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A Ratio Ergodic Theorem for Superadditive Processes*

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1. Introduction

Let $(X, \mathfrak{F}, \mu)$ be a measure space and T a positive linear operator on $L_1 = L_1(X, \mathfrak{F}, \mu)$, satisfying

$$\int f\,d\mu = \int Tf\,d\mu \qquad f \in L_1^+. \tag{1.1}$$

Such an operator, necessarily a contraction of L_1, is called *Markovian.* The relations below are often defined only modulo sets of measure zero; the words a.e. may or may not be omitted.

We consider a sequence of L_1^+ functions $(f_0, f_1, f_2, \ldots)$, and denote their partial sums by s_n:

$$s_n = f_0 + \cdots + f_{n-1}, \qquad n \geqq 1; \qquad s_0 = 0. \tag{1.2}$$

(f_n) is called a *superadditive sequence* or *process*, and (s_n) a *superadditive sum*, iff

$$s_{k+n} \geqq s_k + T^k s_n, \qquad k, n \geqq 0, \tag{1.3}$$

or equivalently

$$\sum_{i=k}^{k+n-1} f_i \geqq T^k s_n, \tag{1.3'}$$

and

$$\gamma = \sup_n \int \frac{1}{n} s_n\,d\mu < \infty. \tag{1.4}$$

γ is the *time constant* of the process. (1.1) and (1.3) imply that

$$\int s_{k+n}\,d\mu \geqq \int s_k\,d\mu + \int s_n\,d\mu,$$

hence $\lim_n \frac{1}{n} \int s_n\,d\mu = \gamma$ (see e.g. [8], p. 244).

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(f_n) is called *subadditive* iff $(-f_n)$ is superadditive; *additive* iff it is both superadditive and subadditive. J.F.C. Kingman developed in a series of articles (see [9, 10, 11]) the theory of subadditive processes, in particular proving the ergodic theorem in the case when $\mu(X)=1$ and T is induced by a measure-preserving invertible point-transformation τ by the relation $Tf=f\circ\tau$. Other proofs were given by Burkholder who used the theorem of Komlos (cf. contribution to the discussion of [10]), Del Junco [4], and Derriennic [5]. Here we consider superadditive rather than subadditive processes for the following simple reason: A not necessarily positive superadditive process (f_n) obviously dominates the additive sequence $(T^n f_0)$. If this sequence is subtracted from (f_n), then the problem of convergence is studied in the pleasing context of positive operators acting on positive, rather than negative, functions. Thus e.g. maximal lemmas remain maximal rather than becoming "minimal". But the two theories, superadditive and subadditive, are of course entirely equivalent.

In Section 2 we prove that (f_n) is dominated by an additive process $(T^n\delta)$ with $\int\delta\,d\mu=\gamma$, where γ is the time constant of (f_n). δ is called an *exact dominant* for (f_n) or (s_n). In Section 3 we show that the asymptotic behavior of $\sum_0^{n-1} T^i\delta$ determines that of (s_n), and we derive from this a ratio ergodic theorem generalizing at the same time Kingman's theorem, the Chacon-Ornstein theorem, and Chacon's ratio Theorem [3] (case $T\geqq 0$).

Kingman's elegant proof of this ergodic theorem depends on weak* compactness of L_∞^* (hence on the Hahn-Banach theorem), which appeared natural to Professor Kingman but was much deplored by authors of alternate proofs. The main arguments given below are measure-theoretic rather than functional-analytic[1]. We also give an elementary version of Kingman's original proof.

Kingman's theory is remarkable for its beautiful applications. We do not have as yet any probabilistic applications of our generalization; the most applicable seems Corollary 1 to Theorem 3.2, since conservative and ergodic Markovian operators arise naturally in the theory of nullrecurrent Markov chains, and Harris processes (cf. e.g. Orey [14]). But one also obtains a superadditive process if one restricts any additive (or superadditive) process to the conservative part of the space (cf. (3.7) below). Theorem 2.1 may therefore be used to extend the fundamental Chacon-Ornstein theorem from the conservative case, where the proof is easier (cf. Neveu [13]), to the general case. In fact, the superadditive ratio theorem in Section 3 depends only on the conservative case of Chacon-Ornstein, but implies the general case, and the case $T\geqq 0$ of the Chacon ratio theorem.

