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The Asymptotic Behavior of the Principal Eigenvalue in a Singular Perturbation Problem with Invariant Boundaries

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Summary. We consider diffusion random perturbations of a dynamical system S^t in a domain $G \subset R^m$ which, in particular, may be invariant under the action of S^t. Continuing the study of [K 1–K 4] we find the asymptotic behavior of the principal eigenvalue of the corresponding generator when the diffusion term tends to zero.

1. Introduction

In a connected bounded domain $G \subset R^m$ with a C^2-class smooth boundary ∂G consider a nondegenerate elliptic differential operator

$$L = \tfrac{1}{2} \sum_{i,j \leq m} a^{ij}(x) \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i \leq m} b^i(x) \frac{\partial}{\partial x_i} \tag{1.1}$$

and a first order operator

$$\langle B(x), \nabla \rangle = \sum_{i \leq m} B^i(x) \frac{\partial}{\partial x_i}; \tag{1.2}$$

both operators have C^2-coefficients extended smoothly into the entire space R^m so that they remain bounded functions with bounded first and second derivatives and $(a^{ij}(x))$ becomes uniformly positive definite in R^m. The operator $L_\varepsilon = \varepsilon^2 L + \langle B, \nabla \rangle$ generates a Markov diffusion process $X_\varepsilon(t,x)$ satisfying the stochastic integral equation.

$$X_\varepsilon(t,x) = x + \int_0^t (B(X_\varepsilon(s,x)) + \varepsilon^2 b(X_\varepsilon(s,x))ds + \varepsilon \int_0^t \sigma(X_\varepsilon(s,x))dw(s)$$

$$\text{where} \quad b(x) = (b^1(x), \ldots, b^m(x)), \sigma(x) \tag{1.3}$$

is a matrix satisfying $\sigma(x)\sigma^*(x) = (a^{ij}(x)) \equiv a(x)$ and $w(t)$ is the m-dimensional Wiener process starting at zero (see, for instance [Fri 1]).

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The process $X_\varepsilon(t,x)$ is considered as a small random perturbation of the dynamical system S^t determined by the ordinary differential equation

$$\frac{d(S^t x)}{dt} = B(S^t x), \qquad S^0 x = x. \tag{1.4}$$

Let $\tau_\varepsilon(x,G)$ be the first exit time from G for the process $X_\varepsilon(t,x)$, i.e.,

$$\tau_\varepsilon(x,G) = \inf\{t : X_\varepsilon(t,x) \notin G\}. \tag{1.5}$$

Denote by $\lambda_\varepsilon(G)$ the principal eigenvalue of L^ε corresponding to zero Dirichlet data on ∂G, i.e., the eigenvalue with the greatest real part. It turns out (see [K 1]) that $\lambda_\varepsilon(G)$ is real, negative and can be represented as follows

$$\lambda_\varepsilon(G) = \lim_{t\to\infty} \frac{1}{t} \ln P\{\tau_\varepsilon(x,G) > t\} = \lim_{t\to\infty} \frac{1}{t} \ln \Phi_\varepsilon(t,G), \tag{1.6}$$

where

$$\Phi_\varepsilon(t,G) \equiv \sup_{x\in G} P\{\tau_\varepsilon(x,G) > t\} \tag{1.7}$$

and, as usual, $P\{\cdot\}$ means the probability of the event in brackets. The main purpose of the present paper is the study of the limiting behavior of λ_ε when $\varepsilon \to 0$.

As one learns from [K 4], $\lambda_\varepsilon \to -\infty$ as $\varepsilon \to 0$ unless there exists a closed S^t-invariant set which is contained in $\bar{G} \equiv G \cup \partial G$. The asymptotic behavior of $\lambda_\varepsilon(G)$ was investigated previously for several types of S^t-invariant sets in G such as hyperbolic points and circles (see [K 1]) and general hyperbolic sets (see [K 2]).

In the present paper we shall prove first a general localization theorem which enables one to treat separately different S^t-invariant sets. Namely, one ought to study the limit (1.6) for $\tau_\varepsilon(x,U)$ being the exit time from a small neighborhood U of an S^t-invariant set in place of the whole G, then to let $\varepsilon \to 0$ and, finally, to take the maximum over all S^t-invariant sets. This will be the desired limit of $\lambda_\varepsilon(G)$ as $\varepsilon \to 0$. The above procedure enables us to enrich from time to time the collection of "permitted" S^t-invariant sets in $\bar{G}$ which we can take care about. The main addition to this collection provided by this paper is the case when the whole boundary ∂G or its part are S^t-invariant which was not allowed in [K 1] and [K 2]. We shall consider both cases of attracting and repelling boundaries. Some other cases of S^t-invariant sets which one can treat by this approach will be discussed in the concluding Sect. 8. In Appendix, we consider the asymptotic behavior of the exit time and the exit distribution in an attracting boundary case giving, in particular, an example of divergence of the exit distribution when $\varepsilon \to 0$.

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2. Assumptions and Main Result

A sequence of points $x_0, \ldots, x_n \in \bar{G}$ will be called δ-pseudo-orbit if

$$|S^1 x_i - x_{i+1}| \leqq \delta \quad \text{for} \quad i = 0, 1, \ldots, n, \tag{2.1}$$

where $|a|$ is the length of a vector a and $\bar{Q}$ denotes the closure of Q. For a pair of points $x, y \in \bar{G}$ we shall write $x \to y$ if for any $\delta > 0$ there exist a non-negative $t \leqq 1$ and a δ-pseudo-orbit $x_0, \ldots, x_n \in \bar{G}$ such that $S^t x = x_0$ and $x_n = y$. Since we require for a δ-pseudo-orbit to stay in $\bar{G}$ the above relation "$\to$" may not be transitive. Its extension "$\succ$" to a transitive relation is defined in the following way: $z \succ x$ iff there exists a sequence of points $y_0, \ldots, y_k \in \bar{G}$ such that $y_0 = x$, $y_k = z$, and $y_i \to y_{i+1}$ for all $i = 0, 1, \ldots, k-1$. If $x \succ y$ and $y \succ x$ then we shall write $x \sim y$. It is easy to see that "$\sim$" is an equivalence relation. As usual, any maximal set of equivalent points in $\bar{G}$ will be called an equivalence class. One concludes from the definition that any equivalence class is a close set.

As usual, a closed set K is called S^t-invariant if $S^t K = K$ for all $t \geqq 0$. Suppose that there exist a finite collection of S^t-invariant disjoint equivalence classes $K_1, \ldots, K_\nu \subset G \cup \partial G$ satisfying the following Assumption A.

(A1) $\bigcup_i K_i$ contains the limit set of the dynamical system S^t in $\bar{G}$ i.e. for any $x \in \bar{G}$ all limit points of $S^t x$ as $t \to \pm\infty$ which belong to $\bar{G}$ belong also to $\bigcup_i K_i$;

(A2) one can choose open disjoint sets $U_i \subset G$, $i = 1, \ldots, \nu$ with smooth boundaries ∂U_i such that $U_i \supset K_i \cap G$; the relative interior of $\partial U_i \cap \partial G$ in ∂G contains $K_i \cap \partial G$; the limit

$$\Lambda(K_i) \equiv \lim_{\varepsilon \to 0} \lim_{t \to \infty} \frac{1}{t} \ln \Phi_\varepsilon(t, U_i) \tag{2.2}$$

exists (with Φ_ε defined by (1.7)), and for some positive $\beta_0 < 1$ and each $\delta > 0$ there is $\varepsilon(\delta) > 0$ so that if $\varepsilon \leqq \varepsilon(\delta)$ then one can find a positive $t(\varepsilon, \delta) \leqq \varepsilon^{-2(1-\beta_0)}$ satisfying

$$\Phi_\varepsilon(t(\varepsilon, \delta), U_i) \leqq \exp(\Lambda(K_i) + \delta) t(\varepsilon, \delta)) . \tag{2.3}$$

Now we can formulate the "localization theorem".

Theorem 2.1. *Under Assumption A*

$$\lim_{\varepsilon \to 0} \lambda_\varepsilon(G) = \max_{1 \leqq i \leqq \nu} \Lambda(K_i) \tag{2.4}$$

and the numbers $\Lambda(K_i)$ defined by (2.2) *are determined by compacts K_i only, i.e. they do not depend on the choice of U_i.*

Remark 2.1. One can check that the proof of the Theorem 2.1 does not actually require that the compacts K_i are equivalence classes. It suffices to assume the following two conditions:

(i) $x, y \in K_i$ and $y \succ z \succ x$ imply $z \in K_i$;

(ii) if we write $K_j \succ K_i$ provided $y \succ x$ for some $x \in K_i$ and $y \in K_j$, then in any chain $K_{i_1} \succ K_{i_2} \succ \ldots \succ K_{i_l}$ each index may appear only once up to trivial repetitions.

Remark 2.2. We define the order "$\succ$" in the way different from [K 1] but both definitions are equivalent. In [K 1] we wrote $K_j \succ K_i$ if one could find indices $i_0 = i$, $i_1, \ldots, i_{l-1}$, $i_l = j$ and points $z_1, \ldots, z_l$ such that $S^t z_k$ approaches K_{k-1} when $t \to -\infty$ and it approaches K_k when $t \to \infty$. Clearly this definition implies the definition given in Remark 2.1 above. The opposite is also true which follows from Corollary 3.1 below.

Theorem 2.1 enables one to reduce the problem to the study of principal eigenvalues for the operator L_ε restricted to small neighborhoods of compacts K_i. In order to do this by probabilistic means one needs to know escaping rates for the process $X_\varepsilon(t)$ from small neighborhoods of K_i. This study was accomplished for $K_i \subset G$ being a hyperbolic fixed point or a hyperbolic invariant circle in [K 1] and for a general hyperbolic invariant set in [K 2]. In all these cases Assumption (A 2) is satisfied and the number $\Lambda(K_i)$ can be obtained as

$$\Lambda(K_i) = \lim_{T \to \infty} \frac{1}{T} \ln \text{ volume} \left\{x: \sup_{0 \leqq t \leqq T} \operatorname{dist}(S^t x, K_i) \leqq \delta\right\} \tag{2.5}$$

provided $\delta > 0$ is small enough. The conjecture is, that (2.5) remains true under very general circumstances provided $K_i \subset G$. If K_i is an isolated fixed point (not necessarily hyperbolic) then the number $\Lambda(K_i)$ can be easily calculated by the formula (2.5) which gives

$$\Lambda(K_i) = -\sum_i \max(\operatorname{Re}\alpha_i, 0) \tag{2.6}$$

where $\alpha_i, i = 1, \ldots, m$ are all eigenvalues of the matrix Π such that

$$B(x) = \Pi(x - K_i) + O(|x - K_i|^2)$$

i.e. Π is the linear part of $B(x)$ near the fixed point K_i. When K_i is a general hyperbolic set (see [K 2]) then $\Lambda(K_i)$ turns out to be the, so called topological pressure corresponding to the differential expanding rate along unstable directions.

In all above cases K_i was supposed to be strictly inside of G. Next we are going to discuss the situation when K_i is an S^t-invariant connected component Γ of the boundary ∂G of G. Thus Γ is a closed smooth surface of the codimension one. It is easy to see that one can pick up an open neighborhood U of Γ in R^m such that any point $x \in U$ has a unique representation

$$x = \gamma(x) + \varrho(x) n(x) \tag{2.7}$$

where $\gamma(x) \in \Gamma$,

$$|\varrho(x)| = |x - \gamma(x)| = \operatorname{dist}(x, \Gamma), \tag{2.8}$$

and $n(x) = n(\gamma(x))$ is the interior unit normal to Γ in the sense that it points out into the interior of G i.e. $\varrho(x) > 0$ if $x \in U \cap G$. Characterizing any point $x \in U$ by the pair $(\gamma(x), \varrho(x))$ we get a system of coordinates in U. In these coordinates the normal component $q(x)$ of the vector field $B(x)$ satisfies

$$\frac{d\varrho(S^t x)}{dt} = q(S^t x) = q(\gamma(S^t x), \varrho(S^t x)). \tag{2.9}$$

For each $\gamma \in \Gamma$ define

$$\alpha(\gamma) = \left.\frac{\partial q(\gamma, \varrho)}{\partial \varrho}\right|_{\varrho = 0}. \tag{2.10}$$

Then for $x \in U$ one can write

$$q(x) = \alpha(\gamma(x))\varrho(x) + \psi(x)\varrho^2(x), \tag{2.11}$$

where ψ is a bounded function in U. Regarding the asymptotic behavior of the dynamical system S^t on Γ we shall need also the following Assumption B: uniformly in $\gamma \in \Gamma$ the limit

$$\lim_{t \to \infty} \frac{1}{t} \int_0^t \alpha(S^u \gamma) du = \alpha_0 \tag{2.12}$$

exists and it is independent of γ.

Remark 2.3. The above condition is satisfied if the dynamical system S^t restricted to the S^t-invariant surface Γ is uniquely ergodic i.e. it has a unique invariant measure on Γ. This follows from the continuous time version of Theorem 6.19 in [W].

Now we are able to formulate our result concerning invariant boundaries.

Theorem 2.2. *Let Γ be an S^t-invariant connected component of the boundary ∂G and let U be an open neighborhood of Γ with a smooth boundary ∂U such that $U \cup \partial U$ contains no closed S^t-invariant set except for Γ. Suppose that Assumption B holds and put $U^G = U \cap G$.*

(i) *If $\alpha_0 < 0$ (the case of an attracting boundary) then the limit*

$$\Lambda(\Gamma) = \lim_{\varepsilon \to 0} \lim_{t \to \infty} \frac{1}{t} \ln \Phi_\varepsilon(t, U^G) \tag{2.13}$$

exists and $\Lambda(\Gamma) = \alpha_0$;

(ii) *If $\alpha(\gamma) \equiv 0$ on Γ (the case of a neutral boundary) then the limit* (2.14) *exists and $\Lambda(\Gamma) = 0$;*

(iii) *If $\alpha_0 > 0$ (the case of a repulsing boundary), and, in addition, the dynamical system S^t restricted to Γ has an invariant measure on Γ possessing a smooth positive density with respect to the volume on Γ, then $\Lambda(\Gamma) = -2\alpha_0$.*

Furthermore, in the above cases, for any $\delta, \beta > 0$,

$$\Phi_\varepsilon(t, U^G) \leqq \exp((\Lambda(\Gamma) + \delta)t) \tag{2.14}$$

provided $\varepsilon \leqq \varepsilon(\delta)$ and $t \geqq \left(\ln \frac{1}{\varepsilon}\right)^{1+\beta}$, and so, Γ and U^G can play the role of a pair K_i and U_i in assumption A.

Remark 2.4. One of examples we have in mind which satisfies the assumptions of Theorem 2.2 is the case when the flow S^t on Γ is diffeomorphically conjugate to an irrational rotation on an $(m-1)$-dimensional torus. According to [L] this will be the case if S^t restricted to Γ is a dynamical system with a discrete (pure-point) spectrum and smooth eigenfunctions.

Remark 2.5. We are not able to prove (ii) of the above Theorem 2.2 under the weaker condition $\alpha_0 = 0$. Still, the assumption $\alpha \equiv 0$ on Γ can be relaxed to $\alpha_0 = 0$ and $\alpha(\gamma) \leqq 0$ for all $\gamma \in \Gamma$, or if one has instead a fast convergence to zero of the average in (2.12) when $t \to \infty$.

Remark 2.6. It is not clear, if the additional condition imposed on S^t in (iii) to have a smooth invariant measure on Γ is necessary. We shall need this condition due to our method of deriving (iii) from (i) by means of an adjoint operator. The use of direct probabilistic estimates may eliminate this condition.

3. The "Localization" Theorem

We shall outline, first, the strategy of the proof of Theorem 2.1. The probability in the right hand side of (1.6) can be expressed through iterated integrals of the transition function of the diffusion X_ε stopped on ∂G. It turns out that up to a negligible error the probability is concentrated on paths which are δ-pseudo-orbits provided ε is small. Now if a δ-pseudo-orbit starts close to an equivalence class K and ends close to an equivalence class K' where δ is small enough then $K' \succ K$ (Lemma 3.1). This prevents δ-pseudo-orbits from cycling through neighbourhoods of the K_i, which are in finite number. So a δ-pseudo-orbit visits at most v of these neighbourhoods and remains for almost all time in some of them since the travel time between such neighbourhoods is bounded (Lemma 3.2). These lead to the estimation of the iterated integrals via probabilities of staying in small neighbourhoods of equivalence classes which leads to (2.4) in view of (1.6), (1.7), and (2.2).

Next we pass on to details.

