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Probability Theory and Related Fields

A Strong Renewal Theorem for Generalized Renewal Functions in the Infinite Mean Case

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Summary. Let $F(x)$ be a nonarithmetic c.d.f. on $(0, \infty)$ such that $1-F(x) = x^{-\alpha} L(x)$, where $L(x)$ is slowly varying and $0 \leqq \alpha \leqq 1$. Let $a(x)$ be regularly varying with exponent $\beta \geqq -1$. A strong renewal theorem (of Blackwell type) for generalized renewal functions of the form $G(t) \equiv \sum_{n=0}^{\infty} a(n) F^n(t)$ is proved here, thus extending the recent work of Embrechts, Maejima and Omey [1] and that of Erickson [4].

1. Introduction

Let $\{X_i; i \geqq 1\}$ be a sequence of positive independent random variables with common nonarithmetic distribution function $F(x)$ and write $S_0=0$, $S_n = X_1 + X_2 + \ldots + X_n$ for $n \geqq 1$.

Generalized renewal functions $G(t)$ of the form

$$G(t) \equiv \sum_{n=0}^{\infty} a(n) P(S_n \leqq t) = \sum_{n=0}^{\infty} a(n) F^n(t), \tag{1}$$

where $\{a(n); n \geqq 0\}$ is a sequence of nonnegative constants, have of late received an increasing amount of attention. For example, see papers [1] and [2] and references contained therein; papers [3, 5, 6, 8] consider special cases such as subordinated probability distributions and harmonic renewal measures. A good deal of this work is concerned with the asymptotic behavior of $G(t)$ as $t \to \infty$ (weak renewal theorems) under various assumptions on the moments of $F(x)$ and the constants $a(n)$. The purpose of this paper is to extend some of the recent work of Embrechts, Maejima and Omey [1] to the infinite mean case.

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In particular, we extend their following Blackwell type theorem for generalized renewal functions which have $a(n)$ regularly varying with exponent $\beta \geqq -1$.

Theorem A Embrechts, Maejima and Omey [1]. *Let $F(x)$ be nonarithmetic and have finite mean μ. Let $a(x)=x^{\beta}L(x)$, where $L(x)$ is slowly varying. If $\beta>-1$, then, for every $y>0$,*

$$(G(t+y)-G(t)) \sim y\mu^{-\beta-1}a(t) \quad as \quad t\to\infty. \tag{2}$$

When $\beta=-1$, if $L(x)$ is monotone and, for some $K\geqq 0$,

$$(1-F(x)) \sim K a(x) \quad as \quad x\to\infty$$

or if $x^{1+\delta}(1-F(x))=o(1)$ for some $\delta>0$ as $x\to\infty$, then (2) *also holds.*

When the mean μ is infinite, the asymptotic behavior of $G(t)$ as $t\to\infty$ in the special case $a(n)=1$ has been well-studied. In this case, $G(t)$ becomes the renewal function $U(t)$ and the following theorems are well-known (see Erickson [4]). Theorems B and B′ are equivalent and yield the analogue of the weak renewal theorem, while Theorem C yields the analogue of the strong renewal theorem.

Theorem B. *Let $L(x)$ be slowly varying. Then for $0\leqq\alpha<1$,*

$$(1-F(x))=x^{-\alpha}L(x) \tag{3}$$

if, and only if,

$$U(t)\sim t^{\alpha}(L(t)\,\Gamma(1-\alpha)\,\Gamma(1+\alpha))^{-1} \quad as \quad t\to\infty.$$

Further, the truncated mean function

$$m(t)\equiv\int_0^t (1-F(u))\,du \sim L(t) \quad as \quad t\to\infty$$

if, and only if,

$$U(t)\sim t(L(t))^{-1} \quad as \quad t\to\infty.$$

An alternative form is due to Erickson [4].

