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Probability Theory and Related Fields

Stopping a Two Parameter Weak Martingale

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Summary. This paper deals with the following problem, given a two parameter stochastic process, under what conditions is it possible to stop the process at any stopping line? It is shown that the class of stoppable processes is strictly larger than the class of two parameter integrators. Sufficient conditions for a weak martingale to be stoppable are derived and the stopped r.v. is represented as a one parameter optional dual projection.

I. Introduction

Let $X=\{X_t, t\in\mathbb{R}_+\}$ be a stochastic process defined on some probability space and indexed by the real numbers and let T be a stopping time satisfying $T<\infty$ a.s. or just any positive a.s. finite random variable. In this case the problem of determining "the value of X at time T" is trivial, namely it is X_T which is always a well defined random variable. Moreover if X is an integrator then $X_T=X_o+\int 1_{(0,T]}dX$ where $(0,T]$ is the stochastic interval $\{(\omega,t): 0<t\leqq T(\omega)\}$. When processes are indexed by a set which is only partially-ordered, such as the plane with the partial-order induced by the Cartesian coordinates, two kinds of generalizations of the stopping time occur: the stopping point and the stopping line and these two concepts are necessary for the development of the theory (cf. e.g. [4, 8, 9]). This leads to the following problem. Let $X=\{X_z, z\in\mathbb{R}^2_+\}$ be a two-parameter stochastic process and L be a (random or even deterministic) decreasing line which splits the positive quadrants into two regions, is it possible to define X_L as the process stopped to the line L? Unlike the one parameter case, it is not always possible to do that, but if X is an integrator and X vanishes on the axes ($X_{0,t}=X_{s,0}=0$) then a satisfactory definition is given by $X_L=\int I_{((0,0),L]}dX$ where $((0,0),L]$ is the stochastic region $\{(\omega,z):(0,0)<z\leqq L(\omega)\}$ [9]. Till now, no solution to the problem of stopping a process which is not an integrator seems to have been known. In this paper we

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consider this problem and derive sufficient conditions under which the value of a stopped weak martingale is well defined. This is of interest in the theory of two parameter processes in view of the importance of stopping, of weak martingales and the fact that not all weak martingales are integrators [2, 7].

In the next section we study the deterministic problem for real functions of two variables. We show that the class of functions that can be stopped at a decreasing line is strictly larger than the class of functions of two parameters which are of bounded variation. In Section 3 the stochastic problem is studied, it is shown that the class of processes which can be stopped at any stopping line is strictly larger than the class of stochastic integrators. The main result of this paper is that the class of all stoppable processes contains all the weak martingales which are one parameter semi-martingales in each direction and furthermore, the collection of random variables obtained by stopping when suitably parametrized are a one parameter semimartingale (Theorem 3.1 and Corollary 3.4). Similar results go over directly the case where the stopping lines are replaced by an optional increasing path (Proposition 3.5). In view of Theorem 3.1, Corollary 3.4 and Proposition 3.5 the construction of line integrals with respect to martingales (Cairoli and Walsh, Sect. 4 of [3]) goes over directly to line integrals along nonrandom and random paths and the integration being with respect to weak martingales satisfying the conditions of Theorem 3.1.

The usual notation is followed: The processes are indexed by points of $\mathbb{R}_+^2$, or by points of a rectangle $[(0,0), z_a] = R_{z_a}$ in the positive quadrant $\mathbb{R}_+^2$, in which a partial order induced by the cartesian coordinates is defined: let $z=(s,t)$ and $z'=(s',t')$, then $z \leqq z'$ if $s \leqq s'$ and $t \leqq t'$ and $z<z'$ if $s<s'$ and $t<t'$; we denote $z \wedge z'$ if $s \leqq s'$ and $t \geqq t'$. A probability space $(\Omega, \underline{\underline{F}}, \mathbb{P})$ is given equipped with an increasing right continuous filtration $\{\underline{\underline{F}}_z, z \in \mathbb{R}_+^2\}$ of sub-σ-algebras of $\underline{\underline{F}}$, denote $z=(s,t)$: $\underline{\underline{F}}_z^1 = \underline{\underline{F}}_{s\infty}$ and $\underline{\underline{F}}_z^2 = \underline{\underline{F}}_{\infty t}$. The conditional independence property ($F-4$ of [3]): for every z, $\underline{\underline{F}}_z^1$ and $\underline{\underline{F}}_z^2$ are conditionally independent given $\underline{\underline{F}}_z$, will be assumed throughout the paper.