Notice that $\tau_\varepsilon(x, G) \geqq \tau_\varepsilon(x, U_i)$ for any $x \in G$ provided $G \supset U_i$. Hence,

$$P\{\tau_\varepsilon(x, G) > t\} \geqq P\{\tau_\varepsilon(x, U_i) > t\}$$

and so by (1.6),

$$\lambda_\varepsilon(G) \geqq \lambda_\varepsilon(U_i) = \lim_{t \to \infty} \frac{1}{t} \ln \Phi_\varepsilon(t, U_i). \tag{3.1}$$

Thus by (2.2),

$$\liminf_{\varepsilon \to 0} \lambda_\varepsilon(G) \geqq \max_{1 \leqq i \leqq v} \Lambda(K_i). \tag{3.2}$$

Therefore it remains only to estimate $\lambda_\varepsilon(G)$ from above. Notice that by Markov property (see [Fri 1]) of the process $X_\varepsilon(t, x)$ for any open domain D one has

$$\begin{aligned}\Phi_\varepsilon(t+s, D) &= \sup_{x \in D} P\{\tau_\varepsilon(x, D) > t+s\} \\ &= \sup_{x \in D} E\chi_{\tau_\varepsilon(x, D) > t} E\chi_{\tau_\varepsilon(X_\varepsilon(t, x), D) > s} \leqq \Phi_\varepsilon(t, D)\Phi_\varepsilon(s, D),\end{aligned} \tag{3.3}$$

where E denotes the expectation and χ_A is the indicator function of an event A i.e. $\chi_A = 1$ if A occurs and $\chi_A = 0$ for otherwise.

Thus if $D = G$ then by (1.6) and the standard subadditivity argument (see, for instance, [W, Theorem 4.9]) it follows that

$$\lambda_\varepsilon(G) = \inf_{t > 0} \frac{1}{t} \ln \Phi_\varepsilon(t, G). \tag{3.4}$$

Let $X_\varepsilon^G(t,x)$ be the process $X_\varepsilon(t,x)$ stopped at the exit time $\tau_\varepsilon(x,G)$ i.e. X_ε^G is the process with absorption on ∂G. Denote by $P_\varepsilon^G(t,x,V)=P\{X_\varepsilon^G(t,x)\in V\}$ the transition probability of X_ε^G. As usual, the transition density $P_\varepsilon^G(t,x,y)$ of X_ε^G is the Radon-Nikodim derivative $\frac{p_\varepsilon^G(t,x,dy)}{dy}$. Recall, that $p_\varepsilon^G(t,x,y)$ turns out to be the fundamental solution of the equation

$$\frac{\partial p_\varepsilon^G}{\partial t}=L_\varepsilon p_\varepsilon^G,\ p_\varepsilon^G(t,x,y)\Big|_{x\in\partial G}=0, \tag{3.5}$$

where the operator L_ε is applied in the variable x. From Aronson's estimates [A] it follows that there exist constants $C_1,\beta_1>0$ such that

$$p_\varepsilon^G(1,x,y)\leqq C_1\varepsilon^{-m}\exp\left(-\beta_1\frac{|y-S^1x|^2}{\varepsilon^2}\right)\quad\text{for any } \varepsilon>0. \tag{3.6}$$

By (3.6) and the Chapman-Kolmogorov equality there is $C_2>0$ such that for every integer $n>0$ and any $\varepsilon,\delta>0$, one has,

$$\begin{aligned}P\{\tau_\varepsilon(x,G)>n\}&=P_\varepsilon^G(n,x,G)\\&=\int_G\dots\int_G p_\varepsilon^G(1,x,z_1)p_\varepsilon^G(1,z_1,z_2)\dots p_\varepsilon^G(1,z_{n-1}z_n)dz_1\dots dz_n\\&\leqq I_\varepsilon^{(1)}(\delta,n,x)+C_2\varepsilon^{-m}n\exp(-\beta_1\delta^2\varepsilon^{-2}),\end{aligned} \tag{3.7}$$

where

$$\begin{aligned}I_\varepsilon^{(1)}(\delta,n,x)=\int_{Q_\delta^G(S^1x)}\int_{Q_\delta^G(S^1z_1)}\dots\int_{Q_\delta^G(S^1z_{n-1})}&p_\varepsilon^G(1,x,z_1)\\ &p_\varepsilon^G(1,z_1,z_2)\dots p_\varepsilon^G(1,z_{n-1},z_n)dz_1\dots dz_n\end{aligned} \tag{3.8}$$

and we put $Q_\delta^G(y)=\{z\in G\cup\partial G:|z-y|<\delta\}$.

The integration in (3.8) is over δ-pseudo-orbits starting at x and staying in $G\cup\partial G$. This motivates our next step which is the study of possible behaviors of δ-pseudo-orbits under Assumption (A1).

Let K_i, $i=1,\dots,\nu$ be compacts introduced in Assumption A. We shall write $K_j\succ K_i$ if there exists a pair of points $x\in K_i$ and $y\in K_j$ such that $y\succ x$. Since K_i and K_j are equivalence classes than $K_j\succ K_i$ means that $y\succ x$ for any $x\in K_i$ and $y\in K_j$. Thus $K_j\succ K_i$ and $K_i\succ K_j$ implies $K_i=K_j$, i.e. $i=j$. The following result generalizes Lemma 4.2 from [K 1].

Lemma 3.1. *For any sufficiently small $\theta>0$ there exists a positive $\delta(\theta)<\theta$ such that, if for some $i_1,i_2\leqq\nu$ one can find a $\delta(\theta)$-pseudo-orbit $x_0,\dots,x_n\in G\cup\partial G$ satisfying*

$$\operatorname{dist}(x_0,K_{i_1})\leqq\delta(\theta),\quad \operatorname{dist}(x_j,K_{i_1})\geqq\theta,\quad\textit{and}\quad \operatorname{dist}(x_n,K_{i_2})\leqq\delta(\theta), \tag{3.9}$$

with $1<j\leqq n$, then $i_1\neq i_2$ and $K_{i_2}\succ K_{i_1}$.

Proof. Suppose that for any $\delta>0$ there exists a δ-pseudo-orbit $x_0^{(\delta)},\dots,x_{n(\delta)}^{(\delta)}$ such that

$$\operatorname{dist}(x_0^{(\delta)},K_{i_1})\leqq\delta\quad\text{and}\quad\operatorname{dist}(x_n^{(\delta)},K_{i_2})\leqq\delta. \tag{3.10}$$

Then one can pick up points $y^{(\delta)} \in K_{i_1}$ and $z^{(\delta)} \in K_{i_2}$ satisfying

$$|y^{(\delta)} - x_0^{(\delta)}| \leqq \delta \quad \text{and} \quad |z^{(\delta)} - x_{n(\delta)}^{(\delta)}| \leqq \delta .$$

Since K_{i_1}, K_{i_2} are S^t-invariant it follows that $y^{(\delta)}, x_1^{(\delta)}, \ldots, x_{n(\delta)-1}^{(\delta)}, z^{(\delta)}$ is a $C_3\delta$-pseudo-orbit where

$$C_3 = \sup_{|t| \leqq 1} \sup_x \|DS^t x\| + 2, \tag{3.11}$$

DS^t denotes the differential of S^t at x i.e. its Jacobian matrix and $\|\cdot\|$ is the Euclidean norm of matrices. If we assume this to be true for any $\delta > 0$ small enough then by the definition $K_{i_2} \succ K_{i_1}$. Hence if $K_{i_2} \succ K_{i_1}$ does not hold true then a δ-pseudo-orbit $x_0^{(\delta)}, \ldots, x_{n(\delta)}^{(\delta)}$ satisfying (3.10) may only exist for δ bigger than some $\tilde{\delta} > 0$. In other words, the existence of a δ-pseudo-orbit satisfying (3.10) with $\delta \leqq \tilde{\delta}$ implies already that $K_{i_2} \succ K_{i_1}$.

Now it remains to discuss the case $i_1 = i_2$. Fix $\theta > 0$. It suffices to show that there exists $\tilde{\tilde{\delta}} > 0$ such that any δ-pseudo-orbit $x_0^{(\delta)}, \ldots, x_{n(\delta)}^{(\delta)}$ satisfying (3.10) with $\delta \leqq \tilde{\tilde{\delta}}$ and $i_1 = i_2$ has no points whose distance from K_{i_1} is more than θ. Suppose that, on the contrary, one can find a sequence $\delta_l \to 0$ as $l \to \infty$ and corresponding δ_l-pseudo-orbits $x_0^{(\delta_l)}, \ldots, x_{n(\delta_l)}^{(\delta_l)} \in \bar{G}$ satisfying (3.10) with $\delta = \delta_l$, $i_1 = i_2$ and $\operatorname{dist}(x_{j(\delta_l)}^{(\delta_l)}, K_{i_1}) \geqq \theta$ for some index $j(\delta_l)$. Since the sequence $x_{j(\delta_l)}^{(\delta_l)}$ stays in a compact set and K_{i_1} is compact, as well, we can choose a subsequence, which we denote again by δ_l, such that $x_0^{(\delta_l)} \to x$, $x_{n(\delta_l)}^{(\delta_l)} \to y$ and $x_{j(\delta_l)}^{(\delta_l)} \to z$ as $l \to \infty$.

Then it follows from the definition that $y \succ z \succ x$. But this is impossible since K_{i_1} is the equivalence class, $x, y \in K_{i_1}$ and $\operatorname{dist}(z, K_{i_1}) \geqq \theta$. This completes the proof of Lemma 3.1. □

For any set V we shall use the notations

$$Q_\delta(V) = \{z \in R^m : \operatorname{dist}(z, V) < \delta\} \quad \text{and} \quad Q_\delta^G(V) = Q_\delta(V) \cap (G \cup \partial G). \tag{3.12}$$

Choose θ_0 such that

$$Q_{2C_3\theta_0}^G(K_i) \subset U_i \cup \partial U_i \text{ for all } i = 1, \ldots, v \text{ and put } \delta_0 = \delta(\theta_0) C_3^{-1} \tag{3.13}$$

with $\delta(\theta)$ given by Lemma 3.1 and C_3 defined by (3.11). Since the limit set of the dynamical system S^t restricted to $\bar{G}$ is closed and, according to Assumption (A1), it is disjoint with $\partial G \setminus \bigcup_i K_i$, then there exists $\delta_1 > 0$ small enough such that the set

$$\overline{Q_{\delta_1}(G)} \setminus \bigcup_{1 \leqq i \leqq v} Q_{\frac{1}{2}\delta_0}(K_i)$$

has no common points with this limit set. Thus the number

$$t(x) \equiv \inf \left\{ u \geqq 0 : S^u x \notin \overline{Q_{\delta_1}(G)} \setminus \bigcup_{1 \leqq i \geqq v} Q_{\frac{1}{2}\delta_0}(K_i) \right\} \tag{3.14}$$

is finite for any $x \in G \cup \partial G$. Furthermore, it is easy to see that $t(x)$ is upper semi-continuous, i.e. $t(x) \geqq \limsup_{y \to x} t(x)$ and so

$$T_0 \equiv \sup_{x \in G \cup \partial G} t(x) < \infty . \tag{3.15}$$

Clearly, if $x_0, \ldots, x_n$ is a δ-pseudo-orbit then

$$\max_{0 \leqq k \leqq n} |x_k - S^k x_0| \leqq C_3^{n-1}\delta. \tag{3.16}$$

Notice that if $S^t x \in Q_{\frac{1}{2}\delta_0}(K_i)$ for some x, t, and i then by the S^t-invariance of K_i it follows that

$$S^{[t]+1}x \in Q_{\frac{1}{2}C_3\delta_0}(K_i),$$

where $[t]$ denotes the integral part of t. This together with (3.14)–(3.16) imply

Lemma 3.2 *Any δ-pseudo-orbit $x_0, \ldots, x_n \in G \cup \partial G$ with $n \geqq T_0 + 1$ and $\delta \leqq \frac{1}{2}\delta_0 C_3^{-(T_0+1)}$ has at least one point in $\bigcup_{1 \leqq i \leqq v} Q^G_{\delta(\theta_0)}(K_i)$.*

From this we can derive the following result implying that the order relation among the compacts K_i considered in this paper is, actually, the same as in [K 1].

Corollary 3.1. *Suppose that $K_j \succ K_i$ then one can find indices $r_1 = i, r_2, \ldots, r_s = j$ and points $y_1, \ldots, y_{s-1}$ such that for all $k = 1, \ldots, s$,*

$$\operatorname{dist}(S^{-t}y_k, K_{r_k}) + \operatorname{dist}(S^t y_k, K_{r_{k+1}}) \to 0 \quad \text{as} \quad t \to \infty. \tag{3.17}$$

Proof. For any indices $1 \leqq j_1, \ldots, j_l \leqq v$ we shall introduce a set $\{j_1, \ldots, j_l\}$ of δ-pseudo-orbits $\omega = (x_0, \ldots, x_n)$ with $\delta \leqq \delta(\theta_0)$ such that $\omega \in \{j_1, \ldots, j_l\}$ if for all $i = 1, \ldots, n$,

$$x_i \in \left(\bigcup_{r=1}^{l} Q^G_{\delta(\theta_0)}(K_{j_r}) \right) \cup \left(\bar{G} \setminus \bigcup_{q=1}^{v} Q^G_{\delta(\theta_0)}(K_q) \right)$$

and there exist indices

$$k_0(\omega) = 0 \leqq i_1(\omega) \leqq k_1(\omega) \leqq \ldots \leqq i_l(\omega) \leqq k_l(\omega) \leqq i_{l+1}(\omega) = n$$

such that for $q = 1, \ldots, l$

$$i_q(\omega) = \inf\{r \geqq k_{q-1}(\omega) : x_r \in Q^G_{\delta(\theta_0)}(K_{j_q})\},$$
$$k_q(\omega) = \inf\{r > i_q(\omega) : x_r \notin Q^G_{\theta_0}(K_{j_q})\},$$

and if $k_l(\omega)$ is not defined by the last relation, i.e. if $x_r \in Q^G_{\theta_0}(K_{j_l})$ for all $r \geqq i_l(\omega)$ then we put $k_l(\omega) = n$. From Lemma 3.1 it follows that if $\{j_1, \ldots, j_l\}$ is not empty then $K_{l_l} \succ K_{j_{l-1}} \succ \ldots \succ K_{j_1}$ and all these compacts are different. Furthermore by Lemma 3.2 if $\delta \leqq \frac{1}{2}\delta_0 C_3^{-(T_0+1)}$ then $i_{q+1}(\omega) - k_q(x) \leqq T_0$.

To prove Corollary 3.1 it suffices to consider the case when for any $\delta > 0$ there exists a δ-pseudo-orbit $\omega^{(\delta)} = (x_0^{(\delta)}, \ldots, x_{n(\delta)}^{(\delta)})$ such that $x_0^{(\delta)} \in K_i$ and $x_{n(\delta)}^{(\delta)} \in K_j$. Taking into account the above arguments it is easy to see that one can choose a sequence $\delta_r \to 0$ as $r \to \infty$ and indices $j_1 = i$, $j_2, \ldots, j_l = j$ such that for r big enough $\omega^{(\delta_r)} \in \{j_1, \ldots, j_l\}$ and there exist limits $z_q = \lim_{r \to \infty} x^{(\delta_r)}_{k_q(\omega^{(\delta_r)})}$ for all $q = 1, \ldots, l-1$. Since the limit set of the dynamical system S^t in each $Q^G_{\theta_0}(K_{j_q})$ must be contained in K_{j_q}, then either $\operatorname{dist}(S^{-t}z_q, K_{j_q}) \to 0$ as $t \to \infty$ or there exists a positive $t_q < \infty$ such that

$S^{-t_q}z_q = z_{q-1}$. Similarly, either $\mathrm{dist}(S^t z_q, K_{j_{q+1}}) \to 0$ as $t \to \infty$ or one can find a positive $\tilde{t}_q < \infty$ such that $S^{\tilde{t}_q} z_q = z_{q+1}$. Since $x_0^{(\delta_r)} \in K_i$ and $x_{n(\delta_r)}^{(\delta_r)} \in K_j$ for the whole sequence $\delta_r \to 0$ then it follows that

$$\mathrm{dist}(S^{-t}z_1, K_{j_1}) + \mathrm{dist}(S^t z_{l-1}, K_{j_l}) \to 0 \quad \text{as} \quad t \to \infty .$$

Now put $y_1 = z_1$, $K_{r_1} = K_{j_1}$, and then define successively $y_{k+1} = z_q$, $K_{r_{k+1}} = K_{j_q}$ provided

$$\mathrm{dist}(S^t y_k, K_{j_q}) \to 0 \quad \text{as} \quad t \to \infty .$$

It is easy to see that the points $\{y_k\}$ and the compacts $\{K_{r_k}\}$ satisfy (3.17). □