2. Existence of Exact Dominants

An *exact dominant* of a super-additive process (f_n) is an L_1^+ function δ such that $\int\delta\,d\mu=\gamma$ and

$$\sum_{i=0}^{n-1} T^i\delta\geqq s_n \qquad n=1,2,\ldots. \tag{2.1}$$

[1] They depend on the countable axiom of choice, but not on the stronger maximum ideal principle, required for the proof of the Hahn-Banach theorem

Theorem 2.1. *Let T be a Markovian operator. A superadditive process admits at least one exact dominant.*

Theorem 2.1 in terms of subadditive processes may be stated as a decomposition theorem: A subadditive process is a sum of an additive process and a positive *purely subadditive* process: one which does not dominate any positive additive process. Thus this theorem generalizes Kingman's decomposition (Theorem 1.6 [11]). In the case when T is generated by a measure-preserving point-transformation, the following lemma is due to Kingman.

Lemma 1. *There exists a sequence (φ_m) of L_1^+ functions such that $\int \varphi_m \, d\mu \leqq \gamma$ for all $m \geqq 1$, and for $1 \leqq n < m$,*

$$\sum_{i=0}^{n-1} T^i \varphi_m \geqq \left(1 - \frac{n-1}{m}\right) s_n. \tag{2.2}$$

Proof. Let

$$\varphi_m = \frac{1}{m} \sum_{i=1}^{m} (s_i - T s_{i-1}).$$

By (1.3), $T s_{i-1} \leqq s_i$, hence $\varphi_m \geqq 0$. Also,

$$\int \varphi_m \, d\mu = \frac{1}{m} \sum_{i=1}^{m} \int (s_i - T s_{i-1}) \, d\mu = \frac{1}{m} \int s_m \, d\mu \leqq \gamma$$

by (1.1). Finally, if $1 \leqq n \leqq m$, then

$$\begin{aligned} m \sum_{i=0}^{n-1} T^i \varphi_m &= (I - T^n) \sum_{i=1}^{m-1} s_i + \sum_{i=0}^{n-1} T^i s_m \\ &= \sum_{i=1}^{n} s_i + (m-n) s_n + \sum_{i=1}^{n-1} T^{n-i} s_i \geqq (m-n+1) s_n. \quad \square \end{aligned}$$

We will now assume that μ is a σ-finite measure and $\mathfrak{F}$ is generated by a countable class $\mathscr{A}$. Routine arguments show that this is no loss of generality. Let $h \in L_1^+$ be a fixed function such that $h > 0$ a.e. Let (φ_m) be the sequence defined in Lemma 1. We may assume, if necessary choosing a subsequence of (φ_m) by the diagonal procedure, that

$$\lim_{m \to \infty} \int_A [(T^i \varphi_m) \wedge (jh)] \, d\mu$$

exists for each $A \in \mathscr{A}$ and for each integer $i, j \geqq 0$. For a fixed j, the sequence $((T^i \varphi_m) \wedge (jh))$ is dominated by the integrable function jh. Since $\mathscr{A}$ generates $\mathfrak{F}$, this means that for each $i, j \geqq 0$ there is an L_1^+-function λ_{ij} such that $w\text{-lim}\,(T^i \varphi_m) \wedge (jh) = \lambda_{ij}$, where w-lim denotes the weak limit in L_1. For a fixed $i \geqq 0$, the sequence (λ_{ij}) is non-decreasing in j and, therefore, $\lim_{j \to \infty} \uparrow \lambda_{ij} = \lambda_i$ exists a.e., and also in the strong topology of L_1, since $\int \lambda_i \, d\mu \leqq \sup \int \varphi_m \, d\mu \leqq \gamma$. Now (2.2) implies that if $n \leqq m$,

$$\sum_{i=0}^{n-1} [(T^i \varphi_m) \wedge (jh)] \geqq \left[\sum_{i=0}^{n-1} T^i \varphi_m\right] \wedge (jh) \geqq \left(1 - \frac{n}{m}\right) s_n \wedge jh,$$

which shows that for each $n \geqq 1$

$$\sum_{i=0}^{n-1} \lambda_{ij} \geqq s_n \wedge j h,$$

hence

$$\sum_{i=0}^{n-1} \lambda_i \geqq s_n \quad \text{for all } n \geqq 1. \tag{2.3}$$