A process $X = \{X_z, z \in R_{z_a}\}$ is called a *martingale* if $z \leqq z'$ implies $E[X_{z'} \mid \underline{\underline{F}}_z] = X_z$; a submartingale if the equality is replaced by $\geqq$ and X is adapted (with respect to the filtration $\underline{\underline{F}}_z$). The increment of X on a rectangle $(z, z']$, where $z=(s,t)$ and $z'=(s',t')$ is $X(z,z'] = X_{z'} - X_{(s,t')} - X_{(s',t)} + X_z$. An adapted process X is called a *weak martingale* if $z<z' \Rightarrow E[X(z,z'] \mid \underline{\underline{F}}_z] = 0$, an *increasing process* if $X(z,z'] \geqq 0$ and a *bounded variation* process if it is the difference of two increasing processes.

Denote by S the set of all decreasing lines, i.e. $L \in S$ if and only if

(i) $\forall z, z' \in L \Rightarrow$ either $z \wedge z'$ or $z' \wedge z$.

(ii) $\forall z \in \mathbb{R}_+^2$ and $z \notin L$, $\exists z' \in L: z<z'$ or $z'<z$.

For every $L \in S$, R_L will denote $\{z: \exists z' \in L, z \leqq z'\}$. We denote $L \leqq L'$ if $R_L \subseteq R_{L'}$ and

$$L \wedge L' = \sup\{L'': L'' \leqq L \quad \text{and} \quad L'' \leqq L'\},$$

$$L \vee L' = \inf\{L'': L'' \geqq L \quad \text{and} \quad L'' \geqq L'\}.$$

For each $z \in \mathbb{R}_+^2$, $z=(s,t)$, $\bar{z}^{(1)}$ and $\bar{z}^{(2)}$ will denote the horizontal and vertical lines starting with z and continuing to infinity, i.e. $\bar{z}^{(1)} = [(s,t), (s,\infty))$, $\bar{z}^{(2)} = [(s,t), (\infty,t))$. Set $\bar{z} = \bar{z}^{(1)} \cup \bar{z}^{(2)}$. Also denote $\underline{z}^{(1)} = [(s,0), (s,t)]$, $\underline{z}^{(2)} = [(0,t), (s,t)]$ and set

$\underline{z}=\underline{z}^{(1)}\cup\underline{z}^{(2)}$. Clearly $\bar{z}$ and $\underline{z}$ belong to S and $R_z=R_{\bar{z}}$. Every decreasing line L can be approximated by a decreasing sequence of "stepped decreasing lines" $\{L_n\}$. This sequence can be defined, for example, via the grid on $\mathbb{R}_+^2$ induced by the dyadic numbers of order n.

A stopping line is a function $L:\Omega\to S\cup\{\infty\}$ such that $\forall z$,

$$\{\omega: R_z \subseteqq R_L(\omega)\}\in \underline{\underline{F}}_z$$

namely

$$\{\omega: \underline{z}\leqq L(\omega)\}\in \underline{\underline{F}}_z.$$

It is known that every stopping line can be approximated by a sequence of stepped stopping lines. A stopping point is a function $z:\Omega\to\mathbb{R}_+^2\cup\{\infty\}$ such that $\bar{z}$ is a stopping line. A sequence of decreasing lines L_n, $n=1,2,\ldots$ is said to converge to L if $d(L,L_n)\to 0$ where $d(L_a,L_b)$ is defined by the Hausdorff topology:

$$d(L_a,L_b)=\sup\{d(a,L_b),d(b,L_a)\}$$

$a\in L_a$, $b\in L_b$ and

$$d(a,L_b)=\inf d(a,b)\,,\qquad b\in L_b$$

and $d(a,b)$ is the Euclidean distance.

All the two parameter functions and processes to be considered here will be assumed to be right-continuous and vanish on the axes, $(s,0)$ and $(0,t)$.

II. The Deterministic Case

There are several extensions of the notion of bounded variation from the one variable case to the two variable case. The simplest are the following. (a) The Vitali definition: $f(x,y)=A(x,y)-B(x,y)$ where A and B are increasing in the sense of measures namely $A(z,z']\geqq 0$, $B(z,z']\geqq 0$ whenever $z<z'$. (b) The Arzela definition: $f(x,y)$ is a one parameter function of bounded variation on every increasing path (cf. [5] for a detailed comparative study of the different notions of bounded variation in the plane).