Theorem B′. *Let $0\leqq\alpha\leqq 1$. The following are equivalent and each implies* (4) *below:*

(i) *$m(t)$ is regularly varying with exponent $1-\alpha$*
(ii) *$U(t)$ is regularly varying with exponent α.*

$$U(t)\sim t(m(t)\,\Gamma(1+\alpha)\,\Gamma(2-\alpha))^{-1} \quad as \quad t\to\infty. \tag{4}$$

Theorem C. *Let $F(x)$ satisfy* (3) *with $1/2<\alpha\leqq 1$. Then, for every $y>0$,*

$$(U(t+y)-U(t))\sim y c_{\alpha}(m(t))^{-1} \quad as \quad t\to\infty.$$

If $0<\alpha\leqq 1/2$, then

$$\lim_{t\to\infty} m(t)(U(t+y)-U(t))=y c_{\alpha},$$

where the constant $c_{\alpha}=(\Gamma(\alpha)\,\Gamma(2-\alpha))^{-1}$.

The extension of Theorem B to generalized renewal functions in the infinite mean case was given in [7] and is stated as Theorem D below.

Theorem D Omey [7]. *Let* $0<\alpha\leqq 1$ *and* $\rho>0$. *Let* $\{\alpha(n);\ n\geqq 0\}$ *be a sequence of nonnegative constants and let* $G(t)$ *be of the form* (1). *Any two of the following implies the third*:

(i) $m(t)$ *is regularly varying with exponent* $1-\alpha$

(ii) $\sum_{k=0}^{n} a(k)$ *is regularly varying with exponent* ρ

(iii) $G(t)$ *is regularly varying with exponent* $\rho\alpha$.

Furthermore, if (i) *holds for* $0\leqq\alpha\leqq 1$ *and*

$$\sum_{k=0}^{n} a(k)\sim n^{\rho} L_1(n) \quad as \quad n\to\infty, \tag{5}$$

where $L_1(x)$ *slowly varying and* $\rho\geqq 0$, *then*

$$G(t)\sim (U(t))^{\rho} L_1(U(t))\, C(\alpha,\rho) \quad as \quad t\to\infty,$$

where the constant $C(\alpha,\rho)=\Gamma(1+\rho)(\Gamma(1+\alpha))^{\rho}(\Gamma(1+\alpha\rho))^{-1}$.

In this paper we extend the strong renewal Theorem A to the infinite mean case. The proof of Theorem 2, which is our main result, makes use of Theorem D and a result proved in Theorem 1, which could be of independent interest. Our results follow.

Theorem 1. *Let* $F(x)$ *satisfy* (3) *with* $1/2<\alpha\leqq 1$. *Let* $Q(t)$ *be a nondecreasing regularly varying function with exponent* $\beta\geqq 0$. *If, for every* $y>0$,

$$(Q(t+y)-Q(t))/Q(t)=O(t^{-1}) \quad as \quad t\to\infty, \tag{6}$$

then, for every $y>0$,

$$((U^* Q)(t+y)-(U^* Q)(t))\sim y A(\alpha,\beta)(m(t))^{-1} Q(t) \quad as \quad t\to\infty, \tag{7}$$

where the constant $A(\alpha,\beta)=\Gamma(1+\beta)(\Gamma(2-\alpha)\,\Gamma(\alpha+\beta))^{-1}$ *and* * *denotes convolution.*

Theorem 2. *Let* $F(x)$ *satisfy* (3) *with* $1/2<\alpha\leqq 1$. *Let* $G(t)$ *be of the form* (1). *Let* $a(x)$ *be regularly varying with exponent* $\beta\geqq -1$ *such that* $n a(n)$ *is nondecreasing in n. If, for every* $y>0$,

$$\limsup_{t\to\infty} t\left(1-\frac{1-F(t+y)}{1-F(t)}\right)<\infty, \tag{8}$$

then, for every $y>0$,

$$(G(t+y)-G(t))\sim y D(\alpha,\beta)(m(t))^{-1} a(U(t)) \quad as \quad t\to\infty,$$

where the constant $D(\alpha,\beta)=\alpha(\Gamma(1+\alpha))^{\beta}\,\Gamma(2+\beta)(\Gamma(1+\alpha(\beta+1)))^{-1}$.

This obvious corollary compares Theorems C and 2.

Corollary 1. *Under the assumptions of Theorem 2, for every $y>0$,*

$$\frac{G(t+y)-G(t)}{U(t+y)-U(t)} \sim C(\alpha, \beta+1)\, a(U(t)) \quad as \quad t\to\infty.$$

The next section gives the proofs, while the last section has a few remarks.

