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Probability Theory and Related Fields

Local Limits and Harmonic Functions for Nonisotropic Random Walks on Free Groups

Peter Gerl[1] and Wolfgang Woess[2]

[1] Institut für Mathematik, Universität Salzburg, Petersbrunnstraße 19, A-5020 Salzburg, Austria
[2] Institut für Mathematik und Angewandte Geometrie, Montanuniversität, A-8700 Leoben, Austria

Summary. Nearest neighbour random walks on the homogeneous tree representing a free group with s generators ($2 \leqq s < \infty$) are investigated. By use of generating functions and their analytic properties a local limit theorem is derived. A study of the harmonic functions corresponding to the random walk leads to properties that characterize the r-harmonic function connected with the local limits.

1. Introduction

Let $\mathbb{F}_s = \langle x_1, \ldots, x_s | \rangle$ denote the free group with unit element e and free generators $x_1, \ldots, x_s$ ($2 \leqq s < \infty$). The Cayley-graph of $\mathbb{F}_s$ is a homogeneous tree of degree $2s$ with vertex set $\mathbb{F}_s$ and edges $[x, x x_i^{\pm 1}]$, where $x \in \mathbb{F}_s$ and x_i is one of the free generators. On this homogeneous tree we shall consider non-isotropic nearest neighbour random walks, i.e. (right) random walks ($X_n, n = 0, 1, 2, \ldots$) generated by a probability measure p on $\mathbb{F}_s$, such that transitions with positive probability only occur from an element to itself or to one of its $2s$ neighbours. This means that p is supported by the set

$$S = \{e, x_i, x_i^{-1} \mid i = 1, \ldots, s\},$$

$$p(e) = p_0, \quad p(x_i) = p_i, \quad p(x_i^{-1}) = p_{-i} \quad \text{are all positive,}$$

$$\sum_{i=-s}^{s} p_i = 1 \quad \text{and} \quad p(x) = 0 \quad \text{if } x \in \mathbb{F}_s - S.$$

The transition probabilities are given by

$$\Pr[X_{n+1} = y \mid X_n = x] = p(x^{-1} y),$$

the n-step transition probabilities are

$$\Pr[X_n = y \mid X_0 = x] = p^{(n)}(x^{-1} y),$$

where $p^{(n)}$ denotes the n'th convolution power of p.

We will derive asymptotic expressions for $p^{(n)}(x)$, $x\in\mathbb{F}_s$, as $n\to\infty$ ("local limit theorem") and exhibit various properties of the r-harmonic function connected with these local limits.

Different aspects of random walks on free groups have been studied by various authors, e.g. Kesten [19], Dynkin and Malyutov [10], Cartier [6], [7], Derriennic [8], Dunau [9], Arnaud [3], Letac [20]. Local limit theorems in the case when p is *radial* and the random walk is *isotropic* (i.e. p is a probability measure with the property that $p(x)$ depends only on the distance between x and the unit element in the tree representing $\mathbb{F}_s$) have been obtained by Gerl [13] (for p on $\mathbb{F}_2$ being supported by the generators and their inverses) and under very general assumptions by Sawyer [23] and Picardello [22]. There, harmonic (spherical) functions appear in the framework of harmonic analysis, which is the apparatus used to deal with these problems: compare Figà-Talamanca and Picardello [11] and the references given there. Nonisotropic random walks on $\mathbb{F}_2$ have been studied by Gerl [15], [16]. In a preliminary version of Sect. 2 of the present paper, Woess [27] discusses aspects of random walks on free groups with infinitely many generators. Harmonic analysis and representations of free groups connected with nonradial probabilities will be studied in the forthcoming Ph. D. Thesis by Tim Steger, Washington University, St. Louis. Finally, after we had submitted this paper, the article of Aomoto [2] was brought to our attention, where nonradial harmonic analysis on free groups is treated from the viewpoint of algebraic geometry.

Our tools will be mainly generating functions and their analytic properties. These methods permit a rather elementary approach to the problems in question and seem to be very fruitful throughout the study of random walks on trees (compare e.g. Gerl and Woess [17]).

In Sect. 2 we give some general properties of the random walk and determine the asymptotic behaviour of the return probabilities $p^{(n)}(e)$, $n\to\infty$. Subsequently, this result is extended to a local limit theorem in Sect. 3. In Sect. 4, we obtain a ratio limit theorem for $p^{(n)}(x)/p^{(n)}(e)$, $n\to\infty$, and our approach yields some very special properties of the r-harmonic function occuring in the ratio limit, where r is the inverse of the "spectral radius" of the random walk. Finally, Sect. 5 is dedicated to some special cases.

2. Asymptotic Evaluation of the Return Probabilities

To obtain informations on $p^{(n)}(x)$, $n\to\infty$, we shall study the generating functions

$$G_x(z)=\sum_{n=0}^{\infty} p^{(n)}(x)\,z^n, \qquad z\in\mathbb{C} \text{ and } x\in\mathbb{F}_s \;\; (p^{(0)}(x)=\delta_{e,x}). \tag{2.1}$$

In the present section we are particularly concerned with $x=e$, in this case we omit the index: $G(z)=G_e(z)$. From now on we shall always write x_{-i} for x_i^{-1}, if x_i is a free generator (resp. its inverse). Besides $p^{(n)}(x)$ we need the following "taboo probabilities" for $i\in I=\{\pm 1, \ldots, \pm s\}$:

$$f_i^{(n)}=\Pr[X_n=e;\; X_1=x_i \text{ and } X_k\neq e \quad \text{for } k=1,\ldots,n-1 \mid X_0=e]. \tag{2.2}$$

That is, $f_i^{(n)}$ is the probability to return to e at the n'th step of the random walk for the first time after starting at e and taking the first step from e to x_i. The corresponding generating functions are

$$F_i(z) = \sum_{n=0}^{\infty} f_i^{(n)} z^n, \qquad z \in \mathbb{C} \quad (f_i^{(0)} = 0). \tag{2.3}$$

First of all, we make the following general observations:

Lemma 1. *a) The radius of convergence r of the Taylor series $G_x(z)$ is independent of $x \in \mathbb{F}_s$.*

b) For all $x \in \mathbb{F}_s$, $G_x(z)$ is convergent at $z = r$.

c) $\lim_{n \to \infty} \frac{p^{(n+1)}(x)}{p^{(n)}(x)} = \frac{1}{r}$.