Next, we come back to the proof of Theorem 2.1. Put $\delta_2 = \frac{1}{2}\delta_0 C_3^{-(T_0+1)}$ and consider the integral $I_\varepsilon^{(1)}(\delta_2, n, x)$ defined by (3.8). It follows from Lemmas 3.1 and 3.2 that any δ_2-pseudo-orbit $\omega = (x_0, \ldots, x_n)$ belongs to a set $\{j_1, \ldots, j_l\}$ with some $l \leqq v$ and the corresponding indices

$$k_{q-1}(\omega) \leqq i_q(\omega) \leqq k_q(\omega) \leqq n, \qquad q = 1, \ldots, l$$

satisfy

$$\sum_{1 \leqq q \leqq l} (k_q(\omega) - i_q(\omega)) \geqq n - T_0(v+1).$$

Since the integration in $I_\varepsilon^{(1)}(\delta_2, n, x)$ is over δ_2-pseudo-orbits then we can write

$$I_\varepsilon^{(1)}(\delta_2, n, x) \leqq \sum_{1 \leqq l \leqq v} \sum_{j_1, \ldots, j_l} \sum_{i_1 \leqq k_1 \leqq \ldots \leqq i_l \leqq k_l} I_\varepsilon^{(2)}(j_1, \ldots, j_l; i_1, \ldots, i_l; k_1, \ldots, k_l), \quad (3.18)$$

where

$$I_\varepsilon^{(2)}(j_1, \ldots, j_l; i_1, \ldots, i_l; k_1, \ldots, k_l) \equiv P\{X_\varepsilon^G(r, x) \in Q_{\theta_0}(K_{j_q}) \text{ for all } r = i_q, \ldots, k_q - 1 \text{ and all } q = 1, \ldots, l\},$$

the second sum in (3.18) is taken over $j_1, \ldots, j_l$ such that $\{j_1, \ldots, j_l\} \neq \phi$ and the third sum is taken over $i_1 \leqq k_1 \leqq \ldots \leqq i_l \leqq k_l$ satisfying

$$\sum_{1 \leqq q \leqq l} (k_q - i_q) \geqq n - T_0(v+1). \quad (3.19)$$

It is clear that the total number of elements in the sum in (3.18) does not exceed $v^v n^{2v}$. Hence this sum can be estimated by $v^v n^{2v}$-times the maximal element in the sum, i.e.

$$I_\varepsilon^{(1)}(\delta_2, n, x) \leqq v^v n^{2v} \max_{l \leqq v; j_1, \ldots, j_l; i_1 \leqq k_1 \leqq \ldots \leqq i_l \leqq k_l} I_\varepsilon^{(2)}(j_1, \ldots, j_l; i_1, \ldots, i_l; k_1, \ldots, k_l), \quad (3.20)$$

where the maximum is taken over the same set of indices as in the sum (3.18). By the Markov property of the process X_ε it follows easily that

$$I_\varepsilon^{(2)}(j_1, \ldots, j_l; i_1, \ldots, i_l; k_1, \ldots, k_l) \leqq \prod_{1 \leqq q \leqq l} \sup_{y \in Q_{\theta_0}(K_{j_q})} I_\varepsilon^{(3)}(j_q, k_q - i_q, y), \quad (3.21)$$

where

$$I_\varepsilon^{(3)}(j,r,y)\equiv P\{X_\varepsilon^G(s,y)\in Q_{\theta_0}(K_j) \text{ for all } s=1,\dots,r-1\}$$
$$=\int_{Q_{\theta_0}(K_j)}\dots\int_{Q_{\theta_0}(K_j)} p_\varepsilon^G(1,y,z_1)p_\varepsilon^G(1,z_1,z_2)\dots p_\varepsilon^G(1,z_{r-1},z_r)dz_1\dots dz_r \quad (3.22)$$

and Π denotes the product. By the strong Markov property (see [Fri 1]) of the process X_ε it follows that for any $v,w\in U_j$ one has

$$p_\varepsilon^G(1,v,w)=p_\varepsilon^{U_j}(1,v,w)+E\chi_{\tau_\varepsilon(v,U_j)<1}p_\varepsilon^G(1-\tau_\varepsilon(v,U_j),X_\varepsilon(\tau_\varepsilon(v,U_j),v),w) \quad (3.23)$$

where U_j is the domain introduced in Assumption (A2) and $p_\varepsilon^{U_j}$ is the transition density of the process $X_\varepsilon^{U_j}$ stopped at the exit time $\tau_\varepsilon(x,U_j)$. If $v\in Q_{\theta_0}(K_j)$ then by (3.13),

$$\inf_{0\leqq t\leqq 1}\operatorname{dist}(S^t v,\partial U_j)\geqq C_3\theta_0 \quad (3.24)$$

and so by the estimates of Sect. 2 from [VF] it follows that

$$E\chi_{\tau_\varepsilon(v,U_j)<1}=P\{\tau_\varepsilon(v,U_j)<1\}\leqq C_4\exp(-\delta_3\varepsilon^{-2}) \quad (3.25)$$

for some $C_4,\delta_3>0$ independent of $\varepsilon>0$ and $j=1,\dots,\nu$. This together with (3.6) and (3.23) give

$$p_\varepsilon^G(1,v,w)\leqq p_\varepsilon^{U_j}(1,v,w)+C_1C_4\varepsilon^{-m}\exp(-\delta_3\varepsilon^{-2}) \quad (3.26)$$

provided $v,w\in Q_{\theta_0}^G(K_j)$.

Substituting this estimate into (3.22) we obtain

$$\begin{aligned} I_\varepsilon^{(3)}(j,r,y)&\leqq C_1C_4r\varepsilon^{-m}\exp(-\delta_3\varepsilon^{-2})\\ &\quad+\int_{Q_{\theta_0}(K_j)}\dots\int_{Q_{\theta_0}(K_j)} p_\varepsilon^{U_j}(1,y,z_1)\dots p_\varepsilon^{U_j}(1,z_{r-1},z_r)dz_1\dots dz_r\\ &\leqq C_1C_4r\varepsilon^{-m}\exp(-\delta_3\varepsilon^{-2})+P\{\tau_\varepsilon(y,U_j)>r\}\\ &\leqq C_1C_4r\varepsilon^{-m}\exp(-\delta_3\varepsilon^{-2})+\Phi_\varepsilon(r,U_j), \end{aligned} \quad (3.27)$$

whereas $y\in Q_{\theta_0}^G(K_j)$. By (3.3) we have also for any $t>0$,

$$\Phi_\varepsilon(r,U_j)\leqq(\Phi_\varepsilon(t,U_j))^{\left[\frac{r}{t}\right]}, \quad (3.28)$$

where, again, $[\cdot]$ denotes the integral part. Finally, collecting (3.7), (3.20), (3.21), (3.27)–(3.28) and taking into account (3.19) we derive

$$\begin{aligned}\Phi_\varepsilon(n,G)&\leqq \nu^\nu n^{2\nu}\left(\max_{1\leqq j\leqq\nu}\Phi_\varepsilon(t,U_j)\right)^{\frac{n-T_0(\nu+1)}{t}-\nu}\\ &\quad+\nu^\nu n^{2\nu}((C_1C_4n\varepsilon^{-m}\exp(-\delta_3\varepsilon^{-2})+1)^\nu-1)\\ &\quad+C_2\varepsilon^{-m}n\exp(-\beta_1\delta_2^2\varepsilon^{-2}).\end{aligned} \quad (3.29)$$

Now if (2.3) is true for some $t=t(\varepsilon,\delta)\leqq\varepsilon^{-2(1-\beta_0)}$ then taking $n=n(\varepsilon)=[\varepsilon^{-2+\beta_0}]$ we obtain for ε small enough that

$$\Phi_\varepsilon(n(\varepsilon),G)\leqq 2v^v(n(\varepsilon))^{2v}\exp\left(\left(\delta+\max_{1\leqq j\leqq v}\Lambda(K_j)\right)(n(\varepsilon)-T_0(v+1)-vt(\varepsilon,\delta))\right). \tag{3.30}$$

By (3.4) and the choice of $n(\varepsilon)$ and $t(\varepsilon,\delta)$ this implies

$$\lambda_\varepsilon(G)\leqq\max_{1\leqq j\leqq v}\Lambda(K_j)+\delta+C_5(\varepsilon^{2-\beta_0}+\varepsilon^{\beta_0})\ln\left(\frac{1}{\varepsilon}\right) \tag{3.31}$$

for some $C_5>0$ independent of ε provided $\varepsilon>0$ is small enough. Letting $\varepsilon\to 0$ we have

$$\limsup_{\varepsilon\to 0}\lambda_\varepsilon(G)\leqq\max_{1\leqq j\leqq v}\Lambda(K_j)+\delta$$

and since $\delta>0$ is arbitrary it follows

$$\limsup_{\varepsilon\to 0}\lambda_\varepsilon(G)\leqq\max_{1\leqq j\leqq v}\Lambda(K_j)$$

which together with (3.2) proves Theorem 2.1. □

4. Auxiliary Gaussian Processes

Similarly to [K 3] our proofs will rely heavily upon comparison of the initial diffusion process X_ε with its Gaussian approximation which we shall study in this section. Let $K(t,s,x)$ be the solution of the matrix integral equation

$$K(t,s,x)=I+\int_s^t H(S^u x)K(u,s,x)du, \tag{4.1}$$

where $H(y)=(h_{ij}(y))=\left(\frac{\partial B^i(y)}{\partial y_j}\right)$ and I is the identity matrix.

Define

$$Y(t,x)=\int_0^t K(t,s,x)\sigma(S^s x)dw(s) \tag{4.2}$$

and $Z_\varepsilon(t,x)=S^t x+\varepsilon Y(t,x)$ which is an approximation of X_ε of order ε^2. It is easy to see that both $Y(t,x)$ and $Z_\varepsilon(t,x)$ are Gaussian processes, and $Z_\varepsilon(t,x)$ satisfies the following stochastic integral equation

$$Z_\varepsilon(t,x)=x+\int_0^t(B(S^u x)+H(S^u x)(Z_\varepsilon(u,x)-S^u x))du+\varepsilon\int_0^t\sigma(S^u x)dw(u). \tag{4.3}$$

Differentiating the equation $S^t x=x+\int_0^t B(S^u x)du$ in space variables one concludes that the solution of (4.1) can be expresses by means of the Jacobian matrices, i.e. the differentials DS^r_y of S^r at y, as

$$K(t,u,x)=DS^{t-u}_{S^u_x}. \tag{4.4}$$

From (4.2) it is clear that $Y(t,x)$ has the Gaussian distribution with zero mean and the covariance matrix

$$V(t,x)=\int_0^t K(t,u,x)a(S^u x)K^*(t,u,x)du\,, \tag{4.5}$$

which we express by

$$Y(t,x)\sim N(0,V(t,x))\,. \tag{4.6}$$

Here $a=(a^{ij})=\sigma\sigma^*$ is the matrix of coefficients in (1.1). By (4.4) it follows also

$$K(t,s,x)K(s,u,x)=K(t,u,x) \text{ which implies } K(t,s,x)K(s,t,x)=I \tag{4.7}$$

and

$$\frac{dK(t,s,x)}{ds}=-K(t,s,x)H(S^s x)\,. \tag{4.8}$$

We shall need the following estimates for probabilities connected with $Y(t,x)$.

Lemma 4.1. *There exists $M_1>0$ such that for $q>r$, $t\geqq 1$, $x\in R^m$, and $w\in R^m$ satisfying $|w|=1$, one has*

$$P\{r<\langle Y(t,x),w\rangle<q\}\geqq(2\pi)^{-1/2}(q-r)\,|V^{1/2}(t,x)w|^{-1}\exp(-M_1\max(|r|^2,|q|^2)) \tag{4.9}$$

and

$$|P\{\langle Y(t,x),w\rangle<r\}-\tfrac{1}{2}|\leqq M_1|r|\,, \tag{4.10}$$

where $\langle\cdot,\cdot\rangle$ denotes the inner product in R^m and $V^{1/2}$ is the square root of V. Furthermore,

$$P\{\langle Y(s,S^{t-s}x),w\rangle>r\}\leqq P\{\langle Y(t,x),w\rangle>r\} \text{ for any } t\geqq s\geqq 0\,. \tag{4.11}$$

Proof. Denote by $V^{1/2}(t,x)$ the unique smooth positive definite self-adjoint square root of $V(t,x)$ (see [Fre, Sect. 3.2]). Using the change of variables $z=V^{-1/2}y$ we derive from (4.6),

$$\begin{aligned}
&P\{r_1<\langle Y(t,x),w\rangle<r_2\}\\
&=\frac{1}{(2\pi)^{n/2}(\det V(t,x))^{1/2}}\int_{\{y:r_1<\langle y,w\rangle<r_2\}}\exp(-\tfrac{1}{2}\langle V^{-1}(t,x)y,y\rangle)dy\\
&=\frac{1}{(2\pi)^{n/2}}\int_{\{z:r_1<\langle z,V^{1/2}(t,x)w\rangle<r_2\}} e^{-|z|^2dz}\\
&=(2\pi)^{-1/2}\int_{\{s:r_1\leqq s|V^{1/2}(t,x)w|\leqq r_2\}} e^{\frac{-s^2}{2}}ds\,,
\end{aligned} \tag{4.12}$$

where the last equality is obtained by the direct integration taking an orthonormal basis in R^m whose first vector coincides with $V^{1/2}(t,x)w|V^{1/2}(t,x)w|^{-1}$. Since the matrix $a(z)=(a^{ij}(z))$ is uniformly positive definite then for some $\tilde{\beta}>0$ one derives from (3.11), (4.4), (4.5), and (4.7) that for any $t \geqq 1$ the following holds

$$|V^{1/2}(t,x)w|^2 \geqq \tilde{\beta} \int_{t-1}^{t} |K^*(t,u,x)w|^2 du$$

$$= \tilde{\beta} \int_0^1 |K^*(u,0,S^{t-u}x)w|^2 du \geqq \tilde{\beta}\Big(\sup_{z,|u|\geqq 1} \|K(0,u,z)\|\Big)^{-1} \geqq \tilde{\beta} C_3^{-1} \quad (4.13)$$

which together with (4.12) imply (4.9) and (4.10). To prove (4.11) notice that by (4.4),

$$\|V^{1/2}(s,S^{t-s}x)w\|^2 = \int_0^s \langle K(s,u,S^{t-s}x)a(S^{t-s+u}x)K^*(s,u,S^{t-s}x)w,w\rangle du$$

$$= \int_{t-s}^{t} \langle K(t-u,0,S^u x)a(S^u x)K^*(t-u,0,S^u x)w,w\rangle du$$

$$\leqq \int_0^t \langle K(t-u,0,S^u x)a(S^u x)K^*(t-u,0,S^u x)w,w\rangle du$$

$$= |V^{1/2}(t,x)w|^2, \quad (4.14)$$

which together with (4.12) gives (4.11). □

We shall need also the following result.

Lemma 4.2. *For any integer $k>0$ there exist $M_2(k), M_3(k)>0$ such that whenever $t,\varepsilon,\delta>0$ one has*

$$P\Big\{ \sup_{0\leqq u\leqq t} |S^u x - X_\varepsilon(u,x)| \geqq \delta \Big\} \leqq M_2(k)\Big(\frac{\varepsilon}{\delta}\Big)^{2k} \exp(M_4 kt), \quad (4.15)$$

and

$$P\Big\{ \sup_{0\leqq s\leqq t} |X_\varepsilon(s,x) - Z_\varepsilon(s,x)| \geqq \delta \Big\} \leqq M_3(k)\Big(\frac{\varepsilon^2}{\delta}\Big)^{2k} \exp(M_5 kt) \quad (4.16)$$

for some $M_4, M_5>0$ independent of t, ε, δ, and k.

Proof. Taking into account that the coefficients in (1.3) are bounded together with their derivatives we derive from (1.3) and (1.4) that

$$\sup_{0\leqq u\leqq t} |S^u x - X_\varepsilon(u,x)| \leqq C_6 \int_0^t \sup_{0\leqq u\leqq s} |S^u x - X_\varepsilon(u,x)| ds + C_6 \varepsilon^2 t$$

$$+ \varepsilon \sup_{0\leqq s\leqq t} \Big| \int_0^s \sigma(X_\varepsilon(u,x)) dw(u) \Big| \quad (4.17)$$

for some $C_6>0$ independent of ε, x, and t.

Thus, by Gronwall's inequality (see, for instance, [H, Chap. 3, Sect. 1]) one has

$$\sup_{0\leqq u\leqq t} |S^u x - X_\varepsilon(u,x)| \leqq e^{C_6 t}\Big(C_6 \varepsilon^2 t + \varepsilon \sup_{0\leqq s\leqq t} \Big| \int_0^s \sigma(X_\varepsilon(u,x)) dw(u) \Big| \Big). \quad (4.18)$$

Now taking the $2k$-th power of both sides, using the inequality

$$(a+c)^{2k} \leqq 2^{2k}(|a|^{2k}+|c|^{2k})$$

and applying the expectation we get

$$E \sup_{0\leqq u\leqq t} |S^u x - X_\varepsilon(u,x)|^{2k}$$
$$\leqq \varepsilon^{2k} e^{2k(C_6+\ln 2)}\left((C_6\varepsilon t)^{2k} + E \sup_{0\leqq s\leqq t}\left|\int_0^s \sigma(X_\varepsilon(u,x))dw(u)\right|^{2k}\right) \tag{4.19}$$

Employing the standard martingal estimates of moments of stochastic integrals (see [Fri 1, Chap. 4, Sect. 6]) one obtains

$$E \sup_{0\leqq u\leqq t} |S^u x - X_\varepsilon(u,x)|^{2k} \leqq M_2(k)\varepsilon^{2k} e^{M_4 kt} \tag{4.20}$$

which together with Chebyshev's inequality prove (4.15).

