contained in the jumps of V, and V' is purely discontinuous, so it follows that

$$E[V', V']_t = E \sum_{s \leq t} (\Delta V'_s)^2 \leq E \sum_{s \leq t} (\Delta V_s)^2 = E[V, V]_t.$$

We now see that $E[M',M']_t \leqq E[M,M]_t$ for $t \geqq 0$. Since M is square integrable $E[M,M]_\infty < \infty$, and the proof is concluded by applying Lemma 1.6.

Corollary 3.9. *If* $M \in \mathscr{M}^2_{\mathrm{loc}}(\mathscr{F})$ *then* $M' \in \mathscr{M}^2_{\mathrm{loc}}(\mathscr{G})$.

Theorem 3.10. *If* X *is an* $\{\mathscr{F}_t\}$*-semimartingale then* X *is a* $\{\mathscr{G}_t\}$*-semimartingale.*

Proof. We can write $X = X_0 + M + B$, where $M \in \mathscr{M}_{\mathrm{loc}}(\mathscr{F})$ and $B \in \mathscr{A}(\mathscr{F})$, and $M_0 = B_0 = 0$. It is enough to show that X^{T_n} is a $\{\mathscr{G}_t\}$-semimartingale, for some sequence T_n of $\{\mathscr{G}_t\}$-optional times, with $\sup_n T_n = +\infty$; see [9, IV, T33].

Recall the decomposition of Gundy, [9, IV, T8]: there exists a sequence (T_n) of $\{\mathscr{F}_t\}$-stopping times with $\lim T_n = +\infty$, such that $M^{T_n} = U^n + V^n$, where $U^n \in \mathscr{M}^2(\mathscr{F})$ and $V^n \in \mathscr{V}(\mathscr{F})$. Thus we may write

$$X^{T_n} = X_0 + U^n + V^n + B = X_0 + (U^n)' + (V^n + B - H \cdot \langle U^n, \ddot{A} \rangle)$$

completing the proof.

Corollary 3.11. *If* T *is totally inaccessible relative to* $\{\mathscr{F}_t\}$ *then* T *is totally inaccessible relative to* $\{\mathscr{G}_t\}$.

Proof. Set $Y_t = I\{t \geqq T\}$. The dual previsible projection of Y relative to $\{\mathscr{G}_t\}$ is given by $\tilde{Y} - H \cdot \langle \tilde{Y}, \ddot{A} \rangle$, which is continuous.

Note. Stricker [12], has proved that if $\{\mathscr{F}_t\}$ is any filtration, and $\{\mathscr{H}_t\}$ is a subfiltration of $\{\mathscr{F}_t\}$, then any $\{\mathscr{F}_t\}$-semimartingale adapted to $\{\mathscr{H}_t\}$ is an $\{\mathscr{H}_t\}$-semimartingale.

4. Structure of $\{\mathscr{G}_t\}$-Progressive Processes

In this section we establish some results which will be used in the proof of the representation theorem for $\{\mathscr{G}_t\}$-martingales.

Define $\{\mathscr{F}_t\}$-optional times α_{nm}, β^*_{nm}, β_{nm} for $n \geqq 1$, $m \geqq 0$, and and some fixed $r \geqq 1$, as follows

$$\beta_{n0} = 0,$$

$$\alpha_{nm} = \inf\left\{t \geqq \beta_{n,m-1} : A^o_t > \frac{1}{n}\right\},$$

$$\beta^*_{nm} = \inf\left\{t \geqq \alpha_{nm} : A^o_t < \frac{1}{2nr}\right\},$$

$$\beta_{nm} = \beta^*_{nm} \wedge \inf\{t > \alpha_{nm} : \Delta \hat{A}_t > 0\}.$$

By Lemma 2.6(ii), and Lemma 2.7, we see that

$$\beta_{nm} = \beta^*_{nm} \wedge \inf\left\{t > \alpha_{nm} : \Delta \hat{A}_t \geqq \frac{1}{2nr}\right\}.$$

Since $\hat{A}$ is increasing, and bounded for almost all ω, $\Delta \hat{A}_t \geqq \frac{1}{2nr}$ for only finitely many t, a.s. Thus, for each n, $\beta_{nm} = \beta^*_{nm}$ for all but finitely many m, a.s.