Proof. It is obvious that $p^{(n)}(x) > 0 \ \forall n \geqq k$ if x has the reduced representation

$$x = x_{i_1} \dots x_{i_k}, \quad \text{where} \quad i_j \in I \quad \text{and} \quad i_j + i_{j+1} \neq 0. \tag{2.4}$$

In Markov chain-terminology this means that the random walk is irreducible and aperiodic. Now [26] and [14] yield a), resp. c). Again by [26], convergence or divergence of $G_x(r)$ is independent of $x \in \mathbb{F}_s$. Divergence means that the random walk is r-recurrent which is impossible as $\mathbb{F}_s$ – being nonamenable – is not a recurrent group ([18], p. 85). Thus b) holds. (It is also possible to derive b) from the proof of Proposition 3 below.) □

In the next Lemma we derive equations describing how $G(z)$ and the functions $F_i(z)$, $i \in I$ are related. The tree structure of $\mathbb{F}_s$ stands behind the arguments of the proof.

Lemma 2. *a)* $G(z) = \dfrac{1}{1 - p_0 z - \sum_{j \in I} F_j(z)}$

b) $F_i(z) = \dfrac{p_i p_{-i} z^2}{1 - p_0 z - \sum_{j \in I, j \neq -i} F_j(z)}, \qquad i \in I.$

Proof. a) Let

$$f^{(n)} = \Pr[X_n = e;\ X_k \neq e \quad \text{for } k = 1, \dots, n-1 \mid X_0 = e], \quad f^{(0)} = 0$$

denote the probability of first returning to e at the n'th step and let

$$F(z) = \sum_{n=0}^{\infty} f^{(n)} z^n, \qquad z \in \mathbb{C}.$$

Then the identity

$$p^{(n)}(e) = \sum_{k=0}^{n} f^{(k)} p^{(n-k)}(e) \qquad \text{for } n \geqq 1$$

yields

$$G(z) = \frac{1}{1 - F(z)}.$$

On the other hand, $f^{(1)}=p_0$ and $f^{(n)}=\sum_{j\in I} f_j^{(n)}$ for $n>1$, i.e.

$$F(z)=p_0\,z+\sum_{j\in I} F_j(z).$$

b) Denote for $i\in I$

$$q_i^{(n)}=\Pr\left[X_n=e;\ X_k\neq x_{-i}\quad \text{for } k=1,\ldots,n-1\,|\,X_0=e\right],\qquad q_i^{(0)}=1.$$

Then, similar to a)

$$\sum_{n=0}^{\infty} q_i^{(n)} z^n=\frac{1}{1-p_0\,z-\sum\limits_{j\in I,\, j\neq -i} F_j(z)},$$

and observing $f_i^{(n)}=p_i\,q_i^{(n-2)}\,p_{-i}$, we obtain the desired formula. □

As corollaries of Lemma 2 we get the following two Propositions which are essential for all that follows:

Proposition 1.

$$F_i(z)=\frac{\sqrt{1+4p_i\,p_{-i}\,z^2\,G(z)^2}-1}{2G(z)}\qquad \textit{for } i\in I.$$

Proof. From Lemma 2 we see that $F_i(z)=p_i\,p_{-i}\,z^2\Big/\left(\frac{1}{G(z)}+F_{-i}(z)\right)$. Thus $F_i(z)\,F_{-i}(z)+\frac{1}{G(z)}F_i(z)-p_i\,p_{-i}\,z^2=0$, and exchanging i and $-i$ we find that $F_i(z)=F_{-i}(z)$ and that $F_i(z)$ satisfies the quadratic equation

$$G(z)\,F_i(z)^2+F_i(z)-p_i\,p_{-i}\,z^2\,G(z)=0.$$

Among the two solutions we have to take the branch where the root has positive sign, as $G(0)=1$ and $F_i(0)=0$. □

Proposition 2. $G(z)=P(zG(z))$, *where*

$$P(t)=1+p_0\,t+\sum_{i=1}^{s}\left(\sqrt{1+4p_i\,p_{-i}\,t^2}-1\right)\qquad (t\in\mathbb{C}).$$

Proof. By Lemma 2 and Proposition 1,

$$G(z)=\frac{1}{1-\sum\limits_{i\in I}\left(\sqrt{1+4p_i\,p_{-i}\,z^2\,G(z)^2}-1\right)/2G(z)}.\qquad \square$$

So far we have only assured the existence of the common radius of convergence r of $G_x(z)$, $x\in\mathbb{F}_s$. Our next step is to determine r by the solution of a sufficiently simple equation.

Proposition 3. *Let θ denote the unique positive real solution of*

$$tP'(t)=P(t).$$

Then the radius of convergence r of $G(z)$ is given by

$$r=1/P'(\theta).$$

Proof. As $G(z)$ is a Taylor series with positive coefficients, r is a singularity of $G(z)$ and no pole by Lemma 1b). So what we are looking for is the smallest positive singularity of $G(z)$.