Next we can write

$$B(X_\varepsilon(u,x)) = B(S^u x) + H(S^u x)(X_\varepsilon(u,x) - S^u x) + \Psi(u,x,X_\varepsilon(u,x))\,|X_\varepsilon(u,x)-S^u x|^2, \tag{4.21}$$

where ψ is a bounded vector function. Thus (1.3) and (4.3) yield

$$\sup_{0\leqq s\leqq t} |X_\varepsilon(s,x) - Z_\varepsilon(s,x)| \leqq C_7 \int_0^t \sup_{0\leqq u\leqq s} |X_\varepsilon(u,x) - Z_\varepsilon(u,x)| ds$$
$$+ C_7 t \sup_{0\leqq u\leqq t} |X_\varepsilon(u,x) - S^u x|^2$$
$$+ C_6 \varepsilon^2 t + \varepsilon \sup_{0\leqq s\leqq t}\left|\int_0^s (\sigma(X_\varepsilon(u,x)) - \sigma(S^u x))dw(u)\right| \tag{4.22}$$

for some $C_7>0$ independent of ε, x, and t.

Employing again Gronwall's inequality we obtain

$$\sup_{0\leqq s\leqq t} |X_\varepsilon(s,x) - Z_\varepsilon(s,x)| \leqq e^{C_7 t}\Bigg(C_7 t \sup_{0\leqq u\leqq t} |X_\varepsilon(u,x) - S^u x|^2 + C_6\varepsilon^2 t$$
$$+ \varepsilon \sup_{0\leqq s\leqq t}\left|\int_0^s (\sigma(X_\varepsilon(u,x)) - \sigma(S^u x))dw(u)\right|\Bigg). \tag{4.23}$$

In the same way as in the proof of (4.19), we take the $2k$-th power of both sides in (4.23), then employ the standard moment estimates of the stochastic integral above, and, finally, use (4.20) to derive that

$$E \sup_{0\leqq s\leqq t} |X_\varepsilon(s,x) - Z_\varepsilon(s,x)|^{2k} \leqq M_3(k)\varepsilon^{4k} e^{M_5 kt} \tag{4.24}$$

for some $M_3(k)$, $M_5>0$ independent of ε, x, t. This together with Chebyshev's inequality give (4.16). □

The formula (4.9) shows that the estimates for probabilities connected with $Y(t,x)$ rely upon behavior of $|V^{1/2}(t,x)w|$. The following result provides necessary estimates near the attracting boundary.

Lemma 4.3. *Let Γ be a closed C^2-class smooth S^t-invariant $(m-1)$-dimensional surface satisfying Assumption B with $\alpha_0<0$. Then there exists an open set $U\supset\Gamma$ such that*

$$\operatorname{dist}(S^t x,\Gamma)\leqq C_8 e^{\frac{1}{2}\alpha_0 t}\operatorname{dist}(x,\Gamma) \tag{4.25}$$

for any $x\in U$ and

$$\sup_{z\in\Gamma}|V^{1/2}(t,z)n(S^t z)|\leqq C_9 \tag{4.26}$$

for any $t>0$ with $C_8, C_9>0$ independent of t.

Proof. By (2.9) and (2.11),

$$\ln\frac{\varrho(S^t x)}{\varrho(x)}=\int_0^t \alpha(\gamma(S^u x))du+\int_0^t \psi(S^u x)\varrho(S^u x)du. \tag{4.27}$$

Since the convergence in (2.12) is uniform then there exists $t_0>0$ such that

$$\int_0^{t_0}\alpha(S^u\gamma(x))du\leqq\tfrac{4}{3}\alpha_0 t_0<0 \tag{4.28}$$

for all x from a small neighbourhood of Γ. Furthermore,

$$|\varrho(S^u x)|=\operatorname{dist}(S^u x,\Gamma)\leqq|S^u\gamma(x)-S^u x|\leqq C_3^{u+1}|\varrho(x)| \tag{4.29}$$

where C_3 is defined by (3.11). Since the function α is smooth then for some $C_{10}>0$,

$$\begin{aligned}&|\alpha(\gamma(S^u x))-\alpha(S^u\gamma(x))|\leqq C_{10}|\gamma(S^u x)-S^u\gamma(x)|\\&\leqq C_{10}(|\gamma(S^u x)-S^u x|+|S^u x-S^u\gamma(x)|\leqq C_{10}C_3^{u+1}|\varrho(x)|.\end{aligned} \tag{4.30}$$

Thus, if $\varrho(x)$ is small enough then

$$\int_0^{t_0}|\alpha(\gamma(S^u x))-\alpha(S^u\gamma(x))|du+\int_0^{t_0}|\psi(S^u x)\varrho(S^u x)|du\leqq\frac{|\alpha_0|t_0}{4},$$

which together with (4.27) and (4.28) give

$$|\varrho(S^{t_0}x)|\leqq|\varrho(x)|e^{\frac{1}{2}\alpha_0 t_0}. \tag{4.31}$$

This means that one can choose an open neighborhood U of Γ such that (4.31) holds true for any $x\in U$. By (4.29) and (4.31),

$$|\varrho(S^t x)|\leqq C_3^{t_0}e^{\frac{1}{2}\alpha_0(t-t_0)} \tag{4.32}$$

proving (4.25). Hence we can pick up $T_1=T_1(U)$ such that

$$\operatorname{dist}(S^{T_1}x,\Gamma)=|\varrho(S^{T_1}x)|\leqq\tfrac{1}{2}\operatorname{dist}(x,\Gamma)=\tfrac{1}{2}|\varrho(x)| \tag{4.33}$$

for all $x\in\bar{U}$.
Next, by (4.5) for any $x\in\Gamma$ and $t>0$,

$$|V^{1/2}(t,x)n(S^t x)|^2\leqq C_{11}\int_0^t|K^*(t,u,x)n(S^t x)|^2du \tag{4.34}$$

for some $C_{11}>0$ independent of t. To estimate the integral above remark that $K(t,u,x)$ as the differential given by (4.4) maps tangent vectors of the S^t-invariant surface Γ at the point $S^u x$ to tangent vectors at $S^t x$. Thus, if $\langle v, n(S^u x)\rangle = 0$ then

$$\langle K^*(t,u,x)n(S^t x), v\rangle = \langle K(t,u,x)v, n(S^t x)\rangle = 0$$

and so

$$K^*(t,u,x)n(S^t x) = \varphi(t,u,x)n(S^u x) \tag{4.35}$$

for some scalar function $\varphi(t,u,x)$. Furthermore, for $x\in\Gamma$ and any number ζ with $|\zeta|$ small enough it follows from (4.4) that

$$|S^{t-u}(S^u x+\zeta n(S^u x)) - S^t x - \zeta K(t,u,x)n(S^u x)| \leqq C_{12}(t,x)\zeta^2, \tag{4.36}$$

where $C_{12}(t,x)>0$ is independent of ζ.

Next, it is easy to see [cf. (5.9) below] that

$$\begin{aligned} |\langle S^{t-u}(S^u x+\zeta n(S^u x)) - S^t x, n(S^t x)\rangle| &\leqq \varrho(S^{t-u}(S^u x+\zeta n(S^u x)) + C_{13}(t,x)\zeta^2 \\ &\leqq C_8 e^{\frac{1}{2}\alpha_0(t-u)}|\zeta| + C_{13}(t,x)\zeta^2, \end{aligned} \tag{4.37}$$

where $C_{13}(t,x)>0$ is independent of ζ and the last inequality follows from (4.25).

Since Γ is S^t-invariant then by (4.4) we see that $K^*(t,u,x)n(S^t x)$ is normal to Γ at $S^u x$, and so by (4.35)–(4.37),

$$\begin{aligned} |K^*(t,u,x)n(S^t x)| &= |\langle K(t,u,x)n(S^u x), n(S^t x)\rangle| \\ &\leqq C_8 e^{\frac{1}{2}\alpha_0(t-u)} + C_{14}(t,x)\,|\zeta|, \end{aligned} \tag{4.38}$$

where $C_{14}(t,x) = C_{12}(t,x) + C_{13}(t,x)$.

Letting $\zeta\to 0$ in (4.38) and using (4.34) we obtain (4.26) with

$$C_9 = (2C_8 C_{11}|\alpha_0|^{-1})^{1/2}, \quad \text{since} \quad \alpha_0<0. \quad \square$$

5. Attracting Boundary: An Upper Bound

First, we shall outline briefly the strategy of proofs of the upper and the lower bounds in this and the next sections. After some time $u(\varepsilon)$ the process X_ε will be very close to the attracting boundary Γ. Breaking the time into intervals of length $T(\varepsilon)$ of order $\left(\ln\frac{1}{\varepsilon}\right)^\beta$, $0<\beta<1$ we can approximate $X_\varepsilon(t,z)$ on each such interval by the Gaussian process Z_ε from the previous section. The probability of exit during the time $T(\varepsilon)$ for the process Z_ε starting very close to Γ can be estimated via a kind of reflection principle (see (5.11) and [Va, Sect. 7]) for the process $\varepsilon Y(t,z) = Z_\varepsilon(t,z) - S^t z$. The difference $S^t z$ between Z_ε and εY gives rise to the probability for $\varepsilon Y(T(\varepsilon),z)$ to stay in a narrow strip of width dist $(S^{T(\varepsilon)}z,\Gamma)$ with z already very close to Γ. Such probability has the order of this width which is about $\exp\left(\int_0^{T(\varepsilon)} \alpha(\gamma(S^u z))du\right)$. This together with (2.12) lead to the assertions (i) and (ii) of Theorem 2.2.

Next we pass to details. In this section we shall estimate the limit (2.13) from above. The same arguments as in Sect. 3 show that the limit (2.13) does not depend on a neighborhood U of an attracting boundary Γ provided $\bar{U}$ contains no closed invariant set except for Γ. Thus, we can consider $U \supset \Gamma$ described in Lemma 4.3. In this and in the next section we shall study only the local behavior of the process $X_\varepsilon(t,x)$ in the neighborhood U and so we shall simplify the notations writing

$$\tau_\varepsilon(x) \equiv \tau_\varepsilon(x, U \cap G), \qquad \tilde{X}_\varepsilon(t,x) \equiv X_\varepsilon^{UG}(t,x) = X_\varepsilon(\min(t, \tau_\varepsilon(x)), x), \tag{5.1}$$

and

$$q_\varepsilon(t,x,y) = p_\varepsilon^{UG}(t,x,y),$$

where, recall, p^{UG} is the transition density of the process X_ε^{UG} and $U^G = U \cap G$.

Denote

$$\delta_4(\varepsilon) = \varepsilon\left(\ln\frac{1}{\varepsilon}\right)^4, \qquad \delta_5(\varepsilon) = \varepsilon\left(\ln\frac{1}{\varepsilon}\right)^2,$$

$$\Gamma(\varepsilon) = \{x : 0 < \varrho(x) < \delta_4(\varepsilon)\}, \qquad k(\varepsilon) = \left[\left(\ln\frac{1}{\varepsilon}\right)^{1-\frac{\beta}{2}}\right],$$

$$T(\varepsilon) = \left[\left(\ln\frac{1}{\varepsilon}\right)^{\beta}\right] T_1, \qquad l(\varepsilon) = \left[\frac{1}{\ln 2}\left(\ln\frac{1}{\varepsilon} + \ln \sup_{x \in U \cap G} \varrho(x)\right)\right] + 1,$$

$u(\varepsilon) = T_1 l(\varepsilon)$, where T_1 was introduced in (4.33), $[\cdot]$ means the integral part and $0 < \beta < 1$ is an arbitrary fixed number. Then we can write,

$$\begin{aligned} &\sup_{x \in \overline{U \cap G}} P\{\tau_\varepsilon(x) > u(\varepsilon) + k(\varepsilon)T(\varepsilon)\} \\ &\quad = \sup_{x \in \overline{U \cap G}} \int_{U^G} q_\varepsilon(u(\varepsilon), x, z) P\{\tilde{X}_\varepsilon(k(\varepsilon)T(\varepsilon), z) \in U^G\} dz \\ &\quad \leqq \sup_{z \in \Gamma(\varepsilon)} P\{\tau_\varepsilon(z) > k(\varepsilon)T(\varepsilon)\} \\ &\qquad + \sup_{x \in \overline{U \cap G}} P\{\tilde{X}_\varepsilon(u(\varepsilon), x) \in U^G \backslash \Gamma(\varepsilon)\}. \end{aligned} \tag{5.3}$$

By the Markov property, similarly to (3.3), one can see that

$$\begin{aligned} \sup_{z \in \Gamma(\varepsilon)} P\{\tau_\varepsilon(z) > k(\varepsilon)T(\varepsilon)\} &\leqq \left(\sup_{z \in \Gamma(\varepsilon)} P\{\tau_\varepsilon(z) > T(\varepsilon)\}\right)^{k(\varepsilon)} \\ &\quad + k(\varepsilon) \sup_{z \in \Gamma(\varepsilon)} P\{\tilde{X}_\varepsilon(T(\varepsilon), z) \in U^G \backslash \Gamma(\varepsilon)\}. \end{aligned} \tag{5.4}$$

If the process $\tilde{X}_\varepsilon(t,z)$ starts in $\Gamma(\varepsilon)$ and it turns out to be in $U^G \backslash \Gamma(\varepsilon)$ at the time $t = T(\varepsilon)$, then $\tilde{X}_\varepsilon$ must pass from $\Gamma(\varepsilon)$ into $U^G \backslash \Gamma(\varepsilon)$ during one of the time intervals $(iT_1, (i+1)T_1)$, $i = 0, 1, \ldots, \left[\left(\ln\frac{1}{\varepsilon}\right)^{\beta}\right]$.

Thus, by the Markov property and the definition of T_1 in (4.33) one has,

$$\begin{aligned}
&\sup_{z\in\Gamma(\varepsilon)} P\{\tilde{X}_\varepsilon(T(\varepsilon),z)\in U^G\backslash\Gamma(\varepsilon)\}\\
&\leqq\left(\ln\frac{1}{\varepsilon}\right)^\beta \sup_{z\in\Gamma(\varepsilon)} P\{\tilde{X}_\varepsilon(T_1,z)\in U^G\backslash\Gamma(\varepsilon)\}\\
&\leqq\left(\ln\frac{1}{\varepsilon}\right)^\beta \sup_{z\in\Gamma(\varepsilon)} \int_{\{y:|S^{T_1}z-y|\geqq\frac{1}{2}\delta_4(\varepsilon)\}} q_\varepsilon(T_1,z,y)dy\\
&\leqq\exp\left(-\left(\ln\frac{1}{\varepsilon}\right)^4\right),
\end{aligned}\tag{5.5}$$

provided $\varepsilon>0$ is small enough, where the last inequality above follows from (3.6). Next, by the Chapman-Kolmogorov equality

$$\begin{aligned}
&\sup_{x\in\overline{U\cap G}} P\{\tilde{X}_\varepsilon(u(\varepsilon),x)\in U^G\backslash\Gamma(\varepsilon)\}\\
&=\sup_{x\in\overline{U\cap G}}\int_{U^G}\dots\int_{U^G}\int_{U^G\backslash\Gamma(\varepsilon)} q_\varepsilon(T_1,x,z_1)\dots q_\varepsilon(T_1,z_{l(\varepsilon)-1},z_{l(\varepsilon)})dz_1\dots dz_{l(\varepsilon)}\\
&\leqq l(\varepsilon)\sup_{y\in\overline{U\cap G}}\int_{\{z:|z-S^{T_1}y|\geqq\delta_5(\varepsilon)\}} q_\varepsilon(T_1,y,z)dz\\
&\quad+\sup_{y\in\overline{U\cap G}} I_\varepsilon^{(4)}(y),
\end{aligned}\tag{5.6}$$

where

$$I_\varepsilon^{(4)}(y)=\int\underset{\Xi}{\dots}\int q_\varepsilon(T_1,y,z_1)\dots q_\varepsilon(T_1,z_{l(\varepsilon)-1},z_{l(\varepsilon)})dz_1\dots dz_{l(\varepsilon)}$$

and the integration in $I_\varepsilon^{(4)}(y)$ is over the set Ξ of sequences $z_0=x, z_1,\dots,z_{l(\varepsilon)}$ such that $|z_{i+1}-S^{T_1}z_i|\leqq\delta_5(\varepsilon)$ for all $i=0,\dots,l(\varepsilon)-1$ and $z_{l(\varepsilon)}\in U^G\backslash\Gamma(\varepsilon)$. From the definitions of T_1, $\Gamma(\varepsilon)$, and $\delta_2(\varepsilon)$ in (4.33) and (5.2) it follows easily that Ξ is empty and so $I_\varepsilon^{(4)}(y)=0$. Estimating also the integral of $q_\varepsilon(T_1,y,z)$ over the set $\{z:|z-S^{T_1}y|\geqq\delta_5(\varepsilon)\}$ using (3.6), we obtain from (5.6) that for $\varepsilon>0$ small enough,

$$\sup_{x\in U\cap G} P\{\tilde{X}_\varepsilon(u(\varepsilon),x)\in U^G\backslash\Gamma(\varepsilon)\}\leqq\exp\left(-\left(\ln\frac{1}{\varepsilon}\right)^3\right).\tag{5.7}$$

Next, it remains to estimate $\sup_{z\in\Gamma(\varepsilon)} P\{\tau_\varepsilon(z)>T(\varepsilon)\}$ from above.