We now claim that $T\lambda_i \leqq \lambda_{i+1}$ for each $i \geqq 0$. To show this, it is enough to show that $T\lambda_{ij} \leqq \lambda_{i+1}$ for each $j \geqq 0$. For a fixed $j \geqq 0$ and for a given $\varepsilon > 0$ we can find an integer $k \geqq 0$ such that $T(jh) \leqq kh + g$, where $g \in L_1^+$ and $\int g \, d\mu < \varepsilon$. Now T is continuous in the strong, hence also weak, topology of L_1, and therefore

$$\begin{aligned} T\lambda_{ij} &= T\{w\text{-lim}\,(T^i \varphi_m \wedge jh)\} = w\text{-lim}\, T[(T^i \varphi_m) \wedge jh] \\ &\leqq w\text{-lim}\,[(T^{i+1} \varphi_m) \wedge T(jh)] \\ &\leqq w\text{-lim}\,[(T^{i+1} \varphi_m) \wedge (kh+g)] \\ &\leqq w\text{-lim}\,[(T^{i+1} \varphi_m) \wedge (kh) + (T^{i+1} \varphi_m) \wedge g] \\ &\leqq \lambda_{(i+1),k} + g, \end{aligned}$$

which shows that $T\lambda_{ij} \leqq \lambda_{i+1}$.

Hence we have

$$\lambda_i = (\lambda_i - T\lambda_{i-1}) + T(\lambda_{i-1} - T\lambda_{i-2}) + \cdots + T^{i-1}(\lambda_1 - T\lambda_0) + T^i \lambda_0$$

with all the summands positive. Now

$$\gamma \geqq \int \lambda_i \, d\mu = \int [(\lambda_i - T\lambda_{i-1}) + (\lambda_{i-1} - T\lambda_{i-2}) + \cdots + (\lambda_1 - T\lambda_0) + \lambda_0] \, d\mu.$$

Define

$$\delta = \lambda_0 + \sum_{i=0}^{\infty} (\lambda_{i+1} - T\lambda_i), \tag{2.4}$$

then $\int \delta \, d\mu \leqq \gamma$, and $\sum_{i=0}^{n-1} T^i \delta \geqq \lambda_0 + \lambda_1 + \cdots + \lambda_{n-1} \geqq s_n$ by (2.3). This shows that $\int \delta \, d\mu \geqq \int \frac{1}{n} s_n \, d\mu$ for each $n \geqq 1$. Since $\int \frac{1}{n} s_n \, d\mu \to \gamma$, $\int \delta \, d\mu = \gamma$. δ is an exact dominant for the process. □

We also give another proof of Theorem 2.1. Consider at first the case $Tf = f \circ \tau$ where τ is an invertible, measure-preserving point-transformation of $(X, \mathfrak{F}, \mu)$. We show that there exists an elementary version of Kingman's original argument (cf. [11]). Let $\mathscr{A}'$ be a countable algebra generating $\mathfrak{F}$ and set $\mathscr{A} = \bigcup_{i=-\infty}^{+\infty} \tau^{-i} \mathscr{A}'$. Replacing if necessary φ_m by a subsequence and using diagonal procedure, we may assume that

$$\lim \int_{\tau^{-i}A} \varphi_m \, d\mu = \psi_i(A) \qquad i=0, \pm 1, \pm 2, \dots \tag{2.5}$$

exists for each $A \in \mathscr{A}$. Now recall the Yosida-Hewitt theorem ([16], [6]): If ψ is a *charge* (a finite, finitely additive, non-negative set-function) on an algebra $\mathscr{A}$, then there exists a unique maximal measure dominated by ψ, denoted $m(\psi)$, and given by

