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Probability Theory and Related Fields

Ergodicity Conditions for Two-dimensional Markov Chains on the Positive Quadrant

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Summary. The basic problem considered in this paper is that of determining conditions for recurrence and transience for two dimensional irreducible Markov chains whose state space is $Z_+^2 = Z_+ \times Z_+$. Assuming bounded jumps and a homogeneity condition Malyshev [7] obtained necessary and sufficient conditions for recurrence and transience of two dimensional random walks on the positive quadrant. Unfortunately, his hypothesis that the jumps of the Markov chain be bounded rules out for example, the Poisson arrival process. In this paper we generalise Malyshev's theorem by means of a method that makes novel use of the solution to Laplace's equation in the first quadrant satisfying an oblique derivative condition on the boundaries. This method, which allows one to replace the very restrictive boundedness condition by a moment condition and a lower boundedness condition, is of independent interest.

1. Introduction

The basic problem considered in this paper is that of determining conditions for recurrence and transience for two dimensional irreducible Markov chains whose state space is $Z_+^2 = Z_+ \times Z_+$, where Z_+ denotes the nonnegative integers. We shall denote the Markov chain by $x(t) = (x_1(t), x_2(t))$, where $x_i(t)$, $i = 1, 2$ are nonnegative integer valued random variables and denote the associated filtration by $\mathscr{F}(t) = \sigma\{x(s): 0 \leqq s \leqq t\}$. As is to be expected the criteria for recurrence and transience depend on the interior and boundary behaviour of the drift vector of the Markov chain. Let us define the interior and boundaries of Z_+^2 as the sets Z_0, Z_1, Z_2, where

$$Z_0 = \{(i,j):\ i > 0, j > 0\}; \quad Z_1 = \{(i,0):\ i > 0\}; \quad Z_2 = \{(0,j):\ j > 0\}.$$

Let $A_i(x) = x(t+1) - x(t)$, $x(t) = x \in Z_i$, and define the drift vectors

$$\vec{d}_i(x) = E\{x(t+1) - x(t) \,\|\, x(t) = x\} = E\{A_i(x)\}, \quad x \in Z_i.$$

Here $x=(x_1, x_2)$ with the norm $\|x\|=\sqrt{x_1^2+x_2^2}$. Later we shall switch to polar coordinates with $r=\|x\|$ and $\theta=\arctan(x_2/x_1)$. The expectation operator E is applied to each component separately, so $E(x(t))=(Ex_1(t), Ex_2(t))$, etc. To simplify the calculations we shall assume *the homogeneity condition*:

$$\vec{d}_i=\vec{d}_i(x). \tag{1}$$

In other words the drift vectors are constant in the interior and on the boundaries. We denote the x and y components of the drift vectors as follows: $\vec{d}_i=(d_{ix}, d_{iy})$. We assume that $d_{1y}>0$, $d_{2y}>0$ for otherwise the process after reaching the boundary Z_1, say, will stay there. Assuming *the boundedness condition*:

$$\|A_i(x)\| \quad \text{is uniformly bounded in } x \text{ and} \quad i=0,1,2. \tag{2}$$

Malyshev proved the following theorem (see [7]):

Theorem 1. *Assume the conditions* 1, 2 *hold.*

A) *If $d_{0x}>0$, $d_{0y}\geqq 0$, or $d_{0x}\geqq 0$, $d_{0y}>0$, then $x(t)$ is transient.*

B) *If $d_{0x}<0$, $d_{0y}<0$, then $x(t)$ is positive recurrent if and only if*

$$d_{0x}d_{1y}-d_{0y}d_{1x}<0, \qquad d_{0y}d_{2x}-d_{0x}d_{2y}<0; \tag{3}$$

null recurrent if (3) *is weakened to*:

$$d_{0x}d_{1y}-d_{0y}d_{1x}\leqq 0, \qquad d_{0y}d_{2x}-d_{0x}d_{2y}\leqq 0 \tag{4}$$

and transient in the remaining cases.