Now A^o is right continuous, so that $\alpha_{nm} > \beta^*_{n,m-1}$ whenever $\beta_{n,m-1} = \beta^*_{n,m-1}$ and $\beta^*_{nm} > \alpha_{nm}$. Consequently, because A^o has left limits, the sequences $(\alpha_{nm})_{m=1}^{\infty}$, $(\beta_{nm})_{m=1}^{\infty}$, have no accumulation point in $[0, \infty)$.

Define a sequence γ_n of $\{\mathscr{G}_t\}$-optional times by setting $\gamma_n = \inf\left\{t \geqq L: A^o_t > \frac{1}{n}\right\}$.

Lemma 4.1. (i) *Suppose that T is an $\{\mathscr{F}_t\}$-optional time, that $T \leqq L$ [respectively: $T < L$] on $\{T < \infty\}$, and that $T > \alpha_{nm}[T \geqq \alpha_{nm}]$. Then $T \geqq \beta_{nm}$.*

(ii) *$L \geqq \beta_{nm}$ on $\{L > \alpha_{nm}\}$.*

(iii) $[\![\gamma_n]\!] \subseteq \bigcup_{m \geqq 1} [\![\alpha_{nm}]\!]$.

(iv) $((L, \infty)) \subseteq \bigcup_{n \geqq 1, m \geqq 1} [\![\alpha_{nm}, \beta_{nm}))$.

Proof. (i) By Lemma 2.6(ii) $A^o_T = \Delta \hat{A}_T$ on $\{T < \infty\}$. The definition of β_{nm} now ensures that $T \geqq \beta_{nm}$. If $T < L$ on $\{T < \infty\}$ then $A^o_T = 0$ on $\{T < \infty\}$, and hence $T > \alpha_{nm}$ a.s.

(ii) Set $T = \inf\{u > \alpha_{nm}: C_u = u\}$. Then $T \leqq L$ on $\{T < \infty\}$, so that $A^o_T = \Delta \hat{A}_T$ on $\{T < \infty\}$. The right-continuity of A^o now ensures that $T > \alpha_{nm}$, and by (i) we have $L \geqq T \geqq \beta_{nm}$ on $\{L > \alpha_{nm}\}$.

(iii) Choose $\omega \in \Omega$, $n \geqq 1$. By (ii) $L(\omega)$ does not lie in any interval of the form $((\alpha_{nm}, \beta_{nm}))$, so that for some $m \geqq 0$, $\beta_{nm-1}(\omega) \leqq L(\omega) \leqq \alpha_{nm}(\omega)$. Consequently $\gamma_n(\omega) = \alpha_{nm}(\omega)$.

(iv) This follows from Lemma 2.4, and the inclusion

$$\left\{(t, \omega): A^o_t(\omega) > \frac{1}{n}\right\} \subseteq \bigcup_{m \geqq 1} [\![\alpha_{nm}, \beta_{nm})).$$

We now turn our attention to the structure of $\{\mathscr{G}_t\}$-optional processes, and $\{\mathscr{G}_t\}$-optional times.

Lemma 4.2. *Let T be an $\{\mathscr{F}_t\}$-optional time, and ξ a $\mathscr{G}_T$ measurable random variable. Then there exist $\mathscr{F}_T$ measurable random variables η and v such that for every ω*

$$\xi(\omega) = A_T(\omega)\,\eta(\omega) + (1 - A_T(\omega))\,v(\omega).$$

Proof. This is a simple consequence of Lemma 2.2.

Proposition 4.3. *Let X be a right-continuous $\{\mathscr{G}_t\}$-progressive process. Then there exist $\{\mathscr{F}_t\}$-progressive processes H and K, such that the processes X and $(1-A)H + AK$ are identical.*

Proof. For $p \geqq 0$ we may choose H'_p, K'_p such that $X_p = (1 - A_p) H'_p + A_p K'_p$. Define H and K by setting $H_t = \liminf_{p \downarrow t} H'_p$, $K_t = \liminf_{p \downarrow t} K'_p$: H and K are then $\{\mathscr{F}_t\}$-progressive, by [3, IV, T17]. It is easily verified that for every (t, ω) $X_t(\omega) = (1 - A_t(\omega))\,H_t(\omega) + A_t(\omega)\,K_t(\omega)$.

