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Probability Theory and Related Fields

A Class of Not Local Rank One Automorphisms Arising from Continuous Substitutions

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Summary. A simple class of not local rank one, loosely Bernoulli transformations is constructed. These examples have simple spectra. It is studied when inverse limits of local rank one transformations are again of local rank one.

I. Introduction

Let $T:(X,\mathscr{B},m)\circlearrowleft$ be an ergodic automorphism on a Lebesque space.

By a *partition* we mean a set $Q=(Q_0,\dots,Q_{n-1})$, where $Q_i\cap Q_j=\emptyset$, $i\neq j$, $Q_i\in\mathscr{B}$. A partition Q is a *partition of* X if $\bigcup_{i=0}^{n-1} Q_i=X$. Let Q and R be partitions with the same number of atoms, say n, then

$$\varrho(Q,R)=\sum_{i=0}^{n-1} m(Q_i\triangle R_i). \tag{1}$$

Let Q, R be partitions. We write $Q\leqq R$ if every atom of Q is a union of atoms of R.

Let $A\subset X$, $m(A)>0$ and Q be a partition. Then by $Q|_A$ we mean the partition of A, $Q|_A=(Q_0\cap A,\dots,Q_{n-1}\cap A)$.

Following [4] T is said to have *local rank* 1 if there is a number $a>0$ such that for every $\delta>0$ and for every partition Q of X there exist a set $F\subset X$, an integer n and a partition R of $A=\bigcup_{i=0}^{n-1} T^iF$ such that the set

$$\{F,TF,\dots,T^{n-1}F\} \tag{2}$$

is a T-stack, i.e. $T^iF\cap T^jF=\emptyset$, $i\neq j$,

$$m(A)\geqq a, \tag{3}$$

$$\varrho(Q|_A,R)<\delta \quad \text{where} \quad R\leqq\{F,TF,\dots,T^{n-1}F\}. \tag{4}$$

For the definitions of *finite rank* and *loosely Bernoulli* (LB) *automorphisms* we refer to [9].
From [4] it follows that the following inequalities hold:
finite rank $\subsetneq$ local rank one $\subsetneq$ LB$\cap$ zero entropy.

To obtain transformations that are LB and not local rank 1 Ferenczi [4] proved that Vershik processes are never local rank 1 (but they can enjoy LB property [13]). Since, local rank 1 transformations have finite spectral multiplicity, this gives another way to produce LB transformations that are not local rank 1. In the present paper we provide a new large class of LB and zero entropy transformations that are not local rank 1. These examples are of the form $T\times\tau$ where T is a continuous substitution on two symbols [2] and τ has a rational pure point spectrum.

The second problem we deal with is connected with an inverse limit of local rank one transformations. We point out the only reason which can make $T=\lim\operatorname{inv} T_n$ not be of local rank one. It turns out that even an inverse limit of finite rank automorphisms need not be of local rank 1 (but is always LB [9]). The examples presented allow us to write the following sequence of inequalites finite rank $\subsetneq$ local rank one $\subsetneq$ inverse limit of local rank 1 $\subsetneq$ LB. In particular, the class of local rank 1 processes is not closed in $\bar{d}$ metric.

One of the famous problems in ergodic theory is the problem of spectral multiplicity. For some account on this subject we refer to [12]. We mention here that Baxter proved in [1] that rank one implies simple spectrum (see also [3]). It was asked whether the inverse implication was true. This supposition was turned out to be false. Del Junco in [6] showed that the classical Morse sequence had rank 2 and simple spectrum. The result was generalized in [8]. In particular, all continuous substitutions on two symbols have rank 2 and simple spectra. In [5] Ferenczi constructed an automorphism with simple spectrum which is not LB. So this example has a simple spectrum and infinite rank. We prove that our examples have simple spectra and therefore we get a class of automorphisms with simple spectra which are LB but not local rank 1.

II. Notations

Throughout this paper we will assume that T is an ergodic automorphism of a Lebesque space $(X,\mathscr{B},m)$, unless it states otherwise.

Let $x\in X$, $m\in N$ and $Q=(Q_0,\ldots,Q_{n-1})$ be a partition of X. $Q-m$*-name of* x is a string $\xi=(\xi_0,\xi_1,\ldots,\xi_{m-1})$ where $\xi_i=0,\ldots,n-1$ and $\xi_i=s$ iff $T^ix\in Q_s$.