C) *If $d_{0x}\geqq 0$, $d_{0y}<0$, then $x(t)$ is positive recurrent if and only if*

$$d_{0x}d_{1y}-d_{0y}d_{1x}<0, \tag{5}$$

null recurrent if

$$d_{0x}d_{1y}-d_{0y}d_{1x}=0, \tag{6}$$

and transient in the remaining cases.

D) *is symmetric to case C.*

Remark 1. Each of these conditions has a geometric interpretation. For example, the condition (5) implies that $d_{0x}/d_{0y}>d_{1x}/d_{1y}$, so $\vec{d}_1$ points to the left of $-\vec{d}_0$ out of the wedge determined by $-\vec{d}_0$ and the positive x-axis.

In proving Theorem 1, Malyshev used a *Lyapounov* function $F(x)$ with the property that $\nabla F(x)\cdot\vec{d}_i$ be of constant sign for $x\in Z_+^2$, or a suitable subset thereof. For example, $\nabla F\cdot\vec{d}_i\leqq 0$ would imply that the Markov process is recurrent (positive or null), with the reverse inequality implying transience. Malyshev's construction of the function F is a geometrical one and therefore one cannot differentiate if since no explicit analytic form of it is given. Indeed, after carefully examining his construction of the function F, I concluded that $F\in C^1$ but F is not in C^2. And this is why his proof only works for Markov chains with bounded jumps. It is worth pointing out that the class of Markov chains studied by Fayolle, et al., see [2], do not satisfy the homogeneity condition (1) while the boundedness condition (2) rules out the Poisson arrival process, say, to a two queue system of the kind studied by Baccelli, see [1].

The main purpose of this paper is to give an alternative proof of Malyshev's theorem which utilises a Lyapounov function $\Psi(r, \theta)$ which is smooth in the interior and satisfies an oblique derivative condition on the boundaries. (For the purposes of constructing a Lyapounov function it is much more convenient to switch to polar coordinates.) To this end we introduce the auxiliary function $\Phi(r, \theta)=r^{\alpha}\cos(\alpha\theta-\theta_1)$, where α and θ_1 depend in an explicit way on the angles that the vectors $\vec{d}_i$ make with $\vec{n}_i$, the inward pointing normals to the coordinate axes Z_i, $i=1, 2$.

The function Ψ is defined as follows:

$$\Psi=\Phi \text{ if } \alpha>0, \quad \text{and} \quad \Psi=\Phi^{-1} \text{ if } \alpha<0.$$

As a consequence of this construction we are able to drop the unnecessarily strong condition (2) and replace it with: (i) a moment condition (29) and (ii) a lower boundedness condition (28) which is potentially more useful. Under these weaker conditions, however, portions of Theorem (1) must be modified as follows:

Theorem 2. *Assume the Markov chain satisfies conditions* (29) *and* (28).

A) *If* $d_{0x}>0$, $d_{0y}\geqq 0$ (*or* $d_{0x}\geqq 0$, $d_{0y}>0$), *then* $x(t)$ *is transient. Thus part A of Theorem* (1) *goes through without change.*

B) *If* $d_{0x}<0$, $d_{0y}<0$, *then part B of Theorem* (1) *is modified as follows:*

(i) *If*

$$d_{0x}d_{1y}-d_{0y}d_{1x}<0, \qquad d_{0y}d_{2x}-d_{0x}d_{2y}<0 \tag{7}$$

then $x(t)$ *is recurrent, but not necessarily positive recurrent. If, in addition the parameter* α, *see Definition* (2), *lies in the range* $1\leqq\alpha<2$ *then* $x(t)$ *is positive recurrent.*

(ii) *If either*

$$d_{0x}d_{1y}-d_{0y}d_{1x}>0, \quad \text{or} \quad d_{0y}d_{2x}-d_{0x}d_{2y}>0 \tag{8}$$

hold then $x(t)$ *is transient.*

(iii) *The only remaining exceptional cases are where* $d_{0x}d_{1y}-d_{0y}d_{1x}\leqq 0$, $d_{0y}d_{2x}-d_{0x}d_{2y}\leqq 0$ *and one of these holds with equality.*