Remark. It is not always possible to choose H and K to be $\{\mathscr{F}_t\}$-optional, even if X is $\{\mathscr{G}_t\}$-optional. Consider the following example. Let B be Brownian motion, with $B_0 = 0$, $\tau = \inf\{t \geqq 0: |B_t| = 1\}$, and L be the last exit from 0 of the process B^τ.

Let $X_t = I\{t \geqq L\}\,\mathrm{sgn}(B_\tau)$. Then X is right-continuous, and $\{\mathcal{G}_t\}$-adapted, so $\{\mathcal{G}_t\}$-optional. We can choose H, K as follows: H is 0, and $K_t = \mathrm{sgn}(B_{t+})$, if this exists, 0 otherwise. If an $\{\mathcal{F}_t\}$-optional choice of K existed then a previsible choice of K would exist ([2, V, T22]), so that $\Delta X_L = K_L$ would be $\mathcal{G}_{L-}$ measurable, contradicting the fact that X is a $\{\mathcal{G}_t\}$-martingale.

All the trouble here arises at L, so that we do have the following positive result. Let $\tau_n = \inf\left\{t \geqq 0 \colon A_t^o > 1 - \frac{1}{n}\right\}$.

Lemma 4.4. *Let X be a right-continuous $\{\mathcal{G}_t\}$-optional process.*

(i) *If X is constant on $[\![\tau_n, \infty))$, then H can be chosen to be right-continuous.*

(ii) *If X is zero on $[\![0, \alpha_{nm}))$, and constant on $[\![\beta_{nm}, \infty))$, then K may be chosen to be right-continuous.*

Proof. (i) We may take H to be constant on $[\![\tau_n, \infty))$. Define the $\{\mathcal{F}_t\}$-progressive processes H', H'' by setting $H'_t = \limsup_{p \downarrow t} H_p$, $H'' = \liminf_{p \downarrow t} H_p$, and set $T = \inf\{t \geqq 0 \colon H''_t < H'_t\}$. Then T is the debut of an $\{\mathcal{F}_t\}$-progessive set, and is therefore an $\{\mathcal{F}_t\}$-optional time. For each ω the path $H.(\omega)$ follows $X.(\omega)$ up to $L(\omega)$, so that, by the right-continuity of X, we have $T \geqq L$. But then, by Lemma 2.5(i), $A_T^o = 1$, so that $T \geqq \tau_n$. Since H is constant on $[\![\tau_n, \infty))$, it follows that H is right-continuous.

(ii) We may take K to be zero on $[\![0, \alpha_{nm}))$, and constant on $[\![\beta_{nm}, \infty))$. Define K', K'' and T as in (i). Now $T < L$, and $T \geqq \alpha_{nm}$, hence, by Lemma 4.1, $T \geqq \beta_{nm}$. Thus K is right-continuous.

Theorem 4.5. *Let T be a $\{\mathcal{G}_t\}$-stopping time. Then there exists a sequence $(S_n)_{n=1}^{\infty}$ of disjoint $\{\mathcal{F}_t\}$-stopping times such that*

$$[\![T]\!] \subseteq [\![L]\!] \cup \bigcup_{n=1}^{\infty} [\![S_n]\!].$$

Proof. Let T_n, T_{nm} be the restrictions of T to $\{T < L\} \cap \{T < \tau_n\}$, $\{T > L\} \cap \{\alpha_{nm} \leqq T < \beta_{nm}\}$ respectively. Then by Lemma 4.1(iv)

$$[\![T]\!] \subseteq [\![L]\!] \cup \Big(\bigcup_{n \geqq 1} [\![T_n]\!]\Big) \cup \Big(\bigcup_{n \geqq 1, m \geqq 1} [\![T_{nm}]\!]\Big).$$

Set $X = I_{[\![T_n, \infty))}$, and choose H as in Lemma 4.4(i). If $R_n = \inf\{t \geqq 0 \colon H_t = 1\}$ then R_n is $\{\mathcal{F}_t\}$-optional, and $[\![T_n]\!] \subseteq [\![R_n]\!]$. Similarly, if $Y = I_{[\![T_{nm}, \infty))}$ then K may be chosen as in Lemma 4.4(ii), and if $R_{nm} = \inf\{t \geqq 0 \colon K_t = 1\}$ then $[\![T_{nm}]\!] \subseteq [\![R_{nm}]\!]$.