Finite strings of symbols $0,\ldots,n-1$ we will call *blocks*. Let $\xi=(\xi_0,\ldots,\xi_{m-1})$ be such a one. Then $m=|\xi|$ is the *lenght* of ξ (we will also say ξ is an m-block). Suppose $|\xi|=|\eta|=m$, then

$$d(\xi,\eta)=\operatorname{card}\{i:0\leqq i\leqq m-1,\ \xi_i\neq\eta_i\}/m. \tag{5}$$

Now, let ξ, η be blocks and $\delta>0$. We denote

$$\operatorname{fr}(\xi,\eta)=\operatorname{card}\{i:\eta[i,i+|\xi|-1]=\xi\}, \tag{6}$$

where $\eta[i,j]=(\eta_i,\eta_{i+1},\ldots,\eta_j)$, $\eta[i,i]=\eta[i]$,

$$\delta-\operatorname{fr}(\xi,\eta)=\operatorname{card}\{i:d(\eta[i,i+|\xi|-1],\xi)<\delta\}. \tag{7}$$

If $d(\eta[i, i+|\xi|-1], \xi)<\delta$ then we say ξ *appears in* η *at* i *within* δ.

Recall that if $Q=(Q_0, \ldots, Q_{n-1})$ is a partition of X then we obtain a measure P on blocks on $\{0, \ldots, n-1\}$ given by the process (T, Q) as follows

$$P\{\xi : \xi \text{ is an } m\text{-block}\} = m\{x \in X : Q-m\text{-name } x = \xi\}. \tag{8}$$

Now, we consider blocks on $\{0, 1\}$. Let $B=(b_0, \ldots, b_{k-1})$ and $C=(c_0, \ldots, c_{m-1})$ be such ones. Then

$$B \times C = B^{c_0} B^{c_1} \ldots B^{c_{m-1}}, \tag{9}$$

where $B^0 = B$, $B^1 = \tilde{B} = (1-b_0, \ldots, 1-b_{k-1})$.

Let us put

$$x = B \times B \times B \times \ldots, \tag{10}$$

where $B[0]=0$, $|B|=\lambda \geqq 2$ and $B \neq 0\ldots0$, $01\ldots010$. Such sequences can be considered as constant Morse sequences [7] or as continuous substitutions in normal form [2]. Let $\mathcal{O}_x = \overline{\{T^i x : i \in Z\}}$ be the closed orbit under the shift transformation T. It is known that $(\mathcal{O}_x, T)$ is strictly ergodic [7]. The unique T-invariant measure (ergodic) we shall denote by μ_x. This measure is mirror-invariant, i.e.

$$\mu_x(C) = \mu_x(\tilde{C}) \quad \text{for any block } C. \tag{11}$$

Denote $c_t = B \times \ldots \times B_{t+1 \text{ times}}$ and call c_t *or* $\tilde{c}_t$ *t-symbols*. By strict ergodicity of $\mathcal{O}_x$ we have

$$\mu_x(B) = \lim_{m \to \infty} \frac{1}{m} \operatorname{fr}(B, x[0, m-1]) = \lim_{t \to \infty} \frac{1}{\lambda^{t+1}} \operatorname{fr}(B, c_t) \quad \text{for any block } B. \tag{12}$$

III. Results

Before formulating some equivalence condition T to be of local rank one we mark out some subclass of finite blocks of symbols. Namely, we say that a block η *has an* $a-\delta$*-structure of a* $\delta-\xi_0$*-block* ($\delta>0$, $0<a<1$, ξ_0-block) if

$$\eta = \varepsilon_0 \xi_1 \varepsilon_1 \ldots \zeta_l \varepsilon_l, \qquad d(\xi_i, \xi_0) < \delta, \qquad \sum_{i=0}^{l} |\varepsilon_i| < (1+\delta)(1-a)|\eta|. \tag{13}$$

Proposition 1. *T has local rank 1 iff*

$$(\exists a>0)(\forall \delta>0)(\forall Q\text{-partition})(\exists \xi_0\text{-block})(\exists N>0)(\forall n \geqq N)$$

$$P\{\xi - n\text{-block}: \xi \text{ has an } a-\delta\text{-structure of a } \delta-\xi_0\text{-block}\} > 1-\delta.$$

Proposition 1 is similar to Theorem 3.1 in [6] and the proof can be obtained slightly modyfying Del Junco's proof.

Corollary 1. *Let T be of local rank 1 with a constant $a>0$ and τ be its factor-automorphism. Then τ is also of local rank 1 with a constant $b>0$ and moreover it can be assumed $b \geqq a$.*

Below we formulate two propositions that will be our tools in the proofs of main results of this paper.