C) *If* $d_{0x}>0$, $d_{0y}<0$, *then part C of Theorem* (1) *is modified as follows:*

(i) *If*

$$d_{0x}d_{1y}-d_{0y}d_{1x}<0, \tag{9}$$

then $x(t)$ *is recurrent. If in addition* $1\leqq\alpha<2$, *then* $x(t)$ *is positive recurrent.*

(ii) *If*

$$d_{0x}d_{1y}-d_{0y}d_{1x}>0, \tag{10}$$

holds then $x(t)$ *is transient.*

(iii) *The case where* $d_{0x}d_{1y}-d_{0y}d_{1x}=0$ *cannot be handled via our methods.*

D) *is symmetric to case C.*

We conclude this section by noting that the methods and some of the results of Malyshev were anticipated 10 years earlier by Kingman, see [5]. This paper also contains two figures which illustrate geometrically the conditions of Theorem 1.

The referee has noted that the Lyapounov functions constructed in this paper could be used to prove conditions for transience/recurrence of reflected diffusions on the positive quadrant (assuming the existence of such processes). This would add to the currently known recurrence criteria for such diffusions, see [4].

2. Construction of the Lyapounov Function

A complete proof of Theorem 2 would require the construction of seven Lyapounov functions, one for each of the cases listed in parts A, B, C of Theorem 2. So, we shall only consider part C of the theorem in detail and content ourselves with giving a list in Sect. 4 of Lyapounov functions that the reader could use to prove parts A and B of Theorem (2). Thus, for the next two sections, we suppose that $d_{0x} \geqq 0$ and $d_{0y} < 0$.

In order to define the function $\Phi(r, \theta)$ we have to define the constants α, θ_1, ψ_1, ψ_2, ψ which are related to the angles made by the vectors $\vec{d}_i$ with respect to the inward pointing normals $\vec{n}_1 = (0, 1)$, $\vec{n}_2 = (1, 0)$.

Definition 1. ψ = angle between $-\vec{d}_0$ and $\vec{n}_1$, ψ_i = angle between $\vec{d}_i$ and $\vec{n}_i$, $i = 1, 2$.

Note that for $i = 1, 2$, $-\frac{\pi}{2} < \psi_i < \frac{\pi}{2}$, and we adopt the convention that $\psi_i > 0$ when the vector d_i is pointing towards the origin O. Note that case C and condition (5) together imply that $0 < \psi < \psi_1$, consequently, there exists an angle θ_1 with the property that $0 < \psi < \theta_1 < \psi_1$. On the other boundary we simply choose θ_2 so that $-\frac{\pi}{2} < \theta_2 < \min\left(\psi_2, \frac{\pi}{2} - \psi\right)$. If condition (6) holds then $\psi = \psi_1 = \theta_1$.

Definition 2. $\alpha = (\theta_1 + \theta_2)\, 2/\pi$, $\vec{v}_1 = (-\sin\theta_1, \cos\theta_1)$, $\vec{v}_2 = (-\sin\theta_2, -\cos\theta_2)$.

Remark 2. Note that $-\frac{\pi}{2} < \theta_1 + \theta_2 < \pi$ implies that $-1 < \alpha < 2$ and that for $\alpha > 0$ the function $\theta \to \alpha\theta - \theta_1$ varies between $-\theta_1$ and θ_2 as θ varies from 0 to $\frac{\pi}{2}$. For $\alpha < 0$ the function decreases from $-\theta_1$ to $\theta_2 < 0$. The possibility of $\alpha = 0$ can be avoided by suitably adjusting θ_2. So we can safely assume that $\alpha \neq 0$ holds.