For real $t, 0\leqq t<\infty$, the curve $y=P(t)$ is strictly increasing and convex, $P(0)=1$, $P'(0)=p_0$, and as $t\to\infty$, $P(t)$ approaches the asymptote $y=dt-(s-1)$, where $d=p_0+\sum_{i=1}^{s} 2\sqrt{p_i p_{-i}}$. Note that $d\leqq 1$ and that equality holds if and only if the probability measure is symmetric.

For real $z, 0<z\leqq r, t=zG(z)$ is of course also positive real, and $G(z)$ can be seen in the real (t,y)-plane as the y-coordinate of the point where $y=P(t)$ and the line $y=\frac{1}{z}t$ intersect. For $0<z\leqq 1/d$, there is exactly one point of intersection, for larger z there are two (by continuity of $G(z)$ we have to take the left one) until we reach the line through the origin that is tangent to $y=P(t)$, which is the case for $t=\theta$. For z still larger, $\frac{1}{z}t=P(t)$ has no real solution t. We have

$$Q(\theta)=1, \quad \text{where} \quad Q(t)=\sum_{i=1}^{s}\left(1-1\Big/\sqrt{1+4p_i p_{-i}t^2}\right).$$

Because of the shape of $y=P(t)$, θ is uniquely determined.

Writing

$$\mathscr{F}(z,w)=P(zw)-w \tag{2.5}$$

we have $\mathscr{F}(z,G(z))\equiv 0$. Let z be real, $0<z<1/P'(\theta)$. Then $\mathscr{F}_w(z,G(z))=zP'(zG(z))-1\neq 0$, as otherwise $zG(z)P'(zG(z))=G(z)=P(zG(z))$, i.e. $zG(z)=\theta$ contradicting $z<1/P'(\theta)$. It follows from the theorem on implicit functions that $G(z)$ is analytic. On the other hand, $\mathscr{F}_w(z,G(z))=0$ for $z=1/P'(\theta)$ which is therefore the required singularity. □

Remark. The number $1/r$ is often called the *spectral radius* of the random walk, being the smallest real number c admitting a *positive* solution σ of the convolution equation

$$p*\sigma=c\sigma,$$

where σ is a real-valued function on $\mathbb{F}_s$ (compare [18]). The result of Proposition 3 can also be written as

$$\frac{1}{r}=\min\left\{p_0+t+\sum_{i=1}^{s}(\sqrt{t^2+4p_i p_{-i}}-t)\,|\,t>0\right\}. \tag{2.6}$$

This formula is similar to the formula given by Akemann and Ostrand [1] for the norm of p as a convolution operator on $l^2(\mathbb{F}_s)$: If p is symmetric or if $p_0=0$

then we have

$$\|p\| = \min\left\{p_0 + t + \frac{1}{2}\sum_{i=\pm 1}^{\pm s}\left(\sqrt{t^2+4p_i^2} - t\right) \,\middle|\, t>0\right\} \tag{2.7}$$

(in [1] the corresponding formula is given for the case $p_0=0$). In particular, (2.6) and (2.7) coincide if p is symmetric as it has to be [5], [19].

We are now ready to obtain the main result of this section:

Theorem 1. *a) Near $z=r$, $G(z)$ has an expansion by powers of $\sqrt{r-z}$ (Puiseux series),*

$$G(z) = a_0 - b_0\sqrt{r-z} + c_0(r-z) - d_0\sqrt{r-z}^{\,3} + \ldots,$$

where

$$a_0 = P(\theta) \quad \text{and} \quad b_0 = \sqrt{\frac{2P(\theta)\,P'(\theta)^3}{P''(\theta)}}.$$

b) $p^{(n)}(e) = \dfrac{b_0\sqrt{r}}{2\sqrt{\pi}}\, r^{-n} n^{-3/2} + O(r^{-n} n^{-5/2})$.

Proof. Consider the space $l^2(\mathbb{F}_s, \psi)$ with the weighted inner product $\langle f, g\rangle_\psi = \sum f(x)\overline{g(x)}\,\psi(x)$, where $\psi(x_{i_1}\ldots x_{i_k}) = \prod_{j=1}^{k}(p_{-i_j}/p_{i_j})$. Using the fact that $p\cdot\sqrt{\psi}$ is symmetric on $\mathbb{F}_s$, it is easy to see that p acts on this space as a self-adjoint convolution operator and has as such, in particular, a real spectrum. Hence, the resolvent $(\lambda\cdot\delta - p)^{-1}(x) = G_x(1/\lambda)/\lambda$ is analytic for λ in $\mathbb{C}-\mathbb{R}$, and $G(z)$ has only real singularities. As $p_0>0$, $z=-r$ cannot be a singularity by the shape of $y=P(t)$. Therefore $z=r$ is the only singularity on the circle of convergence $|z|=r$. Furthermore, for the function $\mathscr{F}(z,w)$ given in (2.5) we have

$$\mathscr{F}_z(r, G(r)) = P'(\theta)\,P(\theta) \neq 0 \quad \text{and}$$
$$\mathscr{F}_{ww}(r, G(r)) = r^2 P''(\theta) \neq 0.$$

Now the method of Darboux implies both a) and b) (see Bender [4], Ths. 4 and 5 and Szegö [25], Th. 8.4). □

Remark. The $n^{-5/2}$ in the remainder term is better than the usual n^{-2} given in [13], [15], [22]. The exponent $-5/2$ is obtained by a thorough use of Darboux's theorem in [25].