Turning to the stopping problem, let $f:\mathbb{R}_+^2\to\mathbb{R}$ and $L\in S$ and assume that L is bounded ($\sup(s:(s,t)\in L)<\infty$ and $\sup(t:(s,t)\in L)<\infty$). Further assume that L is stepped of order n and z_i, $i=0,1,\ldots,n$ denote the corners of L counted from the upper left to the lower right, namely, $z_i=(s_i,t_i)$, $z_0=(0,t_0)$, $z_n=(s_n,0)$ and for i odd $s_{i+1}=s_i$, $t_{i+1}<t_i$ and for i even $s_{i+1}>s_i$, $t_{i+1}=t_i$. Then $f(L)$, the value of the f at the line L is clearly well defined to be

$$f(L)=\sum_{i\,\text{odd}}(-f(z_{i+1})+f(z_i))=\sum_{i\,\text{even}}f(z_{i+1})-f(z_i)\,. \tag{2.1}$$

Definitions. Let $f:\mathbb{R}_{+}^2\to\mathbb{R}$ be a right-continuous function and $L\in S$. f is said to be stoppable at L if for any sequence of stepped decreasing lines $\{L_n\}_n$ converging to L, the limit $f(L_n)$ exists and does not depend on the chosen sequence. The limit is denoted $f(L)$ and is called the value of f at the line L. f is said stoppable if it is stoppable at each $L\in S$. Note that if f is a function of bounded variation in the Vitali sense then f is stoppable since $f(L)=\int I_{R_L}df$ where the integral is a two dimensional Stieltjes integral and functions of bounded variation in the Vitali sense are those which induce a bounded measure on the Borel sets of the plane.

Proposition 2.1. (a) *The class of stoppable processes is strictly larger than the class of functions of bounded variation in the Vitali sense.* (b) *The class of functions which are of bounded variation in the Arzela sense is not included in the class of stoppable functions.*

Proof. (a) The following example exhibits a continuous and stoppable function which is not of Vitali bounded variation. It is adapted from [5]. Let us divide the unit square into four quarter squares and let σ_1 denote the upper-right quarter square. Next divide the lower-left quarter square into four quarter squares and let σ_2 denote the quarter square nearest to σ_1 etc. We obtain in this way an infinite sequence of squares converging towards the origin $(0,0)$. In each σ_j, let $f(x,y)$ be defined by the surface of a regular pyramid whose base is σ_j and height $1/j$; let $f(x,y)$ vanish over the rest of $R_{(1,1)}$. The function $f(x,y)$ is of unbounded variation in the Vitali sense [and also $f(x,x)$ is a one parameter function of unbounded variation] however any $L \in S$ intersects with only one of the squares σ_j and consequently $f(x,y)$ is stoppable. Turning to (b), let $f(x,y) = I_{\{x+y \geqq 1\}}(x,y)$. This is a right-continuous function increasing on every increasing path which is not stoppable. Similarly, let $g(x,y) = f(1-x,y)$ where $f(x,y)$ is as defined in the proof of part (a) then $g(x,y)$ is continuous but not stoppable.

Remark. Note that in general, even if f is stoppable it need not be of bounded variation on decreasing paths (e.g., $f(s,t) = g(s)$ where $g(\cdot)$ is not of bounded variation then $f((z,z']) = 0$ for every rectangle $(z,z']$). However, if f is stoppable and if for every decreasing path L the one parameter functions $f(L \wedge (x,0))$ and $f(L \wedge (0,y))$ are of bounded variation in x and y respectively then f is of bounded variation on all decreasing paths. The proof is straightforward noting that if z_1 and z_2 are two points such that $z_1 \wedge z_2$, then

$$\begin{aligned} &f(z_1) + f(L \wedge (\bar{z}_2^{(1)} \cup \underline{z}_2^{(1)})) - f(L \wedge (\bar{z}_1^{(1)} \cup \underline{z}_1^{(1)})) \\ &\quad = f(z_2) + f(L \wedge (\bar{z}_1^{(2)} \cup \underline{z}_1^{(2)})) - f(L \wedge (\bar{z}_2^{(2)} \cup \underline{z}_2^{(2)})), \end{aligned}$$

which yields an upper bound for $|f(z_1) - f(z_2)|$.