2. Proofs

Proof of Theorem 1. We first prove (7) for $\beta>0$. Fix $y>0$ and choose δ such that $1/2<\delta<1$. Write

$$\begin{aligned}((U^*Q)(t+y)-(U^*Q)(t)) &= \int_{0^-}^{t} (U(t+y-z)-U(t-z))\,Q(dz)\\ &\quad -\int_{t}^{t+y} U(t+y-z)\,Q(dz) \equiv I(t)+J(t),\end{aligned}$$

say. Define I_1, I_2, I_3 and I_4 by

$$\begin{aligned}I(t) &= \left(\int_{0^-}^{\delta t} - \int_{[\delta t]}^{\delta t} + \int_{[\delta t]}^{[t]} + \int_{[t]}^{t}\right)(U(t+y-z)-U(t-z))\,Q(dz)\\ &\equiv I_1(t)-I_2(t)+I_3(t)+I_4(t),\end{aligned}$$

where $[x]$ is the greatest integer function. We now prove a series of lemmas which approximate the integrals I_1, I_2, I_3, I_4 and J. We return to the proof of Theorem 1 after Lemma 4.

Lemma 1. *$I_2(t)$, $I_4(t)$ and $J(t)$ are $o(Q(t)/m(t))$ as $t\to\infty$.*

Proof. These estimates easily follow from (6), Theorem C and the monotonicity and regular variation of $U(t)$, $Q(t)$ and $m(t)$.

Lemma 2. *If $\beta>0$,*

$$\lim_{t\to\infty} I_1(t)\, m(t)/Q(t) = y c_\alpha \beta \int_0^{\delta} u^{\beta-1}(1-u)^{\alpha-1}\,du.$$

Proof. Choose $\varepsilon>0$. By Theorem C there exists a $t_0=t_0(\varepsilon)$ such that

$$(y c_\alpha - \varepsilon) \leqq m(t)(U(t+y)-U(t)) \leqq (y c_\alpha + \varepsilon)$$

for $t>t_0$, where $c_0=(\Gamma(\alpha)\,\Gamma(2-\alpha))^{-1}$. Write

$$I_1(t) = \int_{0^-}^{\delta t} (U(t+y-z)-U(t-z))\, m(t-z)/m(t-z)\, Q(dz).$$

We have for $t>t_0(1-\delta)^{-1}$

$$(y c_\alpha - \varepsilon)\, I_{11}(t) \leqq I_1(t)\, m(t)/Q(t) \leqq (y c_\alpha + \varepsilon)\, I_{11}(t), \tag{9}$$

where

$$I_{11}(t) = \int_{0^-}^{\delta t} (m(t)/m(t-y))\, Q(dz)/Q(t) = \int_{0^-}^{\delta} (m(t)/m(t(1-u)))\, Q(t\,du)/Q(t),$$

since $t>t_0(1-\delta)^{-1}$ and $z<\delta t$ imply that $t-z>t_0$.

Since $m(t)/m(t(1-u))$ converges uniformly to $(1-u)^{\alpha-1}$ for u in $[0,\delta]$ and the measures $Q(t\,du)/Q(t)$ converge weakly to $\beta u^{\beta-1}du$ as $t\to\infty$,

$$\lim_{t\to\infty} I_{11}(t)=\beta\int_0^{\delta}(1-u)^{\alpha-1}u^{\beta-1}\,du.$$

Letting $t\to\infty$ followed by $\varepsilon\to 0^+$ in (9) proves the lemma.

Lemma 3. $\limsup_{t\to\infty} I_3(I)\,m(t)/Q(t)=o(1)$ *as* $\delta\to 1^-$.

Proof. By the monotonicity of $U(t)$ and $Q(t)$,

$$\begin{aligned}I_3(t)&=\sum_{k=[\delta t]}^{[t]-1}\int_k^{k+1}(U(t+y-z)-U(t-z))\,Q(dz)\\&\leqq\sum_{k=[\delta t]}^{[t]-1}(U(t+y-k)-U(t-k-1))(Q(k+1)-Q(k))\\&\leqq Q(t)\sum_{k=[\delta t]}^{[t]-1}(U(t+y-k)-U(t-k-1))(Q(k+1)-Q(k))/Q(k).\end{aligned}$$

For large t and for some constant C independent of δ.