We now proceed to compute $\nabla\Phi$ in polar coordinates:

$$\nabla\Phi(r, \theta) = \alpha r^{\alpha - 1}(\cos(\alpha\theta - \theta_1), -\sin(\alpha\theta - \theta_1)). \tag{11}$$

In particular then

$$\nabla\Phi(r,0)=\alpha r^{\alpha-1}(\cos(\theta_1),\sin(\theta_1))$$

and

$$\nabla\Phi(r,\pi/2)=\alpha r^{\alpha-1}(\cos(\theta_2),-\sin(\theta_2)).$$

Consequently, $\nabla\Phi(r,0)\cdot\vec{v}_1=0$ and $\nabla\Phi\left(r,\frac{\pi}{2}\right)\cdot\vec{v}_2=0$. From this one easily sees that for $\alpha>0$, $\nabla\Phi(r,0)\cdot\vec{d}_1<0$ and that $\nabla\Phi\left(r,\frac{\pi}{2}\right)\cdot\vec{d}_2<0$, with the inequalities reversed for $\alpha<0$. It is a routine calculation to verify that the angle $a(\theta)$ between $\alpha^{-1}\nabla\Phi(r,\theta)$ and $\vec{d}_0$ is given by $a(\theta)=\theta_1-\alpha\theta-\left[\left(\psi-\frac{\pi}{2}\right)-\theta\right]$, where the first term is the angle between $\alpha^{-1}\nabla\Phi$ and $\vec{r}$, and the second term is the angle between $\vec{r}$ and $\vec{d}_0$. Since $a(\theta)$ is a linear function it assumes its minimum and maximum values at 0 or $\frac{\pi}{2}$. Now $a(\theta)=\pi/2+(\theta_1-\psi)>\pi/2$, since we are assuming $\theta_1>\psi$, and $a(\pi/2)=\pi-(\psi+\theta_2)>\pi/2$, since we have chosen $\theta_2<\min(\psi_2,\pi/2-\psi)<\pi/2-\psi$. Similarly, it can be shown that $a(\theta)<3\pi/2$, and therefore $\cos(a(\theta))<0$ for $0\leqq\theta\leqq\pi/2$. Therefore,

$$\nabla\Phi(r,\theta)\cdot\vec{d}_0<0,\qquad\text{for all }\theta\in\left[0,\frac{\pi}{2}\right]\text{ and }\alpha>0.\tag{12}$$

$$\nabla\Phi(r,\theta)\cdot\vec{d}>0,\qquad\text{for all }\theta\in\left[0,\frac{\pi}{2}\right]\text{ and }\alpha<0.\tag{13}$$

On the other hand if condition (6) holds then $\nabla\Phi(r,0)\cdot\vec{d}_i=0$, $i=0,1$ consequently we can only assert

$$\nabla\Phi(r,\theta)\cdot\vec{d}_0\leqq 0,\qquad\alpha>0.\tag{14}$$

and

$$\nabla\Phi(r,\theta)\cdot\vec{d}_0\geqq 0,\qquad\alpha<0.\tag{15}$$

Definition 3. If condition (5) or (6) holds we set

$$\Psi=\Phi,\quad\text{for }\alpha>0,\qquad\Psi=\Phi^{-1}\quad\text{for }\alpha<0.\tag{16}$$

It is clear that Ψ is a Lyapounov function in the sense that $\Psi\geqq 0$,

$$\nabla\Psi\cdot\vec{d}_i<0,\qquad\text{on }Z_i,\ i=0,1,2\text{ assuming condition (5)}\tag{17}$$

and that

$$\nabla\Psi\cdot\vec{d}_i\leqq 0,\qquad\text{on }Z_i\text{ assuming condition (6) holds.}\tag{18}$$

We conclude this section by deriving some estimates on the function Ψ from which we will be able to deduce an inequality of the following form when (5) holds.

$$E\{\Psi(x(t+1))-\Psi(x(t))\,|\,x(t)=x\}\leqq -\varepsilon, \tag{19}$$

provided $\|x\|\geqq K$, for some $K>0$ and $\varepsilon\geqq 0$.

Remark 3. If $\varepsilon=0$ then we can only conclude that the process is recurrent, whereas $\varepsilon>0$ implies that the process is positive recurrent. These facts will be established in the next section of this paper.