To do this notice that by the strong Markov property for any $z\in\Gamma(\varepsilon)$ one has

$$\begin{aligned}
P\{X_\varepsilon(T(\varepsilon),z)\in Q_{\varepsilon^{7/4}}(U^G)\}\geqq{}&P\{\tau_\varepsilon(z)>T(\varepsilon)\}\\
&+P\{\tau_\varepsilon(z)\leqq T(\varepsilon)\}\inf_{\zeta\in\Gamma,\,0\leqq t\leqq T(\varepsilon)} P\{X_\varepsilon(t,\zeta)\in Q_{\varepsilon^{7/4}}(U^G)\},
\end{aligned}\tag{5.8}$$

where $Q_\delta(V)$ was defined by (3.12), and, recall, $X_\varepsilon(t,x)$ without "wave" is the process in the whole R^m. Since Γ is of C^2-class then there exists $C_{15}>0$ such that for any

$x, y \in U$,

$$|\varrho(x) - \langle x - \gamma(y), n(y)\rangle| = |\langle x - \gamma(x), n(x)\rangle - \langle x - \gamma(y), n(y)\rangle|$$
$$= |\langle \gamma(y) - \gamma(x), n(y)\rangle + \langle \gamma(x) - x, n(y) - n(x)\rangle|$$
$$\leqq \tfrac{1}{2} C_{15} |x-y| (|x-y| + |\varrho(x)|)$$
$$\leqq C_{15} |x-y| (|x-y| + |\varrho(y)|) \tag{5.9}$$

since

$$|\varrho(y)| + |x-y| \geqq |x - \gamma(y)| \geqq |\varrho(x)|,$$

where, recall, $n(z)$ is the interior unit normal to Γ at $\gamma(z)$. Since $0 < \beta < 1$ in (5.2) then from (5.9), and Lemmas 4.1 and 4.2 it follows that for any $\zeta \in \Gamma$ and $0 \leqq t \leqq T(\varepsilon)$ one has,

$$P\{\tilde{X}_\varepsilon(t,\zeta) \in Q_{\varepsilon^{7/4}}(U^G)\} \geqq P\{\varrho(Z_\varepsilon(t,\zeta)) > -\tfrac{1}{2}\varepsilon^{7/4}\} - P\{|X_\varepsilon(t,\zeta) - Z_\varepsilon(t,\zeta)| \geqq \tfrac{1}{2}\varepsilon^{7/4}\}$$
$$\geqq P\{\langle Y(t,\zeta), n(S^t\zeta)\rangle \geqq 0\}$$
$$- P\{|Z_\varepsilon(t,\zeta) - S^t\zeta| \geqq \varepsilon^{8/9}\}$$
$$- P\{|X_\varepsilon(t,\zeta) - Z_\varepsilon(t,\zeta)| \geqq \tfrac{1}{2}\varepsilon^{7/4}\} \geqq \tfrac{1}{2} - \varepsilon^k$$

for any $k \geqq 1$ and $\varepsilon \leqq \varepsilon(k)$. Now (5.8) and (5.10) imply for $z \in \Gamma(\varepsilon)$,

$$P\{\tau_\varepsilon(z) > T(\varepsilon)\} \leqq 2(P\{X_\varepsilon(T(\varepsilon), z) \in Q_{\varepsilon^{7/4}}(U^G)\} - \tfrac{1}{2}) + \varepsilon^2 \tag{5.11}$$

provided $\varepsilon > 0$ is small enough. Remark that this inequality has some similarity with the well known reflection principle for the Brownian motion (see, for instance, [Va, Sect. 7]). Since for each $t > 0$,

$$\langle Z_\varepsilon(t,z) - \gamma(S^t z), n(S^t z)\rangle = \varepsilon\langle Y(t,z), n(S^t z)\rangle + \varrho(S^t z) \tag{5.12}$$

then applying (5.9), and using Lemmas 4.1 and 4.2 one obtains for any $z \in \Gamma(\varepsilon)$ and $\varepsilon > 0$ small enough,

$$P\{X_\varepsilon(T(\varepsilon), z) \in Q_{\varepsilon^{7/4}}(U^G)\} \leqq \mathrm{P}\{\varrho(\mathrm{Z}_\varepsilon(\mathrm{T}(\varepsilon), z) \geqq -2\varepsilon^4\}$$
$$+ P\{|X_\varepsilon(T(\varepsilon), z) - Z_\varepsilon(T(\varepsilon), z)| \geqq \varepsilon^{7/4}\}$$
$$\leqq P\{\varepsilon\langle Y(T(\varepsilon), z), n(S^{T(\varepsilon)}z)\rangle + \varrho(S^{T(\varepsilon)}z) \geqq -3\varepsilon^{7/4}\}$$
$$+ P\{|Z_\varepsilon(T(\varepsilon), z) - S^{T(\varepsilon)}z| \geqq \varepsilon^{8/9}\}$$
$$+ P\{|X_\varepsilon(T(\varepsilon), z) - Z_\varepsilon(T(\varepsilon), z)| \geqq \varepsilon^{7/4}\}$$
$$\leqq \tfrac{1}{2} + \varepsilon^{-1}|\varrho(S^{T(\varepsilon)}z)| + \varepsilon^{1/2}. \tag{5.13}$$

From (4.27) and (4.32) it follows that there is $C_{16} > 0$ such that for any $x \in \bar{U}$,

$$C_{16}^{-1}\varrho(x) \leqq \varrho(S^t x) \exp\left(-\int_0^t \alpha(\gamma(S^u x))du\right) \leqq C_{16}\varrho(x), \tag{5.14}$$

and so, by (5.11)–(5.13) for any $z \in \Gamma(\varepsilon)$,

$$P\{\tau_\varepsilon(z) > T(\varepsilon)\} \leqq 2C_{16}\left(\ln\frac{1}{\varepsilon}\right)^4 \exp\left(\int_0^{T(\varepsilon)} \alpha(\gamma(S^u x))du\right) + 3\varepsilon^{1/2} \tag{5.15}$$

provided $\varepsilon > 0$ is small enough.

By (4.30), (4.32), Assumption B and the definitions (5.2) with $0 < \beta < 1$ one can see that for $\delta > 0$ there exists $\varepsilon(\delta) > 0$ such that for any $x \in \Gamma(\varepsilon)$ and $\varepsilon < \varepsilon(\delta)$,

$$(\alpha_0 - \delta) \leqq \frac{1}{T(\varepsilon)} \int_0^{T(\varepsilon)} \alpha(\gamma(S^u x))du \leqq (\alpha_0 + \delta). \tag{5.16}$$

Finally, by (1.7), (3.3), (5.2)–(5.5), (5.7), (5.15), and (5.16), taking into account that $\Phi_\varepsilon(t, U^G)$ decreases in t, we obtain

$$\frac{1}{t}\ln \Phi_\varepsilon(t, U^G) \leqq \alpha_0 + \delta + \left(\ln\frac{1}{\varepsilon}\right)^{-(1-\beta)}$$

provided $\varepsilon > 0$ is small enough and $t \geqq \left(\ln\frac{1}{\varepsilon}\right)^{1+\beta}$. Thus

$$\limsup_{\varepsilon\to 0} \lambda_\varepsilon(U^G) = \limsup_{\varepsilon\to 0} \lim_{t\to\infty} \frac{1}{t} \Phi_\varepsilon(t, U^G) \leqq \alpha_0 + \delta. \tag{5.18}$$

Since $\delta > 0$ is arbitrary then

$$\Lambda(\Gamma) \equiv \limsup_{\varepsilon\to 0} \lambda_\varepsilon(U^G) \leqq \alpha_0. \tag{5.19}$$

6. Lower Bound for Attracting and Neutral Boundaries

In this section we shall complete the proof of Assertions (i) and (ii) of Theorem 2.2. Both assertions require only lower bounds for the limit (2.13) since the upper bound is already established for the case of attracting boundary in the previous section, and in the case of neutral boundary we only need zero upper bound which is always true.

We shall start with the case of attracting boundary. Denote

$$\Omega(\varepsilon) = \{x \in U^G : \varepsilon < \varrho(x) < \varepsilon^{7/8}\}.$$

Then for any $x \in \Omega(\varepsilon)$, $t > 0$ and an integer $k > 0$,

$$\begin{aligned} P\{\tau_\varepsilon(x) > tk\} &\geqq \int_{\Omega(\varepsilon)} \dots \int_{\Omega(\varepsilon)} q_\varepsilon(t, x, z_1) \dots q_\varepsilon(t, z_{k-1} z_k) dz_1 \dots dz_k \\ &\geqq \left(\inf_{x\in\Omega(\varepsilon)} P\{\tau_\varepsilon(x) > t \text{ and } X_\varepsilon(t, x) \in \Omega(\varepsilon)\}\right)^k. \end{aligned} \tag{6.1}$$

For $x \in \Omega(\varepsilon)$ and $t > 0$ define

$$\tilde{\Omega}_\varepsilon(t, x) = \{y \in U^G : \langle y - \gamma(S^t x), n(S^t x)\rangle \geqq 2\varepsilon \text{ and } |y - S^t x| < \tfrac{1}{2}\varepsilon^{7/8}\}.$$

If $t \geqq T_1$ then by (4.33) and (5.9)

$$\tilde{\Omega}_\varepsilon(t,x) \subset \Omega(\varepsilon) \tag{6.2}$$

provided ε is small enough. Similarly to (5.8), by the strong Markov property one has

$$\begin{aligned} P\{X_\varepsilon(t,x) \in \tilde{\Omega}_\varepsilon(t,x)\} \leqq & P\{\tau_\varepsilon(x) > t \text{ and } X_\varepsilon(t,x) \in \Omega(\varepsilon)\} \\ & + P\{\tau_\varepsilon(x) \leqq t \text{ and } X_\varepsilon(\tau_\varepsilon(x),x) \notin \Gamma\} \\ & + \sup_{y \in \Gamma, 0<s<t} P\{X_\varepsilon(s,y) \in \tilde{\Omega}_\varepsilon(t,x)\}. \end{aligned} \tag{6.3}$$

From (4.15) and (4.25) it follows the existence $C_{17}(t) > 0$ such that for any $x \in \Omega(\varepsilon)$, and an integer $l > 0$,

$$P\{\tau_\varepsilon(x) \leqq t \text{ and } X_\varepsilon(\tau_\varepsilon(x),x) \notin \Gamma\} \leqq C_{17}(t)\varepsilon^l \tag{6.4}$$

provided $\varepsilon > 0$ is small enough. Furthermore, by (5.12) and Lemma 4.2 for any $x \in \Omega(\varepsilon)$ and $t \geqq T_1$ one has

$$\begin{aligned} P\{X_\varepsilon(t,x) \in \tilde{\Omega}_\varepsilon(t,x)\} \geqq & P\{\langle Z_\varepsilon(t,x) - \gamma(S^t x), n(S^t x)\rangle \geqq 2\varepsilon + \varepsilon^{3/2}\} \\ & - P\{|Z_\varepsilon(t,x) - X_\varepsilon(t,x)| \geqq \varepsilon^{3/2}\} \\ & - P\{|X_\varepsilon(t,x) - S^t x| \geqq \tfrac{1}{2}\varepsilon^{7/8}\} \\ \geqq & P\left\{\langle Y(t,x) n(S^t x)\rangle \geqq 2 - \frac{\varrho(S^t x)}{\varepsilon} + \varepsilon^{1/2}\right\} \\ & - C_{18}(t)\varepsilon^{1/2}, \end{aligned} \tag{6.5}$$

where $C_{18}(t) > 0$ is independent of x and ε, provided ε is small enough.

Now suppose that $y \in \Gamma$, $0 \leqq s \leqq t$, and $|S^s y - S^t x| \geqq 2\varepsilon^{7/8}$ then by (4.15),

$$P\{X_\varepsilon(s,y) \in \tilde{\Omega}_\varepsilon(t,x)\} \leqq P\{|S^s y - X_\varepsilon(s,y)| \geqq \varepsilon^{7/8}\} \leqq C_{19}(t)\varepsilon^k \tag{6.6}$$

for some $C_{19}(t) > 0$ independent of ε, provided ε is small enough. Next, consider the case when

$$y \in \Gamma, \quad 0 \leqq s \leqq t, \quad \text{and} \quad |S^s y - S^t x| < 2\varepsilon^{7/8} \tag{6.7}$$

which implies $|y - S^{t-s}x| \leqq 2C_3^{t+1}\varepsilon^{7/8}$. Then for some $C_{20}(t) > 0$ independent of x and s,

$$E|Y(s,y) - Y(s, S^{t-s}x)|^2 \leqq C_{20}(t)\varepsilon^{7/4} \tag{6.8}$$

which follows from (4.2), the smooth dependence of $\sigma(x)$ and $K(s,u,x)$ on x and from the properties of stochastic integrals. Now by (5.9), (6.7), (6.8) and Lemma 4.2

it follows

$$\begin{aligned}P\{X_\varepsilon(s,y)\in\tilde{\Omega}_\varepsilon(t,x)\} &\leqq P\{\langle X_\varepsilon(s,y)-\gamma(S^tx), n(S^tx)\rangle\geqq 2\varepsilon\}\\ &\leqq P\{\langle S^sy-\gamma(S^tx)+\varepsilon Y(s,S^{t-s}x), n(S^tx)\rangle>2\varepsilon-\varepsilon^{3/2}\}\\ &\quad+P\{|X_\varepsilon(s,y)-Z_\varepsilon(s,y)|\geqq\tfrac{1}{2}\varepsilon^{3/2}\}\\ &\quad+P\{|Y(s,y)-Y(s,S^{t-s}x)|\geqq\tfrac{1}{2}\varepsilon^{1/2}\}\\ &\leqq P\{\langle Y(s,S^{t-s}x), n(S^tx)\rangle>2-2\varepsilon^{1/2}\}\\ &\quad+C_{21}(t)\varepsilon^{1/2},\end{aligned} \tag{6.9}$$

where $C_{21}(t)>0$ is independent of x, y, s, and ε. Now by (4.11), (6.3)–(6.6), and (6.9) for any $x\in\Omega(\varepsilon)$,

$$\begin{aligned}&P\{\tau_\varepsilon(x)>t \text{ and } X_\varepsilon(t,x)\in\Omega(\varepsilon)\}\\ &\leqq P\left\{2-2\varepsilon^{1/2}\geqq\langle Y(t,x), n(S^tx)\rangle\geqq 2-\frac{\varrho(S^tx)}{\varepsilon}+\varepsilon^{1/2}\right\}-C_{22}(t)\varepsilon^{1/2},\end{aligned} \tag{6.10}$$

where $C_{22}(t)>0$ is independent of x and ε.

By (5.14) for any $x\in\Omega(\varepsilon)$,

$$\frac{\varrho(S^tx)}{\varepsilon}\geqq C_{16}^{-1}\exp\left(\int_0^t\alpha(\gamma(S^ux))du\right). \tag{6.11}$$

Assumption (B2) implies that for any $\delta>0$ there exists $t(\delta)>0$ such that whenever $t\geqq t(\delta)$ and $x\in U$, one has

$$\int_0^t\alpha(S^u\gamma(x)du\geqq(\alpha_0-\delta)t.$$

Thus by (4.30) for any $x\in\Omega(\varepsilon)$ and $t\geqq t(\delta)$,

$$\int_0^t\alpha(\gamma(S^ux))du\geqq\int_0^t\alpha(S^u\gamma(x))du-C_{23}(t)\varrho(x)\geqq(\alpha_0-\delta)t-C_{23}(t)\varepsilon^{7/8}, \tag{6.12}$$

where $C_{23}(t)>0$ is independent of x and ε. Since $|V^{1/2}(x,t)n(S^tx)|$ is continuous function of x then by (4.26),

$$\sup_{x\in\tilde{\Omega}_\varepsilon(t,x)}|V^{1/2}(x,t)n(S^tx)|\leqq 2C_9 \tag{6.13}$$

provided $\varepsilon<\varepsilon(t)$. Finally, from (4.9), (6.1), and (6.10)–(6.13), we get

$$\frac{1}{tk}\ln P\{\tau_\varepsilon(x)>tk\}\geqq\alpha_0-\delta-(2C_9+C_{24})t^{-1}-C_{25}(t)\varepsilon^{1/2}, \tag{6.14}$$

where

$$C_{24}=2M_1+\ln(C_8+1)+\ln(C_{16}+1),$$

$C_{25}(t)>0$ is independent of ε and x, and we suppose that

$$\exp((\alpha_0-\delta)t+C_{23}(t)\varepsilon^{7/8})\geqq 3\varepsilon^{1/2}.$$

Letting $k\to\infty$ one obtains by (6.14),

$$\lambda_\varepsilon(U^G)\geqq\alpha_0-\delta-(2C_9+C_{24})t^{-1}-C_{25}(t)\varepsilon^{1/2}. \tag{6.15}$$

Now passing in (6.15) to the limit when, first, $\varepsilon\to 0$, then $t\to\infty$, and, finally $\delta\to 0$ we derive

$$\liminf_{\varepsilon\to 0}\lambda_\varepsilon(U^G)\geqq\alpha_0 \tag{6.16}$$

which together with (5.19) proves the assertion (i) of Theorem 2.2.