A suitable non-disjoint sequence (S_n) therefore exists and by [2, IV, T17] a disjoint sequence can be found.

5. Representation of Continuous $\{\mathcal{G}_t\}$-Martingales

We make the following assumption: there exists a family $\{M^i \colon i \in I\}$ of continuous elements of $\mathcal{M}^2(\mathcal{F})$ with the martingale representation property for continuous $\{\mathcal{F}_t\}$-martingales. By this we mean that if Z is a continuous element of $\mathcal{M}^2_{\mathrm{loc}}(\mathcal{F})$, and $\langle M^i, Z\rangle = 0$ for every $i \in I$, then $Z = 0$.

Theorem 5.1. *The family $\{M^{i'}: i \in I\}$, consists of continuous elements of $\mathcal{M}^2(\mathcal{G})$, and has the martingale representation property for continuous $\{\mathcal{G}_t\}$-martingales.*

We shall need the following simple application of Itô's lemma for semimartingales.

Lemma 5.2. *Suppose that X, Y, Z are optional processes, that X and Y are semimartingales (r), and that $X = YZ$. Let $T = \inf\{t \geqq 0: Y_t < \varepsilon\}$. Then, if $Y_0 > 0$, and $\Delta Z_T I\{T < \infty\}$ is integrable, Z^T is a semimartingale (r).*

Proposition 5.3. *Let Z be a continuous element of $\mathcal{M}^2(\mathcal{G})$. If Z is zero on $[\![0, L]\!]$, and $\langle Z, M^{i'}\rangle = 0$ for every $i \in I$, then Z is null.*

Proof. Recall the definitions of the random times (α_{nm}), (β_{nm}), and (γ_n) from Section 4. Define $\{\mathcal{G}_t\}$-optional times (δ_{ns}), $n \geqq 1$, $s \geqq 1$, by setting $\delta_{ns} = \inf\left\{t \geqq \gamma_n: A^o_t < \frac{1}{2ns}\right\}$. Since Z is continuous, it is sufficient to prove that $I_{(\!(\gamma_n, \delta_{ns}]\!]} \cdot Z$ is null for every $n \geqq 1$, $s \geqq 1$. Fix n and s, and take the r in the definitions of (α_{nm}) and (β_{nm}) to be s. To simplify the notation we shall drop the subscripts n and s, and refer to γ, δ, (α_m), (β_m): we shall also assume that $Z = I_{(\!(\gamma, \delta]\!]} \cdot Z$.

By Lemma 4.1 we have $[\![\gamma]\!] \subseteq \bigcup_{m \geqq 1} [\![\alpha_m]\!]$. Set $U^m_t = I\{t > \gamma\} I\{\gamma = \alpha_m\}$, a $\{\mathcal{G}_t\}$-previsible process, and $Z^m = U^m \cdot Z$. Then $Z = \sum_{m \geqq 1} Z^m$, and for each m, Z^m is a continuous $\{\mathcal{G}_t\}$-martingale, constant except on $[\![\alpha_m, \beta_m]\!]$, and $\langle Z^m, M^{i'}\rangle = U^m \cdot \langle Z, M^{i'}\rangle = 0$.

We may now apply Lemma 4.4 to Z^m: there exists a right-continuous $\{\mathcal{F}_t\}$-optional process K^m, zero on $[\![0, \alpha_m]\!]$ and constant on $[\![\beta_m, \infty)\!)$, such that $Z^m = AK^m$. An argument similar to that in Lemma 4.4(ii) shows that if T is the time of the first jump of K^m, then $T \geqq \beta_m$. K^m is $\{\mathcal{F}_t\}$-optional; so taking optional projections we have $(Z^m)^o = A^o K^m$. Now $(Z^m)^o$ and A^o both have left limits at β_m, and $A^o_{\beta_m -} \geqq \frac{1}{2ns}$: thus K^m also has a left limit at β_m. Since Z^m is continuous we may take $K^m_t = K^m_{\beta_m -}$ for $t \geqq \beta_m$, ensuring that K^m is continuous everywhere.