3. The Local Limit Theorem

Before we can give asymptotic expressions for all the sequences $p^{(n)}(x)$, $n\to\infty$ $(x\in\mathbb{F}_s)$, we have to introduce further "taboo probabilities" and their generating functions:

For $i \in I$ let

$$u_i^{(n)} = \Pr[X_n = x_i;\ X_k \neq x_i \quad \text{for } k=1, \ldots, n-1 \,|\, X_0 = e] \tag{3.1}$$

and

$$U_i(z) = \sum_{n=0}^{\infty} u_i^{(n)} z^n, \qquad z \in \mathbb{C} \ (u_i^{(0)} = 0). \tag{3.2}$$

$u_i^{(n)}$ is the probability of arriving at x_i at the n'th step for the first time after starting at e.

Lemma 3. *a)* $U_i(z) = F_{-i}(z)/p_{-i} z$

b) If $x \in \mathbb{F}_s$ *has the reduced representation* (2.4), $x = x_{i_1} \ldots x_{i_k}$ $(x \neq e)$, *then*

$$G_x(z) = U_{i_1}(z) \ldots U_{i_k}(z)\, G(z).$$

Proof. a) is clear from $f_{-i}^{(n)} = p_{-i} u_i^{(n-1)}$.

b) (Compare [6]) Let $y = x_{i_2} \ldots x_{i_k}$. By the tree structure of $\mathbb{F}_s$, the random walk must pass through x_{i_1} on the way from e to x, and thus

$$p^{(n)}(x) = \sum_{k=0}^{n} u_{i_1}^{(k)} p^{(n-k)}(y), \qquad n = 0, 1, 2, \ldots, \text{ and}$$

$$G_x(z) = U_{i_1}(z)\, G_y(z).$$

The formula is now obtained inductively. □

Because of Lemma 3b) we shall study analytic properties of $U_i(z)$, $i \in I$, before we proceed to the local limit theorem.

Proposition 4. *a) The radius of convergence of* $U_i(z)$ *is* r, *and* $z = r$ *is the only singularity on the circle of convergence.*

b) Near $z = r$, $U_i(z)$ *has an expansion by powers of* $\sqrt{r-z}$:

$$U_i(z) = a_i - b_i \sqrt{r-z} + c_i(r-z) - d_i \sqrt{r-z}^{\,3} + \ldots,$$

where

$$a_i = \frac{\sqrt{1 + 4p_i p_{-i} \theta^2} - 1}{2p_{-i}\theta} \quad \text{and} \quad b_i = \frac{b_0}{a_0} \frac{a_i}{\sqrt{1 + 4p_i p_{-i} \theta^2}}.$$

Proof. a) From $u_i^{(n)} \leqq p^{(n)}(x_i)$ and Lemma 1a) it follows that the radius of convergence of $U_i(z)$ cannot be smaller than r. We have $G_{x_i}(z) = U_i(z)\, G(z)$. The formula for $U_i(z)$ (Lemma 3a) and Proposition 1) and the arguments of the proof of Theorem 1 show that $z = r$ is the only singularity of $G_{x_i}(z)$ on its circle of convergence. This means that for $|z| = r$, $z \neq r$ both numerator and denominator of $G_{x_i}(z)/G(z) = U_i(z)$ are analytic implying that a singularity of $U_i(z)$ could be only a pole which is impossible because of

$$|U_i(z)| \leqq U_i(r) \leqq G_{x_i}(r) < \infty \qquad \text{(Lemma 1b))}.$$

Thus only $z=r$ can be a singularity of $U_i(z)$ on the circle $|z|=r$, and that it is in fact one will follow from b).

b) By Lemma 3a) and Proposition 1 we have $U_i(z)=f_i(zG(z))$, where

$$f_i(t)=\frac{\sqrt{1+4p_i p_{-i} t^2}-1}{2p_{-i} t}, \qquad t\in\mathbb{C}.$$

$f_i(t)$ is analytic near $t=\theta$, $\theta=rG(r)$, and

$$f_i'(\theta)=\frac{f_i(\theta)}{\theta\sqrt{1+4p_i p_{-i}\theta^2}}\neq 0.$$

Therefore $f_i(t)$ carries the simple branching point $z=r$ of $G(z)$, resp. $zG(z)$, to a simple branching point of $U_i(z)$, and $U_i(z)$ can also be expressed as a Puiseux series,

$$U_i(z)=a_i-b_i\sqrt{r-z}+c_i(r-z)-\ldots.$$

We get $a_i=U_i(r)$ and

$$b_i=\lim_{z\to r-}\frac{f_i(\theta)-f_i(zG(z))}{\sqrt{r-z}}=rb_0 f_i'(\theta),$$

yielding the proposed formulas for a_i and b_i. □

Now we can easily derive the following result:

Theorem 2. *Let $x\in\mathbb{F}_s$ have the reduced representation $x=x_{i_1}\ldots x_{i_k}$. Then*

$$p^{(n)}(x)=\frac{b_x\sqrt{r}}{2\sqrt{\pi}}r^{-n}n^{-3/2}+O_x(r^{-n}n^{-5/2}),$$

where

$$b_x=a_{i_1}\ldots a_{i_k}\left(b_0+a_0\sum_{j=1}^{k} b_{i_j}/a_{i_j}\right).$$

Proof. By Lemma 3b), Theorem 1a) and Proposition 4 we have near $z=r$ (which is the only singularity of $G_x(z)$ for $|z|=r$)

$$\begin{aligned}G_x(z)&=(a_0-b_0\sqrt{r-z}+c_0(r-z)-d_0\sqrt{r-z}^3+\ldots)\\ &\quad\cdot\prod_{j=1}^{k}(a_{i_j}-b_{i_j}\sqrt{r-z}+c_{i_j}(r-z)-d_{i_j}\sqrt{r-z}^3+\ldots)\\ &=a_x-b_x\sqrt{r-z}+c_x(r-z)-d_x\sqrt{r-z}^3+\ldots,\end{aligned}$$

where

$$a_x=a_0 a_{i_1}\ldots a_{i_k}$$

and

$$b_x=b_0 a_{i_1}\ldots a_{i_k}+a_0 b_{i_1} a_{i_2}\ldots a_{i_k}+\ldots+a_0 a_{i_1}\ldots a_{i_{k-1}} b_{i_k}.$$