Before stating the next theorem we shall have to introduce some notation so we can distinguish between $D_i\,\Psi=\Psi_{x_i}$ and Ψ_r and Ψ_θ, the symbol D_i denoting the partial derivative with respect x_i of the function $\Psi(r,\theta)$, where r,θ are the polar coordinates of the point $x=(x_1,x_2)$. Similarly, $D_{ij}\,\Psi=\Psi_{x_i x_j}$ each of which is computed via the chain rule. For example to compute $D_1\,\Psi$ we proceed as follows: $D_1\,\Psi=\nabla\Psi(r,\theta)\cdot\vec{i}$, where $\vec{i}=(\cos\theta,\ -\sin\theta)$, which equals $\Psi_r\cos(\theta)$ $-r^{-1}\,\Psi_\theta\sin(\theta)$, and the higher order derivatives are computed in a similar fashion. Thus

$$\begin{aligned} D_{11}\,\Psi=\Psi_{rr}\cos^2(\theta)+2r^{-2}\,\Psi_0\sin\theta\cos\theta \\ +r^{-2}\,\Psi_{\theta\theta}\sin^2\theta-2r^{-1}\,\Psi_{r\theta}\cos\theta\sin\theta+r^{-1}\,\Psi_r\sin^2\theta. \end{aligned} \tag{20}$$

Using the explicit formula for Ψ given in (16) it is a routine calculation to check that $D_{ij}\,\Psi=O(r^{|\alpha|-2})$ whereas $\|\nabla\Psi(r,\theta)\|=O(r^{|\alpha|-1})$. Recalling the fact that the angle $a(\theta)$ between $\nabla\Psi(r,\theta)$ and $\vec{d}_0$ lies in the range $(\pi/2,\ 3\pi/2)$ when (5) holds it is easy to show that $r^{-|\alpha|+1}\,\|\nabla\Psi(r,\theta)\|\cdot\|\vec{d}_0\|\,\cos(a(\theta))\leqq K_0<0$, for $\theta\in[0,\pi/2]$. More precisely we have the following theorem:

Theorem 3. *Suppose* (5) *holds. Then there are constants* K_0 *and* K_1 *such that*

$$\limsup_{r\to\infty} r^{-|\alpha|+1}\,\nabla\Psi(r,\theta)\cdot\vec{d}_0\leqq K_0<0, \tag{21}$$

$$\limsup_{r\to\infty} r^{2-|\alpha|}\,|D_{ij}\,\Psi|\leqq K_1<\infty \tag{22}$$

$$\limsup_{r\to\infty} \Psi(r,\theta)=\infty, \tag{23}$$

uniformly in $\theta\in\left[0,\dfrac{\pi}{2}\right]$. *On the boundaries we have:*

$$\limsup_{r\to\infty} r^{-|\alpha|+1}\,\nabla\Psi(r,0)\cdot\vec{d}_1\leqq K_0<0, \tag{24}$$

$$\limsup_{r\to\infty} r^{-|\alpha|+1}\,\nabla\Psi\left(r,\frac{\pi}{2}\right)\cdot\vec{d}_2\leqq K_0<0. \tag{25}$$

Proof. The first and last two assertions follow from Eq. (16), the explicit formula for $\nabla\Psi$, see (11), $\cos(a(\theta))<0$ for $\theta\in\left[0, \frac{\pi}{2}\right]$, and the remark preceeding Theorem (3). The second assertion is established by verifying that each term in the expression for $D_{ij}\,\Psi$ is $O(r^{|\alpha|-2})$ uniformly in $\theta\in\left[0, \frac{\pi}{2}\right]$. The assertion (23) follows from (16) and the fact that $\cos(\alpha\theta-\theta_1)>0$.

By a Taylor's series expansion

$$\Psi(x+h)-\Psi(x)=\nabla\Psi(x)\cdot h+R(x,h), \tag{26}$$

where $h=(h_1, h_2)$ and the remainder term is given by

$$R(x,h)=\tfrac{1}{2}\sum_{i,j=1,2} D_{ij}\,\Psi(\eta_{ij})\,h_i\,h_j$$

and η_{ij} is a point that lies on the line joining x to $x+h$.