It follows from the equation $(Z^m)^o = A^o K^m$, and Lemma 5.2, that K^m is an $\{\mathcal{F}_t\}$-semimartingale (r). However K^m is also a $\{\mathcal{G}_t\}$-semimartingale, by Theorem 3.10. Consider the process $[K^m, M^i]$. Now $Z^m = I\{\gamma = \alpha_m\} K^m$, and so, on the set $\{\gamma = \alpha_m\}$, we have

$$[K^m, M^i] = [K^m, M^{i'}] = [Z^m, M^{i'}] = \langle Z^m, M^{i'}\rangle = 0.$$

Set $T = \inf\{s \geqq 0: [K^m, M^i]_s > 0\}$. Then $T < L$ on $\{T < \infty\}$, and $T \geqq \alpha_m$, so that by Lemma 4.1(i), $T \geqq \beta_m$. But K^m is constant on $[\![\beta_m, \infty)\!)$, and $[K^m, M^i]$ is continuous, and therefore $T = \infty$ a.s., and $[K^m, M^i] = 0$.

As K^m is an $\{\mathcal{F}_t\}$-semimartingale (r) we may write $K = N + B$, where $N \in \mathcal{M}^2(\mathcal{F})$ and $B \in \mathcal{V}(\mathcal{F})$. Now $\langle N^c, M^i\rangle = [K, M^i] = 0$, for every $i \in I$, therefore, since $\{M^i: i \in I\}$ has the martingale representation property for $\{\mathcal{F}_t\}$-martingales, $N^c = 0$. Consequently, by Lemma 1.3, $K^m \in \mathcal{V}_{\text{loc}}(\mathcal{F})$.

However $Z^m = AK^m$: it follows that $Z^m \in \mathscr{V}_{\mathrm{loc}}(\mathscr{G})$, and therefore that Z^m is null, since Z^m is a continuous $\{\mathscr{G}_t\}$-martingale. Now $Z = \sum_{m \geq 1} Z^m$, so that Z is also null, completing the proof.

Proposition 5.4. *Let Z be a continuous element of $\mathscr{M}^2(\mathscr{G})$. If $Z = Z^L$, and $\langle Z, M^{i'} \rangle = 0$ for every $i \in I$, then Z is null.*

The proof is very similar in idea to that of 5.3, and is therefore omitted.

Proof of Theorem 5.1. Note that if $Z \in \mathscr{M}^2_{\mathrm{loc}}(\mathscr{G})$, and $\langle Z, M^{i'} \rangle = 0$, then $\langle Z^T, M^{i'} \rangle = 0$ for any $\{\mathscr{G}_t\}$-optional time T. The result now follows from Propositions 5.3 and 5.4.

6. Representation of Purely Discontinuous $\{\mathscr{G}_t\}$-Martingales

In this section we shall assume that $\mathscr{F}$ is the P-completion of a countably generated σ-field $\mathscr{F}^0$. Let μ be an $\{\mathscr{F}_t\}$-optional, integer-valued random measure defined on $(0, \infty) \times E$, where E is a Lusin space, such that M_μ is $\tilde{\mathscr{P}}$-σ-finite.

By Lemma 1.5 the σ-field $\mathscr{G}_L$ is the P-completion of a countably generated σ-field $\mathscr{G}^0_L$: let $(G_n)_{n=1}^\infty$ be a sequence of $\mathscr{G}^0_L$ measurable sets generating $\mathscr{G}^0_L$. Recall the definitions of D, an $\{\mathscr{F}_t\}$-optional subset of $\Omega \times [0, \infty)$, and α, an $\{\mathscr{F}_t\}$-optional E-valued process, from Section 1. Define the $\{\mathscr{G}_t\}$-optional process β on $E' = E \cup 2^{\mathbb{N}}$ by setting $\beta_s(\omega) = \alpha_s(\omega)$ if $s \neq L(\omega)$, and $\beta_L(\omega) = (I_{G_n}(\omega))_{n=1}^\infty \in 2^{\mathbb{N}}$. Let $D' = D \cup [\![L]\!]$, so that D' is a $\{\mathscr{G}_t\}$-optional subset of $\Omega \times [0, \infty)$. We may now define a $\{\mathscr{G}_t\}$-optional random measure μ' on $(0, \infty) \times E'$ by setting

$$\mu'(\omega; dt, dx) = \sum_{s > 0} I_{D'}(s, \omega)\, \varepsilon_{(s, \beta_s(\omega))}(dt, dx).$$

Note that μ' is integer valued, and that $M_{\mu'}$ is $\tilde{\mathscr{P}}(\mathscr{G})$-$\sigma$-finite: consequently μ' has a $\{\mathscr{G}_t\}$-dual previsible projection v', and there exists the space $\mathscr{G}^2_{\mathrm{loc}}(\mu', \mathscr{G})$ of $\tilde{\mathscr{P}}(\mathscr{G})$-measurable functions such that if $U \in \mathscr{G}^2_{\mathrm{loc}}(\mu', \mathscr{G})$, the stochastic integral $U * (\mu' - v')$ is a purely discontinuous element of $\mathscr{M}^2_{\mathrm{loc}}(\mathscr{G})$.