$$\begin{aligned}I_3(t)&=CQ(t)\sum_{k=[\delta t]}^{[t]-1}(U(t+y-k)-U(t-k-1))/k\\&\leqq 3CQ(t)\,t^{-1}\sum_{k=[\delta t]}^{[t]-1}(U(t+y-k)-U(t-k-1))\end{aligned}$$

by assumption (6). We allow C to be perhaps different values upon subsequent appearances for convenience. We approximate the summations by integrals using the monotonicity of $U(t)$:

$$\sum_{k=[\delta t]}^{[t]-1}U(t+y-k)\leqq\int_{t+y-[t]+1}^{t-[\delta t]+y+1}U(u)\,du\leqq\int_{y+1}^{t(1-\delta)+y+2}U(u)\,du$$

and

$$\sum_{k=[\delta t]}^{[t]-1}U(t-k-1)\geqq\int_0^{t-[\delta t]-1}U(u)\,du\geqq\int_0^{t(1-\delta)-1}U(u)\,du.$$

Hence, for large t,

$$\begin{aligned}I_3(t)&\leqq CQ(t)\,t^{-1}\left(\int_{y+1}^{t(1-\delta)+y+2}U(u)\,du-\int_0^{t(1-\delta)-1}U(u)\,du\right)\\&\leqq CQ(t)\,t^{-1}\left(\int_{t(1-\delta)-1}^{t(1-\delta)+y+2}U(u)\,du-\int_0^{y+1}U(u)\,du\right)\\&\leqq CQ(t)\,t^{-1}\int_{t(1-\delta)-1}^{t(1-\delta)+y+2}U(u)\,du\leqq CQ(t)\,t^{-1}\,U(t(1-\delta)+y+2)(y+3).\end{aligned}$$

By the regular variation of $U(t)$ and Theorem B′,

$$I_3(t) \leqq C(1-\delta)^\alpha Q(t)/m(t)$$

for large t. This completes the proof.

Returning now to the proof of Theorem 1, we proceed to establish (7) for $\beta > 0$. Let $B(a, b)$ be the Beta function. By the triangle inequality,

$$\begin{aligned}
&|((U^*Q)(t+y)-(U^*Q)(t))\, m(t)/Q(t) - y c_\alpha \beta B(\beta,\alpha)| \\
&\leqq |I_1(t)\, m(t)/Q(t) - y c_\alpha \beta \int_0^\delta u^{\beta-1}(1-u)^{\alpha-1}\, du| \\
&\quad + |I_2(t)\, m(t)/Q(t)| + |I_3(t)\, m(t)/Q(t)| + |I_4(t)\, m(t)/Q(t)| \\
&\quad + |J(t)\, m(t)/Q(t)| + y c_\alpha \beta |B(\beta,\alpha) - \int_0^\delta u^{\beta-1}(1-u)^{\alpha-1}\, du|.
\end{aligned}$$

Letting $t \to \infty$ and applying the above lemmas yields

$$\begin{aligned}
&\limsup_{t\to\infty} |((U^*Q)(t+y)-(U^*Q)(t))\, m(t)/Q(t) - y c_\alpha \beta B(\beta,\alpha)| \\
&\leqq C(1-\delta)^\alpha + y c_\alpha \beta |B(\beta,\alpha) - \int_0^\delta u^{\beta-1}(1-u)^{\alpha-1}\, du|.
\end{aligned}$$

Since $c_\alpha \beta B(\beta, \alpha) = A(\alpha, \beta)$ and the right side of the above goes to zero as $\delta \to 1^-$, we have proved (7) for $\beta > 0$.

The $\beta = 0$ case requires only the reexamination of integral I_1.

Lemma 4. *If* $\beta = 0$, $\lim_{t\to\infty} I_1(t)\, m(t)/Q(t) = y c_\alpha$.

Proof. It suffices to show that $I_{11}(t) \sim 1$ as $t \to \infty$ because (9) is still valid. Since the measures $Q(t\,du)/Q(t)$ converge weakly to the measure which gives mass 1 to the origin as $t \to \infty$ and $m(t)/m(t(1-u)) = 1$ when $u = 0$, the limit of $I_{11}(t)$ as $t \to \infty$ is indeed 1.