Proposition 2 [8]. *There are $L>0$, $\delta_0>0$ such that for every $t\in N$ if $c_t\times v$, $|v|=L$ appears in x at i within δ_0 then $c_t\times v$ appears exactly at i and $\lambda^{t+1}|i$.*

Proposition 3. *There is a $K>0$ such that for every block η*

$$\mu_x(\eta)\leqq K/|\eta|. \tag{14}$$

Proof. Let K_1 be a number satysfying

$$\mu_x(v)\leqq K_1/L \quad \text{for any } v, \quad |v|=L. \tag{15}$$

Let us notice that Proposition 2 implies

$$\mathrm{fr}(c_{t_0}\times v, c_t)=\mathrm{fr}(v, c_{t-t_0+1}) \tag{16}$$

for t large enough, $|v|=L$.

Hence

$$\frac{1}{\lambda^{t+1}}\,\mathrm{fr}(c_{t_0}\times v, c_t)=\frac{1}{\lambda^{t-t_0+2}}\,\mathrm{fr}(v, c_{t-t_0+1})\,\frac{1}{\lambda^{t_0-1}}$$

which implies

$$\mu_x(c_{t_0}\times v)=\frac{1}{\lambda^{t_0-1}}\,\mu_x(v)\leqq L\lambda^2K_1/|c_{t_0}\times v|. \tag{17}$$

Now, let η be any block with $|\eta|\geqq\lambda^2L$. Let t_0 be the greatest natural number such that $c_{t_0}\times v$ is contained in η for some v, $|v|=L$. Thus from (17)

$$\mu_x(\eta)\leqq\mu_x(c_{t_0}\times v)\leqq K_2/|\eta|.$$

Let K_3 be a number satysfying $\mu_x(\eta)\leqq K_3/|\eta|$ for $|\eta|<\lambda^2L$. So $K=\max(K_2,K_3)$ satisfies (14).

Let us take $Q=(Q_0,Q_1)$ the natural generator for T on $\mathcal{O}_x$,

$$Q_i=\{y\in\mathcal{O}_x : y[0]=i\}, \quad i=0,1.$$

Next let us fix $q\in N$

$$(q,\lambda)=1 \tag{18}$$

and let $Z_q=\{0,\dots,q-1\}$ be the group of integer mod q equipped with the uniform measure ν_q and $\tau_q : Z_q \circlearrowleft$, $\tau_q(i)=i+1$.

We shall examine the product transformations $T\times\tau_q$. By (18) these products are ergodic.

Denote by $\hat{Q}=Q\times(\{0\},\dots,\{q-1\})=\{\hat{Q}_0,\dots,\hat{Q}_{2q-1}\}$ the partition of $\mathcal{O}_x\times Z_q$ given by the formula

$$\hat{Q}_{2i+k}=Q_k\times\{i\}, \quad 0\leqq i\leqq q-1, \quad k=0,1. \tag{19}$$

As an immediate consequence of (19) and by the definition of τ_q, $\hat{Q}-n$-names of points $(y,i)\in\mathcal{O}_x\times Z_q$ one can describe as follows

$$Q-n\text{-name}(y,i)=(2i+y[0],2(i+1)+y[1],\ldots,2(i+n-1)+y[n-1])\quad(\bmod 2q)\,. \tag{20}$$

So, for any $y\in\mathcal{O}_x$, $i,j\in Z_q$, $i\neq j$ and $n\geqq 1$

$$d(\hat{Q}-n\text{-name}(y,i),\hat{Q}-n\text{-name}(y,j))=1\,. \tag{21}$$

Proposition 4. *If a, $0<a<1$ satisfies* $(\forall\delta>0)$ $(\exists\xi_0-Q-r\text{-}block)$ $(\exists N>0)$ $(\forall n\geqq N)$

$$P\{\eta-n\text{-}block\text{: } \eta \text{ } has\ an\ a-\delta\text{-}structure\ of\ a\ \delta-\xi_0\text{-}block\}>1-\delta \tag{22}$$

then $a<2K/q$ (K is defined in Proposition 3). Here P is the measure given by the process $(T\times\tau_q,\hat{Q})$ and the measure $\mu_x\times\nu_q$.