Theorem 6.1. *If $\{M^i : i \in I; \mu - v\}$ has the martingale representation property for $\{\mathscr{F}_t\}$-martingales, then $\{M^{i'} : i \in I; \mu' - v'\}$ has the martingale representation property for $\{\mathscr{G}_t\}$-martingales.*

If $X, Y \in \mathscr{M}^2_{\mathrm{loc}}(\mathscr{G})$, and X is continuous, and Y is purely discontinuous, then $\langle X, Y \rangle = 0$. Theorem 6.1 is therefore an immediate consequence of Theorem 5.3 and 6.2, since if $\langle Z, M^{i'} \rangle = 0$ for all $i \in I$, then $Z^c = 0$, and if $\langle Z, U * (\mu' - v') \rangle = 0$ for all $U \in \mathscr{G}^2_{\mathrm{loc}}(\mu', \mathscr{G})$, then $Z^d = 0$.

Theorem 6.2. *If $\mu - v$ has the martingale representation property for purely discontinuous $\{\mathscr{F}_t\}$-martingales, then $\mu' - v'$ has the martingale representation property for purely discontinuous $\{\mathscr{G}_t\}$-martingales.*

Lemma 6.3 (Jacod [5]). *Let T be a $\{\mathscr{G}_t\}$-optional time. A necessary and sufficient condition for all purely discontinous elements M of $\mathscr{M}^2(\mathscr{G})$ whose jumps are contained in $[\![T]\!]$ to be of the form $M = U * (\mu' - v')$ for some $U \in \mathscr{G}^2_{\mathrm{loc}}(\mu', \mathscr{G})$ is that $\mathscr{G}_T = \mathscr{G}_{T-} \vee \sigma(\beta_T)$.*

Lemma 6.4. *Let S be an $\{\mathscr{F}_t\}$-optional time. Then $\mathscr{G}_S = \mathscr{G}_{S-} \vee \sigma(\beta_S)$.*

Proof. Set $\mathscr{H} = \mathscr{G}_{S-} \vee \sigma(\beta_S)$: it is clear that $\mathscr{H} \subseteq \mathscr{G}_S$. Now $\{L \leqq S\} \in \mathscr{H}$, since $\{L < S\} \in \mathscr{G}_{S-}$, and $\{L = S\} = \{\beta_S \in 2^{\mathbb{N}}\} \in \sigma(\beta_S)$.

Let $F \in \sigma(\alpha_S)$: then $F \cap \{S \neq L\} \in \sigma(\beta_S)$, as $\beta_S = \alpha_S$ on $\{S \neq L\}$. Now $\sigma(\alpha_S) \subseteq \mathscr{F}_S$, so that $F \cap \{S = L\} \in \mathscr{G}_L$. Thus $F \cap \{S = L\} = G \Delta \Lambda$, where $G \in \sigma(\beta_L)$, and $P(\Lambda) = 0$, for $\mathscr{G}_L$ is the P-completion of $\sigma(\beta_L)$ by the definition of β. We also have $G \cap \{S = L\} \in \sigma(\beta_S)$: hence F belongs to the P-completion of $\sigma(\beta_S)$.

Since $\mu - \nu$ has the martingale representation property with respect to $\{\mathscr{F}_t\}$, we have, from Lemma 5.2, that $\mathscr{F}_S = \mathscr{F}_{S-} \vee \sigma(\alpha_S)$. Now $\mathscr{F}_{S-} \subseteq \mathscr{G}_{S-}$, and we have proved that $\sigma(\alpha_S) \subseteq \mathscr{F}_0 \vee \sigma(\beta_S)$, hence $\mathscr{F}_S \subseteq \mathscr{H}$. Therefore, by Lemma 2.2(i), $\mathscr{G}_S \subseteq \mathscr{H}$.