We can now complete the proof of Theorem 1. The triangle inequality gives

$$\begin{aligned}
|((U^*Q)(t+y)-(U^*Q)(t))\, m(t)/Q(t) - y c_\alpha| &\leqq |I_1(t)\, m(t)/Q(t) - y c_\alpha| + |I_2(t)\, m(t)/Q(t)| \\
&\quad + |I_3(t)\, m(t)/Q(t)| + |I_4(t)\, m(t)/Q(t)| + |J(m(t)/Q(t)|.
\end{aligned}$$

Since $A(\alpha, 0) = c_\alpha$, replacing Lemma 2 with Lemma 4 and letting first $t \to \infty$ and then $\delta \to 1^-$ establishes (7) for $\beta = 0$ and we have proved Theorem 1.

Proof of Theorem 2. Let $G_1(t) \equiv \int_0^t u\, G(du)$ and $Q(t) \equiv \int_0^t u\, F(du)$ in what follows. Fix $y > 0$. A simple calculation shows that

$$t(G(t+y)-G(t)) \leqq G_1(t+y) - G_1(t) \leqq (t+y)(G(t+y)-G(t)).$$

Rearranging yields

$$(t+y)^{-1}(G_1(t+y)-G_1(t)) \leqq G(t+y)-G(t) \leqq t^{-1}(G_1(t+y)-G_1(t)).$$

Therefore,

$$(G(t+y)-G(t)) \sim t^{-1}(G_1(t+y)-G_1(t)) \quad \text{as } t \to \infty. \tag{10}$$

Let $b(n) \equiv (n+1)\, a(n+1) - n\, a(n)$ and

$$R(t) \equiv \sum_{n=0}^{\infty} b(n)\, F^n(t).$$

Since $\sum_{k=0}^{n} b(k) = (n+1)\, a(n+1)$ is $(\beta+1)$-varying,

$$R(t) \sim U(t)\, a(U(t))\, C(\alpha, \beta+1) \quad \text{as} \quad t \to \infty \tag{11}$$

by Theorem D.

We give a series of lemmas which enable us to determine the asymptotic behavior of $G_1(t+y) - G_1(t)$ as $t \to \infty$.

Lemma 5. *The truncated mean function* $Q(t) = \int_0^t u F(du)$ *is a nondecreasing* $(1-\alpha)$-*varying function such that* $Q(t) \sim \alpha m(t)$ *as* $t \to \infty$.

Proof. This is a standard result from regular variation.

Lemma 6. *If, for every* $y > 0$, (8) *holds, then, for every* $y > 0$,

$$(Q(t+y) - Q(t))/Q(t) = O(t^{-1}) \quad \textit{as} \quad t \to \infty$$

Proof. By the definition of $Q(t)$,

$$\begin{aligned} 0 < t(Q(t+y) - Q(t))/Q(t) &\leqq t(t+y)(F(t+y) - F(t))/Q(t) \\ &\leqq ((t+y)(1-F(t))/Q(t))\, t\left(1 - \frac{(1-F(t+y))}{(1-F(t))}\right). \end{aligned}$$

Since $Q(t) \sim \alpha m(t)$ and $t(1-F(t))/m(t) \sim 1-\alpha)$ as $t \to \infty$, $(t+y)(1-F(t))/Q(t) \sim (1-\alpha)/\alpha$ as $t \to \infty$. Hence, by (8),

$$\limsup_{t \to \infty} t(Q(t+y) - Q(t))/Q(t) < \infty.$$

Lemma 7. $G_1(t) = (R * U * Q)(t)$.

Proof. The proof involves manipulations of convolutions and is omitted.

Lemma 8. *For every* $y > 0$,

$$(G_1(t+y) - G_1(t)) \sim \alpha y R(t) \quad \textit{as} \quad t \to \infty.$$

Proof. Choose δ such that $1/2 < \delta < 1$. Write

$$\begin{aligned} G_1(t+y) - G_1(t) &= \int_{0^-}^{t} ((U*Q)(t+y-z) - (U*Q))(t-z)\, R(dz) \\ &\quad + \int_{t}^{t+y} (U*Q)(t+y-z)\, R(dz) \\ &= \left(\int_{0^-}^{\delta t} + \int_{\delta t}^{t}\right) ((U*Q)(t+y-z) - (U*Q)(t-z))\, R(dz) \\ &\quad + \int_{t}^{t+y} (U*Q)(t+y-z)\, R(dz) \\ &\equiv J_1(t) + J_2(t) + K(t) \end{aligned}$$

by Lemma 7. We proceed to examine the integrals J_1, J_2, and K. First, it follows from Lemmas 5 and 6 and Theorem 1 that

$$((U^*Q)(t+y)-(U^*Q)(t))\sim\alpha y \quad \text{as} \quad t\to\infty. \tag{12}$$