Proof. First, observe that from the definition of P it follows that

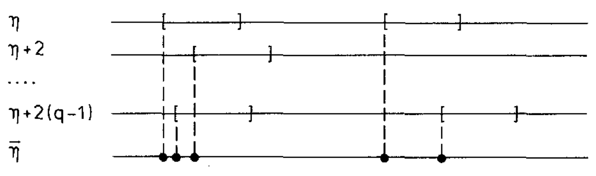
$$P(y[0,n-1]\times\{i\})=P(y[0,n-1]\times\{j\})\,,\qquad i,j\in Z_q\,. \tag{23}$$

Furthermore from (20)

$$\hat{Q}-n\text{-name}(y,i)=\hat{Q}-n\text{-name}(y,j)+2(i-j)(\bmod 2q)\,. \tag{24}$$

Therefore, if we consider blocks η having an $a-\delta$-structure of a $\delta-\xi_0$-block and $|\eta|\geqq N$ then there must exist an η such that $\eta,\eta+2,\ldots,\eta+2(q-1)$ all have an $a-\delta$-structure of a $\delta-\xi_0$-block [this is a consequence of (22), (23), and (24) if $\delta<1/q$].

From (21) we have the Fig. 1:



We see that if ξ_0 appears in η at i within δ then ξ_0 cannot appear in $\eta+2j, j\neq 0$ at i within δ, for a suitable $\delta>0$. Therefore if $\bar{\eta}$ denotes the block obtained from η (and $\eta+2,\ldots,\eta+2(q-1)$) by writing 0 instead of even numbers and 1 instead of odd numbers then

$$d((\eta+2j)[i,i+r-1],\xi_0)<\delta\quad\text{implies}\quad d(\bar{\eta}[i,i+r-1],\bar{\xi}_0)<\delta\,, \tag{25}$$

where $\bar{\xi}_0$ is the block obtained from ξ_0 by the way above.

We have by (13)

$$\eta+2j=\varepsilon_0^{(j)}\xi_1^{(j)}\varepsilon_1^{(j)}\ldots\xi_{l_j}^{(j)}\varepsilon_{l_j}^{(j)}\,,$$
$$\sum_{s=0}^{l_j}|\varepsilon_s^{(j)}|<(1+\delta)(1-a)n\,,\qquad d(\xi_s^{(j)},\xi_0)<\delta\,,\qquad s=1,\ldots,l_j\,. \tag{26}$$

This implies

$$\delta-\mathrm{fr}(\xi_0,\eta+2j)\geqq(1-2\delta)\ \mathrm{an}\,,\qquad j=0,\ldots,q-1\,. \tag{27}$$

Hence

$$\delta-\mathrm{fr}(\bar{\xi}_0,\bar{\eta})\geqq(1-2\delta)\ \mathrm{qan}\,. \tag{28}$$

Without loss of generality we may assume that $\bar{\xi}_0$ appears in x. Next, we choose the greatest natural number t_0 such that

$$\bar{\xi}_0=\xi_1\zeta\xi_2\,,$$

where ζ is a concatenation of L t_0-symbols. It is not difficult to see that $|\zeta|>C(\lambda,L)r$, where $C(\lambda,L)$ is a constant depending only on λ and L. Thus for a suitable choice of δ and by (28) and Proposition 2 we get

$$\mathrm{fr}(\zeta,\bar{\eta})\geqq(1-C'(\lambda,L)\delta)\cdot\mathrm{qan}\,.$$

So, applying ergodic theorem to possibly larger n and other $\bar{\eta}$ we have $\mu_x(\zeta)\geqq(1-C''(\lambda,L)\delta)\,qa$ and therefore for a suitable $\delta>0$ our assertion is satisfied.

Let $\{n_t\}$ be a sequence of natural numbers $n_t<n_{t+1}$, $n_t|n_{t+1}$. We let $G\{n_t:t\geqq0\}$ denote the subgroup of roots of unity generated by $\{\exp 2\pi i/n_t,\ t\geqq0\}$.

As an immediate consequence of Corollary 1 and Proposition 4 we get

Theorem 1. *Let τ be an ergodic automorphism with discrete spectrum,* $\mathrm{Sp}(\tau)=G\{q^t:t\geqq0\}$, $(q,\lambda)=1$, *where* $\mathrm{Sp}(\tau)$ *is the group of all eigenvalues of τ. Then $T\times\tau$ is not of local rank* 1.

Let us observe, however, that the examples obtained from Theorem 1 are always ergodic Z_2-extensions over discrete spectrum and therefore they are LB automorphisms [9].