If the components h_i, $i=1, 2$ are bounded from below by a constant c, that is $h\in H_c=\{h\in Z^2\colon\ h_1\geqq c,\ h_2\geqq c>-\infty\}$, then it follows from Theorem 3 that

$$|D_{ij}\,\Psi(\eta_{ij})|=O(\|x\|^{|\alpha|-2}) \qquad \text{as } \|x\|\to\infty \text{ uniformly in } h\in H_c. \tag{27}$$

Remark 4. For a model of a queueing network where each server serves one customer at a time, $c=-1$.

Now for $i\in\{0, 1, 2\}$ and $x\in Z_i$, we replace h by the random vector

$$A_i(x)=x(t+1)-x(t), \qquad x(t)=x \quad \text{so} \quad \vec{d}_i(x)=E\{A_i(x)\}.$$

We make the following assumptions on $A_i(x)$:

$$\exists K>0, \qquad c>-\infty \quad \text{so that } A_i(x)\in H_c, \quad \forall\,\|x\|\geqq K, \tag{28}$$

and there exists $K_2>0$ such that

$$E\{\|A_i(x)\|^2\}<K_2<\infty, \qquad i=0,1,2. \tag{29}$$

We now substitute $A_i(x)$ for h in the Taylor expansion above and take expectations of both sides obtaining

$$E\{\Psi(x(t+1))-\Psi(x(t))\,|\,x(t)=x\}=\nabla\Psi(x)\cdot\vec{d}_i(x)+E\{R(x,A_i(x))\}. \tag{30}$$

Next observe that $\|x\|\geqq K+c\sqrt{2}$ implies $\|x+A_i(x)\|\geqq K$. Consequently, $\sup\{|D_{ij}\ \Psi(\eta_{ij})|\colon i,j=1,2\}=O(\|x\|^{|\alpha|-2})$, where η_{ij} is a point on the line joining x to $x+A_i(x)$. Therefore

$$E\{|R(x,A_i(x)|\}=O(\|x\|^{|\alpha|-2}).$$

Consequently,

$$E\{\Psi(x(t+1))-\Psi(x(t))\,|\,x(t)=x\in Z_i\}=\nabla\Psi(x)\cdot\vec{d}_i(x)+O(\|x\|^{|\alpha|-2}). \tag{31}$$

Case 1. $1 \leqq \alpha < 2$, and condition (5) holds.

In this case $\limsup_{\|x\| \to \infty} \nabla\Psi(x) \cdot \vec{d}_i < 0$ and therefore

$$E\{\Psi(x(t+1)) - \Psi(x(t)) \mid x(t) = x\} \leqq -\varepsilon \quad \text{for some } \varepsilon > 0 \text{ and } \forall \|x\| \geqq K. \tag{32}$$

Case 2. $-1 < \alpha < 1$, and condition (5) holds.

In this case we can only conclude that

$$E\{\Psi(x(t+1)) - \Psi(x(t)) \mid x(t) = x\} \leqq 0 \quad \text{for } \|x\| \geqq K. \tag{33}$$

Case 3. $d_{0x} d_{1y} - d_{0y} d_{1x} > 0$, i.e. neither condition 5 nor condition 6 holds. Geometrically this means $\psi_1 < \psi$ and therefore $\exists$ an angle $\theta_1 > 0$ such that $\psi > \theta_1 > \psi_1$. On the other boundary we choose $\theta_2 > \max\left(\psi_2, \frac{\pi}{2} - \psi\right) > 0$, so $\alpha > 0$ in this case. It is now easy to check that with these choices for θ_1, θ_2 that the function Ψ satisfies the condition $\nabla\Psi(x) \cdot \vec{d}_i > 0$ on Z_i, $i = 0, 1, 2$. Hence $\Gamma = \Psi^{-1}$ satisfies

$$\nabla\Gamma \cdot \vec{d}_i < 0, \quad \text{on } Z_i, \; i = 0, 1, 2, \tag{34}$$

and by similar reasoning to that above there is a $K > 0$ such that

$$E\{\Gamma(x(t+1)) - \Gamma(x(t)) \mid x(t) = x\} \leqq 0 \tag{35}$$

whenever $\|x\| \geqq K$.