Next, we shall pass to the neutral boundary case. Denote

$$\tilde{\tilde{\Omega}}(\varepsilon)=\{x\in U^G:\varepsilon^{7/8}<\varrho(x)<3\varepsilon^{7/8}\}.$$

Then, similarly to (6.3)–(6.5) using the strong Markov property one obtains for any $t>0$ and $x\in\tilde{\tilde{\Omega}}(\varepsilon)$,

$$\begin{aligned}&P\{\tau_\varepsilon(x)>t \text{ and } X_\varepsilon(t,x)\in\tilde{\tilde{\Omega}}(\varepsilon)\}\\ &\geqq P\{X_\varepsilon(t,x)\in\tilde{\tilde{\Omega}}(\varepsilon)\}-\sup_{y\in\Gamma,\,0<s<t}P\{X_\varepsilon(s,y)\in\tilde{\tilde{\Omega}}(\varepsilon)\}\\ &\quad -P\{\tau_\varepsilon(x)\leqq t \text{ and } X_\varepsilon(\tau_\varepsilon(x),x)\notin\Gamma\}.\end{aligned} \tag{6.17}$$

Since in our case $\alpha(\gamma)\equiv 0$ on Γ then by (4.27),

$$|\varrho(S^t x)-\varrho(x)|\leqq C_{26}(t)\varrho^2(x) \tag{6.18}$$

for some $C_{26}(t)>0$ independent of x. This together with (4.15) imply that the last terms in (6.17) are of order $C_{27}(t)\varepsilon$ for some $C_{27}(t)>0$ independent of x and ε. Thus by (4.16) we can write

$$\begin{aligned}P\{\tau_\varepsilon(x)>t \text{ and } X_\varepsilon(t,x)\in\tilde{\tilde{\Omega}}(\varepsilon)\}&\geqq P\{X_\varepsilon(t,x)\in\tilde{\Omega}(\varepsilon)\}-C_{27}(t)\varepsilon\\ &\geqq R_\varepsilon(t,x)-2C_{27}(t)\varepsilon,\end{aligned} \tag{6.19}$$

provided $\varepsilon\to 0$ is small enough, where

$$R_\varepsilon(t,x)\equiv P\{\varepsilon^{7/8}+\varepsilon^{3/2}<\varrho(Z_\varepsilon(t,x))<3\varepsilon^{7/8}-\varepsilon^{3/2}\}.$$

Now we obtain from (5.9), (5.12), (6.18) and Lemma 4.2 that

$$\begin{aligned}R_\varepsilon(t,x)&\geqq P\{\varepsilon^{7/8}+2\varepsilon^{3/2}<\langle Z_\varepsilon(t,x)-\gamma(S^t x),n(S^t x)\rangle<3\varepsilon^{7/8}-2\varepsilon^{3/2}\}\\ &\quad -P\{|Z_\varepsilon(t,x)-S^t x|>\varepsilon^{7/8}\}\\ &\geqq P\left\{\frac{\varepsilon^{7/8}-\varrho(x)}{\varepsilon}+3\varepsilon^{1/2}<\langle Y(t,x),n(S^t x)\rangle<\frac{3\varepsilon^{7/8}-\varrho(x)}{\varepsilon}-3\varepsilon^{1/2}\right\}\\ &\quad -C_{28}(t)\varepsilon,\end{aligned} \tag{6.20}$$

for some $C_{28}(t)>0$ independent of x and ε. Thus, for any $x\in\tilde{\tilde{\Omega}}(\varepsilon)$ satisfying $\varrho(x)\geqq 2\varepsilon^{7/8}$ it follows by (4.9) and (4.10) that

$$\begin{aligned} R_\varepsilon(t,x) &\geqq P\{-\varepsilon^{1/8}+3\varepsilon^{1/2}<\langle Y(t,x),n(S^t x)\rangle<-\varepsilon^{1/8}-3\varepsilon^{1/2}\} \\ &=\tfrac{1}{2}-C_{29}(t)\varepsilon^{1/2}-r_\varepsilon(t,x), \end{aligned} \tag{6.21}$$

where $C_{29}(t)>0$ is independent of x and ε, and

$$r_\varepsilon(t,x)\equiv P\{\langle Y(t,x),n(S^t x)\rangle>-\varepsilon^{-1/8}+3\varepsilon^{1/2}\} \tag{6.22}$$

Similarly, for any $x\in\tilde{\tilde{\Omega}}$ satisfying $\varrho(x)<2\varepsilon^{7/8}$ one has

$$\begin{aligned} R_\varepsilon(t,x) &\geqq \{3\varepsilon^{1/2}<\langle Y(t,x),n(S^t x)\rangle<\varepsilon^{-1/8}-3\varepsilon^{1/2}\} \\ &=\tfrac{1}{2}-C_{29}(t)\varepsilon^{1/2}-r_\varepsilon(t,x), \end{aligned} \tag{6.23}$$

where we took into account that $Y(t,x)$ and $-Y(t,x)$ have the same distribution. It is easy to see from (4.12) and (6.22) that

$$\sup_x r_\varepsilon(t,x)\to 0 \quad \text{as} \quad \varepsilon\to 0. \tag{6.24}$$

The final steps of the proof are the same as in the case of an attracting boundary. We use (6.1) with $\tilde{\tilde{\Omega}}(\varepsilon)$ in place of $\Omega(\varepsilon)$ together with (6.19)–(6.23) to obtain

$$\frac{1}{tk}\ln P\{\tau_\varepsilon(x)>tk\}\geqq\frac{1}{t}\ln\left(\tfrac{1}{2}-2C_{27}(t)\varepsilon-C_{29}(t)\varepsilon^{1/2}-r_\varepsilon(t,x)\right). \tag{6.25}$$

Letting, first, $k\to\infty$, then $\varepsilon\to 0$, and, finally, $t\to\infty$ we conclude from (6.24) that

$$\liminf_{\varepsilon\to 0}\lambda_\varepsilon(U^G)\geqq 0. \tag{6.26}$$

Since, always, $\lambda_\varepsilon(U^G)\leqq 0$ then we obtain $\Lambda(\Gamma)=0$ proving Assertion (ii) of Theorem 2.2. □

7. Repulsing Boundary

We shall obtain the result for the repulsing boundary case as, essentially, a non-probabilistic consequence of the first part of Theorem 2.2 concerning an attracting boundary. Consider again a neighborhood U of Γ with a smooth boundary ∂U, such that the representation (2.7) is valid for any $x\in U$. We shall write the operator L_ε restricted to U using the coordinates (γ,ϱ) and preserving the same notations for its coefficients

$$L_\varepsilon=\varepsilon^2\left(\tfrac{1}{2}\langle a(\gamma,\varrho)\nabla,\nabla\rangle+\langle b(\gamma,\varrho),\nabla\rangle\right)+\langle B(\gamma,\varrho),\nabla\rangle, \tag{7.1}$$

where, recall, $a=(a^{ij})$ is a matrix and $b=(b^i)$, $B=(B^i)$ are vector functions. Introduce the smooth measure $d\mu=d\gamma\, d\varrho$ where $d\gamma$ is the volume element of Γ. Considering the action of L_ε on the space of twice differentiable functions with zero data on ∂U^G we can write its formal adjoint operator L_ε^* (see [Fri 1, v. 1, p. 142]) in $U^G = U \cap G$ with respect to the inner product generated by the measure μ in the following form

$$L_\varepsilon^* = \varepsilon^2(\tfrac{1}{2}\langle a(\gamma,\varrho)\nabla, \nabla\rangle + \langle \hat{b}(\gamma,\varrho), \nabla\rangle) - \langle B(\gamma,\varrho), \nabla\rangle$$
$$+ \varepsilon^2 c(\gamma,\varrho) + \langle \nabla, B(\gamma,\varrho)\rangle \equiv \hat{L}_\varepsilon + \varepsilon^2 c(\gamma,\varrho) - \langle \nabla, B(\gamma,\varrho)\rangle\,, \tag{7.2}$$

where $\hat{b} = \frac{1}{2}\langle \nabla a, \nabla\rangle - \langle b, \nabla\rangle$, $c = \frac{1}{2}\langle \nabla, \nabla a\rangle - \nabla b$ and the operator

$$\hat{L}_\varepsilon = \varepsilon^2(\tfrac{1}{2}\langle a\nabla, \nabla\rangle + \langle \hat{b}, \nabla\rangle) - \langle B, \nabla\rangle \tag{7.3}$$

meets the conditions of the first part of Theorem 2.2 since Γ becomes an attracting boundary for the dynamical system $\hat{S}^t \equiv S^{-t}$ satisfying

$$\frac{d(\hat{S}^t x)}{dt} = -B(\hat{S}^t x)\,. \tag{7.4}$$

Thus we can apply to the operator $\hat{L}_\varepsilon$, and to the corresponding diffusion process $\hat{X}_\varepsilon$ generated by $\hat{L}_\varepsilon$, the results proved in Sects. 5 and 6.

Let, as in the previous two sections, $\tilde{X}_\varepsilon = X_\varepsilon^{U^G}$ be the process X_ε stopped at the moment $\tau_\varepsilon(x) = \tau_\varepsilon(x, U^G)$ of exit from $U^G = G \cap U$ and let $q_\varepsilon(t,x,y) = p_\varepsilon^{U^G}(t,x,y)$ be its transition density with respect to the measure μ. Then, (see [Fri 1, v. 1, p. 149]) as a function of x, $q_\varepsilon(t,x,y)$ satisfies the equation $\frac{\partial u}{\partial t} = L_\varepsilon u$, and as a function of y it satisfies the equation $\frac{\partial v}{\partial t} = L_\varepsilon^* v$. On the space $\mathscr{H}_0(U^G)$ of bounded functions f in U^G with zero data on ∂U^G consider the following operators

$$P_\varepsilon^t f(x) = \int_{U^G} q_\varepsilon(t,x,y) f(y)\,d\mu(y) \quad \text{and} \quad (P_\varepsilon^t)^* f(y) = \int_{U^G} f(x) q_\varepsilon(t,x,y)\,d\mu(x)\,. \tag{7.5}$$

In the same way as in Lemma 3.1 of [K 1] it is easy to see that the supermum norms of these operators can be expressed by the following formulas

$$\|P_\varepsilon^t\| = \sup_{x \in U^G} \int_{U^G} q_\varepsilon(t,x,y)\,d\mu(y)$$

and

$$\|(P_\varepsilon^t)^*\| = \sup_{y \in U^G} \int_{U^G} q_\varepsilon(t,x,y)\,d\mu(x)\,. \tag{7.6}$$

Thus, by (3.6) and the Chapman-Kolmogorov formula,

$$\|P_\varepsilon^{t+1}\| = \sup_{x \in U^G} \int_{U^G} \int_{U^G} q_\varepsilon(1,x,z) q_\varepsilon(t,z,y)\,d\mu(z)\,d\mu(y)$$
$$\leqq \sup_{x,z \in U^G} q_\varepsilon(1,x,z) \int_{U^G} \int_{U^G} q_\varepsilon(t,v,w)\,d\mu(v)\,d\mu(w)$$
$$\leqq C_1 \varepsilon^{-m} \mu(U^G)\, \|(P_\varepsilon^t)^*\|\,. \tag{7.7}$$

Similarly,

$$\|(P_\varepsilon^{t+1})^*\| = \sup_{y\in U^G} \int_{U^G}\int_{U^G} q_\varepsilon(t,x,z)q_\varepsilon(1,z,y)d\mu(x)d\mu(z)$$
$$\leqq C_1\varepsilon^{-m}\mu(U^G)\,\|P_\varepsilon^t\|. \tag{7.8}$$

The relations (7.7) and (7.8) yield that if $\lambda_\varepsilon(U^G)$ is the principal eigenvalue of the operator L_ε in U^G, and so $\exp(t\lambda_\varepsilon(U^G))$ is the principal eigenvalue (the spectral radius) of P_ε^t, then L_ε^* and $(P_\varepsilon^t)^*$ have the same principal eigenvalues $\lambda_\varepsilon(U^G)$ and $\exp(t\lambda_\varepsilon(U^G))$, respectively. Thus, we can derive estimates for $\lambda_\varepsilon(U^G)$ by studying the operator $(P_\varepsilon^t)^*$. Notice also that

$$\|P_\varepsilon^t\| = \Phi_\varepsilon(t, U^G), \tag{7.9}$$

where Φ_ε was defined in (1.7).

From (7.2) and the Feynman-Kac formula (see [Fre, Sect. 2.1]) it follows that

$$(P_\varepsilon^t)^* f(z) = Ef(\hat{X}_\varepsilon(t,z))\exp\left(\int_0^t (\varepsilon^2 c(\hat{X}_\varepsilon(s,z)) - \langle \nabla, B(\hat{X}_\varepsilon(s,z))\rangle)ds\right). \tag{7.10}$$

According to (2.9)–(2.11) we can write

$$\langle \nabla, B(\gamma,\varrho)\rangle = \alpha(\gamma) + \tilde{\psi}(\gamma,\varrho)\varrho + \operatorname{div} B^\Gamma(\gamma), \tag{7.11}$$

where $\tilde{\psi}(\gamma,\varrho)$ is a bounded function, $B^\Gamma(\gamma)\equiv B(\gamma,0)$ and $\operatorname{div} B^\Gamma(\gamma) = \langle \nabla, B^\Gamma(\gamma)\rangle$ denotes the divergence. Since we suppose that the dynamical system S^t preserves a measure having a smooth positive density $r(\gamma)$ with respect to the volume on Γ, then by the Liouville theorem (see [CFS, p. 48])

$$0 = \operatorname{div}(r(\gamma)B^\Gamma(\gamma)) = r(\gamma)\operatorname{div} B^\Gamma(\gamma) + \langle B^\Gamma(\gamma), \nabla \Gamma(\gamma)\rangle,$$

and so $\operatorname{div} B^\Gamma(\gamma) = -\langle B^\Gamma(\gamma), \nabla \ln r(\gamma)\rangle$.

Thus by (1.4),

$$\int_0^t \operatorname{div} B^\Gamma(S^u\gamma)du = -\int_0^t \frac{d}{du}(\ln r(S^u\gamma)du = \ln r(\gamma) - \ln r(S^t\gamma). \tag{7.12}$$

This is the only place where we use the assumption about smooth invariant measure, which we need to assure boundedness of the integral in the left hand side of (7.12).

Notice that by (7.5) and (7.6) it follows

$$\|(P_\varepsilon^t)^*\| = \sup_{x\in U^G} (P_\varepsilon^t)^*\chi_{U^G}(x) \tag{7.13}$$

where χ_{U^G} denotes the function identically equal one in U^G and equal zero outside of U^G.

In the remaining part of this section we shall use the notations (5.2) with T_1 chosen to satisfy (4.33) for the dynamical system $\hat{S}^t = S^{-t}$ in place of S^t. By (7.10) and the Markov property of the process $\hat{X}_\varepsilon$ similarly to (5.3)–(5.7) we can write for any $x \in U^G$ that

$$
\begin{aligned}
&(P_\varepsilon^{u(\varepsilon)+k(\varepsilon)T(\varepsilon)})^* \chi_{U^G}(x) \\
&= E\left(\exp\left(\int_0^{u(\varepsilon)} (\varepsilon^2 c(\hat{X}_\varepsilon(s,x)) - \langle \nabla, B(\hat{X}_\varepsilon(s,x))\rangle) ds\right)\right)(P_\varepsilon^{k(\varepsilon)T(\varepsilon)})^* \chi_{U^G}(\hat{X}(u(\varepsilon),x)) \\
&\leqq e^{C_{30}u(\varepsilon)}\left(\sup_{z\in\Gamma(\varepsilon)} (P_\varepsilon^{T(\varepsilon)})^* \chi_{U^G}(z)\right)^{k(\varepsilon)} \\
&\quad + \exp\left(2C_{30}(u(\varepsilon)+k(\varepsilon)T(\varepsilon)) - \left(\ln\frac{1}{\varepsilon}\right)^3\right)
\end{aligned}
\tag{7.14}
$$

provided $\varepsilon > 0$ is small enough, where $C_{30} > 0$ is independent of ε and x. Applying (4.15), (4.30), and (5.16) to the dynamical system $\hat{S}^t$ and to the process $\hat{X}_\varepsilon$ one obtains from (7.10)–(7.12) that for any $z \in \Gamma(\varepsilon)$,

$$(P_\varepsilon^{T(\varepsilon)})^* \chi_{U^G}(z) \leqq C_{31} e^{-(\alpha_0-\delta)T(\varepsilon)} P\{\hat{\tau}_\varepsilon(z) > T(\varepsilon)\} + \varepsilon \tag{7.15}$$

provided $\varepsilon < \varepsilon(\delta)$ is small enough, where $C_{31} > 0$ is independent of z and ε, and $\hat{\tau}_\varepsilon(z) \equiv \inf\{t : \hat{X}_\varepsilon(t,z) \notin U^G\}$ is the exit time for $\hat{X}_\varepsilon$ playing here the same role as $\tau_\varepsilon(z)$ played for X_ε in Sects. 5 and 6. Applying (5.15) and (5.16) to the flow $\hat{S}^t$ and the process $\hat{X}_\varepsilon$ we derive from (7.13)–(7.15) that

$$(u(\varepsilon)+k(\varepsilon)T(\varepsilon))^{-1} \ln \|(P_\varepsilon^{u(\varepsilon)+k(\varepsilon)T(\varepsilon)})^*\| \leqq -2(\alpha_0-\delta) + \left(\ln\frac{1}{\varepsilon}\right)^{-\beta/3} \tag{7.16}$$

provided $\varepsilon < \tilde{\varepsilon}(\delta)$ is small enough.