Again, the method of Darboux yields the result. □

Remark. The remainder term $O_x(r^{-n}n^{-5/2})$ can be majorized in the following way:

$$|O_x(r^{-n}n^{-5/2})| \leqq (a_x Q(k) + \varepsilon_x/n)\, r^{-n}n^{-5/2},$$

if x has the reduced representation $x = x_{i_1} \dots x_{i_k}$, where $Q(k)$ is a polynomial of degree 3 in k with coefficients not depending on x and ε_x is a constant depending on x. This can be proved by a more careful use of Darboux's theorem (see [25]), which yields

$$\left| p^{(n)}(x) - \frac{b_x \sqrt{r}}{2\sqrt{\pi}}\, r^{-n} n^{-3/2} \right| = \left| \frac{d_x\, 3\sqrt{r}}{4\sqrt{\pi}} + \frac{\varepsilon_x}{n} \right| r^{-n} n^{-5/2}.$$

d_x is a sum of certain products of the numbers a_0, b_0, c_0, d_0 and $a_{i_j}, b_{i_j}, c_{i_j}, d_{i_j}$ ($j = 1, \dots, k$), and without explicitely calculating the coefficients c_i and d_i ($i \in I$) one can pass to the maxima of the absolute values of expressions like c_i/a_i, etc., to obtain

$$\left| \frac{d_x\, 3\sqrt{r}}{4\sqrt{\pi}} \right| \leqq a_x Q(k).$$

The leading term of $Q(k)$ is αk^3, where $\alpha = \dfrac{\sqrt{r}}{8\sqrt{\pi}} \left(\max\limits_{i \in I} \dfrac{b_i}{a_i} \right)^3$.

4. Ratio Limits and Harmonic Functions

The following ratio limit theorem is a simple consequence of Theorems 1 and 2, but it will lead to interesting properties concerning certain harmonic functions.

Corollary 1. *Let* $\sigma(x) = b_x/b_0$, *where* b_0 *and* b_x *are the numbers given in Theorems* 1 *and* 2. *Then*

a) $\lim\limits_{n\to\infty} \dfrac{p^{(n)}(x)}{p^{(n)}(e)} = \sigma(x)$ *for all* $x \in \mathbb{F}_s$,

b) σ *satisfies the convolution equation*

$$p * \sigma = \frac{1}{r}\, \sigma, \quad \textit{and} \quad \sigma(e) = 1.$$

Statement b) is equivalent with the observation that the function φ_r on $\mathbb{F}_s$, defined by

$$\varphi_r(x) = \sigma(x^{-1}) \tag{4.1}$$

is r-harmonic:

Definition. A complex-valued function φ on $\mathbb{F}_s$ is called *z-harmonic* ($z \in \mathbb{C}$) with respect to the given random walk, if

$$\varphi(x) = z \sum_y p(x^{-1}y)\, \varphi(y) \quad \text{and} \quad \varphi(e) = 1.$$

In other words:

$$\varphi(x) = z p_0 \varphi(x) + z \sum_{i \in I} p_i \varphi(x x_i), \tag{4.2}$$

and for convenience we have chosen the normalization $\varphi(e)=1$. Theorem 2, resp. Corollary 1 yield the following representation of φ_r:

Theorem 3. *If $x \in \mathbb{F}_s$ has the reduced representation $x = x_{i_1} \dots x_{i_k}$ $(x \neq e)$ then*

$$\varphi_r(x) = U_{-i_1}(r) \dots U_{-i_k}(r) \left\{ 1 + \sum_{j=1}^{k} (\varphi_r(x_{i_j}) - U_{-i_j}(r))/U_{-i_j}(r) \right\}.$$

Proof. We have $\varphi_r(x_i) = a_{-i} + \frac{a_0}{b_0} b_{-i}$, thus

$$\varphi_r(x) = b_{x^{-1}}/b_0 = a_{-i_1} \dots a_{-i_k} \left\{ 1 + \sum_{j=1}^{k} (\varphi_r(x_{i_j}) - a_{-i_j})/a_{-i_j} \right\},$$

and $a_i = U_i(r)$. $\square$

From the formulas for a_i and b_i of Proposition 4 we get an explicit form of φ_r:

$$\varphi_r(x) = \left(1 + \sum_{j=1}^{k} \frac{1}{\sqrt{1 + 4 p_{i_j} p_{-i_j} \theta^2}} \right) \prod_{j=1}^{k} \frac{\sqrt{1 + 4 p_{i_j} p_{-i_j} \theta^2} - 1}{2 p_{i_j} \theta}. \tag{4.3}$$

For real z, $0 < z \leqq r$, the positive z-harmonic functions have integral representations (see Cartier [6], Dynkin and Malyutov [10], Derriennic [8], Furstenberg [12]):

Let Ω be the set of all infinite reduced words

$$\omega = x_{j_1} x_{j_2} x_{j_3} \dots, \quad \text{where} \quad j_l \in I \quad \text{and} \quad j_l + j_{l+1} \neq 0 \quad \text{for } l = 1, 2, 3, \dots. \tag{4.4}$$

For $x \in \mathbb{F}_s$ with the reduced representation $x = x_{i_1} \dots x_{i_k}$ denote by E_x the set of all $\omega \in \Omega$ in whose representation (4.4) $x_{j_1} \dots x_{j_k} = x$. The family of sets E_x, $x \in \mathbb{F}_s$, is the basis of a topology making Ω a compact space.