3. Derivation of Ergodicity Conditions

We remind the reader that we are only proving part C of Malyshev's Theorem and we shall begin with the proof of transience. So suppose neither (6) nor (5) holds. Let $A = \{x\colon \|x\| \leqq K\}$ and $A' = \{x\colon \|x\| > K\}$. Then we have the following result.

Lemma 1. *Suppose neither condition* 5 *nor condition* 6 *holds and let* $y(t) = \Gamma(x(t))$ *for all* $t \geqq 0$. *Then* $y(t)$ *is a positive supermartingale on* A', *i.e.*,

$$E\{y(t+1) \mid \mathscr{F}(t)\} \leqq y(t), \quad \textit{whenever } x(t) \in A', \; t \geqq 0, \tag{36}$$

and consequently, for $T = \inf\{t \geqq 0\colon x(t) \in A\}$ $\{y(t \wedge T), \mathscr{F}(t), t \geqq 0\}$ *is a nonnegative supermartingale.*

Proof. Recall that $\Gamma(x) = \Psi(x)^{-1}$ satisfies (35). Since $x(t)$ is a Markov process (35) implies (36) and the last result follows from Doob's optional stopping theorem.

Choose $\|x_0\| \gg K$ so that $\Gamma(x_0) < K_4 = \inf\{\Gamma(x)\colon x \in A\}$ and set $x_0 = x(0)$. To prove that $x(t)$ is transient it suffices to show that $P_{x_0}\{T < \infty\} \neq 1$. Assume to the contrary that $P_{x_0}\{T < \infty\} = 1$ and use the bounded convergence theorem to conclude that $E\{y(T)\}=\lim_{t\to\infty} E\{y(t\wedge T)\}\leqq E\{y(0)\}<K_4$ while on the other hand $y(T)\in A$ implies $y(T)\geqq K_4$, and therefore $E\{y(T)\}\geqq K_4$, a contradiction.

Now suppose condition (5) holds. To prove recurrence we bring in the stochastic process $y(t)=\Psi(x(t))\,I_{\{T>t\}}(x(t))$ and note that $y(t)=0$ on the set $\{T\leqq t\}$, where (T *and* A) have the same definitions as in the transient case discussed above. We now show that $y(t)$ is a nonnegative supermartingale. We first observe that $I_{\{T>t+1\}}\leqq I_{\{T>t\}}$. Then

$$\begin{aligned}E\{y(t+1)\,|\,\mathscr{F}(t)\}&\leqq E\{\Psi(x(t+1))\,I_{\{T>t\}}|\,\mathscr{F}(t)\}\\&=I_{\{T>t\}}(E\{\Psi(x(t+1))\,|\,\mathscr{F}(t)\}\leqq I_{\{T>t\}}\,\Psi(x(t))=y(t),\end{aligned}$$

where we have used Eq. (33) in the second last step. So $y(t)$ is a nonnegative supermartingale as claimed. Indeed, the same argument shows that for every $s\geqq 0$ the process $\{y(s+t),\,\mathscr{F}(s+t),\,t\geqq 0\}$ is also a nonnegative supermartingale. Note also that

$$P\{\limsup_{t\to\infty}\|x(t)\|=\infty\}=1, \tag{37}$$

since we are assuming that the Markov chain is irreducible. We now claim

$$\text{for any }\ s\geqq 0\colon\ P\{y(s+t)=0,\text{for some } t\geqq 0\}=1. \tag{38}$$

For the proof of this, note that by the (super)martingale $y(t)$ we have that $\lim_{t\to\infty} y(t)=0$ for otherwise $\limsup_{t\to\infty} y(t)=\infty$, where we use Eq. (37) to deduce that $\limsup_{t\to\infty}\Psi(x(t))=\infty$ on the set $\{T=\infty\}$. Consequently the finite set A is visited infinitely often with probability one. This argument is due to Lamperti, see [6]. It follows from this that the process is recurrent, for if it were transient, with probability one, the finite set would be visited only a finite (but random) number of times. We have thus shown that when the inequality in condition (5) is reversed then $x(t)$ is transient and when it holds it is recurrent. When can we assert that $x(t)$ is positive recurrent? At this point we make note of the fact that the methods of this paper using a second order Taylor expansion of the Lyapounov function do not allow us to conclude recurrence under condition (6). This is one of several borderline cases which cannot be settled by our methods at this time. We are able, however, to show that $x(t)$ is positive recurrent when condition (5) holds and $1\leqq\alpha<2$. More precisely we state the following result.