III. The Random Case

As in the one parameter case, the theory of integration in the random case is more interesting than in the deterministic case and the class of processes with respect to which an integral can be defined is larger than in the deterministic case.

Definition. A process $X = \{X_z, z \in \mathbb{R}_+^2\}$ is called a 1-martingale (2-martingale) if it is adapted and for each fixed t, $\{X_{(s,t)}, \underline{F}_{(s,t)} s \geqq 0\}$ is a martingale (for each fixed s, $\{X_{(s,t)}, \underline{F}_{(s,t)} t \geqq 0\}$ is a martingale). The predictable σ-field is the σ-field generated by the sets $F \times (z,z']$ where $F \in \underline{F}_z$ in the product space $\Omega \times \mathbb{R}_+^2$.

A two parameter integrator (or semi-martingale) is a right-continuous adapted process X such that the stochastic integral $\int \varphi dX$ is well defined for all the bounded predictable processes $\varphi=\{\varphi_z, z\in\mathbb{R}^2_+\}$ as the limit in probability of the stochastic integral with respect to simple approximations to φ (cf. [2, 7]). Let $z_a=(s_a, t_a)$, $s_a<\infty$, $t_a<\infty$ be fixed; from now on we will consider integration over the finite rectangle R_{z_a}. The class of integrators over R_{z_a} contains the finite variation processes (in the sense of Vitali), the L^p $(p>1)$ martingales and [9] processes of the form $E[B_t \mid \underline{F}^1_s]$ where $\{B_t, 0\leqq t\leqq t_a\}$ is a process of integrable variation and B_t is $\underline{F}_{s_a,t}$ adapted and satisfies a very stringent condition of absolute continuity with respect to a deterministic measure. However the class of integrators cannot be very large since it was shown by Bakry in [1] that in general, processes of the form $E(B_t \mid \underline{F}^1_s)$ need not be integrators. In this section we show that processes of the form $E(B_t \mid \underline{F}^1_s)$ are stoppable and more generally, every weak martingale satisfying some boundary conditions is stoppable.

Let X be a two parameter process vanishing on the axes and λ a stepped stopping line. Then $X(\lambda)$, the process X stopped at λ as defined by (2.1) is well defined for every $\omega\in\Omega$. Turning to general stopping lines, recall that every stopping line can be approximated from above by a decreasing sequence of dyadic stepped stopping lines, we will denote this sequence by $\{\lambda_n\}_{n=1}^{\infty}$.

Definition. Let X be a two parameter process, X is said to be stoppable if for any stopping line λ, the sequence $\{X(\lambda_n)\}_n$ converges in probability. The limit is denoted by $X(\lambda)$ and is called the value of the process stopped at λ.

Remarks. (a) We have clearly $X(\lambda_n)=\int 1(R_{\lambda_n})dX$. Therefore, since R_λ is a predictable set, every integrator is stoppable but the converse does not hold. (b) The term "stoppable process" could perhaps be replaced by "quasi integrator".

Theorem 3.1. *Let $M=\{M_z, z\in R_{z_a}\}$ be a square integrable weak martingale and further assume that the one parameter boundary processes $\{M_{(s_a,t)}, \underline{F}_{(s_a,t)} t\geqq 0\}$ and $\{M_{(s,t_a)}, F_{(s,t_a)} s\geqq 0\}$ are each the sum of a one parameter martingale and a one parameter process of integrable variation, then M is stoppable.*

Remark. The example of Bakry [1] satisfies the conditions of the theorem and is therefore stoppable but as shown in [1] it is not an integrator.

Proof. Note first that under the conditional independence assumption ($F(4)$) a weak martingale satisfying the conditions of the theorem can be decomposed as the sum of a 1-martingale and a 2-martingale both satisfying the condition of the theorem. Furthermore a 1-martingale satisfying the conditions of the theorem can be decomposed into the sum of a square integrable martingale and a 1-martingale of the form

$$M_{s,t}=E(B_t \mid \underline{F}_{s,t}) \tag{3.1}$$

where B_t is a right continuous one parameter process of integrable variation adapted to $\underline{F}_{s_a,t}$ (cf. [7, 9]). Since square integrable martingales are integrators and therefore stoppable, we may assume, without loss of generality that M is a 1-martingale satisfying (3.1). In order to prove the theorem we need the following:

Proposition 3.2. *Let $\{\underline{G}^n_t, 0\leqq t\leqq t_a\}$ be a decreasing sequence of one parameter filtrations and $\underline{G}_t=\bigcap_n \underline{G}^n_t$. Let $A=\{A_t\}$ be a process of integrable variation and* *denote by* ${}^nA=\{{}^nA_t\}$ $({}^\infty A=\{{}^\infty A_t\})$ *its dual optional projection relative to the filtration* $\{\underline{\underline{G}}^n_t, 0\leqq t\leqq t_a\}$ $(\{\underline{\underline{G}}_t, 0\leqq t\leqq t_a\})$. *Then for every bounded, measurable, right continuous, not-necessarily-adapted process* $\{X_t, 0\leqq t\leqq t_a\}$, *as* $n\to\infty$

$$E\left[\int_0^{t_a} X_t d({}^nA_t)\right]\to E\left[\int_0^{t_a} X_t d({}^\infty A_t)\right] \tag{3.2}$$

moreover, as $n\to\infty$, *a.s.*

$${}^nA_{t_a}\to{}^\infty A_{t_a} \tag{3.3a}$$

and

$${}^nA_{t-}\to{}^\infty A_{t-}\,. \tag{3.3b}$$

Proof of Proposition. Let $\underline{\underline{B}}$ denote the Borel σ-field on $[0,t_a]$ and let μ, μ_n, μ_∞ denote the stochastic measures on the product space $([0,t_a]\times\Omega, \underline{\underline{B}}\times\underline{\underline{F}})$ induced by the processes A, ${}^nA$, ${}^\infty A$ respectively. For a bounded and measurable process X we denote by $X^n=\{X^n_t\}$ $(X^\infty=\{X^\infty_t\})$ the optional projection of X with respect to the filtration $\{\underline{\underline{G}}^n_t\}$ $(\{\underline{\underline{G}}_t\})$. Then, (cf. e.g. [6]):

$$\begin{aligned}\mu_\infty(X)&=\mu(X^\infty)=\mu(X^n)+\mu(X^\infty-X^n)\\&=\mu_n(X)+\mu(X^\infty-X^n)\,.\end{aligned} \tag{3.4}$$

Consider $\mu(X^\infty-X^n)$, note that a.s.

$$X^\infty_t=E(X_t\,|\,\underline{\underline{G}}_t)\quad\text{and}\quad X^n_t=E(X_t\,|\,\underline{\underline{G}}^n_t)\,.$$

Therefore, for each t, $\{X^n_t\}_{n=1}^\infty$ is a reverse martingale sequence and by the reverse martingale convergence theorem $X^\infty_t-X^n_t\to0$ a.s. as $n\to\infty$. If the process X is right continuous then its optional projections are right continuous and the convergence holds for all t in $[0,t_a]$ a.s. Hence, by the dominated convergence theorem $\mu(X^\infty-X^n)\to0$. Consequently, by (3.4)

$$\mu_n(X)\to\mu_\infty(X)$$

for all bounded right continuous processes X. This proves (3.2). In order to prove (3.3a) let $\vartheta(0\leqq\vartheta\leqq t_a)$ be fixed and set

$$C_\vartheta=\{\omega: {}^\infty A_\vartheta>\limsup{}^nA_\vartheta\}$$

then by (3.2), setting $X_t=1_{C_{t_a}}$ (i.e., X_t is a random constant), the probability of C_{t_a} is zero. Similarly for $>$ replaced by $<$ and lim sup replaced by lim inf proves (3.3a). Similarly for some fixed ϑ let

$$X_t(\omega)=1_{C\vartheta}(\omega)\cdot 1_{[0,\vartheta)}(t)$$

then (3.3b) follows from (3.2) which completes the proof of the proposition.