If $x = x_{i_1} \dots x_{i_k} \in \mathbb{F}_s$ and $\omega = x_{j_1} x_{j_2} x_{j_3} \dots \in \Omega$ (reduced representations) then there is an index m, $0 \leqq m \leqq k$, such that $i_l = j_l$, $l = 1, \dots, m$ and (if $m < k$) $i_{l+1} \neq j_{l+1}$. Now let

$$K_z(x, \omega) = U_{i_1}(z)^{-1} \dots U_{i_m}(z)^{-1} U_{-i_{m+1}}(z) \dots U_{-i_k}(z). \tag{4.5}$$

Then for fixed z and ω the function

$$x \mapsto K_z(x, \omega)$$

is z-harmonic on $\mathbb{F}_s$. Following [6], Th. 2.1, we have:

Every positive z-harmonic function φ on $\mathbb{F}_s$ $(0 < z \leqq r)$ has a unique integral representation

$$\varphi(x) = \int_{\Omega} K_z(x, \omega) \, d\nu(\omega), \tag{4.6}$$

where ν is a probability measure on Ω.

To see this, we just remark that in the assumptions of Cartier [6], §2 we can assign value zp_i to each oriented edge $[x, xx_i]$ in the Cayley-graph of $\mathbb{F}_s$, where $x_i \in S$; the loops at each vertex, induced by our assumption $p_0 > 0$, leave Cartier's result unchanged. Thus Ω is the Martin boundary of the random walk (see [8]). See also [21] for a boundary representation of *all* z-harmonic functions in the radial case.

A probability measure ν on Ω is given by its values on the basis $\{E_x \mid x \in \mathbb{F}_s\}$ of the topology. We write

$$\nu_{i_1 \dots i_k} = \nu(E_x) \quad \text{if } x = x_{i_1} \dots x_{i_k} \text{ (reduced representation).}$$

Thus, if $x = x_{i_1} \dots x_{i_k}$ then

$$\begin{aligned}\varphi(x) = \sum_{l=0}^{k-1} & U_{i_1}(z)^{-1} \dots U_{i_l}(z)^{-1} U_{-i_{l+1}}(z) \dots U_{-i_k}(z)(\nu_{i_1 \dots i_l} - \nu_{i_1 \dots i_{l+1}}) \\ & + U_{i_1}(z)^{-1} \dots U_{i_k}(z)^{-1} \nu_{i_1 \dots i_k}\end{aligned} \tag{4.7}$$

if φ is a positive z-harmonic function and ν the corresponding probability measure on Ω (compare [10], where – as in [8] – only the case $z=1$ is treated).

Definition. We say that a probability measure ν on Ω is *multiplicative*, if there exist numbers $\mu_i > 0$, $i \in I$, such that for $k \geqq 2$

$$\nu_{i_1 \dots i_k} = \mu_{i_1} \dots \mu_{i_{k-1}} \nu_{i_k}.$$

Compare this definition with the one in [2], p. 311. We remark that

$$\nu_i = \sum_{j \in I, j \neq -i} \nu_{ij} = \sum_{j \in I, j \neq -i} \mu_i \nu_j = \mu_i (1 - \nu_{-i}),$$

therefore

$$\mu_i = \frac{\nu_i}{1 - \nu_{-i}} \quad \text{and} \quad \nu_i = \frac{\mu_i (1 - \mu_{-i})}{1 - \mu_i \mu_{-i}}. \tag{4.8}$$

An example of a multiplicative probability measure is the Poisson measure, i.e. the measure $\tilde{\nu}$ corresponding to the harmonic function having constant value 1 (which is 1-harmonic in our terminology): $\tilde{\nu}$ is the "hitting probability" of the random walk, $\tilde{\nu}(E_x) = \Pr[\lim_{n \to \infty} X_n \in E_x \mid X_0 = e]$. The multiplicative property is satisfied with constants $\tilde{\mu}_i = \Pr[\exists n: X_n = x_i \mid X_0 = e]$.

Theorem 4. *The probability measure ν on Ω corresponding to the r-harmonic function φ_r is multiplicative and for $i \in I$ we have*

$$\nu_i = \frac{\sqrt{1 + 4p_i p_{-i} \theta^2} - 1}{2\sqrt{1 + 4p_i p_{-i} \theta^2}} \quad \textit{and} \quad \mu_i = \frac{(\sqrt{1 + 4p_i p_{-i} \theta^2} - 1)^2}{4p_i p_{-i} \theta^2}$$

(θ is given by Proposition 3).

Proof. If we put $\mu_i = U_i(r) U_{-i}(r)$ then we obtain from (4.7) for $x = x_{i_1} \dots x_{i_k}$:

$$\begin{aligned}\varphi_r(x) = & U_{-i_1}(r) \dots U_{-i_k}(r) \\ & + \sum_{l=1}^{k} U_{i_1}(r)^{-1} \dots U_{i_{l-1}}(r)^{-1} U_{-i_l}(r) \dots U_{-i_k}(r)(\mu_{i_l}^{-1} - 1) \nu_{i_1 \dots i_l}\end{aligned}$$

and thus

$$\varphi_r(x) = U_{-i_1}(r) \dots U_{-i_k}(r) \left\{ 1 + \sum_{l=1}^{k} (\mu_{i_l}^{-1} - 1)\, \mu_{i_1}^{-1} \dots \mu_{i_{l-1}}^{-1} \nu_{i\mathbb{N}\dots i_l} \right\}. \tag{4.9}$$