Proof of Theorem 6.2. Suppose that $Z \in \mathscr{M}^2_{\text{loc}}(\mathscr{G})$, and that $\langle Z, U * (\mu' - \nu')\rangle = 0$ for every $U \in \mathscr{G}^2_{\text{loc}}(\mu', \mathscr{G})$. If T is any $\{\mathscr{G}_t\}$-optional time the same is true of Z^T, so we may take $Z \in \mathscr{M}^2(\mathscr{G})$.

Let Y be the purely discontinuous $\{\mathscr{G}_t\}$-martingale whose sole jump is ΔZ_L at time L. By Lemma 6.3, and the construction of β_L, $Y = U * (\mu' - \nu')$ for some $U \in \mathscr{G}^2_{\text{loc}}(\mu', \mathscr{G})$. It is clear that U can be taken to be zero off $2^{\mathbb{N}}$. Then $[Z, U * (\mu' - \nu')] = A(\Delta Z_L)^2$, which implies that $\Delta Z_L = 0$ a.s.

The process Z is right-continuous with left limits, so we may find a sequence (T_i) of $\{\mathscr{G}_t\}$-optional times which exhaust the jumps of Z. Then, by Theorem 4.5 there exists a disjoint sequence (S_j) of $\{\mathscr{F}_t\}$-optional times which also exhaust the jumps of Z. Let Z^j be the $\{\mathscr{G}_t\}$-martingale generated by the jump of Z at S_j. By Lemma 5.3 there exists $U^j \in \mathscr{G}^2_{\text{loc}}(\mu', \mathscr{G})$ such that $Z^j = U^j * (\mu' - \nu')$. But $0 = \langle Z, U^j * (\mu' - \nu')\rangle = \langle Z, Z^j\rangle = \langle Z^j, Z^j\rangle$, which shows that $Z^j = 0$. Thus Z has no jumps, and is therefore null.

References

1. Azéma, J.: Quelques applications de la théorie générale des processus. I. Invent. Math. **18**, 293–336 (1972)
2. Dellacherie, C.: Capacités et processus stochastiques. Berlin-Heidelberg-New York: Springer 1972
3. Dellacherie, C., Meyer, P.A.: Probabilités et potentiel. Hermann: Paris 1975
4. Jacod, J.: Multivariate point processes: predictable projection, Radon-Nikodym derivatives, representation of martingales. Z. Wahrscheinlichkeitstheorie verw. Gebiete **31**, 235–253 (1975)
5. Jacod, J.: Un théorème de representation pour les martingales discontinues. Z. Wahrscheinlichkeitstheorie verw. Gebiete **34**, 225–244 (1976)
6. Jacod, J.: A general theorem of representation for martingales. Proc. Sympos. Pure Math. **31**, 37–53 (1977)
7. Jacod, J., Yor, M.: Étude des solutions extrémales et représentation intégrale des solutions pour certains problèmes de martingales. Z. Wahrscheinlichkeitstheorie verw. Gebiete **38**, 83–125 (1977)
8. Jeulin, T., Yor, M.: Grossissement d'une filtration et semi-martingales: formules explicites. Sém. Probab. Strasbourg XII, Lect. Notes Math. 649. Berlin-Heidelberg-New York: Springer 1978
9. Meyer, P.A.: Un cours sur les integrales stochastiques. Sém. Probab. Strasbourg X, Lect. Notes Math. 511. Berlin-Heidelberg-New York: Springer 1976
10. Meyer, P.A., Smythe, R.T., Walsh, J.B.: Birth and death of Markov processes. Proc. 6th Berkeley Sympos. Math. Statist. Probab. Univ. Calif. **3**, 293–305 (1972)

11. Millar, P.W.: Random times and decomposition theorems. Proc. Sympos. Pure Math. **31**, 91–103 (1977)
12. Stricker, C.: Quasimartingales, martingales locales, semimartingales et filtration naturelle. Z. Wahrscheinlichkeitstheorie verw. Gebiete **39**, 55–63 (1977)
13. Yor, M.: Grossissement d'une filtration et semi-martingales: theoremes generaux. Sém. Proba. Strasbourg XII, Lect. Notes Math. Berlin-Heidelberg-New York: Springer 1978

Received November 14, 1977