Returning to the proof of the theorem, let λ be a stopping line and let $\{\lambda_n\}_n$ be the decreasing sequence of dyadic stepped stopping lines converging to λ. We assume that $R_\lambda\subset R_{\lambda_n}\subset R_{z_a}$. We reparametrize now the vertical direction by setting $u=t_a-t$, (cf. the figure below). Let $z(n,u)$ be the maximum of all points at the

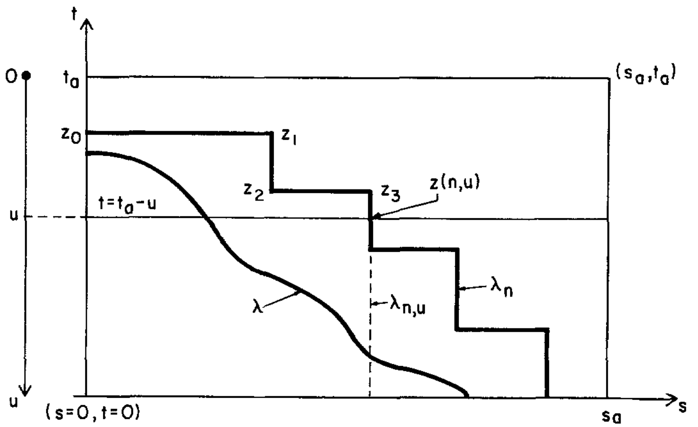

Fig. 1

intersection of λ_n with the horizontal line $t=t_a-u$ namely

$$\begin{aligned} z(n,u) &= \sup_s \{(s,t)\in\lambda_n, t=t_a-u\} \quad \text{if} \quad \{(s,t)\in\lambda_n, t=t_a-u\}\neq\phi \\ &= (0,t_a) \quad \text{otherwise}. \end{aligned}$$

The random point $z(n,u)$ is a stopping point.

Denote by $\lambda_{n,u}$ the stopping line which is the minimum of λ_n and the vertical line passing through $z(n,u)$ $(\lambda_{n,u}=\lambda_n \wedge (\bar{z}^{(1)}(n,u)\cup \underline{z}^{(1)}(n,u)))$ and by $\underline{\underline{G}}^n_u$ the σ-field induced by $\lambda_{n,u}$ namely, the σ-field generated by the random variables $\{M_z\cdot 1_{R(\lambda_{n,u})}(z), z\in R_{z_a}\}$. Note that for each n, $\{\underline{\underline{G}}^n_u\}$ is a one parameter filtration which satisfies the "usual conditions" and for each u, $\{\underline{\underline{G}}^n_u\}_n$ is a decreasing sequence of σ-fields. Set $\underline{\underline{G}}_u=\bigcap_{n=1}^{\infty}\underline{\underline{G}}^n_u$. Let $A_u=-M_{z_a}+M_{(s_a,(t_a-u)-)}$, hence A is right continuous. Let ${}^nA=\{{}^nA_u\}$ $({}^\infty A=\{{}^\infty A_u\})$ denote the dual optional projection of A relative to the filtration $\{\underline{\underline{G}}^n_u, 0\leqq u\leqq t_a\}$ $(\{\underline{\underline{G}}_u, 0\leqq u\leqq t_a\})$. In order to complete the proof of theorem we need the following

Lemma 3.3. $M(\lambda_{n,u})={}^nA_u-E({}^nA_u)={}^nA_u-EA_u$.

Proof of Lemma. Note that $\underline{\underline{G}}^n_u$ is "piecewise constant" assuming that λ_n is a dyadic stopping line with steps of length $t_a\cdot 2^{-n}$ in the t direction then $\underline{\underline{G}}^n_u$ remains constant for

$$k t_a 2^{-n}\leqq u<(k+1)\cdot t_a\cdot 2^{-n}$$

and for $u_1<u_2$ in this interval

$$\begin{aligned} {}^nA_{u_2}-{}^nA_{u_1} &= E(A_{u_2}-A_{u_1}\mid \underline{\underline{G}}^n_{u_1}) \\ &= E(A_{u_2}-A_{u_1}\mid \underline{\underline{F}}^1_{z(n,u)}). \end{aligned}$$

This and Theorem 76, Ch. VI of [6] yield the proof of the lemma. The proof of the theorem follows now from the lemma and from (3.3a).