Furthermore, also by (4.7),

$$\varphi_r(x_i) = U_{-i}(r) + (\mu_i^{-1} - 1)\, \nu_i\, U_{-i}(r),$$

and putting this into the formula of Theorem 3 we get

$$\varphi_r(x) = U_{-i_1}(r) \dots U_{-i_k}(r) \left\{ 1 + \sum_{l=1}^{k} (\mu_{i_l}^{-1} - 1)\, \nu_{i_l} \right\}. \tag{4.10}$$

The multiplicativity of ν now follows by induction on k, comparing (4.9) and (4.10). By Proposition 4, $\mu_i = a_i a_{-i}$, this and (4.8) yield the expressions for μ_i and ν_i given above. □

From Theorems 3 and 4 we see that the pair (φ_r, ν) has the following properties:

Corollary 2. *a) ν is a multiplicative probability measure.*

b) If $x = x_{i_1} \dots x_{i_k}$ is reduced and $(j_1, \dots, j_k)$ is a permutation of $(i_1, \dots, i_k)$ such that $y = x_{j_1} \dots x_{j_k}$ is also reduced, then

$$\varphi_r(x) = \varphi_r(y).$$

c) φ_r is determined by its values on the support S of p.

d) $p_{i_1} \dots p_{i_k} \varphi_r(x_{i_1} \dots x_{i_k}) = p_{-i_1} \dots p_{-i_k} \varphi_r(x_{-i_1} \dots x_{-i_k})$

e) ν is symmetric in the sense that $\nu_i = \nu_{-i}\ \forall i \in I$.

Remark. It can be proved by careful calculations that φ_r is the unique positive r-harmonic function on $\mathbb{F}_s$ having properties a) and b), and in the proof b) can be replaced by the weaker condition

$$\varphi_r(x_{i_1} \dots x_{i_k}) = \varphi_r(x_{i_k} \dots x_{i_1}),$$

which implies that φ_r is also r-harmonic on the left:

$$\varphi_r(x) = r \sum_y \varphi_r(y)\, p(y x^{-1}).$$

(This property is an immediate consequence of Corollary 1a).) The proof of this result and its extension to a whole class of z-harmonic functions $(0 < z \leq r)$ involves some lengthy calculations and is carried out in [28].

Compare the situation of Corollary 2 with the case of commutative groups: in the corresponding ratio limit theorem [24], it is a positive exponential which plays a similar role as the pair (φ_r, ν) on the free group.

5. Special Cases

In Proposition 2 we have described $G(z)$ as the solution of $G(z) = P(zG(z))$ without calculating it explicitely. This can be done easily in the following special case:

A) Assume that $p_i p_{-i}$ is independent of $i \in \{1, \dots, s\}$:

$$p_i p_{-i} = q^2, \quad i = 1, \dots, s. \tag{5.1}$$

Then

$$P(t) = 1 + p_0 t + s(\sqrt{1 + 4q^2 t^2} - 1)$$

and the solution of the quadratic equation into which $G(z) = P(zG(z))$ can be transformed, is

$$G(z) = \frac{-(s-1)(1-p_0 z) + s\sqrt{(1-p_0 z)^2 - 4(2s-1)q^2 z^2}}{(1-p_0 z)^2 - 4s^2 q^2 z^2}. \tag{5.2}$$

We get

$$r = \frac{1}{p_0 + 2q\sqrt{2s-1}}, \qquad \theta = \frac{\sqrt{2s-1}}{2(s-1)q},$$

$$a_0 = \frac{1}{r}\theta, \quad b_0 = \frac{s(q\sqrt{2s-1})^{1/2}}{2(s-1)^2 q^2 r^2}, \quad a_i = \frac{q}{p_{-i}\sqrt{2s-1}}, \quad b_i = \frac{b_0}{a_0} \frac{(s-1)q}{sp_{-i}\sqrt{2s-1}} \tag{5.3}$$

$$\mu_i = \frac{1}{2s-1}, \quad v_{i_1 \dots i_k} = \frac{1}{2s(2s-1)^{k-1}}.$$

For φ_r the following formula is obtained:

$$\varphi_r(x_{i_1} \dots x_{i_k}) = \left(1 + \frac{s}{s-1} k\right) \frac{1}{\sqrt{2s-1^k}} \prod_{j=1}^{k} \sqrt{p_{-i_j}/p_{i_j}}. \tag{5.4}$$

B) The case $p_0 = 0$: In the random walk it is not possible to remain at an element with positive probability, one has to move to one of the neighbours in the tree. Everything works out in the same way with the exception of a slight change in the local limit theorem:

We have $p^{(2n-1)}(e) = u_i^{(2n)} = 0$ for $n = 1, 2, \dots, P(t)$ is an even function and besides $z = r$, also $z = -r$ becomes a singularity of $G(z)$ and $U_i(z)$ $(i \in I)$. Using similar arguments as in Sect. 2 and 3 one can see that there are no further singularities on the circle of convergence. Near $z = -r$ we obtain the following Puiseux series:

$$\begin{aligned} G(z) &= a_0 - b_0\sqrt{r+z} + c_0(r+z) - d_0\sqrt{r+z}^3 + \dots \\ U_i(z) &= -a_i + b_i\sqrt{r+z} - c_i(r+z) + d_i\sqrt{r+z}^3 - \dots . \end{aligned} \tag{5.5}$$

Again using the method of Darboux [25] we get for $x \in \mathbb{F}_s$ having length k in the reduced representation (2.4):

$$p^{(n)}(x) = b_x\sqrt{r/\pi}\, r^{-n} n^{-3/2} + O_x(r^{-n} n^{-5/2}) \quad \text{as} \quad n \to \infty,$$

if n and k have the same parity, and (5.6)

$$p^{(n)}(x) = 0 \quad \text{otherwise} \quad (b_x = b_0 \text{ if } x = e).$$

The formulas for r, θ, a_0, b_0 etc. remain the same. The ratio limit has to be adjusted to the parities of the convolution exponent n and the length k of the occuring group element, but the properties and results concerning φ_r remain the same.