$$[m(\psi)](A) = \inf \sum \psi(A_i), \qquad A \in \mathscr{A}, \tag{2.6}$$

where the infimum is taken over all countable partition $\{A_i\}$ of A, $\{A_i\} \subset \mathscr{A}$. $\pi = \psi - m(\psi)$ is necessarily a *pure* charge, i.e. π does not dominate any non-null measure. As may be surmised from (2.6), there exist completely elementary proofs of this result: it is easy to check directly that $m(\psi)$ defined by (2.6) has the announced properties (see e.g. [15]). It also easily follows from (2.6) that if $\psi_0, \psi_1, \dots, \psi_{n-1}$ are arbitrary charges, $n \geqq 1$, then

$$m(\psi_0 + \dots + \psi_{n-1}) = m(\psi_0) + \dots + m(\psi_{n-1}). \tag{2.7}$$

If $f \in L_1^+$, denote by $f \cdot \mu$ the measure defined by

$$(f \cdot \mu)(A) = \int_A f \, d\mu.$$

Let ψ_i be given by (2.5), $\lambda_i = m(\psi_i)$, then (2.6) and the invariance of $\mathscr{A}$ imply that $\lambda_i = \tau^{-i} \lambda_0$ for all i. From (2.2) it follows that for each $n \geqq 1$

$$\sum_0^{n-1} \psi_i \geqq s_n \mu \quad \text{on } \mathscr{A},$$

hence by (2.7)

$$m\left(\sum_0^{n-1} \psi_i\right) = \sum_0^{n-1} \lambda_i = \sum_0^{n-1} \lambda_0 \circ \tau^{-i} \geqq s_n \cdot \mu \quad \text{on } \mathscr{A}, \text{ hence on } \mathfrak{F}.$$

Therefore

$$\lambda_0(X) = \lim \frac{1}{n} \sum_0^{n-1} (\lambda_0 \circ \tau^{-i})(X) \geqq \gamma.$$

But $\psi_0(X) \leqq \gamma$ implies $\lambda_0(X) \leqq \gamma$. Hence $\lambda_0(X) = \gamma$, $\delta = \dfrac{d\lambda_0}{d\mu}$ is an exact dominant for s_n.

It appears that there is no "elementary" operator version of this argument. A non-elementary one may go as follows: Let L be a Banach limit (cf. [6]), and set

$$\psi_0(A) = L[(\varphi_m \cdot \mu)(A)], \qquad A \in \mathfrak{F}.$$

(ψ_0 is of course a weak* limit point of L_∞^*.) Let T^{**} be the second adjoint of T. Let $\eta_0 = m(\psi_0)$, $\pi_0 = \psi_0 - \eta_0$; then $\eta_0 \geqq s_1 \cdot \mu$. Let $\eta_1 = m(T^{**} \pi_0)$, $\pi_1 = T^{**} \pi_0 - \eta_1$; then $\eta_0 + T\eta_0 + \eta_1 \geqq s_2 \cdot \mu$. In general, given π_n let $\eta_{n+1} = m(T^{**} \pi_n)$, $\pi_{n+1} = T^{**} \pi_n - \eta_{n+1}$.

Set $\eta=\sum_0^\infty \eta_i$, then for each n, $\sum_0^{n-1} T^i\eta \geqq \sum_0^{n-1} T^i\eta_0+\sum_0^{n-2} T^i\eta_1+\cdots+\eta_{n-1}\geqq s_n\cdot\mu$. It follows that $\eta(X)=\gamma$. Set $\delta=d\eta/d\mu$.

3. Ratio Ergodic Theorem

We consider in this section *sub-Markovian operators:* positive linear contractions of L_1. Recall some known facts: The space X decomposes into the *conservative* part C and the *dissipative* part D:

If $f\in L_1, f>0$, then $\sum_0^\infty T^i f=\infty$ on C, $<\infty$ on D. Under T no mass escapes from C to D; hence

$$T^n(\chi_C f)\leqq \chi_C T^n f \quad f\in L_1^+, \quad n=0,1,\ldots. \tag{3.1}$$