Since

$$\|(P_\varepsilon^{r+s})^*\| \leqq \|(P_\varepsilon^r)^*\| \cdot \|(P_\varepsilon^s)^*\|$$

for any $r, s \geqq 0$ then by (7.7)–(7.9) and (7.16), taking into account that $\Phi_\varepsilon(t, U^G)$ decreases in t, we obtain

$$\frac{1}{t} \ln \Phi_\varepsilon(t, U^G) \leqq -2(\alpha_0-\delta) + \left(\ln\frac{1}{\varepsilon}\right)^{-\beta/4} \tag{7.17}$$

for all $t \geqq \left(\ln\frac{1}{\varepsilon}\right)^{1+\beta}$, provided $\varepsilon < \tilde{\tilde{\varepsilon}}(\delta)$ is small enough, proving (2.15). Thus

$$\lambda_\varepsilon(U^G) = \lim_{t\to\infty} \frac{1}{t} \ln \Phi_\varepsilon(t, U^G) \leqq -2(\alpha_0-\delta) + \left(\ln\frac{1}{\varepsilon}\right)^{-\beta/4}. \tag{7.18}$$

Letting, first, $\varepsilon \to 0$, and then $\delta \to 0$ one derives

$$\limsup_{\varepsilon\to 0} \lambda_\varepsilon(U^G) \leqq -2\alpha_0. \tag{7.19}$$

To obtain a lower bound consider again the region $\Omega(\varepsilon)$ introduced at the beginning of Sect. 6. Then by (4.15) applied to $\hat{S}^t$ and $\hat{X}_\varepsilon$, by (7.10)–(7.12), and by the

Markov property of the process $\hat{X}_\varepsilon$ for any $t>0$, $x\in\Omega(\varepsilon)$ and an integer $k>0$ one has

$$(P_\varepsilon^{tk})^*\chi_{U^G}(x) \geqq \left(\inf_{z\in\Omega(\varepsilon)} P\{\hat{\tau}_\varepsilon(z)>t \text{ and } \hat{X}_\varepsilon(t,z)\in\Omega(\varepsilon)\}\right.$$

$$\left.\times\exp\left(-\int_0^t \alpha(\gamma(S^u u))du\right) - C_{32}(t)\varepsilon^{1/2}\right)^k, \tag{7.20}$$

where $C_{32}(t)>0$ is independent of x and ε, provided $\varepsilon>0$ is small enough.

Finally, the same arguments as in (6.10)–(6.15) together with (1.6), (7.7)–(7.9), and (7.13) imply

$$\liminf_{\varepsilon\to 0} \lambda_\varepsilon(U^G) \geqq -2\alpha_0 \tag{7.21}$$

which in combination with (7.19) proves Assertion (iii) of Theorem 2.2. □

Remark 7.1. The adjoint operator method works in some other cases, as well. Suppose that Γ is an S^t-invariant $(m-1)$-dimensional attracting surface in the sense of Assertion (i) of Theorem 2.2 and U is an open neighborhood of Γ containing no other S^t-invariant sets except for Γ and having a smooth boundary ∂U. The difference is that now we consider Γ as an interior and not as a boundary invariant set. Then estimates of [Ve] and [Fri 2] imply that $\lim_{\varepsilon\to 0}\lambda_\varepsilon(U)=0$. Thus if we have, instead, $\Gamma\subset U$ being a repulsing S^t-invariant surface satisfying the conditions of Assertion (iii) in Theorem 2.2, then passing to the adjoint operator in the same way as above we shall obtain that $\lim_{\varepsilon\to 0}\lambda_\varepsilon(U)=-\alpha_0$, where $\alpha_0>0$ is given by (2.12). A similar result holds true if Γ is not necessarily of the co-dimension one. Moreover one can extend the result to the case when Γ is a normally hyperbolic manifold, i.e. when Γ has a hyperbolic structure in transversal to Γ directions (see [HPS]). But this generalization requires much more sophisticated dynamical systems machinery from [HPS] and [K 2] than anything we have employed in this paper.

8. Concluding Remarks

If a connected component Γ of the boundary of ∂G of G is not S^t-invariant, but Γ has S^t-invariant subsets, then, in general, the situation becomes more complicated. Still, combining results of [K 1] with Theorem 2.2 of the present paper we are able to treat some of these cases. Assume, for example, that $\mathcal{O}\in\Gamma$ is a fixed point of the dynamical system S^t isolated from the rest of the limit set. Then we can write

$$B(x)=\Pi(x-\mathcal{O})+0(|x-\mathcal{O}|^2), \tag{8.1}$$

where Π is a matrix. Suppose that Π has an eigendirection ξ which is transversal to Γ at $\mathcal{O}$. Then any vector x can be uniquely represented as $x=x_\xi+x_\Gamma$ where $x_\xi\in\xi$ and x_Γ belongs to the tangent hyperplane $T_{\mathcal{O}}\Gamma$ to Γ at $\mathcal{O}$. Let U be a small neighborhood of $\mathcal{O}$ with a smooth boundary and $U^G = U \cap G$. It is not difficult to understand that the study of the exit time from U^G for the process X_ε needed to determine $\Lambda(\mathcal{O})$ by (2.2), can be carried out, actually, independently for projections of X_ε into ξ and $T_{\mathcal{O}}\Gamma$. The projection into ξ can be treated by means of the one-dimensional version of Theorem 2.2 and the projection into $T_{\mathcal{O}}\Gamma$ is being studied using results of [K 1]. Thus, if γ_1 is an eigenvalue corresponding to ξ and $\gamma_2, \ldots, \gamma_m$ are other eigenvalues of Π then

$$\Lambda(\mathcal{O}) = -\left(|\mathrm{Re}\gamma_1| + \sum_{i=1}^{m} \max(0, \mathrm{Re}\gamma_i) \right). \tag{8.2}$$

One can obtain corresponding results also for other types of S^t-invariant subsets on Γ combining the results of [K 2] with the one-dimensional version of Theorem 2.2. If the matrix Π does not have an eigendirection transversal to Γ at $\mathcal{O}$ then the corresponding asymptotics can not be derived readily from the results of this paper. Still, one can solve the problem using the Gaussian approximation $e^{t\Pi}\left(x + \varepsilon \int_0^t e^{-u\Pi} \sigma(\mathcal{O}) dw(s) \right)$ of the process X_ε near $\mathcal{O}$.

As we have already pointed it out, our assumptions at the beginning of Sect. 2 imply the "nocycle" property of compacts K_i from [K 1], since if an ordered sequence of compacts contains one of them twice, then all of them must belong to the same equivalence class. The following example shows that without the "nocycle" property one can not expect nice and easily formulated results as in Theorem 2.1. On the other hand this is the first example when the principal eigenvalue $\lambda_\varepsilon(G)$ tends to zero as $\left(\ln \frac{1}{\varepsilon}\right)^{-1}$. In all previously known cases this convergence to zero was either polinomial in ε as in [DEF] or exponentially fast in

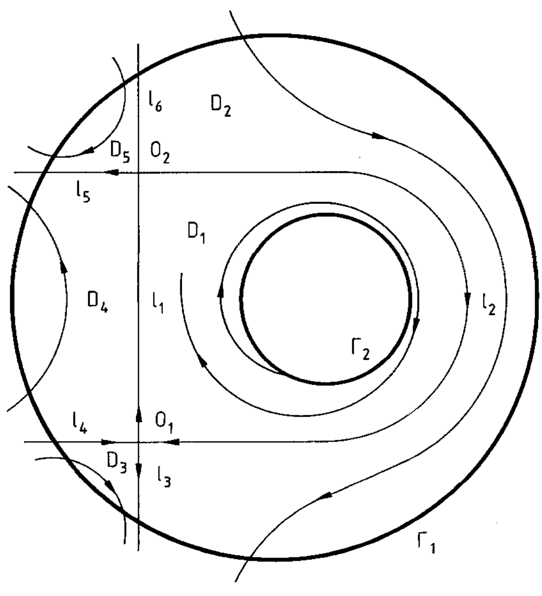

Fig. 1

$\left(-\varepsilon^{-1}\right)$ as in [F 1] and [Ve]. The orbits of the dynamical system S^t in our example are indicated on Fig. 1 by thin lines.
Here G is a ring-type region between two connected components Γ_1 and Γ_2 of the boundary ∂G which are drawn as boldface circles.

All orbits of S^t entering G through Γ_2 approach eventually the loop $\mathscr{L}$ consisting of the points $\mathcal{O}_1, \mathcal{O}_2$ of the saddle type and the orbits l_1, l_2,

$$l_1 = \{z : S^t z \to \mathcal{O}_2 \text{ and } S^{-t} z \to \mathcal{O}_1 \text{ as } t \to \infty\},$$
$$l_2 = \{z : S^t z \to \mathcal{O}_1 \text{ and } S^{-t} z \to \mathcal{O}_2 \text{ as } t \to \infty\}.$$

This loop forms the limit set of the dynamical system S^t in $\bar{G}$. Near the fixed points $\mathcal{O}_1$ and $\mathcal{O}_2$ we have the representation (8.1) with some matrices Π_1 and Π_2, respectively. Let $\gamma_1^{(1)}, \gamma_2^{(1)}$ and $\gamma_1^{(2)}, \gamma_2^{(2)}$ be the eigenvalues of Π_1 and Π_2, correspondingly, such that $\operatorname{Re}\gamma_1^{(1)} < 0 < \operatorname{Re}\gamma_2^{(1)}$ and $\operatorname{Re}\gamma_1^{(2)} < 0 < \operatorname{Re}\gamma_2^{(2)}$. Our picture corresponds to an attracting loop $\mathscr{L}$ which will be satisfied if

$$|\operatorname{Re}\gamma_1^{(1)}| > \operatorname{Re}\gamma_2^{(1)} \quad \text{and} \quad |\operatorname{Re}\gamma_1^{(2)}| > \operatorname{Re}\gamma_2^{(2)}. \tag{8.3}$$

Proposition 8.1. *In the above example*

$$C_{33}^{-1}\left(\ln\frac{1}{\varepsilon}\right)^{-1} \leqq |\lambda_\varepsilon(G)| \leqq C_{33}\left(\ln\frac{1}{\varepsilon}\right)^{-1} \tag{8.4}$$

for some $C_{33} > 0$ *independent of* ε, *provided* $0 < \varepsilon < 1$.

Proof. We shall only sketch the proof leaving details to the reader. Introduce the subdomains $D_i \subset G$, $i = 1, \ldots, 5$ bounded by stable and unstable curves l_i, $i = 1, \ldots, 6$ and the boundaries Γ_1, Γ_2 as it is pointed out on the above picture. We shall start with the upper bound for $\lambda_\varepsilon(G)$. Employing the method of Sect. 5 one can show that there exist constants $\delta_6, C_{34} > 0$ such that

$$\sup_{x \in D_1} P\{X_\varepsilon(C_{34}|\ln\varepsilon|, x) \in D_1\} \leqq 1 - \delta_6. \tag{8.5}$$

On the other hand, combining estimates of Lemmas 4.1 and 4.2 with arguments of [K 3] we conclude that for some δ_7, $C_{35} > 0$,

$$\sup_{x \in G \setminus D_1} P\left\{\tau_\varepsilon(x) > C_{35}\ln\frac{1}{\varepsilon}\right\} \leqq 1 - \delta_7. \tag{8.6}$$

To justify (8.6), remark that, if $x \in l_4 \cup (l_2 \setminus Q_\delta(\mathcal{O}_2))$ where $\delta > 0$ is arbitrary and Q_δ denotes a δ-neighborhood, then according to [K 3] the process $X_\varepsilon(t, x)$ exits near l_3 for the time of order $(\operatorname{Re}\gamma_2^{(1)})^{-1}|\ln\varepsilon|$ with probability close to $\frac{1}{2}$. If $x \in D_3 \cup (D_2 \setminus Q_\delta(\mathcal{O}_2))$ then one can see from the proof in [K 3] that this probability may only increase, and so it is essentially bounded from below by $\frac{1}{2}$. If $x \in D_2 \cap Q_\delta(\mathcal{O}_2)$ one can show that with probability, at least, $\frac{1}{4}$ the process $X_\varepsilon(t, x)$ gets outside of $Q_\delta(\mathcal{O}_2)$ in D_2. Using the strong Markov property and the above arguments we conclude that in this case $X_\varepsilon(t, x)$ exits near l_3 for the time of order

$((\mathrm{Re}\,\gamma_2^{(1)})^{-1}+(\mathrm{Re}\,\gamma_2^{(2)})^{-1})^{-1}\ln\frac{1}{\varepsilon}$ with probability, at least, $\frac{1}{8}$. Similar arguments hold true concerning the fixed point $\mathcal{O}_2$ and the domains D_4 and D_5. Now (3.3), (8.5) and (8.6) together with the Markov property imply

$$\lambda_3 \leqq \frac{\ln(1-\delta_6\delta_7)}{(C_{34}+C_{35})\ln\left(\frac{1}{\varepsilon}\right)} \tag{8.7}$$

To obtain a lower bound for λ_ε we shall use the Gaussian approximation Z_ε in the same way as in the proof of Theorem 2.2. By Lemma 4.2 one can find $\delta_8>0$ small enough such that for $\tilde{T}(\varepsilon)=\delta_8|\ln\varepsilon|$ one has

$$\begin{aligned} &P\{|Z_\varepsilon(T(\varepsilon),x)-X_\varepsilon(\tilde{T}(\varepsilon),x)|>\varepsilon^{3/2}\} \\ &\quad + P\left\{\sup_{0\leqq u\leqq \tilde{T}(\varepsilon)} |S^u x - X_\varepsilon(u,x)|>\varepsilon^{3/4}\right\} < \varepsilon^2 . \end{aligned} \tag{8.8}$$

Define $\tilde{D}=\{x\in D_1 : \mathrm{dist}(x,\Gamma_2)>\delta_9\}$ for some $\delta_9>0$ small enough. Then there is $\delta_{10}>0$ such that

$$\inf_{x\in\tilde{D},\, t>0} \mathrm{dist}(S^t x, \partial G) \geqq \delta_{10} > 0 . \tag{8.9}$$

By the Markov property, similarly to (6.1) it follows that

$$P\{\tau_\varepsilon(x)>\tilde{T}(\varepsilon)k\} \geqq \left(\inf_{x\in\tilde{D}} P\{X_\varepsilon^G(T(\varepsilon),x)\in\tilde{D}\}\right)^k, \tag{8.10}$$

where X_ε^G was defined in (5.1). Next, (8.8) and (8.9) imply for any $x\in\tilde{D}$ that

$$\begin{aligned} P\{X_\varepsilon^G(\tilde{T}(\varepsilon),x)\in\tilde{D}\} &\geqq P\{X_\varepsilon(\tilde{T}(\varepsilon),x)\in\tilde{D}\} - P\{\tau_\varepsilon(x)\leqq\tilde{T}(\varepsilon)\} \\ &\geqq P\{X_\varepsilon(\tilde{T}(\varepsilon),x)\in\tilde{D}\} - \varepsilon^2 \end{aligned} \tag{8.11}$$

provided $\varepsilon>0$ is small enough.

The estimate of the right hand in (8.11) depends on a location of x. For $x\in\tilde{D}$ satisfying $\mathrm{dist}(S^{T(\varepsilon)}x, l_1\cup l_2)>\varepsilon^{3/4}$ we have by (8.8),

$$P\{X_\varepsilon(\tilde{T}(\varepsilon),x)\in\tilde{D}\}\geqq 1-\varepsilon^2 . \tag{8.12}$$

If $x\in\tilde{D}$ satisfies $\mathrm{dist}(S^{\tilde{T}(\varepsilon)}x, l_1\cup l_2)\leqq\varepsilon^{3/4}$, but

$$\mathrm{dist}(S^{\tilde{T}(\varepsilon)}x, \mathcal{O}_1\cup\mathcal{O}_2)\geqq\varepsilon^{2/3}$$

then using the coordinate system (γ,ϱ), where

$$\varrho(x)=\mathrm{dist}(x,l_1\cup l_2)=|x-\gamma(x)| ,$$

one obtains in the same way as in Sect. 6 that

$$P\{X_\varepsilon(\tilde{T}(\varepsilon),x)\in\tilde{D}\}\geqq\tfrac{1}{2}-\varepsilon \tag{8.13}$$

provided $\varepsilon > 0$ is small enough. Similarly, if $|S^{\tilde{T}(\varepsilon)}x - \mathcal{O}_i| < \varepsilon^{2/3}$ for some $i = 1, 2$ then estimating the projections of $X_\varepsilon(\tilde{T}(\varepsilon), x)$, $Z_\varepsilon(\tilde{T}(\varepsilon)x)$, and $S^{\tilde{T}(\varepsilon)}x$ on both eigendirections of the matrix Π_i, we derive

$$P\{X_\varepsilon(\tilde{T}(\varepsilon), x) \in \tilde{D}\} \geqq \delta_{11} \tag{8.14}$$

for some $\delta_{11} > 0$ independent of x and ε, provided $\varepsilon > 0$ is small enough. Finally, (1.6) and (8.10)–(8.14) imply

$$0 \geqq \lambda_\varepsilon(G) \geqq \frac{\ln(\frac{1}{3}\delta_{11})}{|\ln \varepsilon|} \tag{8.15}$$

which together with (8.7) prove (8.4). □

To support our claim that in a general cyclic situation the asymptotic behavior of $\lambda_\varepsilon(G)$ hardly can be described in simple terms, notice that if in (8.3) we have the inequalities in opposite directions then the loop $\mathscr{L} = l_1 \cup l_2$ becomes repulsing, and a combination of methods from [K 1] and from the present paper yields that $\lambda_\varepsilon(G)$ does not converge to zero as $\varepsilon \to 0$ but to some negative number.