C) Equidistribution on the free generators and their inverses: $p_i = \frac{1}{2s}\ \forall i \in I$. This is the "classical" case and a further specialization of cases A) and B). We obtain the formulas that are well known from various authors (Kesten [19], Gerl [14], Figà-Talamanca and Picardello [11], etc.):

$$\begin{aligned} G(z) &= \frac{-(s-1)+\sqrt{s^2-(2s-1)z^2}}{1-z^2}, \qquad r = \frac{s}{\sqrt{2s-1}}, \\ \varphi_r(x) &= \left(1+\frac{s}{s-1}k\right)\frac{1}{\sqrt{2s-1}^k} \quad \text{if } x \in \mathbb{F}_s \text{ has length } k. \end{aligned} \tag{5.7}$$

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References

1. Akemann, Ch.A., Ostrand, Ph.A.: Computing norms in group C^*-algebras. Am. J. Math. **98**, 1015–1047 (1976)
2. Aomoto, K.: Spectral theory on a free group and algebraic curves. J. Fac. Sci. Univ. Tokyo, Sec. IA, **31**, 297–317 (1984)
3. Arnaud, J.P.: Fonctions sphériques et fonctions définies positives sur l'arbre homogène. C.R. Acad. Sci. Paris **290**, 99–101 (1980)
4. Bender, E.A.: Asymptotic methods in enumeration. SIAM Rev. **16**, 485–515 (1974)
5. Berg, Ch., Christensen, J.P.R.: Sur la norme des opérateurs de convolution. Invent. Math. **23**, 173–178 (1974)
6. Cartier, P.: Fonctions harmoniques sur un arbre. Symp. Math. **9**, 203–270 (1972)
7. Cartier, P.: Harmonic analysis on trees. Proc. Symp. Pure Math. **26**, 419–424 (1972)
8. Derriennic, Y.: Marche aléatoire sur le groupe libre et frontière de Martin. Z. Wahrscheinlichkeitstheor. Verw. Geb. **32**, 261–276 (1975)
9. Dunau, J.L.: Étude d'une classe de marches aléatoires sur l'arbre homogène. Publ. Laboratoire de Statistique de l'Univ. Paul Sabatier n° 04-1976, Toulouse (1976)
10. Dynkin, E.B., Malyutov, M.B.: Random walks on groups with a finite number of generators. Sov. Math. Dokl. **2**, 399–402 (1961)
11. Figà-Talamanca, A., Picardello, M.A.: Harmonic analysis on free groups. Lect. Notes Pure Appl. Math. **87**. New York-Basel: Marcel Dekker 1983
12. Furstenberg, H.: Random walks and discrete subgroups of Lie groups. Advs. Probab. **1**, 1–63 (1970)
13. Gerl, P.: Über die Anzahl der Darstellungen von Worten. Monatsh. Math. **75**, 205–214 (1971)
14. Gerl, P.: Wahrscheinlichkeitsmaße auf diskreten Gruppen. Arch. Math. **31**, 611–619 (1978)
15. Gerl, P.: Eine asymptotische Auswertung von Faltungspotenzen in gewissen Gruppen. Sitzungsber. Österr. Akad. Wiss., Math.-Naturw. Klasse, Abt. 2, **186**, 385–396 (1978)
16. Gerl, P.: Ein Gleichverteilungssatz auf F_2. In: Probability Measures on Groups. Lect. Notes Math. **706**, 126–130. Berlin-Heidelberg-New York: Springer 1979
17. Gerl, P., Woess, W.: Simple random walks on trees. (To appear in European J. Comb.)
18. Guivarc'h, Y.: Loi des grands nombres et rayon spectral d'une marche aléatoire sur un groupe de Lie. Astérisque **74**, 47–98 (1980)
19. Kesten, H.: Symmetric random walks on groups. Trans. Am. Math. Soc. **92**, 336–354 (1959)

20. Letac, G.: Problèmes classiques de probabilité sur un couple de Gelfand. In: Analytic Methods in Probability Theory. Lect. Notes Math. **861**, 93–127. Berlin-Heidelberg-New York: Springer 1981
21. Mantero, A.M., Zappa, A.: The Poisson transform and representations of a free group. J. Funct. Anal. **51**, 372–399 (1983)
22. Picardello, M.A.: Spherical functions and local limit theorems on free groups. Ann. Mat. Pura Appl. **33**, 177–191 (1983)
23. Sawyer, S.: Isotropic random walks in a tree. Z. Wahrscheinlichkeitstheor. Verw. Geb. **42**, 279–292 (1978)
24. Stone, Ch.: Ratio limit theorems for random walks on topological groups. Trans. Am. Math. Soc. **125**, 86–100 (1966)
25. Szegö, G.: Orthogonal Polynomials. Providence, Rhode Island: Am. Math. Soc. 1939
26. Vere-Jones, D.: Geometric ergodicity in denumerable Markov chains. Quart. J. Math. (Oxford), Ser. II, **13**, 7–28 (1962)
27. Woess, W.: Puissances de convolution sur les groupes libres ayant un nombre quelconque de générateurs. Publ. Inst. Elie Cartan **7**, 180–190, Preprint (1983)
28. Woess, W.: Harmonic functions on free groups. In "Probabilités sur les Structures Géométriques", Proceedings, Toulouse (1984)

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