From the proof of the theorem it follows directly that:

Corollary 3.4. *Let* $M_{s,t}=E(B_t \mid \underline{\underline{F}}_{(s,t_a)})$ *where B is adapted and of integrable variation and* λ *a stopping line, then*

$$M(\lambda)={}^{\infty}A_{t_a}-EA_{t_a}$$

where $A_u=+B_{(t_a-u)-}-B_{t_a}$ *and* ${}^{\infty}A$ *is the dual optional projection as defined above. Furthermore, for every* $u=t_a-t$

$$M(\lambda_u)={}^{\infty}A_u-EA_u$$

where $\lambda_u=\lim\limits_{n\to\infty}\lambda_{n,u}$. *Therefore for* λ *parametrized by* u, $M(\lambda_u)$ *is a semimartingale in the* u *parameter.*

The definition of a process stopped at a stopping line was given by an approximation from above and is associated with the fact that the process is right-continuous. An approximation from below can also be obtained as follows. Let λ be a predictable stopping line (its graph $[\lambda]=\{(\omega,z): z\in\lambda(\omega)\}$ belongs to the σ-field of the predictable sets). It is known (see for example [4]) that such a stopping line can be approximated from below by an increasing sequence of stepped stopping lines $\{\mu_n\}_n$. Let X be a process, then the sequence $\{X(\mu_n)\}_n$ is well defined. Using the same arguments as before, but taking the predictable projection instead of the optional one, we obtain that the sequence $\{X(\mu_n)\}_n$ converges uniquely under the same assumptions on the process X. The limit is denoted $X_-(\lambda)$. This makes it possible to study regular or left quasi-continuous processes X which satisfy $E[X(\lambda)]=E[X_-(\lambda)]$ for every predictable stopping line.

A companion to the notion of stopping lines is the concept of optional increasing paths introduced by J.B. Walsh (see [8]) which is defined to be an increasing and continuous family of stopping points $\Gamma=\{Z_t, t\geqq 0\}$. An optional increasing path splits the positive quadrant into two regions: the right-below side denoted by $\vec{\Gamma}$ and the left-above side denoted $\overleftarrow{\Gamma}$. Here too, we can define the expressions $X(\vec{\Gamma})$ and $X(\overleftarrow{\Gamma})$ by approximating Γ with stepped optional increasing paths. Such an approximation can be done, for example in $\vec{\Gamma}$, in the following way. Divide the quadrant by the dyadic points of order n. Take the interval from the origin at the point $(2^{-n},0)$ and then go up at the first point of intersection with Γ (this point is not necessarily a dyadic point). Then go right until the first point which is dyadic in the first coordinate, and so on. In this manner, we obtain a sequence $\{\vec{\Gamma}_n\}_n$ of stepped optional increasing paths in $\vec{\Gamma}$ converging to Γ and which is increasing with respect to a lexicographical order. In a symmetric way, we obtain a sequence $\{\overleftarrow{\Gamma}_n\}_n$ in $\overleftarrow{\Gamma}$. By the same construction as in the proof of Theorem 3.1 it follows that:

Proposition 3.5. *If X satisfies* (3.1) *then the sequences* $\{X(\vec{\Gamma}_n)\}$ *and* $\{X(\overleftarrow{\Gamma}_n)\}$ *converge in probability to the random variables denoted* $X(\vec{\Gamma})$ *and* $X(\overleftarrow{\Gamma})$ *respectively. Furthermore,* $X(\vec{\Gamma})$, $(X(\overleftarrow{\Gamma}))$ *is the dual optional (predictable) projection of the process* $\{B_t, 0\leqq t\leqq t_a\}$.

References

1. Bakry, D.: Une remarque sur les semimartingales à deux indices. Lect. Notes in Math. **850**, 671–672 (1980)
2. Bakry, D.: Semi-martingales à deux indices. Lecture Notes in Math. **920**, 355–369 (1982)
3. Cairoli, R., Walsh, J.B.: Stochastic integrals in the plane. Acta Math. **134**, 111–183 (1975)
4. Cairoli, R., Walsh, J.B.: Régions d'arrêt, localisations et prolongements de martingales. Z. Wahrscheinlichkeitstheor. Verw. Geb. **44**, 279–306 (1978)
5. Clarkson, J.A., Adams, C.R.: On definitions of bounded variation for functions of two variables. Trans. Am. Math. Soc., **35**, 824–854 (1933)
6. Dellacherie, C., Meyer, P.A.: Probabilities and potential, Part B. Amsterdam: North Holland 1982
7. Meyer, P.A.: Théorie élémentaire des processus à deux indices. Lecture Notes in Math. **863**, 1–39 (1981)
8. Walsh, J.B.: Optional increasing paths. Lect. Notes in Math. **863**, 172–201
9. Wong, E., Zakai, M.: Weak martingales and stochastic integrals in the plane. Ann. Probab. **4**, 570–586 (1976)

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