Choose $\varepsilon>0$. Hence, for large t, $(\alpha y-\varepsilon)\,R(\delta t)\leqq J_1(t)\leqq(\alpha y+\varepsilon)\,R(\delta t)$. Since $R(\delta t)\sim\delta^{\alpha(\beta+1)}R(t)$ as $t\to\infty$ by (11) and ε is arbitrary,

$$\lim_{t\to\infty} J_1(t)/R(t)=\alpha y\,\delta^{\alpha(\beta+1)}.$$

Secondly, since $(U^*Q)(t)$ is bounded on bounded intervals and (12) holds, there exists a constant C such that $J_2(t)\leqq C(R(t)-R(\delta t))$. Therefore,

$$\limsup_{t\to\infty} J_2(t)/R(t)\leqq C(1-\delta^{\alpha(\beta+1)}).$$

Finally, for K we have

$$K(t)\leqq(U^*Q)(y)(R(t+y)-R(t))$$

by the monotonicity of $(U^*Q)(t)$. Therefore, by the regular variation of $R(t)$, $K(t)=o(R(t))$ as $t\to\infty$.

By the triangle inequality,

$$|(G_1(t+y)-G_1(t))/R(t)-\alpha y|\leqq|J_1(t)/R(t)-\alpha y\,\delta^{\alpha(\beta+1)}|+|J_2(t)/R(t)|+|K(t)/R(t)| + \alpha y\,|1-\delta^{\alpha(\beta+1)}|.$$

Combining the above estimates gives

$$\limsup_{t\to\infty}|(G_1(t+y)-G_1(t))/R(t)-\alpha y|\leqq(C+\alpha y)(1-\delta^{\alpha(\beta+1)}).$$

The lemma follows by letting $\delta\to 1^-$.

The proof of Theorem 2 can now be completed. From Lemma 8 and (10) and (11),

$$(G(t+y)-G(t))\sim\alpha y t^{-1}\,U(t)\,a(U(t))\,C(\alpha,\beta+1).$$

Since $D(\alpha,\beta)=\alpha(\Gamma(1+\alpha)\,\Gamma(2-\alpha))^{-1}\,C(\alpha,\beta+1)$, an application of Theorem C completes the proof.

3. Remarks

The following obvious corollary handles the $0<\alpha\leqq 1/2$ cases and shows that if a strong renewal theorem holds for the renewal function, then a strong renewal theorem holds for the generalized renewal function.

Corollary 2. *Theorems 1 and 2 remain true for $0<\alpha\leqq 1/2$ if it is known that for every $y>0$*

$$\lim_{t\to\infty} m(t)(U(t+y)-U(t))=y\,c_\alpha.$$

Assumption (8) of Theorem 2 was made in order to apply Theorem 3 with $Q(t)=\int_0^t uF(du)$ and is not as restrictive as it might appear. If $F(x)$ has a regularly varying density, then (8) holds. Many different slowly varying functions in (3) satisfy (8). For example, if $L(x)$ is ultimately constant or of the form

$$L(t)=(\log_k(t))^\rho,$$

where ρ is real and $\log_k(t)$ is the k-th iterated logarithm, then (8) is satisfied. The assumption does require more than just the first order asymptotic behavior of the tail of the distribution $F(x)$, however. While Lemma 6 can actually be strenghtened to an "if and only if" result, the verification of (8) should be more immediate than the verification of (6) when given the distribution $F(x)$.

The assumption that $na(n)$ be nondecreasing can be made without loss of generality in the finite mean case (see Embrechts, Maejima and Omey [1]), but must be made explicitly in the infinite mean case, as we have done in Theorem 2. The reason for this is that $x(1-F(x))$ need not converge to zero as $x\to\infty$ in the infinite mean case.

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