Remark 8.1. All results of this paper can be readily modified for the case of a smooth manifold in place of R^m.

Appendix: Exit Time and the Exit Distribution for an Attracting Boundary Case

The methods of this paper enable us to study also the asymptotic behaviour of the expectation of the exit time from a neighbourhood of an attracting boundary Γ, as well as the corresponding exit distribution Via the well known connection these provide the corresponding asymptotics for solutions of the Poisson type and the Dirichlet problems (see [Fri 1]).

Let G, U, Γ, α_0, and $\tau_\varepsilon(x) = \tau_\varepsilon(x, U \cap G)$ be the same as in Theorem 2.2 (i). If $x \in U \cap G = U^G$ and $t(x) = \inf\{t : S^t x \notin U^G\} < \infty$ then, clearly, $E\tau_\varepsilon(x) \to t(x)$ as $\varepsilon \to 0$. Otherwise we have the following:

Theorem A.1. *If $S^t x \in U^G$ for all $t \geqq 0$ then*

$$\lim_{\varepsilon \to 0} (|\ln \varepsilon|^{-1} E\tau_\varepsilon(x)) = |\alpha_0|^{-1} . \tag{A.1}$$

Proof. By an easy refinement of the estimates (4.27)–(4.32) we derive that

$$\varrho(x) e^{(\alpha_0 - \delta)t} \leqq \varrho(S^t x) \leqq \varrho(x) e^{(\alpha_0 + \delta)t} \tag{A.2}$$

for each $x \in V \equiv U^G$ provided $t \geqq t(\delta) \geqq 1$. Then for any $\delta > 0$ small enough, each positive $\varepsilon \leqq \varepsilon(\delta)$ and every sequence of points $x_i \in V$, $i = 0, \ldots, n$ satisfying

$$|S^{t(\delta)} x_i - x_{i+1}| \leqq \varepsilon (\ln \varepsilon)^2 \quad \text{for all } i = 1, \ldots, n , \tag{A.3}$$

the relation (A.2) implies

$$e^{nt(\delta)(\alpha_0-\delta)}\varrho(x_0)-C_{36}\varepsilon(\ln\varepsilon)^2\leqq\varrho(x_n)\leqq e^{nt(\delta)(\alpha_0+\delta)}\varrho(x_0)+C_{36}\varepsilon(\ln\varepsilon)^2 \tag{A.4}$$

where $C_{36}>0$ is independent of ε, δ and the sequence $x_0, \ldots, x_n$. Thus integrating over sequences $x_0, \ldots, x_n$ satisfying (A.3), and similarly to (5.6)–(5.7) employing the Chapman-Kolmogorov equality together with (3.6), one obtains for any $\varepsilon>0$ small enough that

$$P\{\tilde{X}_\varepsilon(n(\delta,\varepsilon)t(\delta),x)\notin\Gamma(\varepsilon)\}\leqq\exp(-|\ln\varepsilon|^3), \tag{A.5}$$

where X_ε and $\Gamma(\varepsilon)$ are the same as in (5.1) and (5.2), and $n(\delta,\varepsilon)=[(t(\delta)|\alpha_0+\delta|)^{-1}|\ln\varepsilon|]$.

Using (5.15) and (5.16) with $T(\varepsilon)=|\ln\varepsilon|^{1/2}$ we obtain by (A.5) and the Markov property that

$$P\{\tau_\varepsilon(x)>n(\delta,\varepsilon)t(\delta)+|\ln\varepsilon|^{1/2}\}\leqq P\{\tilde{X}_\varepsilon(n(\delta,\varepsilon)t(\delta),x)\notin\Gamma(\varepsilon)\}$$
$$+\sup_{z\in\Gamma(\varepsilon)}P\{\tau_\varepsilon(z)>|\ln\varepsilon|^{1/2}\}\leqq\exp(-|\ln\varepsilon|^{1/3}) \tag{A.6}$$

provided $\varepsilon>0$ is small enough.

For an x satisfying the condition of Theorem A.1 define inductively the following sequence of sets $G_0(\varepsilon,x)=\{x\}$, $G_{i+1}(\varepsilon,x)=\{y\in U^G: |S^{t(\delta)}z-y|<\varepsilon(\ln\varepsilon)^2$ for some $z\in G_i(\varepsilon,x)\}$. Clearly, if $y\in G_n(\varepsilon,x)$ then there exists a sequence $x_0=x$, $x_1,\ldots,x_n=y$ satisfying (A.3). Since we assume that $S^tx\in U^G$ for all $t\geqq0$ and so $\operatorname{dist}(S^tx,\Gamma)\to0$ as $t\to\infty$ then

$$\inf_{t\geqq0}(S^tx,R^m\backslash(U^G\cup\Gamma))\equiv\beta(x)>0. \tag{A.7}$$

Put $\tilde{n}(\delta,\varepsilon)=[(1-\delta)(t(\delta)|\alpha_0-\delta|)^{-1}|\ln\varepsilon|]$. Then by (A.2) and (A.4) for any $n=1,\ldots,\tilde{n}(\delta,\varepsilon)$,

$$\inf_{0\leqq t\leqq t(\delta),\,y\in G_n(\varepsilon,x)}\operatorname{dist}(S^ty,\Gamma)\geqq\tfrac{1}{2}\varepsilon^{1-\delta} \tag{A.8}$$

for small enough $\varepsilon>0$. It follows from (4.15), (A.7), and (A.8) that

$$\inf_{1\leqq n\leqq\tilde{n}(\delta,\varepsilon),\,y\in G_{n-1}(\varepsilon,x)}P\{\tilde{X}_\varepsilon(y,t(\delta))\in G_n(\varepsilon,\delta)\}\geqq1-C_{37}(\delta)\varepsilon^2 \tag{A.9}$$

for some $C_{37}(\delta)>0$ and small enough $\varepsilon>0$.

Next, by (A.9) and the Chapman-Kolmogorov equality we derive

$$P\{\tau_\varepsilon(x)>\tilde{n}(\varepsilon,\delta)t(\delta)\}\geqq\int_{G_1(x,\varepsilon)}\cdots\int_{G_{\tilde{n}(\varepsilon,\delta)}(\varepsilon,\delta)}q_\varepsilon(t(\delta),x,z_1)$$
$$\times q_\varepsilon(t(\delta),z_1,z_2)\ldots q_\varepsilon(t(\delta),z_{\tilde{n}(\varepsilon,\delta)-1},z_{\tilde{n}(\varepsilon,\delta)})dz_1\ldots dz_{\tilde{n}(\varepsilon,\delta)}$$
$$\geqq\Big(\inf_{1\leqq n\leqq\tilde{n}(\varepsilon,\delta),\,y\in G_{n-1}(\varepsilon,x)}P\{\tilde{X}_\varepsilon(y,t(\delta))\in G_n(\varepsilon,\delta)\}\Big)\tilde{n}(\varepsilon,\delta)$$
$$\geqq(1-C_{37}(\delta)\varepsilon^2)^{\tilde{n}(\varepsilon,\delta)}\geqq1-\varepsilon \tag{A.10}$$

for $0<\varepsilon<\varepsilon(\delta)$, where q_ε was defined in (5.1). Thus by (5.17), (A.6), and (A.10),

$$\begin{aligned}\left|E\tau_\varepsilon(x)/|\ln\varepsilon|-1/|\alpha_0|\right| &= \left|1/|\ln\varepsilon|\int_0^\infty P\{\tau_\varepsilon(x)>u\}du-1/|\alpha_0|\right|\\
&\leqq 1/|\ln\varepsilon|\int_0^{|\alpha_0|^{-1}|\ln\varepsilon|}|P\{\tau_\varepsilon(x)>u\}-1|du\\
&\quad+1/|\ln\varepsilon|\int_{|\alpha|^{-1}|\ln\varepsilon|}^\infty P\{\tau_\varepsilon(x)>u\}du\\
&\leqq 1/|\ln\varepsilon|\int_0^{\tilde n(\varepsilon,\delta)t(\delta)}|P\{\tau_\varepsilon(x)>u\}-1|du\\
&\quad+2(t(\delta)/|\ln\varepsilon|)|n(\varepsilon,\delta)-\tilde n(\varepsilon,\delta)|+|\ln\varepsilon|^{-1/2}\\
&\quad+\int_{n(\varepsilon,\delta)t(\delta)}^{|\ln\varepsilon|^{3/2}}P\{\tau_\varepsilon(x)>u\}du+\int_{|\ln\varepsilon|^{3/2}}^\infty P\{\tau_\varepsilon(x)>u\}du\\
&<\varepsilon t(\delta)|\alpha_0|^{-1}+2(|\alpha_0+\delta|^{-1}-(1-\delta)|\alpha_0-\delta|^{-1})\\
&\quad+|\ln\varepsilon|^{-1/2}+|\ln\varepsilon|^{3/2}\exp(-|\ln\varepsilon|^{1/3})+\int_{|\ln\varepsilon|^{3/2}}^\infty e^{1/2\alpha_0 t}dt \qquad (A.11)\end{aligned}$$

provided $\varepsilon>0$ is small enough. Letting, first, $\varepsilon\to0$ and then $\delta\to0$ we obtain (A.1). □

The asymptotic behaviour of the exist distribution does not seem to obey a general and simply formulated law. We shall exhibit here an example where this distribution diverge when $\varepsilon\to0$. Notice that another example of a divergence, when the dynamical system S^t has a repulsive type fixed point in G, was presented in [E].

Let $G=\{x:1/4<|x|<1\}\subset R^2$, $\Gamma=\{x:|x|=1\}$, and $L_\varepsilon=1/2\varepsilon^2\Delta+\langle B(x),\nabla\rangle$ where

$$\begin{aligned}B(x)&=\lambda x(1-|x|)/|x|+U_{\pi/2}x,\\ U_\varphi x&=(x_1\cos\varphi+x_2\sin\varphi,\ -x_1\sin\varphi+x_2\cos\varphi)\quad\text{and}\quad\lambda>0.\end{aligned}\qquad (A.12)$$

Then, clearly,

$$S^t x=(1-e^{-\lambda t}(1-|x|))U_t x/|x| \qquad (A.13)$$

and

$$\sup|V^{1/2}(t,x)n(S^t x)|\leqq C_{38}, \qquad (A.14)$$

where V is defined by (4.5) and $C_{38}>0$ is independent of $t>0$ and $\lambda>\lambda_0>0$. Furthermore, we can write

$$\varrho(S^t x)=e^{-\lambda t}(1-|x|) \qquad (A.15)$$

and

$$\gamma(S^t x)=U_t x/|x|. \qquad (A.16)$$

Denote for any $k \geqq 1$,

$$\varepsilon_k \equiv (2M)^{-1} e^{-2\pi k\lambda} \quad \text{and} \quad \tilde{\varepsilon}_k = \varepsilon_k e^{-\pi\lambda}, \tag{A.17}$$

where $M > 0$ will be chosen big enough. Let $\xi = (1,0) \in R^2$,

$$\Gamma_1 = \bigcup_{|\varphi| \leqq \pi/4} U_\varphi \xi \quad \text{and} \quad \Gamma_2 = U_\pi \Gamma_1 . \tag{A.18}$$

Proposition A.1. *Suppose that $\lambda, M > 0$ big enough and $x_0 = \xi/2$ then for each $\delta > 0$ there is $k(\delta) > 0$ such that*

$$P\{X_{\varepsilon_k}(\tau_{\varepsilon_k}(x_0), x_0) \in \Gamma_1\} \geqq 1 - \delta \tag{A.19}$$

and

$$P\{X_{\tilde{\varepsilon}_k}(\tau_{\tilde{\varepsilon}_k}(x_0), x_0) \in \Gamma_2\} \geqq 1 - \delta \tag{A.20}$$

provided $k \geqq k(\delta)$.

Proof. Clearly,

$$\varrho(S^{2\pi k} x_0) = M\varepsilon_k, \; \varrho(S^{\pi(2k+1)} x_0) = M\varepsilon_k \tag{A.21}$$

and

$$\gamma(S^{2\pi k} x_0) = \xi, \qquad \gamma(S^{\pi(2k+1)} x_0) = -\xi . \tag{A.22}$$

If λ is big enough then the estimates of Lemma 4.2 give

$$P\left\{\sup_{0 \leqq u \leqq \pi(2k+1)} |S^u x - X_{\varepsilon_k}(u, x)| \geqq \beta\right\} \leqq \varepsilon_k^5/\beta^6 \tag{A.23}$$

and

$$P\left\{\sup_{0 \leqq u \leqq \pi(2k+1)} |X_{\varepsilon_k}(u, x) - Z_{\varepsilon_k}(u, x)| \geqq \varepsilon_k^{3/2}\right\} \leqq \varepsilon_k \tag{A.24}$$

for each $\beta > 0$, $x \in G$, and $k \geqq N_0$.
Denote $G(\varepsilon) = \{x : \varrho(x) \leqq 2\varepsilon^{3/2}\}$. By the strong Markov property

$$P\{X_{\varepsilon_k}(2\pi k, x_0) \in G(\varepsilon_k)\} \geqq P\{\tau_{\varepsilon_k}(x_0) \leqq 2\pi k\} \inf_{y \in \Gamma,\, 0 \leqq s \leqq 2\pi k} P\{X_{\varepsilon_k}(s, y) \in G(\varepsilon_k)\} \tag{A.25}$$

Using (A.23) and (A.24) in the same way as in (5.10) one derives from (A.14) that

$$\inf_{y \in \Gamma,\, 0 \leqq s \leqq 2\pi k} P\{X_{\varepsilon_k}(s, y) \in G(\varepsilon_k)\} \geqq 1/4 \tag{A.26}$$

for all k big enough. Furthermore, by (4.12) similarly to (5.13) we have

$$\begin{aligned} P\{X_{\varepsilon_k}(2\pi k, x_0) \in G(\varepsilon_k)\} &\leqq P\{|X_{\varepsilon_k}(2\pi k, x_0) - Z_{\varepsilon_k}(2\pi k, x_0)| > \varepsilon_k^{3/2}\} \\ &\quad + P\{|X_{\varepsilon_k}(2\pi k, x_0) - S^{2\pi k} x_0| > \varepsilon_k^{5/8}\} \\ &\quad + P\{\langle Z_{\varepsilon_k}(2\pi k, x_0) - \gamma(S^{2\pi k} x_0), n(S^{2\pi k} x_0)\rangle < \varepsilon_k\} \\ &\leqq 2\varepsilon_k + P\{\langle Y(2\pi k, x_0), n(S^{2\pi k} x_0)\rangle < 1 - M\} \leqq 2\varepsilon_k + \delta^2 \end{aligned} \tag{A.27}$$

provided $M \geqq C_{38}\delta^{-1}$, $\delta > 0$ small enough and $k \geqq \tilde{k}(\delta)$. Now (A.25)–(A.27) imply that

$$P\{\tau_\varepsilon(x_0) \leqq 2\pi k\} \leqq \delta/4 \quad \text{for} \quad k \geqq \tilde{\tilde{k}}(\delta). \tag{A.28}$$

By (A.15) and (A.17)

$$\varrho(S^{2\pi k + \pi/8} x_0) = e^{-\lambda\pi/8} M \varepsilon_k. \tag{A.29}$$

Similarly to (5.11)–(5.15) one can obtain from (A.29) that

$$\begin{aligned} P\{\tau_\varepsilon(x_0) \geqq 2\pi k + \pi/8\} &\leqq \delta/4 + \varepsilon_k^{-1} |p(S^{2\pi k + \pi/8} x_0)| \\ &\leqq e^{-\lambda\pi/8} M + \delta/4 \leqq \delta/2 \end{aligned} \tag{A.30}$$

provided $\lambda \geqq \lambda(\delta) > 0$. Finally, by (A.23), (A.28), and (A.30),

$$\begin{aligned} P\{X_{\varepsilon_k}(\tau_{\varepsilon_k}(x_0), x_0) \notin \Gamma_1\} &\leqq P\{\tau_{\varepsilon_k}(x_0) \notin [2\pi k, 2\pi k + \pi/8]\} \\ &\quad + P\left\{ \sup_{2\pi k \leqq u \leqq 2\pi k + \pi/8} |X_{\varepsilon_k}(\tau_{\varepsilon_k}(x_0), x_0) - S^u x_0| \right. \\ &\quad \left. \geqq \inf_{2\pi k \leqq u \leqq 2\pi k + \pi/8} \operatorname{dist}(S^u x_0, \partial G \backslash \Gamma_1) \right\} \leqq \delta \end{aligned} \tag{A.31}$$

proving (A.19). The proof of (A.20) is going through exactly in the same way, as above. □

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