If T is *conservative*, i.e. $X=C$, then the subsets B of C such that $T^* 1_B=1_B$, called *invariant for* T, form a σ-algebra $\mathfrak{I}$. The Chacon-Ornstein theorem (see e.g. [7], p. 41 or [12]) then asserts that for any two functions $g,h\in L_1^+$, the ratio

$$\frac{\sum_0^{n-1} T^i g}{\sum_0^{n-1} T^i h}=R_n(g,h) \tag{3.2}$$

converges on the set $\left\{\sum_0^\infty T^i h>0\right\}$ to a finite limit $R(g,h)$, measurable with respect to $\mathfrak{I}$. The mapping $g\to R(g,h)\cdot h$ is a Markovian operator on L_1, and

$$\int_B R(g,h)\cdot h\,d\mu=\int_B g\,d\mu \qquad \text{for each } B\in\mathfrak{I}. \tag{3.3}$$

Theorem 3.1. *Let T be a conservative operator and let s_n be superadditive sums with an exact dominant δ. Then*

$$\lim_{n\to\infty}\frac{s_n}{\sum_0^{n-1} T^i\delta}=1 \quad \text{a.e. on} \quad \left\{\sum_0^\infty T^i\delta>0\right\}.$$

Lemma 1. *Let $a_k=\frac{1}{k}s_k$, then*

$$\sum_{i=0}^{n-1} T^i a_k\leqq s_{n+k} \quad \textit{for all } n\geqq 1. \tag{3.4}$$

Proof. We have

$$\sum_{i=0}^{n-1} T^i a_k = \frac{1}{k}\sum_{i=0}^{n-1} T^i s_k$$
$$\leqq \frac{1}{k}\sum_{i=0}^{n-1}(s_{k+i}-s_i) \leqq \frac{1}{k}\sum_{i=0}^{n+k-1} s_i - \frac{1}{k}\sum_{i=0}^{n-1} s_i \leqq s_{n+k},$$

where the first inequality follows from (1.3) and the last inequality from the fact that (s_n) is a non-decreasing sequence.

Proof of the Theorem. The right-hand side of (3.3) remains unchanged when g is replaced by $T^i g$; hence $R(T^i g, h) = R(g, h)$ on the set $\{h>0\}$. More generally, $R(g,h) = R(g', h')$ whenever $g' = \sum_{i=0}^{\infty} \alpha_i T^i g$, $h' = \sum_{i=0}^{\infty} \beta_i T^i h$, where α_i, β_i are constants $\geqq 0$, with $\sum \alpha_i = \sum \beta_i = 1$. It follows that

$$R(g,h) = \lim_{n\to\infty} \frac{\sum_{i=0}^{n-k} T^i g}{\sum_{i=0}^{n-1} T^i h}, \tag{3.5}$$

whenever k is a fixed integer, positive or negative. Now let $\delta' = \sum_{i=0}^{\infty} \alpha_i T^i \delta$ with $\alpha_i > 0$, $\sum \alpha_i = 1$. For each k we have by (3.4) and (2.1)

$$\sum_{i=0}^{n-1-k} T^i a_k \leqq s_n \leqq \sum_{i=0}^{n-1} T^i \delta. \tag{3.6}$$

Let $\underline{R}$ and $\bar{R}$ be, respectively, lim inf and lim sup of $s_n \Big/ \sum_{i=0}^{n-1} T^i \delta$, then by (3.5) and (3.6)

$$R(a_k, \delta') \leqq \underline{R} \leqq \bar{R} \leqq R(\delta, \delta').$$

Finally, again by (3.3),

$$\int a_k d\mu = \int R(a_k, \delta')\, \delta' d\mu \leqq \int \underline{R}\, \delta' d\mu \leqq \int \bar{R}\, \delta' d\mu \leqq \int \delta' d\mu.$$

But $\lim_k \int a_k d\mu = \gamma = \int \delta' d\mu$. Hence $\underline{R} = \bar{R} = 1$ a.e. on the set

$$\{\delta' > 0\} = \left\{\sum_0^{\infty} T^i \delta > 0\right\}. \quad \square$$

Given a set $A \in \mathfrak{F}$ let T_A be the operator defined by $T_A f = \chi_A \cdot T(\chi_A \cdot f)$, $f \in L_1$. Note that T_C is conservative; let $\mathfrak{I}$ be the σ-algebra of its invariant sets.