Let $T:(X,\mathscr{B},m)\circlearrowleft$ and let $\mathscr{B}_n\subset\mathscr{B}_{n+1}$, $n=1,2,\ldots$ be a sequence of T-invariant σ-algebras and such that $\cup\mathscr{B}_n=\mathscr{B}$. In this case we say T is an *inverse limit of* $T_n=T|_{\mathscr{B}_n}$ and we write $T=\lim\mathrm{inv}\,T_n$.

Corollary 2. *An inverse limits of finite rank transformations need not be of local rank* 1 *(but it has LB property* [9]*).*

Proof. Let τ and T be as in Theorem 1. Then $T\times\tau$ is the inverse limit of $(T\times\tau_{q^t})_{t\geqq1}$. It is not hard to see that the latter automorphisms are of finite rank.

It can be seen that in the Corollary above the reason T is not of local rank 1 is that the constants a_t satisfying (2)–(4) and corresponding to $T\times\tau_t$ tend to 0 when t goes to infinity. In fact, this is the only reason which makes $T=\lim\mathrm{inv}\,T_n$ not be of local rank 1. We formulate this fact as the following.

Theorem 2. *Let $T:(X,\mathscr{B},m)\circlearrowleft$ be an inverse limit of ergodic local rank* 1 *transformations $(T_n)_{n\geqq1}$. When the constants a_n satisfying* (2)–(4) *and corresponding to T_n have the property $0<a\leqq a_n$, $n\geqq1$ then T is also of local rank* 1.

We omit the proof as it is more or less obvious.

Our next goal is to prove that the examples arising from Theorem 1 have simple spectra. For spectral theory of unitary operators we refer to [10].

Let $T:(X,m)\circlearrowleft$, $h\in L^2(X,m)$. By $Z(h)$ we mean the cyclic space generated by h, i.e. $Z(h)=\operatorname{span}\{U_T^n(h):n\in Z\}$.

As an immediate consequence of Fubini's theorem we obtain

Lemma 1. *Let $T:(X,m)\circlearrowleft$ be an automorphism with simple spectrum and let $\tau:(Y,\nu)\circlearrowleft$ be ergodic automorphism with discrete spectrum. Then*

$$L^2(X\times Y, m\times\nu)=\bigoplus_{\lambda\in\mathrm{Sp}(\tau)} Z(gf_\lambda), \tag{29}$$

where $L^2(X,m)=Z(g)$, $U_\tau(f_\lambda)=\lambda\cdot f_\lambda$, $\lambda\in\mathrm{Sp}(\tau)$

Theorem 3. *Given T and τ as in Theorem 1, $T\times\tau$ has a simple spectrum.*

Proof. (For notations below we refer to [11].) From (29) it follows that all we have to show is that if μ is the spectral measure of T then $\mu*\delta_z$ and μ are mutually singular for every $z\in\mathrm{Sp}(\tau)$, $z\neq 0$ (here we assume $\mathrm{Sp}(\tau)\subset[0,1]$). Recall that $\mu=\mu_1+\mu_2$ where μ_1 is the type of discrete measure concentrated on $\mathrm{Sp}(T)=G\{\lambda^t:t\geqq 0\}$ and μ_2 is a measure belonging to the Wiener space and given by

$$\gamma(k)=\lim\frac{1}{m}\sum_{j=0}^{m-1}\alpha(j+k)\overline{\alpha(j)},\qquad k>0,\qquad \alpha(s)=(-1)^{x[s]},$$

$$\gamma(-k)=\overline{\gamma(k)}\ [11].$$

From a result of [11, p. 399] it is sufficient to show that there is a sequence b_n, $1\leqq b_n<\lambda$ such that

$$\sum_{n\geqq 1}|(b_n\lambda^n)|^2\,|1-\exp 2\pi i\lambda^n b_n z|^2 \text{ is divergent}. \tag{30}$$

We will prove that one can put $b_n=1$. Indeed, observe that $\gamma(\lambda^t)=\gamma(\lambda)$, $t\geqq 1$ (it is a consequence of the fact $x=c_t\times x$) and therefore (30) can be reduced to the following

$$\sum_{n\geqq 1}|1-\exp 2\pi i\lambda^n z|^2 \text{ is divergent}. \tag{31}$$

From our assumption $0\neq z=s/q^t$ for some $t>0$ and s, $0<s<q^t-1$. To prove (31) it remains to show that $\lim_{n\to\infty}\cos 2\pi\lambda^n z=1$ does not hold and it can be seen when dealing with iterations of the transformation $v\mapsto\lambda v$ on the unit interval.

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