Theorem 4. *Suppose the Markov chain satisfies condition* (5) *and* $1\leqq\alpha<2$. *Then the Markov chain is positive recurrent.*

The proof is a consequence of the fact that Eq. (32) holds and the following well known criterion of Foster, see [3], which we restate in a more general form.

Theorem 5. *Let P_{ij} denote the transition matrix of an irreducible Markov chain (with a countable state space), A a set of states whose complement A' is a finite set, and let $f(j)$ denote a nonnegative function with the property:*

$$\sum_j P_{ij} f < f(i) - \varepsilon, \ \forall i \in A \quad \textit{for some } \varepsilon > 0. \tag{39}$$

Then the Markov chain is positive recurrent.

4. Lyapounov Functions for Parts A and B of Theorem 2

Throughout this section the parameter α and the vectors $\vec{v}_i$, $i=1, 2$ are defined just as they were in Definition (2). The functions Φ, Ψ, Γ, defined in Sect. 2, remain unchanged.

Definition 4. Lyapounov function for part A of Theorem 2: ψ_i equals the angle that $\vec{d}_i$ makes with $\vec{n}_i$, $i=1, 2$. ψ equals the angle that $\vec{d}_0$ makes with $\vec{n}_1$ and note that: (i) $\psi<0$ and (ii) $-\left(\psi+\frac{\pi}{2}\right)$ equals the angle $\vec{d}_0$ makes with $\vec{n}_2$. Choose θ_i, $i=1, 2$ such that $\pi/2>\theta_1>\sup(\psi, \psi_1)$; and $\pi/2>\theta_2>\sup\left(-\left(\psi+\frac{\pi}{2}\right), \psi_2\right)>-\pi/2$. We can assume that $\alpha>0$ because we can always choose $\theta_i>0$. In this case $a(\theta)=(\theta_1-\alpha\theta)-\left(\frac{\pi}{2}+\psi-\theta\right)$. The Lyapounov function in this case is Γ, as defined in the preceeding section.

Definition 5. Lyapounov function for part B(i) of Theorem 2: In this case ψ is the angle between $-\vec{d}_0$ and $\vec{n}_1$. The condition (7) implies $\psi<0$, $\psi_1>\psi$, $\psi_2>-\left(\psi+\frac{\pi}{2}\right)$. Choose θ_i, $i=1, 2$ as follows: $\psi<\theta_1<\psi_1$, $-\left(\psi+\frac{\pi}{2}\right)<\theta_2<\psi_2$.

The Lyapounov function in this case is Ψ as defined at 16.

Definition 6. Lyapounov function for part B(ii) of Theorem 2: In this case $\psi_1<\psi$ and $\psi_2<-\left(\psi+\frac{\pi}{2}\right)$. Choose θ_1 so that $-\pi/2<\psi_1<\theta_1<\psi<0$, and on the other boundary choose $\theta_2>\max\left(\psi_2, \frac{\pi}{2}-\psi, -\theta_1\right)$. The Lyapounov function is Γ.

5. Concluding Remarks

We have thus shown that Theorem (1), with modifications, remains valid when the boundedness condition (2) is replaced by the conditions (28), (29). The homogeneity condition (1) can also be weakened provided one uses Eq. (30) as well as suitable supplementary hypotheses in order to obtain any one of the Eqs. (32), (33), (35).

The function $\Phi(r, \theta)$ appears in the paper of Varadhan and Williams, who used it to solve a submartingale problem, see [8].

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