Theorem 3.2. *Let s_n and s_n' be superadditive sums with respect to the same sub-Markovian operator T. Then* $\lim s_n/s_n'$ *exists (and is finite) a.e. on $C \cap E$ where E*

$=\{\sup_n s'_n>0\}$. *If in addition either*

a) T *is Markovian or*

b) *on* $D\cap E$ s_n *is of the form* $\sum_0^{n-1} T^i\delta$ *for some* $\delta\in L_1^+$, *then* $\lim(s_n/s'_n)$ *also exists on* $D\cap E$.

Proof. $\lim(s_n/s'_n)=(\lim\uparrow s_n)/(\lim\uparrow s'_n)$ clearly exists on $D\cap E$ and is finite, if $\lim\uparrow s_n<\infty$ on D, which holds if T is Markovian by Theorem 2.1.

It now suffices to consider C. By (3.1) and (1.3′)

$$T^k(\chi_C\cdot s_n)\leqq\chi_C(T^k s_n)\leqq\chi_C\left(\sum_{i=k}^{k+n-1} f_i\right), \tag{3.7}$$

hence the sums $\chi_C s_n$ are superadditive with respect to T_C (and T). Therefore we may assume without loss of generality that $X=C$. Now apply Theorem 3.1.

Corollary 1. *Let* $X=C$ *and suppose* $\mathfrak{T}$ *trivial. If* s_n *and* s'_n *are superadditive sums with time-constants* γ *and* γ', $\gamma'>0$, *then* $s_n/s'_n\to\gamma/\gamma'$ *a.e. on* X.

Proof. $\gamma'>0$ implies that $E=\{\sup_n s'_n>0\}=X$. Let δ and δ' be exact dominants for s_n and s'_n. (3.3) implies that $R(\delta,\delta')=\int\delta\,d\mu/\int\delta'\,d\mu=\gamma/\gamma'$. □

We note that the argument following formula (1.4) shows that

$$\lim\frac{1}{n}\int_B s_n\,d\mu=\sup_n\frac{1}{n}\int_B s_n\,d\mu=\sigma(B), \tag{3.8}$$

exists for each $B\in\mathfrak{T}$. The proof of Theorem 2.1 applied to T_C shows that if δ_C is an exact dominant for $\chi_C\cdot s_n$, then

$$\sigma(B)=\int_B\delta_C\,d\mu\qquad B\in\mathfrak{T}. \tag{3.9}$$

The identification of the limit of s_n/s'_n in the general non-ergodic case becomes however more transparent if we assume $\mu(X)=1$, which by the following standard argument does not involve any essential loss of generality: Since μ is σ-finite, $L_1(X,\mathfrak{F},\mu)$ is isomorphic to $L_1(X,\mathfrak{F},\tilde\mu)$, where $\tilde\mu(X)=1$. Under this isomorphism T is mapped on $\tilde T$ as follows. If $\tilde\mu=r\cdot\mu$, $\tilde Tf=\frac{1}{r}T(f\cdot r), f\in L_1(\tilde\mu)$. The identification of the limit in terms of $\tilde T$ on $L_1(\tilde\mu)$ gives one in terms of T on $L_1(\mu)$.

We now identify the limit, and at the same time slightly extend our results so as to connect them with [3]. Call s_n *extended superadditive* if it satisfies (1.3), but not necessarily the boundedness assumption (1.4). A sequence (f_n) in L_1^+ is called *admissible* iff for each g in L_1^+ and for each integer i, $g\leqq f_i$ implies $Tg\leqq f_{i+1}$. Chacon's theorem [3] in the case of positive operators asserts that s_n/s'_n converges a.e. if s_n is additive and s'_n is a partial sum of an admissible sequence. It is easy to see that a partial sum of an admissible sequence is extended super-additive. Therefore this theorem is a particular case of the following:

Theorem 3.3. *Suppose that T is a sub-Markovian operator, s_n is super-additive, s'_n extended super-additive. Let*

$$s = \sup_n \frac{1}{n} E^{\mathfrak{I}}(s_n \chi_C), \qquad s' = \sup_n \frac{1}{n} E^{\mathfrak{I}}(s'_n \chi_C) \qquad \text{on } C.$$

The following limits exist a.e. on C:

$$\lim_n \frac{1}{n} E^{\mathfrak{I}}(s_n \chi_C) = s < \infty, \tag{3.10}$$

$$\lim_n \frac{1}{n} E^{\mathfrak{I}}(s'_n \chi_C) = s' \leqq \infty. \tag{3.11}$$

Let $E = \{\sup_n s'_n > 0\}$; then $\lim(s_n/s'_n) = s/s'$ a.e. on $C \cap E$. $\lim(s_n/s'_n) = \lim \uparrow s_n / \lim \uparrow s'_n < \infty$ exists also a.e. on $D \cap E$ if either

a) *T is Markovian, or*
b) *s_n is additive on D.*

Proof. The convergence on $D \cap E$ is proved as in Theorem 3.2. Now consider C. The restriction of an (extended) super-additive sum to an invariant subset of C, in particular to C itself, is again (extended) super-additive; therefore we may and do assume in the proof that $X = C$. Now $T^* 1_B = 1_B$ for each B in $\mathfrak{I}$ implies $E^{\mathfrak{I}}(Tf) = E^{\mathfrak{I}} f$; therefore (1.3) implies (3.10) (cf. [8], p. 244). Similarly one proves (3.11), and also the analogue for s'_n of (3.8), with $\sigma'(B)$ now $\leqq \infty$ for each $B \in \mathfrak{I}$. By Fatou's lemma $\int_B s \, d\mu \leqq \sigma(B)$ for $B \subset \mathfrak{I}$; the inverse inequality also holds by (3.8). Thus $\sigma = s\mu$ on $\mathfrak{I}$, and similarly $\sigma' = s'\mu$ on $\mathfrak{I}$. Let $F_i = \{i-1 \leqq s' < i\}$ for $i = 1, 2, \ldots, F = \bigcup F_i$, $G = X - F$. If B is invariant and contained in G, then $\sigma'(B) = \infty$. The application of Theorem 3.2 to the Markovian conservative operators T_{F_i} shows that $\lim(s_n/s'_n)$ exists a.e. on $F_i \cap E$, hence on $F \cap E$. To identify this limit, we note that if δ is an exact dominant for s_n, then by (3.9), $s = E^{\mathfrak{I}} \delta$. Proceeding similarly with T_{F_i}, s'_n, σ' and applying Theorem 3.1, one identifies $\lim(s_n/s'_n)$ on $F \cap E$ as s/s'. It remains to consider $G \cap E$.

Let $a'_k = \frac{1}{k} s'_k$; by Theorem 3.1, the primed version of (3.6), and (3.5), one has

$$\lim \frac{s_n}{s'_n} \leqq \lim_n \frac{\sum_0^{n-1} T^i \delta}{\sum_0^{n-1-k} T^i a'_k} = \frac{E^{\mathfrak{I}} \delta}{E^{\mathfrak{I}} a'_k} \to 0 \qquad \text{as } k \to \infty$$

on $G \cap E$. Hence on this set $\lim(s_n/s'_n) = 0$. □

We note that s_n/s'_n need not converge on $D \cap E$ if s_n is not additive and T is not Markovian. To see this, it suffices to consider the case $T = 0$.

Recently Y. Derriennic [5] gave a nice direct proof of Kingman's theorem, not based on the decomposition theorem (i.e., in our context, Theorem 2.1). It is similarly possible to give a direct proof of the ratio Theorem 3.3. We only state the essential proposition which we have established in the course of such a proof. The details are not given, because the proof based on Theorem 2.1 seems shorter. Assume $\mu(X)=l$. Let s and s' be as in Theorem 3.3.

Proposition 1. *Let $E \subset C$ and assume that* $\limsup (s_n - s'_n) \geqq 0$ *a.e. on E. Let $I(E)$ be the minimal invariant set in $\mathfrak{I}$ containing E. Then*

$$\int_{I(E)} s\,d\mu \geqq \int_{I(E)} s'\,d\mu.$$

Proposition 1 may be considered as the superadditive version of Brunel's lemma [2], in the form given to it in [1].

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